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Abstract We prove the existence of a unique pure-strategy Nash equilibrium in nice games with isotone chain-concave best replies and compact strategy sets. We establish a preliminary fixpoint uniqueness argument showing sufficient assumptions on the best replies of a nice game that guarantee the existence of exactly one Nash equilibrium. Then, by means of a comparative statics analysis, we examine the necessity and sufficiency of the conditions on (marginal) utility functions for such assumptions to be satisfied; in particular, we find necessary and sufficient conditions for the isotonicity and chain-concavity of best replies. We extend the results on Nash equilibrium uniqueness to nice games with upper unbounded strategy sets and we present "dual" results for games with isotone chain-convex best replies. A final application to Bayesian games is exhibited.

Keywords Nash equilibrium uniqueness; Chain-concave best replies; Nice games; Comparative statics; Strategic complementarity.
JEL classification: C61 · C72

1 Introduction

Nash equilibrium uniqueness has been a point of interest since the inception of non-cooperative game theory. In his Ph.D. dissertation (see [25]), John Forbes Nash posed the following rhetorical question about a possible interpretation of the solution concept that took name after him:

> 'What would be a "rational" prediction of the behavior to be expected of rational playing the game in question?'

He answered that (Nash) equilibrium uniqueness, together with other conditions of epistemic nature, are sufficient to expect that rational agents end up behaving as prescribed by the solution concept he proposed for noncooperative situations of strategic interaction:

> 'By using the principles that a rational prediction should be unique, that the players should be able to deduce and make use of it, and

that such knowledge on the part of each player of what to expect the others to do should not lead him to act out of conformity with the prediction, one is led to the concept of a solution defined before.'

His reasoning is not a conclusive argument by which one should expect that the Nash' solution concept can be considered the reasonable prediction of players' behavior only in a non-cooperative game with exactly one Nash equilibrium. Indeed, John Nash himself maintained later on in his thesis that in some classes of noncooperative games some subsolutions can shrink the set of reasonable predictions to a singleton; besides, he offered also a mass-action interpretation of his solution concept for which solution multiplicity is not a problem. Nonetheless, the quotation well enlightens about the historical importance of the issue of Nash equilibrium uniqueness in (non-cooperative) game-theoretic thought. The present paper is devoted to analyze such issue.

On Nash equilibrium uniqueness in the class of games under examination

Many games are known to possess a multiplicity of equilibria and one cannot hope to derive general conditions for the existence of a unique Nash equilibrium; thus, in this work, we shall restrict attention to a particular class of games: the class of nice[1] games with isotone best reply functions. The "isotonicity" of best reply correspondences, in some loose sense, is a very general expression of the strategic complementarity among optimal choices of agents. Games with "isotone" best reply correspondences have received a special attention in the economic and game-theoretic literature because of the richness and easy intelligibility of their equilibrium structure and properties. Such a literature—started from [31] and [32]—had been popularized in economics by several articles during the 1990s: just to mention a few, [21], [34], [23] and [22]. Some of the just mentioned articles showed interesting properties implied by the existence of a unique Nash equilibrium in classes of games that admit isotone selections from best reply correspondences. For example, in such classes Nash equilibrium uniqueness was proved to be: equivalent to dominance solvability (see Theorem 5 and the second Corollary at p. 1266 in [21], Theorem 12 in [23] and Proposition 4 in [1]); sufficient to establish an equivalence between the convergence to Nash equilibrium of an arbitrary sequence of joint strategies and its consistency with adaptive learning processes (first Corollary at p. 1270 in [21] and Theorem 14 in [23]); sufficient to infer the existence—and uniqueness—of coalition-proof Nash equilibria (see Theorem A1 and the last Remark at p.127 in [22]).

A new strand of the literature on nice games on networks, started a (still partial) investigation about the conditions on utility functions for the existence of a unique Nash equilibrium: [4], [3], [16] and [13] to mention a few. Except

[1] I.e., games with a finite set of players whose strategy space is a closed proper real interval with a minimum and whose utility function is strictly pseudoconcave and upper semicontinuous in own strategy. The term *nice game* is introduced in [24] and our definition is similar—but not identical—to the one therein.

for [16],[2] in such papers Nash equilibrium uniqueness is guaranteed by a type of fixpoint argument—introduced by [19] in the economic literature—whose application requires the isotonicity of best reply functions.[3] However, the general structures of the primitives of a game with "isotone" best replies ensuring the existence of a unique Nash equilibrium are still unclear, despite a natural interest of economic and game theorists in the understanding thereof; in particular, the possible role played by the "isotonicity" of best-replies in determining Nash equilibrium uniqueness is unclear. Of course, the literature offers conditions on the primitives of a game that ensure the existence of a unique Nash equilibrium, but not many results seem to crucially depend on the condition of "isotonicity" of best replies. Restricting attention to nice games with isotone best reply functions, can we add something to known Nash equilibrium uniqueness results?

Our contribution

In this paper we shall examine the structure of the primitives of nice games with isotone best replies ensuring Nash equilibrium uniqueness. Our investigation will make use of (a slight generalization of) the following fixpoint argument—similar but not identical to the one in [19]—which employs a notion of generalized concavity, defined in Sect. 2, that we shall name chain-concavity.

> *Let f be a self-map of $[0,1]^n$ with no fixpoints on the boundary of $\mathbb{R}^n_+$ (e.g., each f_i could be positive). Then f has exactly one fixpoint if each component function f_i is isotone and chain-concave.*[4]

We shall derive four results—actually one main result, one "dual" and two extensions—on the existence of exactly one Nash equilibrium in nice games. Our main theorems dispense with any differentiability assumption; but in case of compact nice games with "differentiable" utility functions, a corollary of one of our main results—by which the reader might readily gain an insight of our findings—can be stated thus.[5]

> *Let Γ be a smooth compact nice game. Suppose each strategy set S_i has minimum 0. Then Γ has exactly one Nash equilibrium if, for each player i, the marginal utility function $\mathcal{M}_i$:*

- *is quasiincreasing in every argument other than the i-th one;*
- *has a chain-convex upper level set at height zero;*
- *is positive at $(0,\ldots,0)$.*

[2] Nash equilibrium uniqueness in [16] is guaranteed by a result in [18] which does not crucially depend on the isotonicity of best reply functions.

[3] Indeed, [4] provide also alternative arguments which, however, still rely on the isotonicity of best reply functions.

[4] E.g., $f:[0,1]^4 \to [0,1]^4 : x \mapsto (x_4, 1, x_1 + x_2 - x_1 x_2, (1+x_1)/2)$ has exactly one fixpoint by this fixpoint uniqueness argument, and the fixpoint is $(1,1,1,1)$. See also Remark 1.

[5] For the precise definitions of $\mathcal{M}_i$ and of a smooth compact nice game see Sect. 5; for the precise definition of a chain-convex set see Sect. 2.

In fact our main results do not rely on Fréchet differentiability; nevertheless, they are formulated by means of Dini derivatives, which can be economically interpreted as "generalized marginal utility" functions. The main advantage of this analytical approach is that it allows to remain in the line of the marginalist tradition while dealing with nondifferentiable utility functions. It must be clear, however, that the contribution of our main results on the existence of a unique Nash equilibrium is not (only) the lack of differentiability assumptions. In Sect. 7 we shall explain why the previous statement on the uniqueness of Nash equilibria in smooth compact nice games—or some "smooth" variant thereof for unbounded strategy sets—cannot be inferred from three classical theorems of the literature on Nash equilibrium uniqueness: [28]'s Theorem 2, [18]' s Theorem 5.2 and [10]'s Theorem 4.1.

To investigate the necessity and sufficiency of the conditions imposed on each player's "generalized marginal utility" function we shall preliminarily examine the necessity and sufficiency of the conditions for the optimal solution of a Choice Problem (see Sect. 3 for a precise definition) to be an isotone chain-concave function. Such examination is in fact the main contribution of our paper. A Choice Problem is a particular Type A problem—and hence a problem of comparative statics—in the terminology of [26] and [20] where a parametrized (strictly pseudoconcave upper semicontinuous) function is optimized on a fixed choice set (a compact proper real interval) for each possible given value of the parameter. The function that associates with each value of the parameter the optimal solution of a Choice Problem will be called a Choice function.

As it will be shown in Sect. 6 our analysis provides new results in terms of the necessity and sufficiency of the conditions for both the (chain-)concavity and the isotonicity of Choice functions. To the best of our knowledge, the (chain-)concavity of optimal solutions has not been systematically studied in the literature, but results that guarantee the chain-concavity and the concavity of Choice functions can be useful also for some game-theoretic analysis of problems that are not necessarily related to Nash equilibrium uniqueness[6]. The isotonicity of Choice functions has been investigated in the literature; however, our results on this topic do not follow from well-known theorems such as [23]'s Monotonicity Theorem or other similar results of the subsequent literature (see, e.g., like those established in [29], [9], [1] and—tough in a more abstract spirit—[20]). In fact our results on the isotonicity of Choice functions are structurally similar to Theorem 1 in [27] and hold for a class of problems which is properly included in the class of problems for which Theorem 1 in [27] can guarantee the isotonicity of Choice functions; nevertheless, our two differential characterizations for the particular class of problems we consider seems to be nothing similar to the sufficient conditions on derivatives obtained in Sect. 2.4 in [27] and provide two new alternative characterizations for certain classes of IDO families (in the sense of [27]).

[6] See, e.g., also [6] and [7] for an instance of application of this type of results in the analysis of multi-leader multi-follower games.

Structure of the paper

The paper is organized as follows: Sect. 2 presents preliminary definitions and notations; Sect. 3 introduces the concept of a Choice function for a Choice Problem; Sect. 4 examines the necessity and sufficiency of the conditions of a Choice Problem that guarantee the isotonicity and (chain-)concavity of the associated Choice function; Sect. 5 provides four sets of conditions on (generalized marginal) utility functions that guarantee Nash equilibrium uniqueness in nice games; Sect. 6 relates our results of Sect. 4 to [23]'s Monotonicity Theorem and to some results in [27], while Sect. 7 relates our results of Sect. 5 to three classical theorems on the uniqueness of Nash equilibria; Sect. 8 consider games of incomplete information. An Appendix illustrates the fixpoint argument used to establish our main results on the existence of a unique Nash equilibrium and contains some other mathematical facts.

2 Notation and definitions

2.1 Notation

Let f be a real-valued function on a proper real interval I. There are several standard notations for the (four) Dini derivatives of f: ours is the same of [17] (see definitions 3.1.4–7 at p. 56 therein). Thus the right-hand upper (resp. lower) Dini derivative of f at $x_0 \neq \sup I$ is denoted by $D^+ f(x_0)$ (resp. $D_+ f(x_0)$) and the left-hand upper (resp. lower) Dini derivative of f at $x_0 \neq \inf I$ is denoted by $D^- f(x_0)$ (resp. $D_- f(x_0)$).

Let f be a real-valued function on $A \times B$, where A and B are nonempty subsets of Euclidean spaces. Let $(a^*, b^*) \in A \times B$. Sometimes we write $f(\cdot, b^*)$ to denote the function $A \to \mathbb{R} : a \mapsto f(a, b^*)$ and we write $f(a^*, \cdot)$ to denote the function $B \to \mathbb{R} : b \mapsto f(a^*, b)$. Thus, for instance, the expression $f(\cdot, b^*)(a^*)$ is perfectly equivalent to the expression $f(a^*, b^*)$. Such a notation is standard; however, for sake of clarity, we remark that when A is a proper real interval and we write $D^+ f(\cdot, b^*)(a^*)$ (or an analogous expression) we mean to indicate the right-hand upper Dini derivative of $f(\cdot, b^*)$ at a^*. (Obviously, $D^+ f(\cdot, b^*)(a^*)$ is well-defined and is not equivalent to the—not even well-defined—$D^+ f(a^*, b^*)$.)

2.2 Definitions

The following generalized monotonicity notions are standard (and, for instance, can be found at pp. 127-128 in [2] or at p. 1199 in [12]). However, in the definitions, we prefer to use the term "increasing" instead of "monotone" to remark the fact that our definitions are for functions on totally ordered sets.

