



Munich Personal RePEc Archive

# **Diversity and Economic Growth in a Model with Progressive Taxation**

Wang, Wei and Suen, Richard M. H.

University of Connecticut, University of Leicester

30 October 2015

Online at <https://mpra.ub.uni-muenchen.de/67569/>  
MPRA Paper No. 67569, posted 01 Nov 2015 20:33 UTC

# Diversity and Economic Growth in a Model with Progressive Taxation

Wei Wang\*

Richard M. H. Suen†

30th October, 2015.

## Abstract

Is a more heterogeneous population conducive or detrimental to capital accumulation and economic growth? This paper addresses this question using a dynamic general equilibrium model with *ex ante* heterogeneous consumers and progressive taxation. We show that the answer depends crucially on the shape of the marginal tax function. If this function is concave, then a more heterogeneous population will have a lower average marginal tax rate and a higher level of capital accumulation. The opposite is true when the marginal tax function is convex. These results are robust in a variety of models with either exogenous or endogenous economic growth.

*Keywords:* Consumer Heterogeneity, Progressive Taxation, Economic Growth.

*JEL classification:* D31, E62.

---

\*Department of Economics, 365 Fairfield Way, Unit 1063, University of Connecticut, Storrs CT 06269-1063. Email: [siegindwang@hotmail.com](mailto:siegindwang@hotmail.com).

†Corresponding Author: Department of Economics, Astley Clarke Building, University of Leicester, Leicester LE1 7RH, United Kingdom. Phone: +44 116 252 2880. Email: [mhs15@le.ac.uk](mailto:mhs15@le.ac.uk).

# 1 Introduction

Is a more heterogeneous population conducive or detrimental to capital accumulation and economic growth? What role does redistributive policy, such as progressive taxes, play in this matter? In this paper, we address these questions using a dynamic general equilibrium model with *ex ante* heterogeneous consumers. Our main focus is on the relation between *ex ante* heterogeneity and long-term economic performance.<sup>1</sup>

The economic effects of diversity have long been a subject of interest among researchers.<sup>2</sup> Several recent studies have provided empirical evidence on the positive effect of ethnic and cultural diversity on productivity and economic growth (e.g., Ottaviano and Peri, 2006; Ager and Brückner, 2013, Alesina *et al.*, 2013; Trax *et al.*, 2015).<sup>3</sup> One common hypothesis is that a more heterogeneous population brings forth a greater variety of skills which are complementary inputs in the production process, and thus enhances overall productivity.<sup>4</sup> In the present study, we explore a different mechanism through which diversity can affect aggregate economic outcomes. Our approach highlights the role of *ex ante* heterogeneity in determining the distribution of marginal tax rates across individuals. In the context of representative agent models, the negative relation between marginal tax rate and capital accumulation is straightforward and well understood: a decrease in marginal tax rate raises the return of savings which in turn promotes capital accumulation.<sup>5</sup> The novelty of this study is to show that in a heterogeneous economy, a change in the characteristics of the underlying population can lower the effective marginal tax rate, even when there is no change in the tax schedule *per se*. To achieve this in a tractable manner, we adopt a similar deterministic framework as Sarte (1997), Li and Sarte (2004), Carroll and Young (2008, 2011) and Angyridis (2015). In this type of model, *ex ante* consumer

---

<sup>1</sup>This study is concerned with consumers' differences in some fixed, predetermined characteristics that are directly related to their choices, namely preferences and labour productivity. These differences can be due to ethnic, cultural, physiological or other factors. We are agnostic about the origin of these differences. Throughout this paper, we will treat the terms "diversity" and "*ex ante* heterogeneity" as synonymous.

<sup>2</sup>For extensive survey of this literature, see Alesina and La Ferrara (2005) and Alesina *et al.* (2013).

<sup>3</sup>The analyses in Ottaviano and Peri (2006), Ager and Brückner (2013) and Trax *et al.* (2015) are based on micro-level data from developed countries, such as Germany and the United States. Alesina *et al.* (2013), on the other hand, conduct cross-country comparisons using aggregate level data from 195 countries. Other cross-country studies, such as Easterly and Levine (1997) and Collier and Gunning (1999), focus on African countries and report a negative relation between ethnic diversity and economic growth.

<sup>4</sup>See, for instance, Ottaviano and Peri (2006, p.12-13) and Alesina and La Ferrara (2005, Section 2).

<sup>5</sup>Empirical evidence on this is scant, however, mainly because of the difficulty in measuring marginal tax rate. For this reason, many studies focus on the relation between average tax rate and economic growth. One exception is Padovano and Galli (2001) which construct country-wide point estimates of effective marginal tax rate for 23 OECD countries over the period 1951-1990 and show that this measure is negatively correlated with economic growth. The question of how the distribution or dispersion of marginal tax rates would affect economic growth, however, remains unexplored.

heterogeneity provides the source of income and wealth inequality.<sup>6</sup> Progressive taxation comes into play by distorting prices and incentives, which in turn influences how *ex ante* heterogeneity translates into *ex post* economic inequality. The present study adds to this line of research in two ways: First, while *ex ante* heterogeneity plays a central role in this type of model, virtually no attention has been paid to understand how a change in the distribution of consumer characteristics would affect the aggregate economy. This gap is filled in this paper. Second, when it comes to modelling progressive taxation, existing studies typically focus on a specific parametric form of the tax schedule, which confines our understanding of the effects of progressive taxes. We depart from this practice and conduct our analysis based on some generic properties of the progressive tax function.

Using this approach, we show that the economic effects of diversity depend crucially on an often overlooked feature of the progressive tax schedule, namely the concavity and convexity of the marginal tax function. If this function is concave, then a more heterogeneous population will have a lower average marginal tax rate and a higher level of capital accumulation. The opposite is true when the marginal tax function is convex.<sup>7</sup> The intuition of this can be seen by considering the following example: Start with a homogeneous economy in which all consumers are *ex ante* identical, receive the same amount of before-tax income and face the same progressive tax schedule. Suppose now a mean-preserving dispersion in consumer characteristics is introduced. Such dispersion will lead to a non-degenerate distribution in before-tax income and marginal tax rate. In particular, the relatively poor consumers in the heterogeneous economy will pay a lower marginal tax rate than in the homogeneous world, and the relatively rich will pay a higher rate. The shape of the marginal tax function matters when it comes to aggregation. If the marginal tax function is concave, then the decrease in marginal tax rate among the poor will outweigh the increase among the rich. As a result, the heterogeneous economy will have a lower average marginal tax rate than the homogeneous economy.<sup>8</sup> Our main results in Section 3 generalise this comparison to any two heterogeneous economies which are otherwise identical except for the degree of *ex ante* heterogeneity. We also generalise these results to a variety of models with either exogenous or endogenous economic growth.

---

<sup>6</sup>This type of model implicitly assumes the existence of perfect consumption insurance so that individuals' choices are not affected by idiosyncratic risks. Keane and Wolpin (1997) and Huggett *et al.* (2011) argue that predetermined differences are more important than idiosyncratic risks in explaining the dispersion in lifetime wealth and lifetime utility.

<sup>7</sup>If a progressive tax function  $\tau(\cdot)$  is thrice differentiable, then the implied marginal tax function is concave (or convex) if and only if the third-order derivative  $\tau'''(\cdot)$  is negative (or positive).

<sup>8</sup>The effects under a convex marginal tax function are similar but in opposite directions.

It is important to note that almost all of the existing quantitative studies on progressive taxation have adopted a specification which implies a concave marginal tax function (see Section 3 for details). But the relation between this and the distribution of marginal tax rates has not been fully explained until now.

The rest of the paper is organised as follows. Section 2 describes the baseline model. Section 3 presents the main results which are based on a comparison between two economies with different degrees of *ex ante* heterogeneity. Section 4 shows that our main results can be readily extended to a variety of growth models. Section 5 concludes.

## 2 The Baseline Model

### 2.1 Consumers

Time is discrete and is denoted by  $t \in \{0, 1, 2, \dots\}$ . The economy under study is inhabited by  $S > 1$  infinitely lived consumers with different time preference and labour productivity.<sup>9</sup> Let  $\beta_i \in (0, 1)$  be the subjective discount factor of the  $i$ th consumer,  $i \in \{1, 2, \dots, S\}$ , and let  $\varepsilon_i > 0$  denote his labour productivity. Both characteristics are predetermined and constant over time.

There is a single commodity in this economy which can be used for consumption and investment. Let  $c_{i,t}$  be the consumption of the  $i$ th consumer at time  $t$ . All consumers have preferences over consumption sequences, which can be represented by

$$\sum_{t=0}^{\infty} \beta_i^t u(c_{i,t}). \quad (1)$$

All consumers have the same utility function  $u(\cdot)$  which has the following properties.<sup>10</sup>

**Assumption A1** The utility function  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  is twice continuously differentiable, strictly increasing, strictly concave and satisfies the Inada condition, i.e.,  $\lim_{c \rightarrow 0} u'(c) = +\infty$ .

In each period, each consumer is endowed with one unit of time which they supply inelastically to work. The labour income of the  $i$ th consumer at time  $t$  is  $w_t \varepsilon_i$ , where  $w_t$  is the wage rate for an effective unit of labour. Consumers can save and borrow through a single risk-free

<sup>9</sup>Time preference heterogeneity has been previously considered in Sarte (1997), Li and Sarte (2001), Carroll and Young (2011), Suen (2014) and Angyridis (2015) among others. The empirical evidence on this type of heterogeneity is reviewed in Frederick *et al.* (2002).

<sup>10</sup>Allowing for heterogeneity in the utility function would not change our main results. See Footnote 15 and Section 4 for details.

asset. Let  $a_{i,t}$  be the  $i$ th consumer's asset holdings at the beginning of time  $t$ . The consumer is in debt if this variable takes a negative value. The interest income (or interest payment) associated with these assets is  $r_t a_{i,t}$ , where  $r_t$  is the interest rate. The sum of these two types of income, denoted by  $y_{i,t} \equiv w_t \varepsilon_i + r_t a_{i,t}$ , is subject to a progressive tax.<sup>11</sup> The tax schedule is represented by a function  $\tau(\cdot)$ , which has the following properties.

**Assumption A2** The tax function  $\tau : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is continuously differentiable, strictly increasing and strictly convex with  $\tau(0) \leq 0$ . It also satisfies  $\tau(y) < y$  and  $0 < \tau'(y) < 1$  for all  $y > 0$ .

