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Abstract

Krishna and Serrano (1996) show a unique and efficient outcome in a model of multilateral bargaining. We show that the predictability of the model critically depends on the nature of the surplus i.e. whether it is continuous or discrete. We show that the model suffers from multiple equilibria and severe inefficiency when the surplus is discrete, not continuous as assumed in Krishna and Serrano (1996), and players are patient enough.
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1 Introduction

Rubinstein (1982) obtains unique and efficient equilibrium in a model of bilateral bargaining under very reasonable assumptions. Hence, it is natural for studies on multilateral bargaining to model the problem as extension of Rubinstein (1982) game. However, most of these studies have failed to repeat the success of Rubinstein model. For example, Shaked (credited by Sutton (1986)), Cai (2000, 2003) and Roy Chowdhury and Sengupta (2012) analyse multilateral bargaining under

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different protocols[1], yet, they all discover the same result that multilateral bargaining suffers from multiplicity of equilibria and inefficiency.

Krishna and Serrano (1996)'s work, however, stands out. They show unique and efficient equilibrium in a n-player game, in which each player takes turn to offer a distribution of the surplus and others respond. Those who accept, take their share and exit the game. While the rest of the players continue to bargain over the leftover surplus in same fashion. We call this the **"exit"** game.[2]

In this paper, we answer the following question. Does the "exit" game retains its predictability if the surplus is discrete, and not continuous as assumed by Krishna and Serrano (1996)?[3] In real life, value as measured in currency terms is discrete. All the transactions must be an integer multiple of the smallest currency unit. We show that under such realistic constraint, the results of Krishna and Serrano (1996) break down. If players are patient enough, not only any distribution of the surplus can be supported in an equilibrium, any amount of delay including perpetual delay can be sustained as well (see proposition 1, 2 and 3). We get this folk theorem kind of result because we are able to devise punishment strategies to prevent any deviation from the norm. The proposer and the responders are punished for deviating from the specified strategies, hence maintaining the equilibrium.

There is an alternate perspective to look at this paper. While the "exit" game has unique outcome, the "unanimity" game, studied by Shaked, suffers from continuum of equilibria. The only modelling difference between the two games is the following. In the "unanimity" game, the surplus is divided only after all responders unanimously accept a proposal, whereas in the "exit" game, individual players can accept the proposal and exit with their share. This implies that the "exit" feature is responsible for the (significant) improvement in the predictability of the model. Now, Van Damme et al (1990) and Muthoo (1991) show that the Rubinstein (1982) model's predictability vanishes under discrete surplus i.e. it suffers from multiple equilibria. Hence, by modelling two opposite forces, the

[1]The bargaining protocols in these papers have Rubinsteinian features of alternating offers and infinite horizon. At the same time, they have significant differences as well. For example, the bargaining in Cai (2000, 2003) is strictly sequential, but the other two papers allow for simultaneous bargaining as well. In Shaked's model, all players are identical, whereas in the other two models, there is a central player (buyer) who offers in alternate periods.

[2]The labels, "exit" and "unanimity", for the respective games are borrowed from Krishna and Serrano (1996). The label "unanimity" refers to the game studied by Shaked (see introductory paragraph).

[3]Discrete surplus results in multiplicity of equilibria and extreme inefficiency in Rubinstein (1982) game. This justifies our selection of the "exit" model, since other models suffer from those shortcomings even when the surplus is continuous (see introductory paragraph).

"exit" and the discrete surplus, in the basic "unanimity" game, we are able to compare their effect and observe which one is more potent. By demolishing the predictability of the "exit" game, the discrete surplus turns out to be the more forceful one.

In section 2, we present the model, which we analyse in section 3 and finally, we conclude the paper in section 4.