Definition 1 *An extended real-valued function f on a nonempty subset X of the real line is:*

- ***increasing*** *iff $(\underline{x}, \overline{x}) \in X \times X$ and $\underline{x} < \overline{x} \Rightarrow f(\underline{x}) \leq f(\overline{x})$;*

- ***strictly increasing*** *iff* $(\underline{x},\overline{x}) \in X \times X$ *and* $\underline{x} < \overline{x} \Rightarrow f(\underline{x}) < f(\overline{x})$*;*
- ***pseudoincreasing*** *iff*

$$(\underline{x},\overline{x}) \in X \times X,\ \underline{x} < \overline{x} \text{ and } f(\underline{x}) \geq 0 \Rightarrow f(\overline{x}) \geq 0$$

  and

$$(\underline{x},\overline{x}) \in X \times X,\ \underline{x} < \overline{x} \text{ and } f(\overline{x}) \leq 0 \Rightarrow f(\underline{x}) \leq 0;$$

- ***strictly pseudoincreasing*** *iff* $(\underline{x},\overline{x}) \in X \times X$*,* $\underline{x} < \overline{x}$ *and* $f(\underline{x}) \geq 0 \Rightarrow f(\overline{x}) > 0$*;*[7]
- ***quasiincreasing*** *iff* $(\underline{x},\overline{x}) \in X \times X$*,* $\underline{x} < \overline{x}$ *and* $f(\underline{x}) > 0 \Rightarrow f(\overline{x}) \geq 0$*;*[8]
- ***decreasing (strictly decreasing, pseudodecreasing, strictly pseudodecreasing, quasidecreasing)*** *iff* $-f$ *is increasing (strictly increasing, pseudoincreasing, strictly pseudodecreasing, quasiincreasing).*

To dispel any doubts, the standard notion of a quasiincreasing function employed in this paper is very different from that in [19].

Table 1. Relation diagram for an extended real-valued function f on a real interval

$$\begin{array}{ccccc}
\text{incr.} & & \text{pseudoincr.} & & \text{quasiincr.} \\
\text{(decr.)} & \Rightarrow & \text{(pseudodecr.)} & \Rightarrow & \text{(quasidecr.)} \\
\Uparrow & & \Uparrow & & \\
\text{str. incr.} & & \text{str. pseudoincr.} & & \\
\text{(str. decr)} & \Rightarrow & \text{(str. pseudodecr.)} & &
\end{array}$$

Definition 2 *Let* $\{X_i\}_{i=1}^{m}$ *be a family of nonempty subsets of* $\mathbb{R}$*. An extended real-valued function* f *on* $\prod_{i=1}^{m} X_i$ *is* ***isotone*** *(resp.* ***antitone****) iff* f *is increasing (resp. decreasing) in every argument.*

We introduce some generalized convexity notions: all such definitions are standard except for Definitions 3–4 (i.e., chain-convexity and chain-concavity) which are new to the best of our knowledge. The definitions of a convex set, of a concave function and of a strictly concave function are assumed to be known.

Definition 3 *A subset* X *of* $\mathbb{R}^m$ *is* ***chain-convex*** *iff*

$$\gamma \in [0,1],\ (\underline{x},\overline{x}) \in X \times X \text{ and } \underline{x}_i \leq \overline{x}_i \text{ for all } i = 1,...,m \ \Rightarrow\ \gamma\underline{x} + (1-\gamma)\overline{x} \in X.$$

Needless to say, a convex set is also chain-convex.

[7] Or—equivalently—iff $(\underline{x},\overline{x}) \in X \times X$, $\underline{x} < \overline{x}$ and $f(\overline{x}) \leq 0 \Longrightarrow f(\underline{x}) < 0$.
[8] Or—equivalently—iff $(\underline{x},\overline{x}) \in X \times X$, $\underline{x} < \overline{x}$ and $f(\overline{x}) < 0 \Longrightarrow f(\underline{x}) \leq 0$.

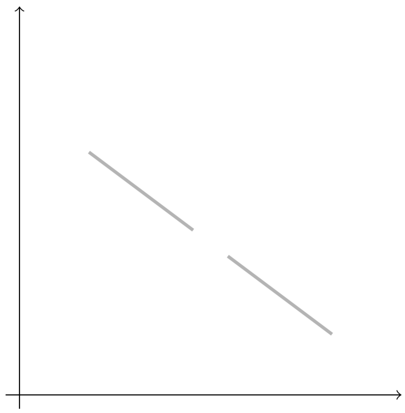

Fig 1. A chain-convex set

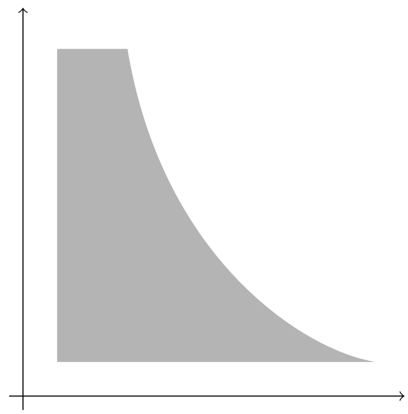

Fig 2. A chain-convex set

Definition 4 *Let X be a chain-convex subset of $\mathbb{R}^m$. A real-valued function f on X is **chain-concave** iff*

$$\gamma \in \left]0,1\right[, \ (\underline{x}, \overline{x}) \in X \times X \text{ and } \underline{x}_i \leq \overline{x}_i \text{ for all } i = 1, ..., m$$
$$\Downarrow$$
$$\gamma f(\underline{x}) + (1-\gamma) f(\overline{x}) \leq f(\gamma \underline{x} + (1-\gamma)\overline{x}).$$

*A real-valued function f on X is **chain-convex** iff $-f$ is chain-concave.*

Lemma 1 *Let Y be a nonempty open subset of $\mathbb{R}^m$, X be chain-convex subset of Y and $f : Y \to \mathbb{R}$ be twice continuously differentiable. Then f is chain-concave on X if the Hessian matrix*

$$\mathrm{H}(x) = \left[\frac{\partial^2 f}{\partial x_i \partial x_l}(x)\right]_{i,l}$$

is conegative at all $x \in X$ (i.e., if $v^T \cdot \mathrm{H}(x) \cdot v \leq 0$ for all $v \in \mathbb{R}^m_+$ at all $x \in X$).

Proof. If $X = \emptyset$ we are done. Suppose $X \neq \emptyset$. Recall that, by the twice continuous differentiability of f, the second-order directional derivative of f at $y \in Y$ along a unit vector $v \in \mathbb{R}^m$ is

$$D_v^2 f(y) = v^T \cdot \mathrm{H}(y) \cdot v.$$

Pick an arbitrary $x \in X$. Suppose $\mathrm{H}(x)$ is conegative. Then $v^T \cdot \mathrm{H}(x) \cdot v \leq 0$ for all $v \in \mathbb{R}^m_+$. Thus $D_v^2 f$ is nonpositive on

$$L_{x,v} := \{(x + tv) \in X : t \in \mathbb{R}\}$$

for every unit vector $v \in \mathbb{R}^m_+$ and hence f is concave on $L_{x,v}$ for every unit vector $v \in \mathbb{R}^m_+$. As x is arbitrary on X, f is chain-concave on X. ■

As every nonnegative matrix is conegative, we readily obtain the following.

Corollary 1 *Under the conditions of Lemma 1, f is chain-concave on X if* $\mathrm{H}(x)$ *is nonnegative at all $x \in X$ (i.e., if*

$$\frac{\partial^2 f}{\partial x_i \partial x_l}(x) \leq 0$$

for all $i = 1, \ldots, m$, all $l = 1, \ldots, m$ and all $x \in X$).

Theorem 1 provides a characterization of chain-concave functions.

Theorem 1 *Let $X \subseteq \mathbb{R}^m$ be nonempty, open and chain-convex and $f : X \to \mathbb{R}$ be twice continuously differentiable. Then f is chain-concave if and only if the Hessian matrix*

$$\mathrm{H}(x) = \left[\frac{\partial^2 f}{\partial x_i \partial x_l}(x)\right]_{i,l}$$

$\mathrm{H}(x)$ *is conegative at all $x \in X$ (i.e., if and only if $v^T \cdot \mathrm{H}(x) \cdot v \leq 0$ for all $v \in \mathbb{R}^m_+$ at all $x \in X$).*

Proof. The *if part* follows from Lemma 1. We prove the *only if part*. Suppose f is chain-concave. Then f is concave on the nonempty convex set

$$L_{x,v} := \{(x + tv) \in X : t \in \mathbb{R}\}$$

for all $(x, v) \in X \times \mathbb{R}^m_+$. Pick an arbitrary $x \in X$. If $v \in \mathbb{R}^m_+ \setminus \{0\}$, the openness of X guarantees that $L_{x,v}$ is open in the topology induced from $\mathbb{R}^m$ on

$$\{(x + tv) \in \mathbb{R}^m : t \in \mathbb{R}\};$$

thus $D^2_v f(x)$ is nonpositive on $L_{x,v}$ for all $v \in \mathbb{R}^m_+ \setminus \{0\}$ and hence $v^T \cdot \mathrm{H}(x) \cdot v \leq 0$ for all $v \in \mathbb{R}^m_+ \setminus \{0\}$. As $v^T \cdot \mathrm{H}(x) \cdot v = 0$ if $v = 0 \in \mathbb{R}^m_+$, $\mathrm{H}(x)$ is conegative. ■

Remark 1 *Clearly, if g and h are chain-concave real-valued functions on a chain-convex subset X of $\mathbb{R}^m$ then so is $g + h$. Besides, a concave function is evidently chain-concave; however the converse is generally false: e.g., every real-valued function defined on the grey set in Fig. 1 is chain-concave but not concave as the domain is not convex. We remark, however, that chain-concavity is strictly more general than concavity—and independent from quasiconcavity—even when the domain is convex: the following four examples clarify.*

1. *The (neither isotone nor antitone) function*

   $$f : \mathbb{R}^2 \to \mathbb{R} : (x_1, x_2) \mapsto -x_1 x_2$$

   is chain-concave but not concave (indeed, not even quasiconcave).

2. *The (antitone) function*

   $$f : [0, 1]^2 \to [-1, 0] : (x_1, x_2) \mapsto -x_1 x_2$$

   is chain-concave but not concave.

3. *The (isotone) function*

$$f:[0,1]^4 \to [0,1]:(x_1,x_2,x_3,x_4) \mapsto x_1+x_2-x_1x_2$$

is chain-concave but not concave.

4. *The (isotone) function*

$$f:\mathbb{R}^2_+ \to [0,1]:(x_1,x_2) \mapsto 1-e^{1-x_1-(x_2+1)^2}$$

is chain-concave but not concave: indeed, f is not even quasiconcave as the upper level set at height $\frac{e}{4}$ of f is not convex.

Through this paper we consider a standard notion of an extended real-valued quasiconcave function. Pseudoconcavity has been subject to several different definitions: here we shall use a strict definition in terms of Dini derivatives obtained from Definition 9 in [8]. (On this see also p. 577 in [14]; see also Definition 2 in [15] for further generalizations of the non-strict version.)

Definition 5 *An extended real-valued function f on a convex subset X of $\mathbb{R}^m$ is **quasiconcave** (resp. **chain-quasiconcave**) iff its upper level sets at finite height*[9] *are convex (resp. chain-convex). A real-valued function f on a convex subset X of $\mathbb{R}^m$ is **strictly quasiconcave** iff*

$$\lambda \in]0,1[\,,\ (\underline{x},\overline{x}) \in X{\times}X \text{ and } \underline{x} \neq \overline{x} \Rightarrow f(\lambda\underline{x}+(1-\lambda)\overline{x}) > \min\{f(\underline{x}),f(\overline{x})\}.$$

*A real-valued function f on a real interval X is **strictly pseudoconcave** iff*

$$(\underline{x},\overline{x}) \in X \times X,\ \underline{x} < \overline{x} \text{ and } f(\underline{x}) \leq f(\overline{x}) \quad \Rightarrow \quad D^+f(\underline{x}) > 0$$

and

$$(\underline{x},\overline{x}) \in X \times X,\ \underline{x} < \overline{x} \text{ and } f(\underline{x}) \geq f(\overline{x}) \quad \Rightarrow \quad D_-f(\overline{x}) < 0.$$

In Remark 2 we recall some known facts: for the proof of (i) see, e.g., Theorem 2.2.3 in [5]; (ii) follows directly from the definitions; (iii) follows from part (ii) of Theorem 14 in [8]; for a proof of (iv) see, e.g., part (i) of Theorem 2.2.1 in [5]; (v) follows from (i) and the definition of strict quasiconcavity; (vi) follows from Corollary 20 in [8]; for the proof of (vii) see the example contained therein; (viii) follows from the definitions of a strictly pseudoconcave function and of a strictly quasiconcave function.

Remark 2 *Let f be a real-valued function on a real interval X. The following facts are true:*

(i) f is quasiconcave if and only if

$$\lambda \in]0,1[\text{ and } (\underline{x},\overline{x}) \in X \times X \Rightarrow f(\lambda\underline{x}+(1-\lambda)\overline{x}) \geq \min\{f(\underline{x}),f(\overline{x})\};$$

[9] The upper level set of f at height $\lambda \in \mathbb{R}\cup\{-\infty,+\infty\}$ is $\{x \in X : f(x) \geq \lambda\}$.