The assumption of a convex tax function (or equivalently an increasing marginal tax function) is often referred to as marginal rate progressivity. This, together with  $\tau(0) \leq 0$ , is equivalent to average rate progressivity, i.e., average tax rate  $\tau(y)/y$  is increasing in  $y$ .

Consumer  $i$ 's budget constraint at time  $t$  is given by

$$c_{i,t} + a_{i,t+1} - a_{i,t} = y_{i,t} - \tau(y_{i,t}) + \theta_t, \quad (2)$$

where  $\theta_t$  is a lump-sum net transfer from the government. Taking prices and government policies as given, each consumer's problem is to choose a sequence of consumption and asset holdings so as to maximize his lifetime utility in (1), subject to the sequential budget constraint in (2) and the initial value of assets  $a_0 > 0$ .<sup>12</sup> There is no other restriction on borrowing except the no-Ponzi-scheme condition, which is implied by the transversality condition stated below. The solution of this problem is completely characterised by the sequential budget constraint in (2), the Euler equation for consumption

$$u'(c_{i,t}) = \beta_i u'(c_{i,t+1}) \{1 + [1 - \tau'(y_{i,t+1})] r_{t+1}\}, \quad (3)$$

and the transversality condition

$$\lim_{T \rightarrow \infty} \left\{ \left[ \prod_{t=1}^T (1 + \psi_{i,t}) \right]^{-1} a_{i,T+1} \right\} = 0,$$

---

<sup>11</sup>This setup implicitly assumes that interests paid on loans are tax deductible. This assumption is adopted mainly for analytical convenience. In most countries, interests paid on personal loans are in general not deductible from taxes. In the United States, for instance, taxpayers can claim deductions on interests paid on student loans and residential mortgages but not on other types of loans (such as credit card debts).

<sup>12</sup>The current framework can be easily extended to allow for heterogeneity in initial wealth. But since we focus on steady-state analysis, this type of heterogeneity is irrelevant for our main results.

where  $\psi_{i,t} \equiv [1 - \tau'(y_{i,t})] r_t$  is the after-tax return from assets.

## 2.2 Production

On the supply side of the economy, there is a large number of identical firms. In each period, each firm hires labour and rents physical capital from the competitive factor markets, and produces output using a neoclassical production technology

$$Y_t = F(K_t, N_t),$$

where  $Y_t$  denotes output at time  $t$ ,  $K_t$  and  $N_t$  denote capital input and labour input, respectively. The properties of the production function are summarised as follows:

**Assumption A3** The production function  $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  is twice continuously differentiable, strictly increasing and strictly concave in  $(K_t, N_t)$ . It also exhibits constant returns to scale (CRTS) in the two inputs and satisfies the Inada conditions.<sup>13</sup>

Let  $R_t$  be the rental price of physical capital at time  $t$ . Then the representative firm solves the following problem

$$\max_{K_t, N_t} \{F(K_t, N_t) - w_t N_t - R_t K_t\},$$

and the first-order conditions are

$$R_t = F_K(K_t, N_t), \quad \text{and} \quad w_t = F_N(K_t, N_t).$$

## 2.3 Government

Tax revenues collected by the government are either spent on “unproductive” government spending ( $G_t$ ) or distributed as transfers to the consumers.<sup>14</sup> The government’s budget is balanced in every period, so that

$$\sum_{i=1}^S \tau(y_{i,t}) = G_t + S\theta_t, \quad \text{for all } t \geq 0. \quad (4)$$

---

<sup>13</sup>Define  $f(k) \equiv F(k, 1)$  as the reduced-form production function. Then the Inada conditions can be expressed as  $\lim_{k \rightarrow 0} f'(k) = +\infty$  and  $\lim_{k \rightarrow \infty} f'(k) = 0$ .

<sup>14</sup>Government spending is called “unproductive” because it has no direct effect on consumers’ utility and the production of goods.

## 2.4 Competitive Equilibrium

To define a competitive equilibrium, we first define  $\mathbf{c}_t = (c_{1,t}, c_{2,t}, \dots, c_{S,t})$  and  $\mathbf{a}_t = (a_{1,t}, a_{2,t}, \dots, a_{S,t})$  as the cross-sectional distributions of consumption and assets at time  $t$ . The exogenous policy instruments include a progressive tax function  $\tau(\cdot)$  and a sequence of government spending  $\{G_t\}_{t=0}^\infty$ . Given these policy variables, a competitive equilibrium consists of sequences of distributions  $\{\mathbf{c}_t, \mathbf{a}_t\}_{t=0}^\infty$ , aggregate inputs  $\{K_t, N_t\}_{t=0}^\infty$ , prices  $\{w_t, r_t, R_t\}_{t=0}^\infty$  and government transfers  $\{\theta_t\}_{t=0}^\infty$  such that

- (i) Given prices and government policies,  $\{c_{i,t}, a_{i,t}\}_{t=0}^\infty$  solves consumer  $i$ 's problem.
- (ii) Given prices,  $\{K_t, N_t\}_{t=0}^\infty$  solves the representative firm's problem in every period.
- (iii) The government's budget is balanced in every period.
- (iv) All markets clear in every period, so that  $K_t = \sum_{i=1}^S a_{i,t}$ , and  $N_t = \sum_{i=1}^S \varepsilon_i$ , for all  $t$ .

In the present study, we focus on the stationary equilibria or steady states of this economy. Both  $G_t$  and  $\theta_t$  are time-invariant in a stationary equilibrium. Define  $k_t \equiv K_t/N_t$  as the capital-labour ratio at time  $t$  and let  $k^*$  denote its value in a steady state. In any stationary equilibrium, the prices are given by  $R^* = F_K(k^*, 1)$ ,  $w^* = F_N(k^*, 1)$  and  $r^* = R^* - \delta$ , and the Euler equation can be expressed as

$$1 = \beta_i \{1 + r^* [1 - \tau'(y_i^*)]\} \Rightarrow \tau'(y_i^*) = 1 - \frac{\rho_i}{r^*}, \quad (5)$$

where  $\rho_i = 1/\beta_i - 1$  is the rate of time preference of the  $i$ th consumer.<sup>15</sup> Equation (5) states that in any stationary equilibrium each consumer faces an after-tax asset return that is equal to his rate of time preference.<sup>16</sup> This condition implicitly defines a one-to-one mapping between  $y_i^*$  and  $\rho_i$ , which forms the basis of our analysis.

To start, define  $\phi : (0, 1) \rightarrow \mathbb{R}_+$  as the inverse of the marginal tax function, i.e.,  $\phi[\tau'(y)] = y$  for all  $y \geq 0$ . Since  $\tau'(\cdot)$  is continuous and strictly increasing, its inverse is a single-valued, continuous, strictly increasing function. Suppose for the moment that  $\tau(\cdot)$  is also thrice differentiable,

<sup>15</sup>Since individual consumption (and hence the marginal utility of consumption) is constant over time in any steady state, equation (5) remains valid even if we allow for heterogeneity in the utility function, i.e.,  $u_i(\cdot) \neq u_j(\cdot)$  for some  $i, j \in \{1, 2, \dots, S\}$  and  $i \neq j$ .

<sup>16</sup>As shown in Sarte (1997), this condition implies a non-degenerate distribution of wealth in the steady state.



then the first and second-order derivatives of  $\phi(\cdot)$  exist and are given by

$$\begin{aligned}\phi'[\tau'(y)] &= [\tau''(y)]^{-1} > 0, \\ \phi''[\tau'(y)] &= -\frac{\phi'[\tau'(y)]\tau'''(y)}{[\tau''(y)]^2} \gtrless 0 \text{ iff } \tau'''(y) \lesseqgtr 0.\end{aligned}$$

The last equation reveals a close connection between the curvature of  $\phi(\cdot)$  and that of  $\tau'(\cdot)$ . Specifically,  $\phi(\cdot)$  is a convex (or concave) function if and only if  $\tau'(\cdot)$  is concave (or convex). The same result can be obtained without using  $\tau''(\cdot)$  and  $\tau'''(\cdot)$ . This is formally stated in Lemma 1. The proofs of all lemmas and propositions can be found in the Appendix.

**Lemma 1** *Suppose Assumption A2 is satisfied. Then  $\phi(\cdot)$  is a convex (or concave) function if and only if  $\tau'(\cdot)$  is concave (or convex).*

Using (5) and the definition of  $y_i^*$ , we can get

$$y_i^* = \phi\left(1 - \frac{\rho_i}{r^*}\right) = w^* \varepsilon_i + r^* a_i^*. \quad (6)$$

Equation (6) implies that, cross-sectionally, the level of before-tax income is inversely related to the rate of time preference. Summing this equation across all consumers gives

$$\sum_{i=1}^S \phi\left(1 - \frac{\rho_i}{r^*}\right) = \bar{\varepsilon} S [F(k^*, 1) - \delta k^*], \quad (7)$$

where  $\bar{\varepsilon}$  is the average labour productivity among the consumers, i.e.,  $\bar{\varepsilon} \equiv \sum_{i=1}^S \varepsilon_i / S$ . The right side of the above equation follows from the CRTS property of the production function. Equation (7) is essentially an accounting identity which states that the sum of individual income equals aggregate output (net of depreciation costs). This equation has two important implications: First, the steady-state value  $k^*$  is independent of the heterogeneity in labour productivity. More specifically,  $k^*$  only depends on the mean of the labour productivity distribution but not other moments or characteristics. Since  $w^*$ ,  $r^*$  and  $F(k^*, 1)$  are uniquely determined by  $k^*$ , it follows that all these variables are also independent of the heterogeneity in labour productivity. Thus, in subsequent analysis, we will focus on the effects of time preference heterogeneity alone. Second, equation (7) provides a direct linkage between the distribution of consumer characteristics  $\boldsymbol{\rho} \equiv (\rho_1, \dots, \rho_S)$  and the aggregate variable  $k^*$ , without explicitly relating to other variables at the

individual level (such as  $y_i^*$ ). This allows us to adopt the following approach in Section 3: First, examine how a change in  $\boldsymbol{\rho}$  would affect  $k^*$ , then analyse their joint effects on the distribution of individual income.

Before proceeding further, we first examine the existence and uniqueness of steady state in this economy. To formulate our next result, some additional notations are necessary. Define  $\rho_{\max} \equiv \max\{\rho_1, \rho_2, \dots, \rho_S\}$ . Then by the strict concavity of  $f(k) \equiv F(k, 1)$  and the Inada conditions of the production function, there exists a unique value  $k_{\max} > 0$  such that  $f'(k_{\max}) = \delta + \rho_{\max}$ . Lemma 2 provides a necessary and sufficient condition under which a unique steady state exists.