2 Model

There are $n(\geq 3)$ players bargaining to distribute a surplus, which is discrete and consists of $\overline{s}$ units. A unit is not further divisible, hence, there are finite number of ways in which the surplus can be distributed. Let the players be denoted by numbers 1 to n. A distribution vector or agreement vector is a vector of units obtained by the n players. We represent it by $x = (x_1, x_2, ..., x_n)$, where x_i is the number of units obtained by the player i. The set of possible distributions, when the number of players is m and the surplus available is s, is defined below.[4]

$$X(m, s) = \{x \in \mathbb{N}_0^m : \sum_{i=1}^{m} x_i = s\} \tag{1}$$

Let $\{1, 2, ..., n\}$ be an exogenously given list of the n players. In period 1, first player in the list i.e. player 1 proposes a distribution $x \in X(n, \overline{s})$, while other players respond. Responders, who accept the proposal, are paid as specified by the proposal , and leave the game forever. In case one or more responders reject the proposal, the game enters second period in which all those who rejected the proposal and the proposer of previous period, bargain over the leftover surplus in similar fashion. Leftover surplus is the original surplus minus the payments made for all the exits in the previous periods. Proposer in a period $t + 1$ is the player next to the proposer of period t in the list and who is still remaining in the game.[5] The new proposer offers one of the possible distributions of the leftover surplus, as defined in (1). The game continues this way till all but one players exit the game, leaving the remaining player with the leftover surplus.

All players are risk neutral. They all discount at a rate $\delta \in (0, 1)$ per period. It is a complete information game, hence, we use subgame perfect nash equilibrium (SPE) to analyse the same. Henceforth, the term equilibrium implies SPE.

[4]The notation $\mathbb{N}_0$ refers to the set of the natural numbers and zero.

[5]Imagine the list to be circular. Hence, after player n, it is player 1's turn to propose if he is still in the game.

We introduce some notations and definitions to facilitate the analysis that follows. Let N be the set of all the n players. The notation $G(m, s)$ represents a game (or subgame) starting with m players and a surplus of s units. Hence, the game described above is represented by $G(n, \overline{s})$. An equilibrium is **efficient** if it ends the game in the first period. If an equilibrium is not efficient, then it is **inefficient**.

3 Analysis

The subgame $G(2, s)$, where $s > 0$, is the Rubinstein (1982) game with discrete surplus, studied by Van Damme et al (1990) and Muthoo (1991). Following three results are reproduced from Van Damme et al (1990), Muthoo (1991) and Krishna and Serrano (1996) for the purpose of comparative analysis. Additionally, lemma V.1 is used in the proofs later on.

Lemma V. 1 *Consider the subgame $G(2, s)$, where $s > 0$. For $\delta > \frac{s-1}{s}$, every possible split of s, including zero for either of the players, can be sustained in an equilibrium.*[6]

Lemma V. 2 *Consider the subgame $G(2, s)$, where $s > 0$. For $\delta > \frac{s-1}{s}$, any amount of delay including perpetual delay can be sustained in an equilibrium.*[7]

Lemma KS. 1 *Consider $G(n, s)$ and, let $s > 0$ be a <u>continuous</u> surplus. For $\delta \in (0, 1)$, the game has a unique and efficient equilibrium.*[8]

We now analyse our model. Following two lemmas fix the limits to the payoff of the player who proposes in the first period i.e. player 1.

Lemma 1 *For $G(n, \overline{s})$ and $\delta > \frac{\overline{s}-1}{\overline{s}}$, there exists an equilibrium such that the game ends in period 1 with the proposer getting all $\overline{s}$ units.*

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[6]See Van Damme et al (1990, proposition 1) and Muthoo (1991, proposition 1). Although Van Damme et al (1990, proposition 1) show any distribution such that each player gets at least one unit is possible in equilibrium, it can be verified that the strategy profile stated in the proposition constitutes an (weak nash) equilibrium even when either of the player gets zero.

[7]See Van Damme et al (1990, proposition 2).

[8]See Krishna and Serrano (1996, theorem 1).

Proof. Following strategy profile, denoted by $E(1)$, constitutes an equilibrium and results into an outcome in which entire surplus goes to the proposer of the first period.

> $E(1)$: When $m \geq 3$ players are present, following strategies are played. A proposer offers $x \in X(m, s)$, and all the responders accept any offer ≥ 0. When $m = 2$, proposer gets zero (this is supported by lemma V.1).