(ii) *if f is strictly concave then f is concave;*

(iii) *if f is strictly concave then f is strictly pseudoconcave;*

(iv) *if f is concave then f is quasiconcave;*

(v) *if f is strictly quasiconcave then f is quasiconcave;*

(vi) *if f strictly pseudoconcave and upper semicontinuous then f is strictly quasiconcave;*

(vii) *the strict pseudoconcavity of f does not imply the quasiconcavity of f (e.g., the lower semicontinuous function and strictly pseudoconcave function*

$$\mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} \frac{1}{1+|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is neither quasiconcave nor upper semicontinuous);

(viii) *if f is either strictly pseudoconcave or strictly quasiconcave then f can possess at most one maximizer.*

Table 2. Relation diagram for an upper semicontinuous real-valued function f on a real interval

conc.		$\Rightarrow$		quasiconc.
$\Uparrow$				$\Uparrow$
str. conc.	$\Rightarrow$	str. pseudoconc.	$\Rightarrow$	str. quasiconc.

We remark a relation between strict pseudoconcavity and strict pseudodecreasingness (see Theorem 1 at p. 1199 in [12] and references therein).

Remark 3 *A real-valued differentiable function f on a proper open real interval is strictly pseudoconcave if and only if Df is strictly pseudodecreasing.*

We remark some simple facts which can be easily proved by the reader.

Remark 4 *Let X be a convex subset of $\mathbb{R}^m$. Let f be an extended real-valued function on X and g be a positive real-valued function on X. If f is quasiconcave (resp. chain-quasiconcave) then $f \cdot g$ has a convex (resp. chain-convex) upper level set at height 0.*[10]

Remark 5 *Let $\{X\}_{i=1}^m$ be a family of nonempty subsets of $\mathbb{R}$. Put $X = \prod_{i=1}^m X_i$. Let f be an extended real-valued function on X. Besides let g be a positive real-valued function on X and h be a nonnegative real-valued function on X.*

- *If f is strictly decreasing in every argument then $f \cdot g$ is strictly pseudodecreasing in every argument.*
- *If f is increasing in every argument (i.e., isotone) then $f \cdot h$ is quasiincreasing in every argument.*

[10] However $f \cdot g$ it need not be quasiconcave (e.g., consider the function $f : [0, 10] \to \mathbb{R}$ defined by $f(x) = \ln(x + 1/2)$ and the function $g : [0, 10] \to \mathbb{R}_{++}$ defined by $g(x) = 2 + \sin x$).

3 CP, C-functions and (D)NC-functions

Here below we shall define a Choice Problem. Such a Choice Problem is nothing but a function f on a real interval A which is "parameterized" by the elements of a subset B of $\mathbb{R}^m$. The set A should be understood as the choice set and the value attained by the function at a point a is interpreted as the "value" of a choice for a given parameter b.

Definition 6 *By a **Choice Problem (CP** in short) we mean a triple (A, B, f) where: (i) A is a compact proper real interval; (ii) B is a nonempty subset of $\mathbb{R}^m$ with $m \in \mathbb{N}$; (iii) f is a function from $A \times B$ into $\mathbb{R}$ such that $f(\cdot, b)$ is strictly pseudoconcave and upper semicontinuous for all $b \in B$.*

Notation $(\mathcal{D}_f, \tilde{\mathcal{D}}_f)$ *With each CP we associate the functions*

$$\mathcal{D}_f : \operatorname{int}(A) \times B \to \mathbb{R} \cup \{-\infty, +\infty\} : (a, b) \mapsto D_- f(\cdot, b)(a)$$

and

$$\tilde{\mathcal{D}}_f : \operatorname{int}(A) \times B \to \mathbb{R} \cup \{-\infty, +\infty\} : (a, b) \mapsto D^+ f(\cdot, b)(a).$$

In the following Sect. 4, we shall analyze how optimal choices change in the parameter. In order to do this we formally define an optimal choice for a given parameter as the maximizer of $f(\cdot, b)$. Clearly, as $f(\cdot, b)$ is an upper semicontinuous strictly pseudoconcave function on a compact set, there exists exactly one maximizer for it, and hence exactly one optimal choice for each parameter $b \in B$. Sometimes, it will be convenient for us to make use of two normalizations of the choice function.

Definition 7 *Given a CP, by the **Choice function (C-function** in short) associated to such a CP we mean the function*

$$\beta : B \to A \text{ such that } \{\beta(b)\} = \arg\max f(\cdot, b) \text{ at all } b \in B,$$

*by the **Normalized Choice function (NC-function** in short) associated to such a CP we mean the function*

$$\beta^* : B \to \mathbb{R}_+ : b \mapsto \beta(b) - \min A,$$

*and by the **Dually Normalized Choice function (DNC-function** in short) associated to such a CP we mean the function*

$$\beta^{**} : B \to \mathbb{R}_+ : b \mapsto \max A - \beta(b).$$

4 Properties of a C-function

We now examine the necessity and sufficiency of the conditions for the isotonicity, concavity and chain-concavity of a C-function. Dual results are provided.

4.1 Isotonicity of a C-function

The following Theorem 2 is the first main result of this Sect. 4. See also Theorem D2 in Appendix D for an extension of Theorem 2 pertaining the special case of a family $\{f(\cdot,b)\}_{b\in B}$ of functions that are continuously differentiable on $\mathrm{int}(A)$.

Theorem 2 *Consider a CP and the associated function β. Suppose B is the Cartesian product of m subsets of $\mathbb{R}$. Then, β is isotone if and only if $\mathcal{D}_f(a,\cdot)$ is quasiincreasing in every argument*[11] *for all $a \in \mathrm{int}(A)$.*

Proof. *If part.* Suppose $\mathcal{D}_f(a,\cdot)$ is quasiincreasing in every argument for all $a \in \mathrm{int}(A)$. Pick $(x,y) \in B \times B$ such that $x \neq y$ and $x_l \leq y_l$ for all $l = 1,...,m$. It suffices to show that $\beta(x) \leq \beta(y)$. If $\beta(x) = \min A$ then $\beta(x) \leq \beta(y)$. Suppose $\beta(x) > \min A$. By the strict pseudoconcavity of $f(\cdot,x)$,

$$D^+ f(\cdot,x)(a) > 0 \text{ for all } a \in [\min A, \beta(x)[\,.$$

Thus, by part (ii) of Theorem 1.13 in [11],

$$\mathcal{D}_f(a,x) = D_- f(\cdot,x)(a) \geq 0 \text{ for all } a \in \,]\min A, \beta(x)[$$

and hence, by Lemma C1 in Appendix C,

$$\mathcal{D}_f(a,y) = D_- f(\cdot,y)(a) \geq 0 \text{ for all } a \in \,]\min A, \beta(x)[$$

because $\mathcal{D}_f(a,\cdot)$ is quasiincreasing in every argument. Hence $\beta(x) \leq \beta(y)$: otherwise $\beta(y) < \beta(x)$ and $D_- f(\cdot,y)(a) \geq 0$ for some $a \in \,]\beta(y), \beta(x)[$ in contradiction with the strict pseudoconcavity of $f(\cdot,y)$.

Only if part. A consequence of Theorem D1 in Appendix D. ■

Corollary 2 *Consider a CP and the associated function β. Suppose B is the Cartesian product of m subsets of $\mathbb{R}$. Then*

(i) β is antitone if and only if $\mathcal{D}_f(a,\cdot)$ is quasidecreasing in every argument for all $a \in \mathrm{int}(A)$;

(ii) β is antitone if and only if $\widetilde{\mathcal{D}}_f(a,\cdot)$ is quasidecreasing in every argument for all $a \in \mathrm{int}(A)$;

(iii) β is isotone if and only if $\widetilde{\mathcal{D}}_f(a,\cdot)$ is quasiincreasing in every argument for all $a \in \mathrm{int}(A)$.

Proof. (i) Reverse the product order of B and apply Theorem 2. (Indeed, as $B = \prod_{i=1}^m B_i$ for some family $\{B_i\}_{i=1}^m$ of m nonempty real intervals, to reverse the order of each B_i one can consider, for instance, $-B_i$.)

(ii) Reverse the order of A and apply Theorem 2. (To reverse the order of A one can consider, for instance, $-A$.)

(iii) Reverse the product order of B and apply part (ii) of Corollary 2. ■

[11] Recall that $\mathcal{D}_f(a,\cdot)$ is a function from $B = \prod_{i=1}^m B$ into $\mathbb{R} \cup \{-\infty,+\infty\}$. Thus the quasiincreasingness of $\mathcal{D}_f(a,\cdot)$ in every argument is—somehow incorrectly—the quasiincreasingness of $\mathcal{D}_f\left(a,(x_l)_{l=1}^m\right)$ in x_l for all $l = 1,...,m$.

4.2 Positivity of a (D)NC-function

Our results on the (chain-)concavity of the C-function β will be established on the subset of B where β is greater than $\min A$ and on the subset of B where β is smaller than $\max A$ (i.e., on the support of the NC-function β^* and of the DNC-function β^{**}). Some simple facts about the necessity and sufficiency of the conditions for B to coincide with the support of β^* (and with the support of β^{**}) are remarked by the following Proposition 1.

Proposition 1 *Consider a CP and the associated functions β^* and β^{**}. The following six facts are true.*[12]

(i) The support of β^* is B if and only if $D^+ f(\cdot, b)(\min A) > 0$ for all $b \in B$.

(ii) Assume that B has a least element, say ω. Besides assume that B is the Cartesian product of m subsets of $\mathbb{R}$ and that β^* is isotone. Then the support of β^* is B if and only if $D^+ f(\cdot, \omega)(\min A) > 0$.

(iii) Assume that B has a greatest element, say α. Besides assume that B is the Cartesian product of m subsets of $\mathbb{R}$ and that β^* is antitone. Then the support of β^* is B if and only if $D^+ f(\cdot, \alpha)(\min A) > 0$.

(iv) The support of β^{**} is B if and only if $D_- f(\cdot, b)(\max A) < 0$ for all $b \in B$.

(v) Assume that B has a greatest element, say α. Besides assume that B is the Cartesian product of m subsets of $\mathbb{R}$ and that β^{**} is antitone. Then the support of β^{**} is B if and only if $D_- f(\cdot, \alpha)(\max A) < 0$.

(vi) Assume that B has a least element, say ω. Besides assume that B is the Cartesian product of m subsets of $\mathbb{R}$ and that β^{**} is isotone. Then the support of β^{**} is B if and only if $D_- f(\cdot, \omega)(\max A) < 0$.

Proof. (i) *If part.* A consequence of the definition of $D^+ f(\cdot, b)$.

Only if part. Suppose the support of β^* is B. If $D^+ f(\cdot, \underline{b})(\min A) \leq 0$ for some $\underline{b} \in B$ then $f(\cdot, \underline{b})(\min A) > f(\cdot, \underline{b})(x)$ for all $x > \min A$ by the the definition of a strictly pseudoconcave function. Hence $\underline{b} \in B$ would not be in the support of β^*: a contradiction.

(ii) *If part.* Suppose $D^+ f(\cdot, \omega)(\min A) > 0$. Then $\beta^*(\omega) > 0$ and ω is in the support of β^*. The isotonicity of β^* implies that B is the support of β^*.

Only if part. The same proof of the 'Only if part' of (i): just put $\underline{b} = \omega$.

(iii) Analogous to the proof of (ii).

(iv)-(vi) Analogous to the proofs of (i)-(iii). ∎

[12] It is perhaps worth to remark that the isotonicity of β is equivalent to the isotonicity of β^* and to the antitonicity of β^{**}.

4.3 Chain-concavity of a C-function

The following Theorem 3 is the other main result of this Sect. 4.

Theorem 3 *Consider a CP and the associated functions β and β^*. Suppose B is the Cartesian product of m real intervals and that β is isotone. Besides suppose β^* is positive. Then β is chain-concave if and only if $\mathcal{D}_f$ has a chain-convex upper level set at height* 0.

Proof. Without loss of generality, we shall put $\min A = 0$. Thus β equals the NC-function β^*.

If part. Assume that D_f has a chain-convex upper level set at height 0. Suppose that x and z are elements of B such that

$$x_i \le z_i \text{ for all } i = 1, ..., m$$

and put

$$\xi := \beta(x) \text{ and } \zeta := \beta(z)$$

By the isotonicity of the positive function β,

$$0 = \min A < \xi \le \zeta.$$

Pick $\gamma \in]0,1[$ and put $y := \gamma x + (1-\gamma) z$. We are done if we prove that

$$\overline{\upsilon} := \gamma\xi + (1-\gamma)\zeta \le \beta(y) =: \upsilon.$$

Case $\min\{\xi,\zeta\} < \max A$. Thus $\xi = \min\{\xi,\zeta\} < \max A$. Suppose, to the contrary, that $\upsilon < \overline{\upsilon}$. Note that

$$D_- f(\cdot, y)(\overline{\upsilon}) < 0 \tag{1}$$

because $f(\cdot, y)$ is a strictly pseudoconcave function maximized at υ, with

$$\min A \le \upsilon < \overline{\upsilon} < \max A.$$

Since ξ and ζ are respectively maximizers of $f(\cdot, x)$ and of $f(\cdot, z)$,

$$D_- f(\cdot, x)(\xi) \ge 0 \le D_- f(\cdot, z)(\zeta)$$

and hence

$$\min\{\mathcal{D}_f(\xi, x), \mathcal{D}_f(\zeta, z)\} \ge 0\,.$$

Therefore (ξ, x) and (ζ, z) belong to the upper level set at height 0 of $\mathcal{D}_f$, and hence so does[13] also $(\overline{\upsilon}, y)$ by the chain-convexity of the upper level set at height 0 of $\mathcal{D}_f$. Therefore

$$\mathcal{D}_f(\overline{\upsilon}, y) = D_- f(\cdot, y)(\overline{\upsilon}) \ge 0,$$

in contradiction with (1).