**Lemma 2** *Suppose Assumptions A1-A3 are satisfied. Then a unique steady state exists if and only if*

$$\bar{\epsilon}S [f(k_{\max}) - \delta k_{\max}] > \sum_{i=1}^S \phi \left( 1 - \frac{\rho_i}{\rho_{\max}} \right). \quad (8)$$

### 3 Main Results

The main results of this paper focus on the effects of consumer heterogeneity on the aggregate variable  $k^*$ , and the role of progressive taxation in shaping these effects. These results are based on a comparison between two economies with different degrees of consumer heterogeneity. Specifically, we consider two economies which have the same size of population  $S$ , average labour productivity  $\bar{\epsilon}$ , production technology  $F(\cdot)$ , and tax function  $\tau(\cdot)$ . The only difference between them is the cross-sectional distribution of time preference, denoted by  $\boldsymbol{\rho} \equiv (\rho_1, \dots, \rho_S)$  and  $\tilde{\boldsymbol{\rho}} \equiv (\tilde{\rho}_1, \dots, \tilde{\rho}_S)$ . Without loss of generality, we assume that the elements in these distributions are ranked in ascending order, i.e.,  $0 < \rho_1 \leq \rho_2 \leq \dots \leq \rho_S$  and  $0 < \tilde{\rho}_1 \leq \tilde{\rho}_2 \leq \dots \leq \tilde{\rho}_S$ . These distributions are also required to satisfy the following assumption.

**Assumption A4** (i) The two distributions share the same mean, i.e.,  $\sum_{i=1}^S \rho_i = \sum_{i=1}^S \tilde{\rho}_i$ . (ii) Both  $\boldsymbol{\rho}$  and  $\tilde{\boldsymbol{\rho}}$  satisfy the condition in (8).

The second part of Assumption A4 ensures that a unique steady state exists in both economies. Our first task here is to address the following question: Suppose one economy has a more heterogeneous population than the other. Then which economy will have a higher level of capital accumulation in the steady state? The answer to this question is given in Proposition 3. To compare the extent of *ex ante* heterogeneity, we adopt the standard Lorenz dominance criterion.

Specifically,  $\boldsymbol{\rho}$  is said to be more heterogeneous than  $\tilde{\boldsymbol{\rho}}$  if

$$\frac{\sum_{i=1}^n \rho_i}{\sum_{i=1}^S \rho_i} \leq \frac{\sum_{i=1}^n \tilde{\rho}_i}{\sum_{i=1}^S \tilde{\rho}_i}, \quad \text{for } n = 1, 2, \dots, S. \quad (9)$$

The rationale for using this criterion is as follows: For any  $\boldsymbol{\rho}$  and  $\tilde{\boldsymbol{\rho}}$  with the same mean, condition (9) is equivalent to saying that  $\boldsymbol{\rho}$  is a mean-preserving spread of  $\tilde{\boldsymbol{\rho}}$ .<sup>17</sup> Thus, in the present context, a more *ex ante* heterogeneous economy is one with a more dispersed distribution of time preference. Let  $k^*$  and  $\tilde{k}^*$  be the unique solution of (7) under  $\boldsymbol{\rho}$  and  $\tilde{\boldsymbol{\rho}}$ , respectively. Then a larger extent of *ex ante* heterogeneity is said to be beneficial (or harmful) to long-term capital accumulation if  $k^* \geq \tilde{k}^*$  (or  $k^* \leq \tilde{k}^*$ ).

**Proposition 3** *Suppose Assumptions A1-A4 are satisfied. Then a larger extent of ex ante heterogeneity is beneficial (or harmful) to long-term capital accumulation if the marginal tax function is concave (or convex).*

One interesting special case of this result is when  $\tilde{\boldsymbol{\rho}}$  is a degenerate distribution at  $\bar{\rho} \equiv \sum_{i=1}^S \rho_i / S$ . In this case, we are comparing a heterogeneous-agent (HA) economy to an identical-agent (IA) economy in which all consumers have the same time preference. Proposition 3 then implies that the HA economy will have a higher (or lower) level of long-run capital accumulation than the IA economy if the marginal tax function is concave (or convex). This result is summarised in Corollary 4.

**Corollary 4** *Suppose Assumptions A1-A4 are satisfied. Then the HA economy will have a higher (or lower) level of long-run capital accumulation than the IA economy if the marginal tax function is concave (or convex).*

The intuition of Proposition 3 and its corollary can be obtained by comparing the distribution of marginal tax rates in the two economies. For ease of explanation, we will focus here on the comparison between an HA economy and its IA counterpart. The general results are presented in Proposition 5. In the IA economy, all consumers have the same before-tax income ( $\bar{y}^*$ ) and face the same marginal tax rate  $\tau'(\bar{y}^*)$ . Introducing a mean-preserving spread in the rate of time preference will create dispersion in both before-tax income and marginal tax rate. Specifically, it will lower the marginal tax rate for those with income less than  $\bar{y}^*$  and raise the marginal tax rate for the others. If the marginal tax function is concave, then the decrease in marginal tax

<sup>17</sup>See Shaked and Shanthikumar (2007, p.116-119) for more details.

rates among the relatively poor will dominate the increase among the relatively rich. To see this more precisely, first recall that  $y_i^*$  is negatively related to  $\rho_i$ . Thus,  $y_1^*$  represents the highest level of income in the HA steady state, and  $y_S^*$  is the lowest. For any  $n \in \{1, 2, \dots, S\}$ , define two groups of consumers according to  $\mathcal{A}_n = \{1, 2, \dots, n\}$  and  $\mathcal{B}_n = \{n, \dots, S\}$ . In words,  $\mathcal{A}_n$  and  $\mathcal{B}_n$  represent the richest and the poorest  $n$  consumers in the HA steady state, respectively. If the marginal tax function is concave, then those in  $\mathcal{B}_n$  will (on average) face a lower marginal tax rate in the HA economy than in the IA economy, i.e.,

$$\frac{1}{S-n+1} \sum_{i=n}^S \tau'(y_i^*) \leq \tau'(\bar{y}^*), \quad \text{for all } n \in \{1, 2, \dots, S\}.$$

Since this is also true for  $\mathcal{B}_1$  (i.e., the entire population), the economy-wide average marginal tax rate in the HA economy is lower than its IA counterpart. Thus, the consumers in the HA economy will in general have a larger incentive to save, which then lead to a higher level of capital accumulation in the steady state.

If the marginal tax function is convex, then the increase in marginal tax rates among the relatively rich will offset the decrease among the relatively poor. As a result, those in  $\mathcal{A}_n$  will (on average) face a higher marginal tax rate in the HA economy than in the IA economy, i.e.,

$$\frac{1}{n} \sum_{i=1}^n \tau'(y_i^*) \geq \tau'(\bar{y}^*), \quad \text{for all } n \in \{1, 2, \dots, S\}.$$

The consumers in the HA economy now have a lower incentive to save, which in turn lead to a lower level of capital accumulation. Proposition 5 generalises the above comparison to any  $\tilde{\rho}$  that satisfies Assumption A4.

**Proposition 5** *Suppose Assumptions A1-A4 are satisfied. Suppose  $\rho$  is more heterogeneous than  $\tilde{\rho}$ .*

(i) *If the marginal tax function is concave, then  $\sum_{i=n}^S \tau'(y_i^*) \leq \sum_{i=n}^S \tau'(\tilde{y}_i^*)$ , for all  $n$ .*

(ii) *If the marginal tax function is convex, then  $\sum_{i=1}^n \tau'(y_i^*) \geq \sum_{i=1}^n \tau'(\tilde{y}_i^*)$ , for all  $n$ .*

Finally, we turn to the distribution of before-tax income in the two economies. As equation (6) makes clear, any difference between  $y_i^*$  and  $\tilde{y}_i^*$  can be attributed to two factors: (i) a direct effect due to the difference between  $\rho_i$  and  $\tilde{\rho}_i$ , and (ii) an indirect, general equilibrium effect due to the difference between  $r^*$  and  $\tilde{r}^*$ . These two forces, however, tend to be counteractive

regardless of whether  $\tau'(\cdot)$  is concave or convex. Thus, in general, it is difficult to compare the distribution of before-tax income in these two economies. Proposition 6 summarises the exact nature of these two forces for the case when  $\tau'(\cdot)$  is concave. Similar results (but in opposite directions) can be obtained when  $\tau'(\cdot)$  is convex.

**Proposition 6** *Suppose Assumptions A1-A4 are satisfied. Suppose  $\boldsymbol{\rho}$  is more heterogeneous than  $\tilde{\boldsymbol{\rho}}$ . If the marginal tax function is concave, then the following results are true:*

(i) *For any  $r \geq \max\{\rho_S, \tilde{\rho}_S\}$ , and for any  $n \in \{1, 2, \dots, S\}$ ,*

$$\sum_{i=1}^n \phi\left(1 - \frac{\tilde{\rho}_i}{r}\right) \leq \sum_{i=1}^n \phi\left(1 - \frac{\rho_i}{r}\right). \quad (10)$$

(ii) *For any  $n \in \{1, 2, \dots, S\}$ ,*

$$\sum_{i=1}^n \phi\left(1 - \frac{\rho_i}{r^*}\right) \leq \sum_{i=1}^n \phi\left(1 - \frac{\rho_i}{\tilde{r}^*}\right) \quad \text{and} \quad \sum_{i=1}^n \phi\left(1 - \frac{\tilde{\rho}_i}{r^*}\right) \leq \sum_{i=1}^n \phi\left(1 - \frac{\tilde{\rho}_i}{\tilde{r}^*}\right). \quad (11)$$

Holding  $r$  constant, a larger extent of *ex ante* heterogeneity is associated with a higher average before-tax income among the relatively rich when  $\tau'(\cdot)$  is concave.<sup>18</sup> On the other hand, a larger extent of *ex ante* heterogeneity will also lead to a lower interest rate (i.e.,  $r^* \leq \tilde{r}^*$ ) when  $\tau'(\cdot)$  is concave as implied by Proposition 3. Holding other things constant, this will lower the average before-tax income among the top earners in both economies. Note that (10) and (11) does not imply any ranking between  $\sum_{i=1}^n \phi\left(1 - \frac{\rho_i}{r^*}\right) \equiv \sum_{i=1}^n y_i^*$  and  $\sum_{i=1}^n \phi\left(1 - \frac{\tilde{\rho}_i}{\tilde{r}^*}\right) \equiv \sum_{i=1}^n \tilde{y}_i^*$  for any  $n \in \{1, 2, \dots, S\}$ , thus it is difficult to make any general statements about the income distribution in these two economies. One way to resolve this is to focus on those  $\boldsymbol{\rho}$  and  $\tilde{\boldsymbol{\rho}}$  that satisfy a “monotone ratio” property.<sup>19</sup>