Clearly, it is optimal for the proposer of period 1 to offer zero to all other players. Given all other responders are accepting zero, a responder accepts zero since rejecting the offers leads to bilateral bargaining in which he gets zero. ■

Lemma 2 *For $G(n, \overline{s})$ and $\delta > \frac{\overline{s}-1}{\overline{s}}$, there exists an equilibrium such that the game ends in period 1 with the proposer getting zero units.*

Proof. Following strategy profile, denoted by $E(0)$, constitutes an equilibrium and results into an outcome in which the proposer gets zero in period 1. Before we state the strategy profile, note that the construction of equilibrium strategy profile in subsequent propositions is very similar to $E(0)$. Hence, understanding $E(0)$ will help in understanding rest of the equilibria.

> $E(0)$: Consider period 1. Player 1 offers a vector $x \in X(n, \overline{s})$ such that $x_1 = 0$. If such a vector is offered, a responder $i \in N \setminus \{1\}$ accepts $\geq x_i$. If player 1, instead, offers $x^{'} \in X(n, \overline{s})$ such that $x_1^{'} > 0$, then a responder $i \neq 2$ accepts $\geq x_i^{'}$, and the responder $i = 2$ (i.e. player 2) rejects $< \overline{s}$. In the subgame starting period 2, following strategies are played. If player 1 offered himself zero in period 1 and only two player are remaining in period 2, then the proposer in period 2 gets zero (this is supported by lemma V.1), otherwise the proposer gets the entire leftover surplus (this is supported by lemma V.1 and lemma 1).

Note that if the player 1 deviates and offers himself $x_1^{'} > 0$, it is profitable for player 2 to reject $0 \leq x_2^{'} < \overline{s}$ and get $x_1^{'} + x_2^{'}$ in the next period since $\delta > \frac{\overline{s}-1}{\overline{s}}$. This results in zero payoff for player 1, rendering the deviation unprofitable. Also, given that player 1 offers himself zero and all other responders accept the proposal, a responder must accept the proposal as well, otherwise he will find himself in a bilateral bargaining in which he is punished by lemma V.1 and obtains zero. ■

Following three propositions prove that any distribution of the surplus is possible and any amount of delay including perpetual delay is possible in equilibrium.

Proposition 1 *Consider $G(n, \overline{s})$ and $\delta > \frac{\overline{s}-1}{\overline{s}}$. For every $x \in X(n, \overline{s})$ there exists an equilibrium such that the distribution x is obtained in period 1.*

Proof. Following strategy profile, denoted by $E(x)$, constitutes an equilibrium and proves the proposition.

> $E(x)$: Consider period 1. Player 1 offers the distribution $x \in X(n, \overline{s})$. A responder $i \in N \setminus \{1\}$ accepts $\geq x_i$. If player 1 deviates and offers a different vector $x^{'} \in X(n, \overline{s})$, then following strategies are played. If $x_1^{'} \leq x_1$, a responder i accepts $\geq x_i^{'}$. However, if $x_1^{'} > x_1$, then a responder $i \neq 2$ accepts $\geq x_i^{'}$. The responder $i = 2$ (i.e. player 2) rejects $< \overline{s}$. In the subgame starting period 2, following strategies are played. If player 1 offered himself $\leq x_1$ in period 1, then the proposer in period 2 gets zero (this is supported by lemma V.1 and lemma 2). If player 1 offered himself $> x_1$ in period 1, then the proposer in period 2 gets entire leftover surplus (this is supported by lemma V.1 and lemma 1).

Can player 1 deviate and offer himself more than the norm x_1 i.e. $x_1^{'} > x_1$? If he does so, it becomes profitable for player 2 to reject $0 \leq x_2^{'} < \overline{s}$ and get $x_1^{'} + x_2^{'}$ in the next period as $\delta > \frac{\overline{s}-1}{\overline{s}}$. This results in zero payoff for player 1. Hence, he does not deviate. ■

Proposition 2 *Consider $G(n, \overline{s})$, $\delta > \frac{\overline{s}-1}{\overline{s}}$ and $t > 0$. For every $x \in X(n, \overline{s})$ there exists an equilibrium such that the distribution x is obtained after a delay of t periods.*

Proof. Following strategy profile, denoted by $E(x, t)$, constitutes an equilibrium and proves the proposition.