[13] Recall—and this is important in this proof—that $x_i \le z_i$ for all $i = 1, ..., m$ and that $\xi \le \zeta$ (as $\xi = \min\{\xi, \zeta\}$).

Case $\min\{\xi,\zeta\} \geq \max A$. Thus $\xi = \zeta = \max A$. By the strict pseudoconcavity of $f(\cdot,x)$ and $f(\cdot,z)$,

$$D^{+}f(\cdot,x)(a) > 0 < D^{+}f(\cdot,z)(a) \text{ for all } a \in \operatorname{int}(A) \cup \{\min A\}.$$

By part (ii) of Theorem 1.13 in [11], $f(\cdot,x)$ and $f(\cdot,z)$ are increasing on $\operatorname{int}(A)$; consequently,

$$\mathcal{D}_f(a,x) = D_{-}f(\cdot,x)(a) \geq 0 \leq D_{-}f(\cdot,z)(a) = \mathcal{D}_f(a,z) \text{ for all } a \in \operatorname{int}(A)$$

and hence

$$\mathcal{D}_f(a,y) = D_{-}f(\cdot,y)(a) \geq 0 \text{ for all } a \in \operatorname{int}(A)$$

by the chain-convexity of the upper level set at height 0 of $\mathcal{D}_f$. Thus we must have $\upsilon = \beta(y) = \max A = \overline{\upsilon}$: otherwise $\beta(y) \in \operatorname{int}(A) \cup \{\min A\}$ and $D_{-}f(\cdot,y)(a) \geq 0$ for some $a \in \,]\beta(y), \max A[$ in contradiction with the strict pseudoconcavity of $f(\cdot,y)$.

Only if part. Assume that β is chain-concave. By way of contradiction, suppose the upper level set of $\mathcal{D}_f$ at height 0 is not chain-convex. Then there exist $(\hat{a},x) \in \operatorname{int}(A) \times B$, $(\check{a},z) \in \operatorname{int}(A) \times B$ and $\gamma \in \,]0,1[$ such that

$$\hat{a} \leq \check{a} \text{ and } x_l \leq z_l \text{ for all } l = 1,...,m,$$

$$D_{-}f(\cdot,x)(\hat{a}) \geq 0 \leq D_{-}f(\cdot,z)(\check{a}), \tag{2}$$

and

$$D_{-}f(\cdot,\gamma x + (1-\gamma)z)(\gamma\hat{a} + (1-\gamma)\check{a}) < 0. \tag{3}$$

By the strict pseudoconcavity of $f(\cdot,x)$ and $f(\cdot,z)$, (2) implies

$$\beta(x) \geq \hat{a} \text{ and } \beta(z) \geq \check{a}.$$

Thus $\gamma\beta(x) \geq \gamma\hat{a}$ and $(1-\gamma)\beta(z) \geq (1-\gamma)\check{a}$, and hence

$$\gamma\hat{a} + (1-\gamma)\check{a} \leq \gamma\beta(x) + (1-\gamma)\beta(z).$$

As $f(\cdot,\gamma x + (1-\gamma)z)(\gamma\hat{a} + (1-\gamma)\check{a})$ is upper semicontinuous and also quasiconcave, Theorem 2.5.2 in [5] and (3) imply that

$$\beta(\gamma z + (1-\gamma)z) < \gamma\hat{a} + (1-\gamma)\check{a}.$$

But then

$$\beta(\gamma x + (1-\gamma)z) < \gamma\beta(x) + (1-\gamma)\beta(z),$$

in contradiction with the chain-concavity of β. ■

Corollary 3 *Consider a CP and the associated functions β, β^{*} and β^{**}.*

(i) Suppose B is the Cartesian product of m real intervals and that β is antitone. Besides suppose β^{*} is positive. Then β is chain-concave if and only if $\mathcal{D}_f$ has a chain-convex upper level set at height 0.

(ii) Suppose B is the Cartesian product of m real intervals and that β is antitone. Besides suppose β^{**} is positive. Then β is chain-convex if and only if $\widetilde{D}_f$ has a chain-convex lower level set at height 0.

(iii) Suppose B is the Cartesian product of m real intervals and that β is isotone. Besides suppose β^{**} is positive. Then β is chain-convex if and only if $\widetilde{D}_f$ has a chain-convex lower level set at height 0.

Proof. (i) Reverse the product order of B and apply Theorem 3.
(ii) Reverse the order of A and apply Theorem 3.
(iii) Reverse the product order of B and apply part (ii) of Corollary 3. ■

4.4 Concavity of a C-function

We prove a variant of Theorem 3 for the concavity of a C-function β on the support of the NC-function β^*; sufficient conditions for the concavity of β can be easily derived by applying Proposition 1. Such a variant is established without preliminary assumptions on the isotonicity of β and the positivity of β^*.

Theorem 4 *Consider a CP and the associated functions β and β^*. Suppose B is convex. Then β^* has convex support and β is concave thereon if and only if $\mathcal{D}_f$ has a convex upper level set at height 0.*

Proof. Without loss of generality, we shall put $\min A = 0$. Thus $\beta = \beta^*$.

If part. Suppose the upper level set of $\mathcal{D}_f$ at height 0 is convex. Choose x and z in B such that

$$\xi := \beta(x) > 0 < \beta(z) =: \zeta.$$

(Therefore $\min\{\xi, \zeta\} > 0 = \min A$.) Pick $\gamma \in]0,1[$ and put $y := \gamma x + (1-\gamma) z$. We are done if we prove that

$$\overline{\upsilon} := \gamma\xi + (1-\gamma)\zeta \leq \beta(y) =: \upsilon.$$

Case $\min\{\xi, \zeta\} < \max A$. Suppose, to the contrary, that $\upsilon < \overline{\upsilon}$. Note that

$$D_{-}f(\cdot, y)(\overline{\upsilon}) < 0 \tag{4}$$

because $f(\cdot, y)$ is a strictly pseudoconcave function maximized at υ, with

$$\min A \leq \upsilon < \overline{\upsilon} < \max A.$$

Since ξ and ζ are respectively maximizers of $f(\cdot, x)$ and of $f(\cdot, z)$,

$$D_{-}f(\cdot, x)(\xi) \geq 0 \leq D_{-}f(\cdot, z)(\zeta)$$

and hence

$$0 \leq \min\{\mathcal{D}_f(\xi, x), \mathcal{D}_f(\zeta, z)\}\ .$$

Therefore (ξ, x) and (ζ, z) belong to the upper level set at height 0 of $\mathcal{D}_f$ and then so does also $(\overline{\upsilon}, y)$ by the convexity of the upper level set at height 0 of $\mathcal{D}_f$. Therefore

$$\mathcal{D}_f\left(\overline{\upsilon}, y\right) = D_- f\left(\cdot, y\right)\left(\overline{\upsilon}\right) \geq 0,$$

in contradiction with (4).

Case $\min\{\xi, \zeta\} \geq \max A$. Thus $\xi = \zeta = \max A$. By the strict pseudoconcavity of $f(\cdot, x)$ and $f(\cdot, z)$,

$$D^+ f\left(\cdot, x\right)(a) > 0 < D^+ f\left(\cdot, z\right)(a) \quad \text{for all } a \in \{\min A\} \cup \operatorname{int}(A).$$

By part (ii) of Theorem 1.13 in [11], $f(\cdot, x)$ and $f(\cdot, z)$ are increasing on $\operatorname{int}(A)$; consequently,

$$\mathcal{D}_f(a, x) = D_- f\left(\cdot, x\right)(a) \geq 0 \leq D_- f\left(\cdot, z\right)(a) = \mathcal{D}_f(a, z) \quad \text{for all } a \in \operatorname{int}(A)$$

and hence

$$\mathcal{D}_f(a, y) = D_- f\left(\cdot, y\right)(a) \geq 0 \quad \text{for all } a \in \operatorname{int}(A)$$

by the convexity of the upper level set at height 0 of $\mathcal{D}_f$. Thus we must have $\upsilon = \beta(y) = \max A = \overline{\upsilon}$: otherwise $\beta(y) \in \{\min A\} \cup \operatorname{int}(A)$ and $D_- f(\cdot, y)(a) \geq 0$ for some $a \in \left]\beta(y), \max A\right[$ in contradiction with the the strict pseudoconcavity of $f(\cdot, y)$.

Only if part. Assume that β has convex support and is concave thereon. By way of contradiction, suppose the upper level set of $\mathcal{D}_f$ at height 0 is not convex. Then there exist $(\hat{a}, x) \in \operatorname{int}(A) \times B$, $(\check{a}, z) \in \operatorname{int}(A) \times B$ and $\gamma \in \left]0, 1\right[$ such that

$$D_- f\left(\cdot, x\right)(\hat{a}) \geq 0 \leq D_- f(\cdot, z)(\check{a}), \tag{5}$$

and

$$D_- f(\cdot, \gamma x + (1-\gamma) z)(\gamma \hat{a} + (1-\gamma)\check{a}) < 0. \tag{6}$$

By the strict pseudoconcavity of $f(\cdot, x)$ and $f(\cdot, z)$, (5) implies

$$\beta(x) \geq \hat{a} > \min A = 0 \text{ and } \beta(z) \geq \check{a} > \min A = 0.$$

Thus $\gamma\beta(x) \geq \gamma\hat{a}$ and $(1-\gamma)\beta(z) \geq (1-\gamma)\check{a}$, and hence

$$\gamma\hat{a} + (1-\gamma)\check{a} \leq \gamma\beta(x) + (1-\gamma)\beta(z).$$

Note that x and z must be in the support of β, and hence that $\gamma x + (1-\gamma) z$ must be in the support of β (because of its convexity). As $f(\cdot, \gamma x + (1-\gamma) z)$ is upper semicontinuous and also quasiconcave, Theorem 2.5.2 in [5] and (6) imply that

$$\beta(\gamma x + (1-\gamma) z) < \gamma\hat{a} + (1-\gamma)\check{a}.$$

Then

$$\beta(\gamma x + (1-\gamma) z) < \gamma\beta(x) + (1-\gamma)\beta(z),$$

in contradiction with the concavity of β on its support. ■

Corollary 4 *Consider a CP and the associated functions β and β^{**}. Suppose B is convex. Then β^{**} has convex support and β is convex thereon if and only if $\tilde{\mathcal{D}}_f$ has a convex lower level set at height* 0.

Proof. Reverse the order of A and apply Theorem 4. ∎

5 Uniqueness of Nash equilibria

This Sect. 5 contains four Nash equilibrium uniqueness theorems. Other four results on some properties of a best reply function—such as its isotonicity, its concavity/convexity and its chain-concavity/chain-convexity—implicitly used in this Sect. 5 are presented in Appendix E as facts of independent interest.

By a **game** Γ we mean a triple $\left(N, (S_i)_{i\in N}, (u_i)_{i\in N}\right)$ where $N = \{1, ..., n\}$ is the **set of players** (i.e., a set whose elements are called players), $S_i \neq \emptyset$ is **player i's strategy set** and $u_i : \prod_{i\in N} S_i \to \mathbb{R}$ is **player i's utility function**. We denote by S the **joint strategy set** $\prod_{i\in N} S_i$. Adopting a standard game-theoretic convention, sometimes we write s_{-i} instead of $(s_l)_{l\in N\setminus\{i\}}$ and (s_i, s_{-i}) instead of s or $(s_l)_{l\in N}$.

Definition 8 *We say that a game Γ is a **nice game** if, for all $i \in N$:*

- *S_i is a proper closed real interval with a minimum;*
- *u_i is upper semicontinuous in the i-th argument;*
- *u_i is strictly pseudoconcave in the i-th argument.*

Definition 9 *We say that a nice game Γ is a **compact nice game** if each S_i is compact and that a nice game Γ is an **unbounded nice game** if each S_i is upper unbounded.*

Notation $(\omega, \alpha, \mathcal{D}_{u_i}, \widetilde{\mathcal{D}}_{u_i})$ *Given a nice game Γ, we shall denote by ω the **least joint strategy** (i.e., $(\min S_i)_{i\in N}$); given a compact nice game Γ, we shall denote by α the **greatest joint strategy** (i.e., $(\max S_i)_{i\in N}$). Given a nice game Γ and $i \in N$, we shall denote by*

$$\mathcal{D}_{u_i} : \operatorname{int}(S_i) \times S_{-i} \to \mathbb{R} \cup \{-\infty, +\infty\} : (s_i, s_{-i}) \mapsto D_{-}u_i(\cdot, s_{-i})(s_i)$$

*player i's "**lower left-hand marginal utility function**" and by*

$$\widetilde{\mathcal{D}}_{u_i} : \operatorname{int}(S_i) \times S_{-i} \to \mathbb{R} \cup \{-\infty, +\infty\} : (s_i, s_{-i}) \mapsto D^{+}u_i(\cdot, s_{-i})(s_i)$$

*player i's "**upper right-hand marginal utility function**".*

Note that in the following Definition 10, the utility function u_i might well be discontinuous in some argument $j \neq i$.