**Definition** *The distributions  $\boldsymbol{\rho} \equiv (\rho_1, \dots, \rho_S)$  and  $\tilde{\boldsymbol{\rho}} \equiv (\tilde{\rho}_1, \dots, \tilde{\rho}_S)$  satisfy the monotone ratio property if  $\{\tilde{\rho}_i/\rho_i\}_{i=1}^S$  is a decreasing sequence.*

An example of this property is as follows: Suppose  $\rho_1 = \varphi\tilde{\rho}_1$ , for some  $\varphi \in (0, 1)$ ;  $\rho_S = (1 + \zeta)\tilde{\rho}_S$ , for some  $\zeta > 0$ ; and  $\rho_i = \tilde{\rho}_i$  for all other  $i$ , then  $\boldsymbol{\rho}$  and  $\tilde{\boldsymbol{\rho}}$  satisfy the monotone ratio property. If  $\boldsymbol{\rho}$  and  $\tilde{\boldsymbol{\rho}}$  have the same mean and satisfy the monotone ratio property, then  $\boldsymbol{\rho}$  is

<sup>18</sup>The condition  $r \geq \max\{\rho_S, \tilde{\rho}_S\}$  in part (i) ensures that  $r \geq \rho_i$  and  $r \geq \tilde{\rho}_i$  for all  $i$ . These conditions are needed because  $\phi(\cdot)$  is only defined on  $(0, 1)$ .

<sup>19</sup>This is not to be confused with the monotone likelihood ratio property which is often used to compare the density function of two random variables. We use the term “monotone ratio property” because of a lack of better alternatives.

a mean-preserving spread of  $\tilde{\rho}$ .<sup>20</sup> Thus, Propositions 3 and 5 remain valid under this property. In addition, we can now directly compare the values of  $y_i^*$  and  $\tilde{y}_i^*$  for all  $i$ . The results are summarised in Proposition 7.

**Proposition 7** *Suppose Assumptions A1-A4 are satisfied. Suppose  $\rho$  and  $\tilde{\rho}$  satisfy the monotone ratio property.*

(i) *If the marginal tax function is concave, then there exists an integer  $q \in \{1, \dots, S - 1\}$  such that  $y_i^* \leq \tilde{y}_i^*$  for  $i = 1, \dots, q$ , and  $y_i^* \geq \tilde{y}_i^*$  for  $i = q + 1, \dots, S$ .*

(ii) *If the marginal tax function is convex, then there exists an integer  $n \in \{2, \dots, S\}$  such that  $y_i^* \geq \tilde{y}_i^*$  for  $i = 1, \dots, n$ , and  $y_i^* \leq \tilde{y}_i^*$  for  $i = n + 1, \dots, S$ .*

Taken together, Propositions 3, 5 and 7 have the following implications: If the marginal tax function is concave, then a larger extent of *ex ante* heterogeneity is associated with a lower before-tax income among the relatively rich consumers. This lowers the marginal tax rate faced by these consumers, raises their incentive to save and in turn promotes capital accumulation at the aggregate level. Another implication is that a smaller share of aggregate income is owned by the top earners in a more *ex ante* heterogeneous world. Thus, if we use this as our measure of income inequality, then income inequality and *ex ante* heterogeneity are negatively related when  $\tau'(\cdot)$  is concave. The opposite is true when  $\tau'(\cdot)$  is convex. In this case, a larger extent of *ex ante* heterogeneity will raise the before-tax income among the relatively rich but lower capital accumulation at the aggregate level. As a result, a larger share of aggregate income is owned by the top earners in a more heterogeneous world. Hence, income inequality and *ex ante* heterogeneity are positively related when  $\tau'(\cdot)$  is convex.

Note that under the monotone ratio condition, income inequality (as measured by the share of aggregate income owned by the top earners) is always negatively related to aggregate capital accumulation, regardless of whether  $\tau'(\cdot)$  is concave or convex.

## ***Discussions***

We now relate our main results to the previous studies on progressive taxation. In the existing literature, two specific forms of progressive tax function are commonly used. The first one is the isoelastic function used by Guo and Lansing (1998), Li and Sarte (2004) and Angyridis (2015).

<sup>20</sup>In the above example,  $\rho$  and  $\tilde{\rho}$  have the same mean if  $(1 - \varphi)\tilde{\rho}_1 = \zeta\tilde{\rho}_S$ .

This type of tax function can be written as

$$\tau(y) = \zeta y^{1+\chi},$$

with  $\zeta > 0$  and  $\chi > 0$ . One distinctive feature of this function is that the ratio between marginal tax rate  $\tau'(y)$  and average tax rate  $\tau(y)/y$  is always equal to the constant  $(1 + \chi)$ . Thus, the parameter  $\chi$  is often interpreted as a measure of tax progressivity. Under this specification, the marginal tax function is given by

$$\tau'(y) = \zeta (1 + \chi) y^\chi,$$

which is concave when  $\chi \leq 1$  and convex when  $\chi \geq 1$ . Using tax returns data in the United States, Li and Sarte (2004) estimate that the value of  $\chi$  in 1985 was 0.88 and the value in 1991 was 0.75, both imply a strictly concave marginal tax function.

Another commonly used tax function is the one proposed and estimated by Gouveia and Strauss (1994),

$$\tau(y) = a_0 \left[ y - (y^{-a_1} + a_2)^{-\frac{1}{a_1}} \right].$$

This functional form was adopted by Sarte (1997), Conesa and Krueger (2006), Erosa and Koreshkova (2007), Carroll and Young (2011) among others. The first, second and third-order derivatives of this tax function are given by

$$\begin{aligned} \tau'(y) &= a_0 \left[ 1 - (1 + a_2 y^{a_1})^{-\left(1 + \frac{1}{a_1}\right)} \right], \\ \tau''(y) &= a_0 a_2 (1 + a_1) (1 + a_2 y^{a_1})^{-\left(2 + \frac{1}{a_1}\right)} y^{a_1 - 1}, \\ \tau'''(y) &= \frac{\tau''(y)}{y} \left[ a_1 - 1 - (2a_1 + 1) \left( \frac{a_2 y^{a_1}}{1 + a_2 y^{a_1}} \right) \right]. \end{aligned} \quad (12)$$

In all existing applications, the parameters  $a_0$ ,  $a_1$  and  $a_2$  are taken to be strictly positive so as to ensure  $\tau''(y) > 0$ . Gouveia and Strauss (1994) report an estimate of 0.768 for  $a_1$  based on U.S. data. Similar values are also used in Sarte (1997) and Conesa and Krueger (2006). From (12), it is obvious that  $0 < a_1 \leq 1$  implies  $\tau'''(\cdot) < 0$ . In their quantitative analysis, Carroll and Young (2011) have also considered counterfactual experiments in which  $a_1 > 1$ . In this case, there exists a unique threshold value of income below which  $\tau'''(y) > 0$  and above which  $\tau'''(y) < 0$ .

In sum, under the conventional specification of progressive tax function, our baseline model will predict a positive relation between *ex ante* heterogeneity and long-run capital accumulation.

## 4 Extensions

### 4.1 Exogenous Growth

In this section we show that the main results in Section 3 are robust to the introduction of exogenous productivity growth. To achieve this, we need to make three changes to the baseline economy. First, in order to be consistent with balanced growth, the utility function is assumed to take the CRRA form, i.e.,

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma},$$

where  $\sigma > 0$  is the inverse of the intertemporal elasticity of substitution (IES). Second, the production technology now includes a labour-augmenting technological factor, which serves as the engine of growth. The production function is rewritten as

$$Y_t = F(K_t, X_t N_t),$$

where  $X_t = \gamma^t$  is the technological factor and  $\gamma > 1$  is the constant growth factor. The production function  $F(\cdot)$  is assumed to have the same properties as in Assumption A3. Finally, we need to ensure that all consumers face a constant marginal tax rate along any balanced growth path. To this end, we assume that the progressive tax function, now denoted by  $T_t(y_{i,t})$ , is changing over time and the marginal tax rates can be expressed as

$$T'_t(y_{i,t}) = \tau' \left( \frac{y_{i,t}}{\gamma^t} \right), \quad \text{for all } t \geq 0.$$

As before, the function  $\tau'(\cdot) : \mathbb{R}_+ \rightarrow (0, 1)$  is strictly increasing.

Define the transformed variables:  $k_t \equiv K_t / (X_t N_t)$  and  $\hat{y}_{i,t} \equiv y_{i,t} / \gamma^t$ . The Euler equation for consumption is now given by

$$\left( \frac{c_{i,t+1}}{c_{i,t}} \right)^\sigma = \beta_i \{ 1 + r_{t+1} [1 - \tau'(\hat{y}_{i,t+1})] \}, \quad (13)$$

where  $r_{t+1} = F_K(k_{t+1}, 1) - \delta$ . A balanced-growth equilibrium is a competitive equilibrium in which  $r_{t+1}$  remains constant over time while all other variables, such as  $K_t$ ,  $c_{i,t}$  and  $y_{i,t}$ , grow by



the same factor  $\gamma$  in every period. Thus, in any balanced-growth equilibrium, we have  $k_t = k^*$ ,  $c_{i,t+1} = \gamma c_{i,t}$  and  $\widehat{y}_{i,t} = \widehat{y}_i^*$  for all  $t$ . Substituting these conditions into (13) gives

$$\gamma^\sigma = \beta_i \{1 + r^* [1 - \tau'(\widehat{y}_i^*)]\} \Rightarrow \tau'(\widehat{y}_i^*) = 1 - \frac{1}{r^*} [\gamma^\sigma (1 + \rho_i) - 1]. \quad (14)$$

Following the same steps as in Section 2.4, we can obtain

$$\sum_{i=1}^S \phi \left( 1 - \frac{1}{r^*} [\gamma^\sigma (1 + \rho_i) - 1] \right) = \bar{\varepsilon} S [F(k^*, 1) - \delta k^*], \quad (15)$$

where  $\phi(\cdot)$  is again the inverse of the marginal tax function. Equation (7) is a special case of (15) with  $\gamma = 1$ , but the main difference between the two is that (15) also depends on the consumers' IES. This opens the door for heterogeneity in IES to affect aggregate capital accumulation. We will defer the discussion of this issue until the end of this section.