> $E(x, t)$: From period 1 to t, following strategies are played. In a subgame with $m < n$ players, the proposer gets entire leftover surplus (this is supported by lemma V.1 and lemma 1). When n players are present, following strategies are played. A proposer offers zero to rest of the players and a responder accepts > 0. If a proposer $i \in N$ deviates and offers a different

vector $x^{'} \in X(n, \overline{s})$, then following strategies are played. If $x_i^{'} = 0$, a responder $j \in N \setminus \{i\}$ accepts $\geq x_j$. However, if $x_i^{'} > 0$, then a responder $j \neq (i+1)$ accepts $\geq x_j^{'}$, and the responder $j = (i+1)$ rejects $< \overline{s}$. At $t+1$, if $m < n$ players are present the proposer gets entire leftover surplus (this is supported by lemma V.1 and lemma 1), otherwise $E(x)$ is played.

Observe that in first t periods and when n players are present, if a proposer i deviates and offers himself $0 < x_i^{'} < \overline{s}$, it is profitable for player $i+1$ to reject $0 \leq x_{i+1}^{'} < \overline{s}$ and get $x_i^{'} + x_{i+1}^{'}$ in the next period since $\delta > \frac{\overline{s}-1}{\overline{s}}$. This results in zero payoff for player i, and hence, rendering the deviation useless. ■

Proposition 3 *For $G(n, \overline{s})$ and $\delta > \frac{\overline{s}-1}{\overline{s}}$, there exists an equilibrium which supports perpetual delay.*

Proof. Consider following strategy profile, denoted by $E(\infty)$, which basically extends the strategies specified for first t periods in $E(x,t)$ to infinite number of periods.

> $E(\infty)$: In a subgame with $m < n$ players, the proposer gets entire leftover surplus (this is supported by lemma V.1 and lemma 1). When n players are present, following strategies are played. A proposer offers zero to rest of the players and a responder accepts > 0. If a proposer $i \in N$ deviates and offers a different vector $x^{'} \in X(n, \overline{s})$, then following strategies are played. If $x_i^{'} = 0$, a responder $j \in N \setminus \{i\}$ accepts $\geq x_j$. However, if $x_i^{'} > 0$, then a responder $j \neq (i+1)$ accepts $\geq x_j^{'}$, and the responder $j = (i+1)$ rejects $< \overline{s}$.

It can be verified that $E(\infty)$ constitutes an equilibrium. ■

Compare above propositions with lemma KS.1. It is clear that the presence of discrete surplus severely deteriorates the predictive ability of the model. Also, observe the similarity in the propositions and the lemma V.1 and V.2. This shows how strong is the effect of presence of discrete surplus. The reason why we get these results is very simple. Lemma V.1 shows that extreme outcomes are possible in a two person subgame i.e. one of the player obtains zero. Using lemma V.1 as a punishment device, we are able to obtain lemma 1 and lemma 2, which itself are extreme results, and hence potential punishment devices. We use these results to punish deviations by proposer and responders from the specified strategy profiles in rest of the results. Hence, forcing the players to follow the specified path along the equilibrium. Such punishment devices do not exist when the surplus is continuous, precisely because of the unique outcome, which is not extreme, in the Rubinstein (1982) type bilateral bargaining.

4 Conclusion

Krishna and Serrano (1996) show that the "exit" feature establishes predictability in the multilateral bargaining, which otherwise suffers from multiplicity of equilibria and inefficiency. We, however, show that the predictability of the "exit" model vanishes if the surplus is discrete and the players are sufficiently patient. Van Damme et al (1990) and Muthoo (1991) show the same effect in bilateral bargaining. Clearly, presence of discrete surplus severely and negatively affects the predictability of a bargaining model. Hence, our work together with Van Damme et al (1990) and Muthoo (1991) raise an important question, which is beyond the scope of this paper. Is there a (reasonable) modelling feature or condition which can counter the effect of the discrete surplus in bargaining models?

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