Definition 10 *We say that a game Γ is a **smooth game** if, for all $i \in N$:*

- *S_i is a proper closed real interval with a minimum;*
- *$u_i(\cdot, s_{-i})$ has a differentiable extension $u_i^{\circ}(\cdot, s_{-i})$, for all $s_{-i} \in S_{-i}$.*

Notation ($\mathcal{M}_i$) *Given a smooth game Γ (and chosen an arbitrary extension $u_i^{\circ}(\cdot, s_{-i})$* for all $s_{-i} \in S_{-i}$, for all $i \in N$) *we shall denote by*

$$\mathcal{M}_i : S_i \times S_{-i} \to \mathbb{R} : (s_i, s_{-i}) \mapsto Du_i^{\circ}(\cdot, s_{-i})(s_i) = \frac{\partial u_i^{\circ}}{\partial s_i}(s)$$

*player i's "**marginal utility function**".*

Remark 6 *Let Γ be a smooth game. Then Γ is also nice if and only if each marginal utility function $\mathcal{M}_i$ is strictly pseudodecreasing in the i-th argument.*

As usual, a (pure strategy) **Nash equilibrium** is a fixpoint of the set-valued **joint best reply function**

$$b : S \to \prod_{i \in N} 2^{S_i} : s \mapsto (\arg\max u_i(\cdot, s_{-i}))_{i \in N}$$

where $u_i(\cdot, s_{-i}) : S_i \to \mathbb{R} : s_i \mapsto u_i(s)$. (I.e., e is a Nash equilibrium for Γ if and only if $e_i \in b_i(e)$ for all $i \in N$.) When **player i's best reply function** b_i is single-valued, such b_i can be understood as a function into S_i: this observation will be often used without further mention in sequel of Sect. 5.

In any nice game player i's best reply b_i can be understood as a partial function $b_i : S \nrightarrow S_i$ defined by $\{b_i(s)\} = \arg\max u_i(\cdot, s_{-i})$ if $\arg\max u_i(\cdot, s_{-i}) \neq \emptyset$ (recall that in any nice game $\arg\max u_i(\cdot, s_{-i})$ is either a singleton or the empty set). Thus, when b_i is nonempty-valued—like, e.g., in compact nice games—such partial function is indeed a function $b_i : S \to S_i$ defined by $\{b_i(s)\} = \arg\max u_i(\cdot, s_{-i})$.

5.1 Bounded strategy sets

The following Theorem 5 provides sufficient conditions for a compact nice game to possess exactly one Nash equilibrium: all additional conditions are imposed only on players' lower left-hand marginal utility functions.

Theorem 5 *Let Γ be a compact nice game. Γ has exactly one Nash equilibrium (and no i-th component of such an equilibrium equals ω_i) if, for all $i \in N$:*

H1 $\mathcal{D}_{u_i}$ is quasiincreasing in the j-th argument, for all $j \in N \setminus \{i\}$;

H2 $\mathcal{D}_{u_i}$ has a chain-convex upper level set at height 0;

H3 $\mathcal{D}_{u_i}(\cdot, \omega_{-i})$ is not nonpositive.[14]

[14] Or—equivalently—$\mathcal{D}_{u_i}(\cdot, \omega_{-i})$ is positive at some strategy in $\operatorname{int}(S_i)$.

Proof. Suppose hypotheses H1–3 hold true. By Theorem 2, by Proposition B1 in Appendix B and part (ii) of Proposition 1 and by Theorem 3, we can conclude that each b_i—understood as a function—is chain-concave, isotone and never equal to ω_i. Theorem A1 in Appendix A guarantees the existence of exactly one Nash equilibrium for Γ. ■

Theorem 5 can be "dually" reformulated as in the following Theorem 6, where all additional conditions are now imposed only on players' upper right-hand marginal utility functions.

Theorem 6 *Let Γ be a compact nice game. Γ has exactly one Nash equilibrium (and no i-th component of such an equilibrium equals α_i) if, for all $i \in N$:*

H1' $\widetilde{\mathcal{D}}_{u_i}$ is quasiincreasing in the j-th argument, for all $j \in N \setminus \{i\}$;

H2' $\widetilde{\mathcal{D}}_{u_i}$ has a chain-convex lower level set at height 0;

H3' $\widetilde{\mathcal{D}}_{u_i}(\cdot, \alpha_{-i})$ is not nonnegative.

Proof. Let $\Gamma^* := (N, (S_i^*)_{i \in N}, (u_i^*)_{i \in N})$ be the game where, for all $i \in N$, $S_i^* = -S_i$ and $u_i^* : s \mapsto u_i(-s)$. Obviously, s is a Nash equilibrium for Γ^* if and only if so is $-s$ for Γ. Putting $S_{-i}^* = \prod_{l \in N \setminus \{i\}} S_l^*$ and noting that

$$D_- u_i^*(\cdot, s_{-i})(s_i) = -D^+ u_i(\cdot, -s_{-i})(-s_i) \text{ for all } (s_i, s_{-i}) \in \operatorname{int}(S_i^*) \times S_{-i}^*$$

for all $i \in N$, it can be easily verified by the reader that Γ^* satisfies all conditions of Theorem 5. This ensures that Γ^* has exactly one Nash equilibrium. We can conclude that Γ has exactly one Nash equilibrium. ■

5.2 Unbounded strategy sets

We shall now consider unbounded nice games and we shall extend Theorems 5 and 6 to the case of upper unbounded strategy sets. The compactness condition on the strategy sets cannot be simply dropped, and additional conditions must be imposed to guarantee the existence of a unique Nash equilibrium.

Theorem 7 *Let Γ be an unbounded nice game. Suppose there exists s^* in the topological interior of S such that*

$$\mathcal{D}_{u_i}\left(s_i^*, s_{-i}^*\right) < 0, \quad \text{for all } i \in N. \tag{7}$$

If conditions H1–3 of Theorem 5 are satisfied for all $i \in N$, then Γ has exactly one Nash equilibrium (and no i-th component of such an equilibrium equals ω_i).

Proof. We shall split the proof into two parts: existence and uniqueness. In the first part we shall construct a new game $\overline{\Gamma}$ which has a common Nash equilibrium with $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$. In the second part we shall prove the existence of at most one Nash equilibrium for Γ. Henceforth suppose H1–3 hold true.

Equilibrium existence. As usual, denote by b the joint best reply for Γ, but consider it as a partial function from S into S: b can be considered as a partial function because Γ is a nice game. Put

$$S_i^* = [\omega_i, s_i^*] \text{ for all } i \in N \text{ and } S^* = \prod_{i \in N} S_i^*.$$

Since Γ is a nice game, (7) ensures that each b_i is nonempty-valued at s^*; in particular $b_i(s^*) \in [\omega_i, s_i^*]$ for all $i \in N$. We can extend the previous conclusion to the entire S^* asserting that each $b_i|_{S^*}$ is a function into S_i^*: to verify this last fact it suffices to note that hypothesis H1 and Lemma C1 in Appendix C imply

$$\mathcal{D}_{u_i}(z) < 0 \text{ for all } z \in S \text{ such that } z_i = s_i^* \text{ and } z_l \leq s_l^* \text{ for all } l \in N \setminus \{i\}$$

and to repeat the reasoning for s^* at any such z. Therefore $b|_{S^*}$ can be understood as a self-map on S^*; this fact in turn implies that $b|_{S^*}$ coincides with the joint best reply, call it b^*, of the game $\Gamma^* = (N, (S_i^*)_{i \in N}, (u_i^*)_{i \in N})$ where

$$u_i^* = u_i|_{S^*} \text{ for all } i \in N.$$

As $b|_{S^*} = b^*$, the fixpoints of b and b^* coincide on S^*; consequently, each Nash equilibrium for Γ^* is also a Nash equilibrium for Γ. It can be easily seen that Γ^* satisfies all conditions of Theorem 5 and hence Γ^* has a (unique) Nash equilibrium, say e with $e_i > \omega_i$ for all $i \in N$; such point e is a Nash equilibrium also for Γ.

Equilibrium uniqueness. Suppose there exist two Nash equilibria for Γ, say $e^\bullet$ and e°. Let $\overline{\Gamma} = (N, (\overline{S}_i)_{i \in N}, (\overline{u}_i)_{i \in N})$ be the game where, for all $i \in N$,

$$\overline{S}_i = [\omega_i, \max\{e_i^\bullet, e_i^\circ\} + 1]$$

and $\overline{u}_i = u_i|_{\overline{S}}$ with $\overline{S} = \prod_{l \in N} \overline{S}_i$. Since $\overline{S}_i \subseteq S_i$ for all $i \in N$, we must have that $e^\bullet$ and e° are Nash equilibria also for $\overline{\Gamma}$ because $e^\bullet \in \overline{S} \ni e^\circ$. But then we have a contradiction, because $\overline{\Gamma}$ satisfies all conditions of Theorem 5 and hence it cannot possess two distinct Nash equilibria. ■

The following Theorem 8 is not a "dual" of Theorem 7, though the structure of their proofs is similar in many parts.

Theorem 8 *Let Γ be an unbounded nice game. Suppose*

$$\widetilde{\mathcal{D}}_{u_i}(t, \ldots, t) < 0 \textit{ for all } i \in N, \textit{ for all sufficiently large } t \in \mathbb{R}_{++}.$$

If conditions H1'–2' of Theorem 6 are satisfied for all $i \in N$, then Γ has exactly one Nash equilibrium.

Proof. The structure of the proof is similar to that of Theorem 7.

Equilibrium existence. By assumption there exists a point, say s^*, in the topological interior of S such that

$$\widetilde{\mathcal{D}}_{u_i}(s^*) < 0 \text{ for all } i \in N.$$

Now the proof is exactly the same proof of that of Theorem 7: just replace "$\mathcal{D}_{u_i}$" with "$\widetilde{\mathcal{D}}_{u_i}$" and "Theorem 5" with "Theorem 6".

Equilibrium uniqueness. Suppose there exist two Nash equilibria for Γ, say $e^{\bullet}$ and e°. Put

$$\underline{t} = \max\left\{\left|e_1^{\bullet}\right|, \ldots, \left|e_n^{\bullet}\right|, \left|e_1^{\circ}\right|, \ldots, \left|e_n^{\circ}\right|\right\}.$$

Choose $t > \underline{t}$ such that $\widetilde{\mathcal{D}}_{u_i}(t, \ldots, t) < 0$ for all $i \in N$ (such a point t can be found by assumption) and put

$$\overline{\alpha} = (t, \ldots, t) \in \mathbb{R}_{++}^n.$$

Thus we have

$$\widetilde{\mathcal{D}}_{u_i}(\overline{\alpha}_i, \overline{\alpha}_{-i}) < 0 \text{ for all } i \in N.$$

Let $\overline{\Gamma} = (N, (\overline{S}_i)_{i \in N}, (\overline{u}_i)_{i \in N})$ be the game where, for all $i \in N$, $\overline{S}_i = [\omega_i, \overline{\alpha}_i]$ and $\overline{u}_i = u_i|_{\overline{S}}$ with $\overline{S} = \prod_{l \in N} \overline{S}_i$. Since $\overline{S}_i \subseteq S_i$ for all $i \in N$, we must have that $e^{\bullet}$ and e° are Nash equilibria also for $\overline{\Gamma}$ because $e^{\bullet} \in \overline{S} \ni e^{\circ}$. However, $\overline{\Gamma}$ satisfies all conditions of Theorem 6 and hence it cannot have two distinct Nash equilibria. ■

6 Relation to other isotonicity theorems

We shall now relate our isotonicity Theorem 2 to other isotonicity theorems of the literature, clarifying the possible differences and what our result can add to the literature. In particular, we shall relate Theorem 2 to the isotonicity Theorem 4 in [23] and to the isotonicity Theorem 1 in [27].

Theorem 4 in [23] provides necessary and sufficient conditions for the isotonicity of (a selection from) the set of maximizers of a parameterized function in both the parameters and the choice sets (which—in [23]—are ordered under Veinott's "strong set order" $\leqslant_S$). The fact that the isotonicity is established in *both* the parameters and the choice sets is an important difference to our Theorem 2 (where the choice set is fixed). Thus, in principle, there is no reason to conjecture that Theorem 4 in [23] implies our Theorem 2, or *vice versa*. This is indeed the case: Theorem 2 is not implied by—and obviously does not imply—Theorem 4 in [23]. Nevertheless, in a CP where B is the Cartesian product of m subsets of $\mathbb{R}$ the quasiincreasingness of $\mathcal{D}_f$ does not imply, but is implied by, the single-crossing property[15] of f in $(a; b)$. The following Proposition 2 formally clarifies the point.