Using the same line of argument as in Lemma 2, we can show that a unique balanced growth equilibrium exists if

$$\bar{\varepsilon} S [F(k_{\max}, 1) - \delta k_{\max}] > \sum_{i=1}^S \phi \left( 1 - \frac{\gamma^\sigma (1 + \rho_i) - 1}{\gamma^\sigma (1 + \rho_{\max}) - 1} \right),$$

where  $k_{\max} > 0$  is the unique value that solves  $F_K(k_{\max}, 1) - \delta = \gamma^\sigma (1 + \rho_{\max}) - 1$ .

The main results of this section are summarised in Proposition 8, which generalise the results in Proposition 3 to this environment.<sup>21</sup>

**Proposition 8** *In the model with exogenous productivity growth, a larger extent of ex ante heterogeneity is beneficial (or harmful) to long-term capital accumulation if the marginal tax function is concave (or convex).*

Suppose now the consumers in this economy differ in both their rate of time preference and intertemporal elasticity of substitution. Let  $(\rho_i, \sigma_i)$  be the characteristics of the  $i$ th consumer. The first thing to note is that heterogeneity in IES is consistent with balanced growth in the presence of progressive taxation. In particular, the consumption of all individuals will again grow by the same factor  $\gamma$  in a balanced-growth equilibrium. Equations (14) and (15) are now modified to become

$$\tau'(\widehat{y}_i^*) = 1 - \frac{1}{r^*} [\gamma^{\sigma_i} (1 + \rho_i) - 1], \quad \text{for all } i,$$

---

<sup>21</sup>The proof of Proposition 8 is essentially identical to that of Proposition 3, hence it is omitted.

$$\sum_{i=1}^S \phi \left( 1 - \frac{1}{r^*} [\gamma^{\sigma_i} (1 + \rho_i) - 1] \right) = \bar{\varepsilon} S [F(k^*, 1) - \delta k^*].$$

It is straightforward to show that this economy with two sources of consumer heterogeneity is observationally equivalent to an economy with time preference heterogeneity alone. To see this, pick any  $\bar{\sigma} > 0$  and define  $\hat{\rho}_i \equiv \gamma^{\sigma_i - \bar{\sigma}} (1 + \rho_i) - 1$  for all  $i$ . Then the above equations can be rewritten as

$$\begin{aligned} \tau'(\hat{y}_i^*) &= 1 - \frac{1}{r^*} [\gamma^{\bar{\sigma}} (1 + \hat{\rho}_i) - 1], \quad \text{for all } i, \\ \sum_{i=1}^S \phi \left( 1 - \frac{1}{r^*} [\gamma^{\bar{\sigma}} (1 + \hat{\rho}_i) - 1] \right) &= \bar{\varepsilon} S [F(k^*, 1) - \delta k^*], \end{aligned}$$

which characterise the equilibrium of an economy in which all consumers share the same IES ( $1/\bar{\sigma}$ ) but have different time preference as captured by  $\hat{\rho} = (\hat{\rho}_1, \dots, \hat{\rho}_S)$ . The results in Proposition 8 now correspond to a change in this distribution.

## 4.2 Endogenous Growth

The main result in Section 3 can also be extended to an environment with endogenous growth. We will illustrate this using the two-sector model in Li and Sarte (2004, Section II). There are two types of commodities in this economy: a consumption good ( $C_t$ ) and an investment good ( $I_t$ ). As in Li and Sarte (2004), the consumption good is produced by a Cobb-Douglas production function

$$C_t = BK_{c,t}^\alpha N_{c,t}^{1-\alpha}, \quad \text{with } B > 0 \text{ and } \alpha \in (0, 1), \quad (16)$$

where  $K_{c,t}$  and  $N_{c,t}$  denote capital input and labour input, respectively. Investment good is produced by a linear technology that uses only physical capital as input, so that

$$I_t = AK_{I,t}, \quad \text{with } A > 0,$$

where  $K_{I,t}$  denote capital input in the investment-good sector. Both goods markets and factor markets are perfectly competitive. Let  $R_t$  be the rental price of physical capital and  $w_t$  be the market wage rate. Then the first-order conditions from the firms' problem are given by

$$R_t = A = \alpha q_t B K_{c,t}^{\alpha-1} N_{c,t}^{1-\alpha} \quad \text{and} \quad w_t = (1 - \alpha) q_t B K_{c,t}^\alpha N_{c,t}^{-\alpha},$$

where  $q_t$  is the price of consumption good relative to investment good at time  $t$ .

The consumers in this economy solve the following problem

$$\max_{\{c_{i,t}, a_{i,t+1}\}_{t=0}^{\infty}} \left[ \sum_{t=0}^{\infty} \beta_i^t \left( \frac{c_{i,t}^{1-\sigma}}{1-\sigma} \right) \right], \quad \text{with } \sigma > 0,$$

subject to the sequential budget constraint

$$q_t c_{i,t} + a_{i,t+1} - a_{i,t} = y_{i,t} - T_t(y_{i,t}) + \theta_t,$$

where  $T_t(\cdot)$  is the tax function at time  $t$ . To ensure that the marginal tax rate is constant along any balanced growth path, we impose the following assumption on the marginal tax function

$$T'_t(y_{i,t}) = \tau' \left( \frac{y_{i,t}}{Y_t} \right), \quad \text{for all } t \geq 0,$$

where  $Y_t = \sum_{i=1}^S y_{i,t}$  denotes aggregate income and  $\tau'(\cdot) : \mathbb{R}_+ \rightarrow (0, 1)$  is a strictly increasing function. The Euler equation for consumption is given by

$$\frac{q_{t+1}}{q_t} \left( \frac{c_{i,t+1}}{c_{i,t}} \right)^{\sigma} = \beta_i \left\{ 1 + r_{t+1} \left[ 1 - \tau' \left( \frac{y_{i,t}}{Y_t} \right) \right] \right\},$$

for all  $i$  and for all  $t \geq 0$ .

In equilibrium, the markets for physical capital and labour are cleared in every period so that

$$K_{c,t} + K_{I,t} = \sum_{i=1}^S a_{i,t} \quad \text{and} \quad N_{c,t} = \sum_{i=1}^S \varepsilon_i.$$

Any balanced-growth equilibrium will have the following properties: First, the net rate of return from asset holdings is given by  $r^* = A - \delta > 0$ . Second, the variables  $\{K_{c,t}, K_{I,t}, Y_t\}$  will grow by the same factor in every period and the common growth factor  $\gamma^*$  is endogenously determined. Finally,  $C_t$  and  $q_t$  will grow by the factor  $(\gamma^*)^\alpha$  and  $(\gamma^*)^{1-\alpha}$  in every period. Substituting these into the Euler equation gives

$$(\gamma^*)^{\tilde{\sigma}} = \beta_i \left\{ 1 + (A - \delta) \left[ 1 - \tau'(\xi_i^*) \right] \right\}, \quad \text{for all } i, \quad (17)$$

where  $\tilde{\sigma} \equiv 1 - \alpha(1 - \sigma)$  and  $\xi_i^* \equiv y_{i,t}/Y_t$ .<sup>22</sup> Using this, we can obtain

$$\sum_{i=1}^S \xi_i^* = \sum_{i=1}^S \phi \left( 1 - \frac{1}{A - \delta} \left[ (\gamma^*)^{\tilde{\sigma}} (1 + \rho_i) - 1 \right] \right) = 1, \quad (18)$$

which is the counterpart of (7). To ensure the existence of a unique solution for (18), it is necessary to impose some restrictions on the parameter values. Specifically, we assume  $\tilde{\sigma} \equiv 1 - \alpha(1 - \sigma) > 0$ , which is satisfied when  $\sigma \geq 1$ , and  $(1 + A - \delta)(1 + \rho_{\min}) > 1 + \rho_{\max}$ , where  $\rho_{\min}$  and  $\rho_{\max}$  are the minimum and maximum elements of  $\boldsymbol{\rho} \equiv (\rho_1, \dots, \rho_S)$ . Then define  $\bar{\gamma}$  and  $\underline{\gamma}$  according to

$$\bar{\gamma} \equiv \left( \frac{1 + A - \delta}{1 + \rho_{\max}} \right)^{\frac{1}{\tilde{\sigma}}} \quad \text{and} \quad \underline{\gamma} \equiv \left( \frac{1}{1 + \rho_{\min}} \right)^{\frac{1}{\tilde{\sigma}}}.$$

It can be shown that any solution of (18) must lie within the range  $(\underline{\gamma}, \bar{\gamma})$ , which is nonempty when  $\tilde{\sigma} > 0$  and  $(1 + A - \delta)(1 + \rho_{\min}) > 1 + \rho_{\max}$ .<sup>23</sup> Finally, define an auxiliary function  $\Psi : (\underline{\gamma}, \bar{\gamma}) \rightarrow \mathbb{R}_+$  according to

$$\Psi(\gamma) \equiv \sum_{i=1}^S \phi \left( 1 - \frac{1}{A - \delta} \left[ \gamma^{\tilde{\sigma}} (1 + \rho_i) - 1 \right] \right).$$

Then  $\Psi(\underline{\gamma}) > 1 > \Psi(\bar{\gamma})$  is both necessary and sufficient for the existence of a unique  $\gamma^*$  that solves (18). This result is formally stated in Lemma 9.

**Lemma 9** *Suppose the following conditions are satisfied:  $A > \delta$ ,  $\tilde{\sigma} \equiv 1 - \alpha(1 - \sigma) > 0$  and  $(1 + A - \delta)(1 + \rho_{\min}) > 1 + \rho_{\max}$ . Then a unique solution of (18) exists if and only if  $\Psi(\underline{\gamma}) > 1 > \Psi(\bar{\gamma})$ .*

Finally, we consider the effects of *ex ante* heterogeneity on the endogenous growth factor  $\gamma^*$ . As in Section 3, we compare two economies which are otherwise identical except for the distribution of time preference. Both economies are assumed to have a unique balanced-growth equilibrium.<sup>24</sup> We say that a larger extent of *ex ante* heterogeneity is beneficial (or harmful) to long-term economic growth if the more heterogeneous economy has a higher value of  $\gamma^*$ . Proposition 10 shows that the growth effect of *ex ante* heterogeneity is again determined by the shape of the marginal tax function.