Proposition 2 *Consider a CP. Suppose B is the Cartesian product of m subsets of $\mathbb{R}$.*

(i) If f satisfies the single-crossing property in $(a; b)$ then $\mathcal{D}_f(a, \cdot)$ is quasiincreasing in every argument, for all $a \in \operatorname{int}(A)$.

(ii) The converse of (i) is generally false.

[15] Note that $f(\cdot, b)$ is obviously quasisupermodular for all $b \in B$.

Proof. (i) By Theorem 4 in [23], β must be isotone. Consequently $\mathcal{D}_f(a,\cdot)$ is quasiincreasing in every argument for all $a \in \operatorname{int}(A)$ by Theorem 2.

(ii) Put $A = B = [0,3]$ and consider the CP (A, B, f) where

$$f : (a,b) \mapsto 3 - (2-a)^2 - (b+1)\max\{0, (a-2)^5\}.$$

Note that $\mathcal{D}_f(a,\cdot)$ is quasiincreasing in every argument, for all $a \in \operatorname{int}(A)$ while f does not satisfy the single-crossing property in $(a;b)$: $\mathcal{D}_f(a,\cdot)$ is obviously quasiincreasing because

$$\mathcal{D}_f(a,b) = \begin{cases} 4 - 2a > 0 & \text{if } a < 2 \\ 0 & \text{if } a = 2 \\ 4 - 2a - 5(a-2)^4(b+1) < 0 & \text{if } a > 2 \end{cases}$$

but the function

$$\Delta : [0,3] \to \mathbb{R} : b \mapsto f(11/4, b) - f(1,b) = \frac{205}{1024} - \frac{243}{1024}b$$

is positive at $b = \frac{1}{10}$ and negative at $b = 2$, and hence f does not not satisfy the single crossing property. ■

The example in the proof of part (ii) of Proposition 2—and in particular Δ—gives evidence also of the fact that even the *if part* of Theorem 2 (resp. Corollary 2) does not follow from any Proposition or Theorem in [20] where at least one of the four conditions (7a), (7b), (7c), (7d) (resp. (8a), (8b), (8c), (8d)) is involved.

Theorem 1 in [27] provides necessary and sufficient conditions for the isotonicity of the C-function β associated to a CP. When attention is restricted to CPs, our Theorem 2 is equivalent to Theorem 1 in [27] in the precise sense that in a CP where B is the Cartesian product of m subsets of $\mathbb{R}$ the quasiincreasingness of $\mathcal{D}_f$ is equivalent to the condition that the family $\{f(\cdot,b)\}_{b\in B}$ satisfies the *interval dominance order* $\succeq_I$ (i.e., that it satisfies the implication

$$b' \in B,\, b'' \in B,\, b''_l \geq b'_l \text{ for all } l = 1, ..., m \;\Rightarrow f(\cdot, b'') \succeq_I f(\cdot, b').)$$

One of the contributions of our Theorem 2 is also the reformulation of the previous implication in terms of (generalized) derivatives. Such a reformulation is of interest because, like Proposition 2 in [27], it can be used to check whether a parametrized family of functions is an IDO family (i.e., a family of functions that obeys the interval dominance order). The following Example 1 shows that there exist cases where our Theorem 2 can be used to check whether a parametrized family of functions is an IDO family and Proposition 2 in [27] cannot.

Example 1 *Put $A = [0, \pi - 1]$ and $B = \{1/2, 1\}$ and consider the CP where*

$$f(\cdot, 1/2) : x \mapsto \sin(x + 1/2) + x \text{ and } f(\cdot, 1) : x \mapsto \sin(x+1) + x.$$

Theorem 2 certainly applies (because $\mathcal{D}_f(\cdot,1/2)$ and $\mathcal{D}_f(\cdot,1)$ are positive on their domain) and hence $\{f(\cdot,b)\}_{b\in B}$ is an IDO family by Theorem 1 in [27]. However, there does not exist any increasing positive function $\alpha : A \to \mathbb{R}$ such that

$$\mathcal{D}_f(\cdot,1)(x) \geq \alpha(x) \cdot \mathcal{D}_f(\cdot,1/2)(x) \text{ for almost all } x \in A$$

(because otherwise

$$\frac{\mathcal{D}_f(\cdot,1)(x)}{\mathcal{D}_f(\cdot,1/2)(x)} \geq \alpha(x) \text{ for a.a. } x \in \operatorname{int}(A) \text{ with } \alpha(x) \geq \alpha(0) > 0 \text{ for all } x \in A$$

in contradiction with the fact that $\mathcal{D}_f(\cdot,1/2)$ and $\mathcal{D}_f(\cdot,1)$ are continuous and positive and that

$$\lim_{x\uparrow\pi-1}\frac{\mathcal{D}_f(\cdot,1)(x)}{\mathcal{D}_f(\cdot,1/2)(x)} = \lim_{x\uparrow\pi-1}\frac{\cos(x+1)+1}{\cos(x+1/2)+1} = 0 = \frac{D_-f(\cdot,1)(\pi-1)}{D_-f(\cdot,1/2)(\pi-1)});$$

thus Proposition 2 in [27] cannot tell us whether $\{f(\cdot,b)\}_{b\in B}$ is an IDO family.

Remark 7 *In the previous example, $f(\cdot,1/2)$ and $f(\cdot,1)$ have continuously differentiable extensions. Clearly, our Theorem 2 can be applied also when $f(\cdot,b)$ is not continuous for some $b \in B$. For example, consider the variant of the CP in Example 1 where only f is modified by letting it be defined by*

$$f(\cdot,1/2) : x \mapsto \sin(x+1/2)+x+\lfloor 2x\rfloor \text{ and } f(\cdot,1) : x \mapsto \sin(x+1)+x+\lfloor 3x\rfloor.$$

Then (A,B,f) is still a CP and the functions $\mathcal{D}_f(\cdot,1/2)$ and $\mathcal{D}_f(\cdot,1)$ are still positive—possibly infinite somewhere, of course—and hence, again by Theorem 2 and by Theorem 1 in [27], we can conclude that $\{f(\cdot,b)\}_{b\in B}$ is an IDO family.

7 Relation to other uniqueness results

Our Theorem 5 does not imply Theorem 2 in [28] and our Theorem 7 does not imply either Theorem 5.1 in [18] or Theorem 4.1 in [10]. To show that our Theorem 5 is not implied by Theorem 2 in [28] and that our Theorem 7 is not implied by Theorem 5.1 in [18], we shall make use of the following elementary—but more restrictive—immediate corollary of Theorems 5 and 7.

Corollary 5 *Let Γ be a smooth nice game. Besides assume that one of the following two conditions holds: (i) S_i is bounded for all $i \in N$; (ii) S_i is unbounded for all $i \in N$ and there exists s^* in $\mathbb{R}^n_{++}$ such that $\mathcal{M}_i(s^*) < 0$ for all $i \in N$. Then Γ has exactly one Nash equilibrium if, for all $i \in N$:*

H1" $\mathcal{M}_i$ is quasiincreasing in the j-th argument, for all $j \in N\setminus\{i\}$;

H2" $\mathcal{M}_i$ is quasiconcave;

H3" $\mathcal{M}_i(\omega) > 0$.

The following Fact will show that in a certain class of symmetric games both the conditions of Theorem 2 in [28] and those of Theorem 5.1 in [18] imply the decreasingness of a certain "balanced" marginal utility function that we shall denote by ξ, while the conditions of our Theorems 5 and 7 are compatible with its strict increasingness on some (infinite) subsets of its domain. Part (vi) will show an example that satisfies all conditions of Theorem 7 but not the "diagonally dominance" condition assumed in Theorem 4.1 in [10]: hence our Theorem 7 is not implied by Theorem 4.1 in [10].

Fact *Suppose Γ is a symmetric smooth nice game where $S_1 = \ldots = S_n = X$.*[16] *Pick an arbitrary $j \in N$ and let*

$$\xi : \mathrm{int}\,(X) \to \mathbb{R} : x \mapsto \mathcal{M}_j\,(x, ..., x)\,.$$

(i) *If Γ satisfies all conditions of [28]'s Theorem 2 then ξ must be decreasing.*

(ii) *If Γ satisfies all conditions of [18]'s Theorem 5.1 then ξ must be decreasing.*

(iii) *If Γ satisfies all conditions of Corollary 5 then ξ must be quasiconcave.*

(iv) *There exists a specification for Γ with $X = [0, 1]$ such that Γ satisfies all conditions of Corollary 5 and ξ is strictly increasing on $X' =]0, 1/12[$.*

(v) *There exists a specification for Γ with $X = \mathbb{R}_+$ such that Γ satisfies all conditions of Corollary 5 and ξ is strictly increasing on $X' =]0, 1/12[$.*

(vi) *There exists a specification for Γ with $X = \mathbb{R}_+$ such that Γ satisfies all conditions of Corollary 5 and*

$$\left|\frac{\partial^2 u_1}{\partial s_1 \partial s_1}\,(1/24)\right| < \left|\frac{\partial^2 u_1}{\partial s_1 \partial s_2}\,(1/24)\right| + \ldots + \left|\frac{\partial^2 u_1}{\partial s_1 \partial s_n}\,(1/24)\right|.$$

Proof. (i) The proof is immediate; however, the following fact must be remarked: if Γ satisfies the conditions of Theorem 2 in [28], then (3.10) in [28] implies

$$\left(x^1 - x^0\right) \xi\left(x^0\right) \sum_{i \in N} r_i + \left(x^0 - x^1\right) \xi\left(x^1\right) \sum_{i \in N} r_i > 0$$

for all $\left(x^0, x^1\right) \in]0, 1[\times]0, 1[$ such that $x^0 < x^1$ (in [28] r is an element of $\mathbb{R}^n_+$ such that $\sum_{i \in N} r_i > 0$)—or equivalently, as $\left(x^1 - x^0\right) \sum_{i \in N} r_i > 0$,

$$\xi\left(x^0\right) > \xi\left(x^1\right)$$

[16] There are various notions of symmetry in the literature. For this Fact one is free to choose any definition that satisfies the following implication:

$$i^\bullet \in N,\ i^\circ \in N \text{ and } x \in X \Longrightarrow u_{i^\bullet}\,(x, ..., x) = u_{i^\circ}\,(x, ..., x)\,.$$

for all $\left(x^0, x^1\right) \in \left]0,1\right[\times \left]0,1\right[$ such that $x^0 < x^1$.

(ii) Analogous to the previous proof.

(iii) A consequence of the assumptions and of the definition of ξ.

(iv) Consider the following symmetric game $\Gamma = \left(N, (S_i)_{i\in N}, (u_i)_{i\in N}\right)$ where $N = \{1,2\}$, $S_1 = S_2 = X = [0,1]$, $u_1(s_1,s_2) = 2s_1(2+2s_2) - 8s_1^3$ and $u_2(s_1,s_2) = 2s_2(2+2s_1) - 8s_2^3$. In this case, Γ satisfies all conditions of Corollary 5 and the function

$$\xi : [0,1] \to \mathbb{R} : t \mapsto -24t^2 + 4t + 4$$

is strictly increasing on $X' = \left]0, 1/12\right[$.

(v) Exactly the same example exhibited in the proof of (iii), but for the specification of X: now put $S_1 = S_2 = X = \mathbb{R}_+$.

(vi) Exactly the modified example in (v): just note that

$$\frac{\partial^2 u_1}{\partial s_1 \partial s_1}(1/24) = -2 < 4 = \frac{\partial^2 u_1}{\partial s_1 \partial s_2}(1/24)$$

and conclude that the also part (vi) is true. ■

The previous Fact has explained the key difference between our uniqueness results and Rosen's and Karamardian's uniqueness theorems (as well as all uniqueness results of the literature—like Theorem 1 in [16]—that follow from those two theorems but do not extend them). Also, by part (vi) of the previous Fact, we can conclude that our Theorem 7 does not follows from Theorem 4.1 in [10]. Finally, we remark that Corollary 5 holds true even when we replace H2" with the weaker condition H2* below:

H2* $\mathcal{M}_i$ has a chain-convex upper level set at height 0.

Clearly, when one replace H2" with the weaker condition H2*, the conclusions of part (iii) of the previous Fact do not generally hold true anymore.

8 Incomplete information

Our results on uniqueness of equilibria extend to some frameworks of incomplete information where type sets are finite. Henceforth, by a **Bayesian game** we mean[17] a quintuple

$$G = \left(M, (Z_l)_{l\in M}, (T_l)_{l\in M}, ((p_l(\cdot|\theta))_{\theta\in T_l})_{l\in M}, (v_l)_{l\in M}\right)$$

where $M = \{1,...,m\}$, with $m > 1$, is a finite set of elements called players and for all $l \in M$:

- Z_l is a nonempty set of elements called player l's strategies;

[17] In fact we are following the interim formulation of the Bayesian game as described in Sect. 3 of [33].

- T_l is a nonempty finite set of elements called player l's types;
- $p_l(\cdot|\theta) : T_{-l} \to [0,1]$ is, for all $\theta \in T_l$, a probability measure[18] on the set

$$T_{-l} := \prod_{k \in M \setminus \{l\}} T_k;$$

- $\upsilon_l : Z_l \times Z_{-l} \times T_l \times T_{-l} \to \mathbb{R}$ is a function that associates a payoff to player l with each joint strategy (z_l, z_{-l}) in $Z_l \times Z_{-l}$ and each joint type $(t_l, t_{-l}) \in T_l \times T_{-l}$, where

$$Z_{-l} := \prod_{k \in M \setminus \{l\}} Z_k.$$

Henceforth we assume that $(l^\bullet, l^\circ) \in M \times M$ and $l^\bullet \neq l^\circ \Rightarrow T_{l^\bullet} \cap T_{l^\circ} = \emptyset$.

A **Bayesian Nash equilibrium** for a Bayesian game G is an m-tuple

$$\sigma = (\sigma_l : T_l \to Z_l)_{l \in M}$$

of functions such that, for all $l \in M$,

$$\sigma_l(\theta) \in \arg\max \sum_{\tau \in T_{-l}} \upsilon_l(\cdot, \sigma_{-l}(\tau), \theta, \tau) \cdot p_l(\tau|\theta) \text{ for all } \theta \in T_l$$

where $\sigma_{-l}(\tau) = (\sigma_k(\tau_k))_{k \in M \setminus \{l\}}$. Henceforth, for all $x \in \bigcup_{l \in M} T_l$, we denote by $\hat{x}$ the (only) element of M such that $x \in T_{\hat{x}}$. Besides, given a set K and a tuple $\tau = (\tau_k)_{k \in K}$, we denote $\{\tau_k : k \in K\}$ by $[\tau]$.

By a **Complete information game** $\Gamma = \left(N, (S_i)_{i \in N}, (u_i)_{i \in N}\right)$ **associated to a Bayesian game** G we mean a game where

$$N = \bigcup_{l \in M} T_l$$

and where, for all $i \in N$,

$$S_i = Z_{\hat{\imath}} \quad \text{and} \quad u_i(s) = \sum_{\tau \in T_{-\hat{\imath}}} \upsilon_{\hat{\imath}}(s_i, (s_k)_{k \in [\tau]}, i, \tau) \cdot p_{\hat{\imath}}(\tau|i).$$

(Thus, above, $k \in [\tau]$ is a shorthand for $k \in \{\tau_1, \ldots, \tau_{\hat{\imath}-1}, \tau_{\hat{\imath}+1}, \ldots, \tau_m\}$.)

A Nash equilibrium for Γ is a joint strategy $s \in S$ such that

$$s_i \in \arg\max \sum_{\tau \in T_{-\hat{\imath}}} \upsilon_{\hat{\imath}}(\cdot, (s_k)_{k \in [\tau]}, i, \tau) \cdot p_{\hat{\imath}}(\tau|i) \text{ for all } i \in N,$$

[18] Henceforth we shall write $p_l(\tau|\theta)$ instead of $p_l(\cdot|\theta)(\tau)$. Clearly, $\sum_{\tau \in T_{-l}} p_l(\tau|\theta) = 1$. One might interpret $p_l(\tau|\theta)$ as the conditional probability for l that the joint type of l's opponents is τ when l's type is θ. However such an interpretation is not very important here.

or equivalently, just substituting the symbol i with θ,

$$s_\theta \in \arg\max \sum_{\tau\in T_{-\hat{\theta}}} \upsilon_{\hat{\theta}}(\cdot, (s_k)_{k\in[\tau]}, \theta, \tau) \cdot p_{\hat{\theta}}(\tau|\theta) \text{ for all } \theta \in N.$$

Thus, since $\{T_1, ..., T_m\}$ is a partition of N, we have that a Nash equilibrium for Γ is a joint strategy $s \in S$ such that, for all $l \in M$,

$$s_\theta \in \arg\max \sum_{\tau\in T_{-l}} \upsilon_l(\cdot, (s_k)_{k\in[\tau]}, \theta, \tau) \cdot p_l(\tau|\theta) \text{ for all } \theta \in T_l.$$

Remark 8 *Let G be a Bayesian game and Γ be the associated complete information game. If $(\overline{\sigma}_l : T_l \to Z_l : \theta \mapsto \overline{s}_\theta)_{l\in M}$ is a Bayesian Nash equilibrium for G then $(\overline{s}_\theta)_{\theta\in N}$ is a Nash equilibrium for Γ. If $(\overline{s}_\theta)_{\theta\in N}$ is a Nash equilibrium for Γ then $(\overline{\sigma}_l : T_l \to Z_l : \theta \mapsto \overline{s}_\theta)_{l\in M}$ is a Bayesian Nash equilibrium for G.*

Corollary 6 *Let G be a Bayesian game. For all $l \in M$ and for all $(y, \theta, \tau) \in Z_{-l} \times T_l \times T_{-l}$, assume that Z_l is a proper compact real interval with minimum ω_l and that the function $\upsilon_l(\cdot, y, \theta, \tau)$ is strictly concave and continuous. Put $\omega = (\omega_l)_{l\in M}$. Then G has exactly one Bayesian Nash equilibrium if, for all $l \in M$ and for all $(\theta, \tau) \in T_l \times T_{-l}$, the function*

$$\mathcal{D}^l_{\theta,\tau} : \operatorname{int}(Z_l) \times Z_{-l} \to \mathbb{R} : (x, y) \mapsto D_-\upsilon_l(\cdot, y, \theta, \tau)(x)$$

is:

(i) increasing in every argument other than the l-th one;

(ii) chain-concave;

(iii) such that $\mathcal{D}^l_{\theta,\tau}(\cdot, \omega_{-l})$ is not nonpositive.

Proof. It is easily verified that, by the definition of u_i, the game Γ satisfies all conditions[19] of Theorem 5. Hence Γ has exactly one Nash equilibrium. By Remark 8, G has exactly one Bayesian Nash equilibrium. ■

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[19] In particular, Γ is a compact nice game such that each u_i is strictly concave in the i-th argument and hence each

$$\mathcal{D}_{u_i} : \operatorname{int}(S_i) \times S_{-i} \to \mathbb{R} : (s_i, s_{-i}) \mapsto D_-u_i(\cdot, s_{-i})(s_i),$$

is defined by

$$D_-u_i(\cdot, s_{-i})(s_i) = \sum_{\tau\in T_{-\hat{\imath}}} D_-\upsilon_{\hat{\imath}}(\cdot, (s_k)_{k\in[\tau]}, i, \tau)(s_i) \cdot p_{\hat{\imath}}(\tau|i),$$

and is: (i) increasing in every argument other than the i-th one (i.e., in every argument $j \in N\setminus\{i\}$); (ii) chain-concave; (iii) such that $\mathcal{D}_{u_i}(\cdot, (\omega_{\hat{\jmath}})_{j\in N\setminus\{i\}})$ is not nonpositive.

Appendix

Appendix A: Fixpoint uniqueness

Theorem A1 *Let I be a finite index set and $\{F_i\}_{i\in I}$ be a family of compact proper real intervals. Let f be a self-map of $F = \prod_{i\in I} F_i$. Assume that each component f_i of f is isotone and chain-concave. Let*

$$F^{\llcorner} := \{t \in F : \min\{t_i - \min F_i : i \in I\} = 0\}.$$

Suppose no fixpoint for f exists in $F^{\llcorner}$. Then f has exactly one fixpoint.

Proof. Each $(F_i, \leq)$ is a complete lattice, where $\leq$ denotes the usual partial order relation on $\mathbb{R}$ induced[20] on F_i. Denote by $\preceq$ the usual product partial order relation on F. Also $(F, \preceq)$ is a complete lattice. By Tarski's fixpoint theorem there exist a least fixpoint for f, say y, and a greatest fixpoint for f, say z. We are done if we prove that $y = z$. By contradiction, suppose $y \neq z$. Note that

$$\min F_i < y_i \leq z_i \text{ for all } i \in I,$$

where the first inequality holds because y is a fixpoint for f and f has not fixpoints in $F^{\llcorner}$ and the second because z is the greatest fixpoint for f. Let

$$y_{\downarrow} := \{t \in F : t \preceq y\} \text{ and } y_{\downarrow\downarrow} := y_{\downarrow} \setminus \{y\},$$

and let $\operatorname{aff}(\{y, z\})$ denote the affine hull of $\{y, z\}$. It can be easily observed that $\operatorname{aff}(\{y, z\}) \cap y_{\downarrow\downarrow} \neq \emptyset$. Pick

$$x \in (\operatorname{aff}(\{y, z\}) \cap y_{\downarrow\downarrow});$$

by construction, we can choose $\gamma \in \,]0, 1[$ such that

$$y = \gamma x + (1 - \gamma) z.$$

By Tarski's fixpoint theorem (see the last equality in the statement of Theorem 1 in [30]), $f(t) \npreceq t$ for all $t \in y_{\downarrow\downarrow}$. Then

$$x_l < f_l(x) \text{ for some } l \in I.$$

Since $f_l(y) - y_l = f_l(z) - z_l = 0 < f_l(x) - x_l$, we have

$$f_l(y) - y_l < \gamma (f_l(x) - x_l) + (1 - \gamma)(f_l(z) - z_l);$$

hence, since $y_l = \gamma x_l + (1 - \gamma) z_l$, we have

$$f_l(y) < \gamma f_l(x) + (1 - \gamma) f_l(z).$$

But the last strict inequality contradicts the chain-concavity of f_l. ■

The "dual" of Theorem A1 is the following Corollary A1 (which implicitly underlies Theorems 6 and 8).

[20] The lack of an index for $\leq$ (i.e., the fact that we write $\leq$ instead of the more correct $\leq_i$) should not be a source of confusion.

Corollary A1 *Let I be a finite index set and $\{F_i\}_{i\in I}$ be a family of compact proper real intervals. Let f be a self-map of $F = \prod_{i\in I} F_i$. Assume that each component f_i of f is isotone and chain-convex. Let*

$$F^{\urcorner} := \{t \in F : \max\{t_i - \max F_i : i \in I\} = 0\}.$$

Suppose no fixpoint for f exists in $F^{\urcorner}$. Then f has exactly one fixpoint.

Appendix B: An equivalence proposition

Proposition B1 *Let I be a proper closed real interval I with a minimum, say w. Suppose $f : I \to \mathbb{R}$ is strictly pseudoconcave and upper semicontinuous. Then $D^+f(w) > 0$ if and only if D_-f is not nonpositive on* $\mathrm{int}(I)$.
Proof. *If part.* Suppose D_-f is not nonpositive on $\mathrm{int}(I)$. Then there exists $k \in \mathrm{int}(I)$ such that $D_-f(k) > 0$. The upper semicontinuous strictly pseudoconcave function $f|_{[w,k]}$ has a unique maximizer: call it x^*. Note that $w < x^*$ (otherwise the strict pseudoconcavity of $f|_{[w,k]}$ would imply $D_-f(k) < 0$). Thus the inequality $D^+f(w) > 0$ follows from the definition of a strictly pseudoconcave function.

Only if part. Suppose $D^+f(w) > 0$. Let $k \in \mathrm{int}(I)$. The upper semicontinuous and strictly pseudoconcave function $f|_{[w,k]}$ has a unique maximizer, say x, and such a maximizer cannot be w, as $D^+f(w) > 0$. Thus $f(w) < f(x)$. By the upper semicontinuity of f, we can well-define

$$y := \min\left\{z \in [w,x] : f(z) \geq f(w) + \frac{f(x) - f(w)}{3} + (z - w)\frac{f(x) - f(w)}{3(x - w)}\right\}.$$

Note that by construction

$$y \in \mathrm{int}(I)$$

and

$$f(z) < f(w) + \frac{f(x) - f(w)}{3} + (z - w)\frac{f(x) - f(w)}{3(x - w)} \text{ for all } z \in [w, y[\,.$$

Since

$$f(y) \geq f(w) + \frac{f(x) - f(w)}{3} + (y - w)\frac{f(x) - f(w)}{3(x - w)},$$

we can conclude that

$$f(y) - f(z) \geq (y - z)\frac{f(x) - f(w)}{3(x - w)} \text{ for all } z \in [w, y].$$

Thus $D_-f(y) = \liminf_{z\uparrow y} \frac{f(y)-f(z)}{y-z} \geq \frac{1}{3}\frac{f(x)-f(w)}{x-w} > 0$. ■

Corollary B1 *Let I be a proper closed real interval I with a maximum, say a. Suppose $f : I \to \mathbb{R}$ is strictly pseudoconcave and upper semicontinuous. Then $D_-f(a) < 0$ if and only if D^+f is not nonnegative on* $\mathrm{int}(I)$.
Proof. Just reverse the order of I and apply Proposition B1. ■

Appendix C: An equivalence lemma

Lemma C1 *Let $A \subseteq \mathbb{R}$ be a proper interval and B be the Cartesian product of m subsets of $\mathbb{R}$. Let $f : A \times B \to \mathbb{R}$ and suppose $f(\cdot, b)$ is strictly pseudoconcave and upper semicontinuous for all $b \in B$. Let*

$$\mathcal{D}_f : \mathrm{int}(A) \times B \to \mathbb{R} \cup \{-\infty, +\infty\} : (a, b) \mapsto D_- f(\cdot, b)(a).$$

Pick an arbitrary $\check{a} \in \mathrm{int}(A)$. Then $\mathcal{D}_f(\check{a}, \cdot)$ is quasiincreasing in every argument if and only if

$$(\underline{x}, \overline{x}) \in B \times B,\ \underline{x}_l \leq \overline{x}_l \text{ for all } l = 1, ..., m \text{ and } \mathcal{D}_f(\check{a}, \cdot)(\underline{x}) \geq 0$$
$$\Downarrow$$
$$\mathcal{D}_f(\check{a}, \cdot)(\overline{x}) \geq 0.$$

Proof. *If part.* The proof is immediate and omitted.