<sup>22</sup>Note that equation (17) can also be obtained in a simple one-sector model with AK production technology or any other models that can be reduced to the AK model. Thus, the following analysis can also be applied to these models.

<sup>23</sup>See the proof of Lemma 9 for more details.

<sup>24</sup>In light of Lemma 9, this can be achieved by a suitable choice of the maximum and minimum elements in  $\boldsymbol{\rho}$  and  $\bar{\boldsymbol{\rho}}$ .

**Proposition 10** *In the model with endogenous growth, a larger extent of ex ante heterogeneity is beneficial (or harmful) to long-term economic growth if the marginal tax function is concave (or convex).*

## 5 Conclusion

In this paper we examine a largely overlooked mechanism through which diversity can affect long-term economic performance. Our approach focuses on the effects of *ex ante* consumer heterogeneity on the distribution of marginal tax rates across individuals. It is shown that the concavity or convexity of the marginal tax function holds the key in determining these effects. One interesting extension of this analysis is to include some form of productive government spending, such as the provision of public consumption good and investment in infrastructure. In this case, the negative impact of a higher effective marginal tax rate can be mitigated or even offset by the benefits of more useful government spending. Another important direction of future research is to characterise the transition dynamics induced by a change in the distribution of consumer characteristics. This type of analysis is important in terms of gauging the welfare effects of such change.

## Appendix

### Proof of Lemma 1

Pick any two positive real numbers  $y_1$  and  $y_2$ , and any  $\alpha \in (0, 1)$ . Then

$$\begin{aligned} \tau'(\alpha y_1 + (1 - \alpha) y_2) &\geq \alpha \tau'(y_1) + (1 - \alpha) \tau'(y_2) \\ \Leftrightarrow \phi[\tau'(\alpha y_1 + (1 - \alpha) y_2)] &\geq \phi[\alpha \tau'(y_1) + (1 - \alpha) \tau'(y_2)] \\ \Leftrightarrow \alpha y_1 + (1 - \alpha) y_2 &\geq \phi[\alpha \tau'(y_1) + (1 - \alpha) \tau'(y_2)] \\ \Leftrightarrow \alpha \phi[\tau'(y_1)] + (1 - \alpha) \phi[\tau'(y_2)] &\geq \phi[\alpha \tau'(y_1) + (1 - \alpha) \tau'(y_2)]. \end{aligned}$$

The second line uses the fact that  $\phi(\cdot)$  is strictly increasing. The third and fourth lines follow from the identity  $\phi[\tau'(y)] = y$ . Hence,  $\phi(\cdot)$  is a convex (or concave) function if and only if  $\tau'(\cdot)$  is concave (or convex). This completes the proof of Lemma 1.

### Proof of Lemma 2

Define the reduced-form production function  $f(k) \equiv F(k, 1)$  and an auxiliary function  $G(k) \equiv \bar{\varepsilon}[f(k) - \delta k]$ , for all  $k \geq 0$ . Since  $f(\cdot)$  is strictly increasing and strictly concave, there exists a unique value  $k_{GR} > 0$  such that  $G'(k) \geq 0$  if and only if  $k \leq k_{GR}$ . Next, consider equations (6) and (7). Since  $\phi(\cdot)$  is only defined on  $(0, 1)$ , equation (6) essentially imposes a restriction on the steady-state interest rate  $r^*$ , which is  $r^* \geq \rho_{\max} \equiv \max\{\rho_1, \rho_2, \dots, \rho_S\}$ . By the strict concavity of  $f(\cdot)$  and the Inada conditions  $\lim_{k \rightarrow 0} f'(k) = \infty$  and  $\lim_{k \rightarrow \infty} f'(k) = 0$ , there exists a unique value  $k_{\max} \in (0, k_{GR})$  such that

$$f'(k_{\max}) = \delta + \rho_{\max}.$$

Note that  $r^* \geq \rho_{\max}$  if and only if  $k^* \leq k_{\max}$ . Thus, any solution of (7) must be contained in the range  $(0, k_{\max})$ . Define the function  $H : (0, k_{\max}) \rightarrow \mathbb{R}_+$  according to

$$H(k) \equiv \frac{1}{S} \sum_{i=1}^S \phi \left[ 1 - \frac{\rho_i}{f'(k) - \delta} \right].$$

Since  $\phi(\cdot)$  is strictly increasing, it follows that  $H(\cdot)$  is strictly decreasing over the range  $(0, k_{\max})$ . In addition,  $H(k) > 0 = G(0)$  for all  $k \geq 0$ . As  $k$  approaches  $k_{\max}$ ,  $H(k)$  becomes

$\frac{1}{S} \sum_{i=1}^S \phi(1 - \rho_i/\rho_{\max}) > 0$ . Thus, a solution of (7) exists if and only if

$$G(k_{\max}) > \frac{1}{S} \sum_{i=1}^S \phi\left(1 - \frac{\rho_i}{\rho_{\max}}\right).$$

In addition, a solution if exists must be unique. A graphical illustration of the unique solution is provided in Figure A1. Once  $k^*$  is known, all other variables in a steady state can be uniquely determined. This completes the proof of Lemma 2.

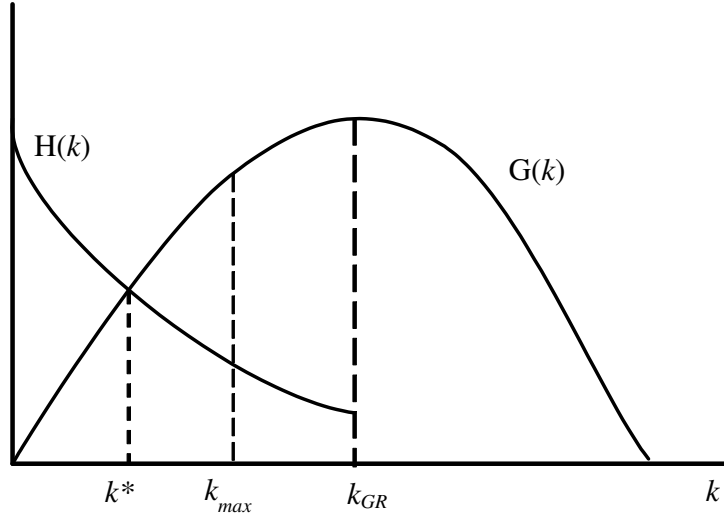


Figure A1: Existence and Uniqueness of Steady State.

### Proof of Proposition 3

The proof of Proposition 3 is built upon Theorem A1, the proof of which can found in Marshall, Olkin and Arnold (2013, Section 2B). Recall that an  $S$ -by- $S$  matrix  $\Pi = [\pi_{i,j}]$  is called *doubly stochastic* if all its elements are non-negative and all rows and columns sum to one, i.e.,  $\sum_{i=1}^S \pi_{i,j} = 1$  for all  $j$  and  $\sum_{j=1}^S \pi_{i,j} = 1$  for all  $i$ .

**Theorem A1** *Let  $\boldsymbol{\rho} \equiv (\rho_1, \dots, \rho_S)$  and  $\tilde{\boldsymbol{\rho}} \equiv (\tilde{\rho}_1, \dots, \tilde{\rho}_S)$  be two vectors of real numbers such that  $0 < \rho_1 \leq \rho_2 \leq \dots \leq \rho_S$  and  $0 < \tilde{\rho}_1 \leq \tilde{\rho}_2 \leq \dots \leq \tilde{\rho}_S$ . Then the following statements are equivalent:*

- (i)  $\sum_{i=1}^n \rho_i \leq \sum_{i=1}^n \tilde{\rho}_i$  for  $n = 1, 2, \dots, S-1$ , and  $\sum_{i=1}^S \rho_i = \sum_{i=1}^S \tilde{\rho}_i$ .
- (ii) *There exists a doubly stochastic matrix  $\Pi$  such that  $\tilde{\boldsymbol{\rho}} = \boldsymbol{\rho}\Pi$ .*

Let  $\boldsymbol{\rho} \equiv (\rho_1, \dots, \rho_S)$  and  $\tilde{\boldsymbol{\rho}} \equiv (\tilde{\rho}_1, \dots, \tilde{\rho}_S)$  be two vectors of real numbers in ascending order that have the same mean. Suppose condition (9) is satisfied so that  $\boldsymbol{\rho}$  is more heterogeneous than  $\tilde{\boldsymbol{\rho}}$ . Then by Theorem A1, there exists a doubly stochastic matrix  $\mathbf{\Pi} = [\pi_{i,j}]$  such that  $\tilde{\boldsymbol{\rho}} = \boldsymbol{\rho}\mathbf{\Pi}$ , or equivalently,

$$\tilde{\rho}_i = \sum_{j=1}^S \pi_{i,j} \rho_j, \quad \text{for all } i \in \{1, 2, \dots, S\}. \quad (19)$$

Suppose the marginal tax function  $\tau'(\cdot)$  is concave. By Lemma 1, this is equivalent to  $\phi(\cdot)$  being a convex function. Pick any  $r$  such that  $r > \max\{\rho_S, \tilde{\rho}_S\}$ . Then we have

$$\phi\left(1 - \frac{\tilde{\rho}_i}{r}\right) = \phi\left(\sum_{j=1}^S \pi_{i,j} \left(1 - \frac{\rho_j}{r}\right)\right) \leq \sum_{j=1}^S \pi_{i,j} \phi\left(1 - \frac{\rho_j}{r}\right),$$

for all  $i \in \{1, 2, \dots, S\}$ . The equality follows from (19). The inequality follows from the convexity of  $\phi(\cdot)$ . Summing the above expression across all consumers gives

$$\begin{aligned} \sum_{i=1}^S \phi\left(1 - \frac{\tilde{\rho}_i}{r}\right) &\leq \sum_{i=1}^S \sum_{j=1}^S \pi_{i,j} \phi\left(1 - \frac{\rho_j}{r}\right) \\ &= \sum_{j=1}^S \left(\sum_{i=1}^S \pi_{i,j}\right) \phi\left(1 - \frac{\rho_j}{r}\right) = \sum_{j=1}^S \phi\left(1 - \frac{\rho_j}{r}\right). \end{aligned} \quad (20)$$

Define  $k_{\max}$  and  $H(\cdot)$  as in the proof of Lemma 2. Similarly define  $\tilde{k}_{\max}$  according to  $f'(\tilde{k}_{\max}) = \delta + \tilde{\rho}_{\max}$  and  $\tilde{H} : (0, \tilde{k}_{\max}) \rightarrow \mathbb{R}_+$  as

$$\tilde{H}(k) \equiv \frac{1}{S} \sum_{i=1}^S \phi\left[1 - \frac{\tilde{\rho}_i}{f'(k) - \delta}\right].$$

Then (20) implies  $\tilde{H}(k) \leq H(k)$  for all  $k$  between zero and  $\min\{k_{\max}, \tilde{k}_{\max}\}$ . In other words, changing the distribution of consumer characteristics from  $\boldsymbol{\rho}$  to  $\tilde{\boldsymbol{\rho}}$  would shift the  $H(k)$  curve in Figure A1 down and to the left. Thus, we have  $k^* \geq \tilde{k}^*$ . A similar argument can be used to show that a convex marginal tax function implies  $k^* \leq \tilde{k}^*$ . This completes the proof of Proposition 3.



## Proof of Proposition 5

First, consider the case when the marginal tax function is concave. Then by Proposition 3, we have  $k^* \geq \tilde{k}^*$  which implies  $r^* \leq \tilde{r}^*$ . Thus, for any  $n \in \{1, 2, \dots, S\}$ , we have

$$\begin{aligned} \sum_{i=n}^S \tau'(y_i^*) &= S - n + 1 - \frac{\sum_{i=n}^S \rho_i}{r^*} \\ &\leq S - n + 1 - \frac{\sum_{i=n}^S \tilde{\rho}_i}{r^*} \leq S - n + 1 - \frac{\sum_{i=n}^S \tilde{\rho}_i}{\tilde{r}^*} = \sum_{i=n}^S \tau'(\tilde{y}_i^*). \end{aligned}$$

The first inequality follows from the fact that (9) and  $\sum_{i=1}^S \rho_i = \sum_{i=1}^S \tilde{\rho}_i$  implies  $\sum_{i=n}^S \rho_i \geq \sum_{i=n}^S \tilde{\rho}_i$ . The second inequality follows from  $r^* \leq \tilde{r}^*$ . This proves the first part of the proposition.

Next, consider the case when the marginal tax function is convex. Now we have  $k^* \leq \tilde{k}^*$  which means  $r^* \geq \tilde{r}^*$ . Starting from (5), we can get

$$\sum_{i=1}^n \tau'(y_i^*) = n - \frac{\sum_{i=1}^n \rho_i}{r^*} \geq n - \frac{\sum_{i=1}^n \tilde{\rho}_i}{r^*} \geq n - \frac{\sum_{i=1}^n \tilde{\rho}_i}{\tilde{r}^*} = \sum_{i=1}^n \tau'(\tilde{y}_i^*).$$

This proves the second part of the proposition.

## Proof of Proposition 6

The proof of part (i) is built upon some established results in statistics. Let  $\mathbf{x} = (x_1, \dots, x_S)$  and  $\mathbf{z} = (z_1, \dots, z_S)$  be two vectors of real numbers. Let  $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[S]}$  denote the elements of  $\mathbf{x}$  in descending order and let  $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(S)}$  denote the same set of elements but in ascending order. Similarly, define  $z_{[i]}$  and  $z_{(i)}$ . Then  $\mathbf{x}$  is said to be *weakly submajorized* by  $\mathbf{z}$  (denoted by  $\mathbf{x} \prec_w \mathbf{z}$ ) if

$$\sum_{i=1}^n x_{[i]} \leq \sum_{i=1}^n z_{[i]}, \quad \text{for all } n \in \{1, 2, \dots, S\}, \quad (21)$$

and  $\mathbf{x}$  is said to be *weakly supermajorized* by  $\mathbf{z}$  (denoted by  $\mathbf{x} \prec^w \mathbf{z}$ ) if

$$\sum_{i=1}^n z_{(i)} \leq \sum_{i=1}^n x_{(i)}, \quad \text{for all } n \in \{1, 2, \dots, S\}. \quad (22)$$

Note that these definitions does not require  $\mathbf{x}$  and  $\mathbf{z}$  to have the same mean. A detailed discussion of these concepts can be found in Marshall, Olkin and Arnold (2013). In particular, the following result is taken from their Theorem A.2 of Chapter 5 (p.167).

**Theorem A2**

(i) For all decreasing convex functions  $g$ ,  $\mathbf{x} \prec_w \mathbf{z}$  implies

$$(g(x_1), \dots, g(x_S)) \prec_w (g(z_1), \dots, g(z_S)). \quad (23)$$

(ii) For all decreasing concave functions  $g$ ,  $\mathbf{x} \prec_w \mathbf{z}$  implies

$$(g(x_1), \dots, g(x_S)) \prec_w (g(z_1), \dots, g(z_S)). \quad (24)$$

This theorem can be applied as follows: First, for any two distributions  $\boldsymbol{\rho}$  and  $\tilde{\boldsymbol{\rho}}$  with elements in ascending order,  $\sum_{i=1}^S \rho_i = \sum_{i=1}^S \tilde{\rho}_i$  and (9) imply  $\tilde{\boldsymbol{\rho}} \prec_w \boldsymbol{\rho}$  and  $\tilde{\boldsymbol{\rho}} \prec_w \boldsymbol{\rho}$ . Second, since  $\phi(1 - \frac{\rho}{r})$  is a decreasing function in  $\rho$ , the ranking  $\rho_1 \leq \rho_2 \dots \leq \rho_S$  implies

$$\phi\left(1 - \frac{\rho_1}{r}\right) \geq \phi\left(1 - \frac{\rho_2}{r}\right) \geq \dots \geq \phi\left(1 - \frac{\rho_S}{r}\right), \quad \text{for any } r \geq \rho_S.$$

Finally, if  $\tau'(\cdot)$  is concave, then  $\phi(1 - \frac{\rho}{r})$  is a decreasing convex function in  $\rho$ . Then using (21) and (23), we can obtain

$$\sum_{i=1}^n \phi\left(1 - \frac{\tilde{\rho}_i}{r}\right) \leq \sum_{i=1}^n \phi\left(1 - \frac{\rho_i}{r}\right), \quad \text{for all } n \in \{1, 2, \dots, S\}.$$

This proves the first part of Proposition 6. If instead  $\tau'(\cdot)$  is convex, then  $\phi(1 - \frac{\rho}{r})$  is a decreasing concave function in  $\rho$ . Then using (22) and (24), we can obtain

$$\sum_{i=n}^S \phi\left(1 - \frac{\rho_i}{r}\right) \leq \sum_{i=n}^S \phi\left(1 - \frac{\tilde{\rho}_i}{r}\right), \quad \text{for all } n \in \{1, 2, \dots, S\}.$$

Next, according to Proposition 3, a concave marginal tax function implies  $r^* \leq \tilde{r}^*$ . Hence, for any  $i \in \{1, 2, \dots, S\}$ , we have

$$\phi\left(1 - \frac{\rho_i}{r^*}\right) \leq \phi\left(1 - \frac{\rho_i}{\tilde{r}^*}\right) \Rightarrow \sum_{i=1}^n \phi\left(1 - \frac{\rho_i}{r^*}\right) \leq \sum_{i=1}^n \phi\left(1 - \frac{\rho_i}{\tilde{r}^*}\right).$$

This proves the second part of Proposition 6. If  $\tau'(\cdot)$  is convex, then we have  $r^* \geq \tilde{r}^*$ . By the same line of argument we can get

$$\sum_{i=n}^S \phi\left(1 - \frac{\rho_i}{\tilde{r}^*}\right) \leq \sum_{i=n}^S \phi\left(1 - \frac{\rho_i}{r^*}\right).$$

### Proof of Proposition 7

Suppose the marginal tax function is concave, which is equivalent to  $\phi(\cdot)$  being a convex function.

Then by Proposition 3, we have  $r^* \geq \tilde{r}^*$ . In addition, the convexity of  $\phi(\cdot)$  implies

$$y_i^* - \tilde{y}_i^* \equiv \phi\left(1 - \frac{\rho_i}{r^*}\right) - \phi\left(1 - \frac{\tilde{\rho}_i}{\tilde{r}^*}\right) \geq \phi'\left(1 - \frac{\tilde{\rho}_i}{\tilde{r}^*}\right) \left(\frac{\tilde{\rho}_i}{\tilde{r}^*} - \frac{\rho_i}{r^*}\right), \quad (25)$$

and

$$y_i^* - \tilde{y}_i^* \leq \phi'\left(1 - \frac{\rho_i}{r^*}\right) \left(\frac{\tilde{\rho}_i}{\tilde{r}^*} - \frac{\rho_i}{r^*}\right). \quad (26)$$

Since  $\boldsymbol{\rho} \equiv (\rho_1, \dots, \rho_S)$  and  $\tilde{\boldsymbol{\rho}} \equiv (\tilde{\rho}_1, \dots, \tilde{\rho}_S)$  share the same mean, we have

$$\sum_{i=1}^S (\tilde{\rho}_i - \rho_i) = \sum_{i=1}^S \left(\frac{\tilde{\rho}_i}{\rho_i} - 1\right) \rho_i = 0.$$

This, together with the monotone ratio property, implies the existence of an integer  $m \in \{1, 2, \dots, S-1\}$  such that  $\rho_i \leq \tilde{\rho}_i$  for  $i = 1, 2, \dots, m$ , and  $\rho_i \geq \tilde{\rho}_i$  for  $i = m+1, \dots, S$ . This also implies the existence of  $q \in \{m, \dots, S-1\}$  such that

$$\frac{\tilde{\rho}_i}{\rho_i} \geq \frac{\tilde{r}^*}{r^*}, \quad \text{for } i = 1, 2, \dots, q, \quad \text{and} \quad \frac{\tilde{\rho}_i}{\rho_i} \leq \frac{\tilde{r}^*}{r^*}, \quad \text{for } i = q+1, \dots, S. \quad (27)$$

Combining (25)-(27) gives  $y_i^* \geq \tilde{y}_i^*$  for  $i = 1, 2, \dots, q$ , and  $y_i^* \leq \tilde{y}_i^*$  for  $i = q+1, \dots, S$ . This proves the first part of the proposition. A similar argument can be used to establish part (ii).

### Proof of Lemma 9

Since the marginal tax rate  $\tau'(\cdot)$  is restricted between zero and one, this essentially imposes an upper bound and a lower bound on the equilibrium growth factor  $\gamma^*$ . To see this, first rewrite (17) as

$$\tau'(\xi_i^*) = 1 - \frac{1}{A - \delta} \left[ (\gamma^*)^{\tilde{\sigma}} (1 + \rho_i) - 1 \right],$$

where  $\tilde{\sigma} \equiv 1 - \alpha(1 - \sigma) > 0$ . Then  $\tau'(\cdot) > 0$  implies

$$\gamma^* < \left( \frac{1 + A - \delta}{1 + \rho_i} \right)^{\frac{1}{\tilde{\sigma}}}, \quad \text{for all } i.$$

Hence, the equilibrium growth factor is bounded above by  $\bar{\gamma}$  as defined in the text. Likewise,  $\tau'(\cdot) < 1$  means that  $\gamma^* > (1 + \rho_i)^{-\frac{1}{\tilde{\sigma}}}$  for all  $i$ . Hence, it is bounded below by  $\underline{\gamma}$ . The conditions  $\tilde{\sigma} > 0$  and  $(1 + A - \delta)(1 + \rho_{\min}) > 1 + \rho_{\max}$  together ensure that  $\bar{\gamma} > \underline{\gamma}$ .

Define the function  $\Psi : (\underline{\gamma}, \bar{\gamma}) \rightarrow \mathbb{R}_+$  according to

$$\Psi(\gamma) \equiv \sum_{i=1}^S \phi \left( 1 - \frac{1}{A - \delta} \left[ \gamma^{\tilde{\sigma}} (1 + \rho_i) - 1 \right] \right).$$

Since  $\phi(\cdot)$  is strictly increasing and  $\tilde{\sigma} > 0$ , it follows that  $\Psi(\cdot)$  is strictly decreasing over the range  $(\underline{\gamma}, \bar{\gamma})$ . If  $\Psi(\underline{\gamma}) > 1 > \Psi(\bar{\gamma})$ , then a unique solution of (18) exists by the intermediate value theorem. Conversely, if there is a unique  $\gamma^*$  in  $(\underline{\gamma}, \bar{\gamma})$  that solves (18), then it must be the case that  $\Psi(\underline{\gamma}) > \Psi(\gamma^*) = 1 > \Psi(\bar{\gamma})$ . This completes the proof of Lemma 9.

### Proof of Proposition 10

Let  $\boldsymbol{\rho} \equiv (\rho_1, \dots, \rho_S)$  and  $\tilde{\boldsymbol{\rho}} \equiv (\tilde{\rho}_1, \dots, \tilde{\rho}_S)$  be two vectors of real numbers in ascending order that have the same mean. Suppose condition (9) is satisfied so that  $\boldsymbol{\rho}$  is more heterogeneous than  $\tilde{\boldsymbol{\rho}}$ . Then by Theorem A1, there exists a doubly stochastic matrix  $\Pi = [\pi_{i,j}]$  such that  $\tilde{\boldsymbol{\rho}} = \boldsymbol{\rho}\Pi$ . Suppose the conditions in Lemma 9 are satisfied in both economies. Consider the case when the marginal tax function  $\tau'(\cdot)$  is concave, or equivalently,  $\phi(\cdot)$  is convex. Pick any  $\gamma$  that satisfies

$$\max \left\{ \left( \frac{1}{1 + \rho_S} \right)^{\frac{1}{\tilde{\sigma}}}, \left( \frac{1}{1 + \tilde{\rho}_S} \right)^{\frac{1}{\tilde{\sigma}}} \right\} < \gamma < \min \left\{ \left( \frac{1 + A - \delta}{1 + \rho_S} \right)^{\frac{1}{\tilde{\sigma}}}, \left( \frac{1 + A - \delta}{1 + \tilde{\rho}_S} \right)^{\frac{1}{\tilde{\sigma}}} \right\}.$$

Then we have

$$\begin{aligned} \phi \left( 1 - \frac{1}{A - \delta} \left[ \gamma^{\tilde{\sigma}} (1 + \tilde{\rho}_i) - 1 \right] \right) &= \phi \left( 1 - \frac{1}{A - \delta} \left[ \gamma^{\tilde{\sigma}} \left( 1 + \sum_{j=1}^S \pi_{i,j} \rho_j \right) - 1 \right] \right) \\ &\leq \sum_{j=1}^S \pi_{i,j} \phi \left( 1 - \frac{1}{A - \delta} \left[ \gamma^{\tilde{\sigma}} (1 + \rho_j) - 1 \right] \right), \end{aligned}$$

for all  $i \in \{1, 2, \dots, S\}$ . The equality follows from (19). The inequality follows from the convexity of  $\phi(\cdot)$ . Summing the above expression across all consumers gives

$$\begin{aligned}
\Psi(\gamma; \tilde{\boldsymbol{\rho}}) &\equiv \sum_{i=1}^S \phi\left(1 - \frac{1}{A-\delta} \left[\gamma^{\tilde{\sigma}} (1 + \tilde{\rho}_i) - 1\right]\right) \\
&\leq \sum_{i=1}^S \sum_{j=1}^S \pi_{i,j} \phi\left(1 - \frac{1}{A-\delta} \left[\gamma^{\tilde{\sigma}} (1 + \rho_j) - 1\right]\right) \\
&= \sum_{j=1}^S \left(\sum_{i=1}^S \pi_{i,j}\right) \phi\left(1 - \frac{1}{A-\delta} \left[\gamma^{\tilde{\sigma}} (1 + \rho_j) - 1\right]\right) \\
&= \sum_{j=1}^S \phi\left(1 - \frac{1}{A-\delta} \left[\gamma^{\tilde{\sigma}} (1 + \rho_j) - 1\right]\right) \equiv \Psi(\gamma; \boldsymbol{\rho}).
\end{aligned}$$

Let  $\gamma^*$  and  $\tilde{\gamma}^*$  be the equilibrium growth factor under  $\boldsymbol{\rho}$  and  $\tilde{\boldsymbol{\rho}}$ , respectively. Then we have

$$\Psi(\gamma^*; \tilde{\boldsymbol{\rho}}) \leq \Psi(\gamma^*; \boldsymbol{\rho}) = 1 = \Psi(\tilde{\gamma}^*; \tilde{\boldsymbol{\rho}}).$$

Since  $\Psi(\gamma; \tilde{\boldsymbol{\rho}})$  is strictly decreasing in  $\gamma$ , we have  $\gamma^* \geq \tilde{\gamma}^*$ . A similar argument can be used to show that a convex marginal tax function implies  $\gamma^* \leq \tilde{\gamma}^*$ . This completes the proof of Proposition 10.

## References

- [1] Ager, P., and Brückner, M. (2013). “Cultural Diversity and Economic Growth: Evidence from the U.S. during the Age of Mass Migration.” *European Economic Review*, vol. 64, pp.76-97.
- [2] Alesina, A., and La Ferrara, E. (2005). “Ethnic Diversity and Economic Performance.” *Journal of Economic Literature*, vol. 43, pp.762-800.
- [3] Alesina, A., Harnoss, J., and Rapoport, H. (2013). “Birthplace Diversity and Economic Prosperity.” *NBER Working Paper* 18699.
- [4] Angyridis, C. (2015). “Endogenous Growth with Public Capital and Progressive Taxation.” *Macroeconomic Dynamics*, vol. 19, pp.1220-1239.
- [5] Carroll, D.R., and Young, E.R. (2009). “The Stationary Wealth Distribution under Progressive Taxation.” *Review of Economic Dynamics*, vol. 12, pp.469-478.
- [6] Carroll, D.R., and Young, E.R. (2011). “The Long Run Effects of Changes in Tax Progressivity.” *Journal of Economic Dynamics & Control*, vol. 35, pp.1451-1473.
- [7] Collier, P., and Gunning, J.W. (1999). “Explaining African Economic Performance.” *Journal of Economic Literature*, vol. 37, pp.64-111.
- [8] Conesa, J.C., and Krueger, D. (2006). “On the Optimal Progressivity of the Income Tax Code.” *Journal of Monetary Economics*, vol. 53, pp.1425–1450.
- [9] Easterly, W., and Levine, R. (1997). “Africa’s Growth Tragedy: Policies and Ethnic Divisions.” *Quarterly Journal of Economics*, vol. 112, pp.1203-1250.
- [10] Erosa, A., and Koreshkova, T. (2007). “Progressive Taxation in a Dynastic Model of Human Capital.” *Journal of Monetary Economics*, vol. 54, pp.667-685.
- [11] Frederick, S., Loewenstein, G., and O’Donoghue, T. (2002). “Time Discounting and Time Preference: A Critical Review.” *Journal of Economic Literature*, vol. 40, pp.351-401.
- [12] Gouveia, M., and Strauss, R.P. (1994). “Effective Federal Individual Tax Functions: An Exploratory Empirical Analysis.” *National Tax Journal*, vol. 47, pp.317-339.

- [13] Guo, J.T., and Lansing, K.J. (1998). "Indeterminacy and Stabilization Policy." *Journal of Economic Theory*, vol. 82, pp.481-490.
- [14] Huggett, M., Ventura, G., and Yaron, A. (2011). "Sources of Lifetime Inequality." *American Economic Review*, vol. 101, pp.2923-2954.
- [15] Keane, M.P., and Wolpin, K.I. (1997). "The Career Decisions of Young Men." *Journal of Political Economy*, vol. 105, pp.473-522.
- [16] Li, W., and Sarte, P.D. (2004). "Progressive Taxation and Long-Run Growth." *American Economic Review*, vol. 94, pp.1705-1716.
- [17] Marshall, A.W., Olkin, I., and Arnold, B.C. (2013). *Inequalities: Theory of Majorization and its Applications* (2nd Edition) Springer.
- [18] Ottaviano, G., and Peri, G. (2006). "The Economic Value of Cultural Diversity: Evidence from U.S. Cities." *Journal of Economic Geography*, vol. 6, pp.9-44.
- [19] Padovano, F., and Galli, E. (2001). "Tax Rates and Economic Growth in the OECD Countries (1950-1990)," *Economic Inquiry*, vol. 39, pp.44-57.
- [20] Sarte, P.D. (1997). "Progressive Taxation and Income Inequality in Dynamic Competitive Equilibrium." *Journal of Public Economics*, vol. 66, pp.145-171.
- [21] Shaked, M., and Shanthikumar, J.G. (2007). *Stochastic Orders*. Springer.
- [22] Suen, R.M.H. (2014). "Time Preference and the Distributions of Wealth and Income," *Economic Inquiry*, vol. 52, pp.364-381.
- [23] Trax, M., Brunow, S., and Suedekum, J. (2015). "Cultural Diversity and Plant-Level Productivity." *Regional Science and Urban Economics*, vol. 53, pp.85-96.