Only if part. By way of contradiction, suppose that $\mathcal{D}_f(\check{a}, \cdot)$ is quasiincreasing in every argument and that there exists a pair $(x^*, x^{**}) \in B \times B$ such that $x_l^* \leq x_l^{**}$ for all $l = 1, ..., m$, $\mathcal{D}_f(\check{a}, \cdot)(x^*) \geq 0$ and $\mathcal{D}_f(\check{a}, \cdot)(x^{**}) < 0$. Then

$$\mathcal{D}_f(\check{a}, \cdot)(x^*) = 0$$

by the quasiincreasingness of $f(\check{a}, \cdot)$ in every argument.

Since $\mathcal{D}_f(\check{a}, \cdot)(x^{**}) < 0$, there exists $\underline{a} \in A$ such that $\underline{a} < \check{a}$ and $f(\cdot, x^{**})(\underline{a}) > f(\cdot, x^{**})(\check{a})$. Consequently, there exists $\hat{a} \in [\underline{a}, \check{a}[$ that maximizes the upper semicontinuous (and strictly pseudoconcave) function $f(\cdot, x^{**})|_{[\underline{a}, \check{a}]}$; hence

$$\mathcal{D}_f(\cdot, x^{**})(a) < 0 \text{ for all } a \in]\hat{a}, \check{a}[$$

by the strict pseudoconcavity of $f(\cdot, x^{**})|_{[\underline{a}, \check{a}]}$.

Since $\mathcal{D}_f(\check{a}, \cdot)(x^*) = 0$, the definition of a strictly pseudoconcave function implies that

$$f(\cdot, x^*)(\hat{a}) < f(\cdot, x^*)(\check{a});$$

therefore

$$\mathcal{D}_f(\check{a}, \cdot)(x^*) = 0 < \frac{f(\cdot, x^*)(\check{a}) - f(\cdot, x^*)(\hat{a})}{\check{a} - \hat{a}},$$

and[21] hence, by part (ii) of Theorem 1.8 in [11], there exists $a^\circ \in]\hat{a}, \check{a}[$ such that

$$\mathcal{D}_f(\cdot, x^*)(a^\circ) \geq \frac{f(\cdot, x^*)(\check{a}) - f(\cdot, x^*)(\hat{a})}{\check{a} - \hat{a}} > 0.$$

Thus

$$\mathcal{D}_f(a^\circ, \cdot)(x^*) > 0 > \mathcal{D}_f(a^\circ, \cdot)(x^{**})$$

[21] Note that $\mathcal{D}_f(\check{a}, \cdot)(x^*) = 0 < \frac{f(\cdot, x^*)(\check{a}) - f(\cdot, x^*)(\hat{a})}{\check{a} - \hat{a}}$ implies that, for some $t \in]\hat{a}, \check{a}[$,

$$f(\cdot, x^*)(t) - f(\cdot, x^*)(\hat{a}) > \frac{f(\cdot, x^*)(\check{a}) - f(\cdot, x^*)(\hat{a})}{\check{a} - \hat{a}}(t - \hat{a}).$$

in contradiction with the quasiincreasingness of $\mathcal{D}_f(a^\circ, \cdot)$ in every argument. ■

The conclusion of Lemma C1 is not "*Then* $\mathcal{D}_f(\check{a}, \cdot)$ *is quasiincreasing in every argument if and only if* $\mathcal{D}_f(\check{a}, \cdot)$ *is pseudoincreasing in every argument*". Example C1 shows that such a conclusion would be wrong.

Example C1 *Consider the CP where* $A = B = [0, 2]$ *and*

$$f : A \times B \to \mathbb{R} : (a, b) \mapsto \begin{cases} 6 - |1 - a| & \text{if } b = 0, \\ a(2 - a) & \text{if } b > 0. \end{cases}$$

Note that $f(\cdot, b)$ *is strictly pseudoconcave and continuous, that*

$$\mathcal{D}_f(1, 0) > 0 = \mathcal{D}_f(1, b) \text{ for all } b > 0$$

and that $\mathcal{D}_f(a, \cdot)$ *is positive for* $a \in \,]0, 1[$ *and negative for all* $a \in \,]1, 2[$. *We can conclude that* $\mathcal{D}_f$ *is quasiincreasing but not pseudoincreasing.*

Appendix D: Theorem 2 revisited

Definition D1 *By a* ***Choice Problem***$^\#$ *(**CP**$^\#$ in short) we mean a triple* (A, B, f) *where: (i)* A *is a compact proper real interval; (ii)* B *is a nonempty subset of* $\mathbb{R}^m$ *with* $m \in \mathbb{N}$*; (iii)* f *is a function from* $A \times B$ *into* $\mathbb{R}$ *such that* $f(\cdot, b)$ *is strictly quasiconcave and upper semicontinuous for all* $b \in B$.

With a CP$^\#$ we associate a C-function β and a function $\mathcal{D}_f$: their definitions are analogous to those of the functions β and $\mathcal{D}_f$ associated to a CP (Sect. 3).

Definition D2 *By a* ***Choice Problem***$^{\#\#}$ *(**CP**$^{\#\#}$ in short) we mean a CP*$^\#$ *where* $f(\cdot, b)$ *is continuously differentiable on* $\operatorname{int}(A)$ *for all* $b \in B$.

Clearly, any CP is also a CP$^\#$ (but the converse is not generally true) and any CP$^{\#\#}$ is also a CP$^\#$ (but the converse is not generally true).

Theorem D1 *Consider a CP*$^\#$ *and the associated C-function* β. *Suppose* B *is the Cartesian product of* m *subsets of* $\mathbb{R}$. *Then,* β *is isotone only if* $D_f(a, \cdot)$ *is quasiincreasing in every argument for all* $a \in \operatorname{int}(A)$.

Proof. Assume that β is isotone and by way of contradiction suppose that $\mathcal{D}_f(\overline{a}, \cdot)$ is not quasiincreasing in the j-th argument for some $\overline{a} \in \operatorname{int}(A)$. Then there exist $\overline{a} \in \operatorname{int}(A)$, $x \in B$ and $y \in B$ such that $x_j < y_j$, $x_l = y_l$ for all $l = \{1, ..., m\} \setminus \{j\}$ and

$$D_- f(\cdot, x)(\overline{a}) > 0 > D_- f(\cdot, y)(\overline{a}).$$

Corollary 2.5.2 in [5] implies that $f(\cdot, x)$ is strictly decreasing on $[\beta(x), \max A]$ and that $f(\cdot, y)$ is strictly increasing on $[\min A, \beta(y)]$. Thus

$$a \in A \text{ and } a > \beta(x) \;\; \Rightarrow D_- f(\cdot, x)(a) \leq 0$$

and

$$a \in A \text{ and } \min A < a \leq \beta(y) \ \Rightarrow D_{-}f(\cdot,y)(a) \geq 0.$$

Therefore $\overline{a} \leq \beta(x)$ and $\overline{a} > \beta(y)$, which implies $\beta(y) < \beta(y)$ in contradiction with the isotonicity of β. ■

If attention is restricted to "continuously differentiable" CPs, Theorem D2 generalizes Theorem 2. We do conjecture, however, that in Theorem 2 the strict pseudoconcavity of each $f(\cdot,b)$ cannot be replaced by the strict quasiconcavity of each $f(\cdot,b)$ unless additional assumptions on either the type of continuity or the degree of differentiability of each $f(\cdot,b)$ are imposed (but, in this regard, we do not have an illuminating example that clearly disproves our conjecture).

Theorem D2 *Consider a CP$^{\#\#}$ and the associated C-function β. Suppose B is the Cartesian product of m subsets of $\mathbb{R}$. Then, β is isotone (resp. antitone) if and only if $\mathcal{D}_f(a,\cdot)$ is*[22] *quasiincreasing (resp. quasidecreasing) in every argument for all $a \in \operatorname{int}(A)$.*

Proof. We shall consider only the case of isotonicity of β: the case of antitonicity of β is nothing but the dual. Also, the *only if* part follows from Theorem D1: thus we shall prove only the *if part*. Assume that $\mathcal{D}_f(a,\cdot)$ is quasiincreasing in every argument for all $a \in \operatorname{int}(A)$. Pick $(x,y) \in B \times B$ such that $x \neq y$ and $x_l \leq y_l$ for all $l = 1,...,m$. By way of contradiction suppose

$$\beta(y) < \beta(x).$$

By Corollary 2.5.2 in [5] we have that $f(\cdot,x)$ is strictly increasing on $[\beta(y),\beta(x)]$. Pick a pair $(\underline{a},\overline{a})$ such that

$$\beta(y) < \underline{a} < \overline{a} < \beta(x).$$

Since $f(\cdot,x)$ is strictly increasing on $[\underline{a},\overline{a}] \subset]\beta(y),\beta(x)[$, a well known result of Real Analysis ensures that

$$Df(\cdot,x)(a^*) > 0 \text{ for some } a^* \in]\underline{a},\overline{a}].$$

Thus, as $f(\cdot,x)$ is continuously differentiable on $]\beta(y),\beta(x)[$, we have that there exists a proper closed interval $I \subset]\underline{a},\overline{a}[$ such that $a^* \in \operatorname{int}(I)$ and

$$Df(\cdot,x)(a) > 0 \text{ for all } a \in I.$$

Therefore, as $f(\cdot,y)$ is differentiable on $\operatorname{int}(A)$ and $\mathcal{D}_f(a,\cdot)$ is quasiincreasing in every argument for all $a \in \operatorname{int}(A)$, we have that

$$Df(\cdot,y)(a) \geq 0 \text{ for all } a \in I.$$

Hence, by part (i) of Theorem 1.13 in [11], $f(\cdot,y)$ must be increasing on $\operatorname{int}(I)$, where

$$\emptyset \neq \operatorname{int}(I) \subset [\beta(y),\max A];$$

but this is in contradiction with Corollary 2.5.2 in [5], by which $f(\cdot,y)$ should be strictly decreasing on $[\beta(y),\max A]$. ■

[22] Clearly, $D_{-}f(\cdot,b)(a) = Df(\cdot,b)(a)$ for all $b \in B$ and all $a \in \operatorname{int}(A)$.

Appendix E: On some properties of best reply functions

To the reader's convenience, we provide some facts that can be readily inferred from the results in Sect. 4 (Corollaries E1–3) and Appendix D (Corollary E4).

Corollary E1 *Let Γ be a compact nice game Γ and $i \in N$. Then player i's best reply function b_i is isotone (resp. antitone) if and only if $\mathcal{D}_{u_i}$ is quasiincreasing (resp. quasidecreasing) in the j-th argument, for all $j \in N \setminus \{i\}$.*

The variant—in terms of the upper right-hand marginal utility function $\widetilde{\mathcal{D}}_{u_i}$—of Corollary E1 that follows from Corollary 2 is left to the reader.

Corollary E2 *Let Γ be a compact nice game Γ and $i \in N$. Assume that $\mathcal{D}_{u_i}(\cdot, \omega_{-i})$ is not nonpositive (resp. that $\widetilde{\mathcal{D}}_{u_i}(\cdot, \alpha_{-i})$ is not nonnegative) and that $\mathcal{D}_{u_i}$ (resp. $\widetilde{\mathcal{D}}_{u_i}$) is quasiincreasing. Then:*

- (i) *player i's best reply function b_i is isotone and greater than ω_i;*
- (ii) *player i's best reply function b_i is chain-concave (resp. chain-convex) if and only if $\mathcal{D}_{u_i}$ (resp. $\widetilde{\mathcal{D}}_{u_i}$) has a chain-convex upper (resp. lower) level set at height 0.*

The variant of Corollary E2 that concerns the case of an antitone best reply function is left to the reader.

Corollary E3 *Let Γ be a compact nice game Γ and $i \in N$. Assume that $\mathcal{D}_{u_i}(\cdot, s_{-i})$ is not nonpositive (resp. $\widetilde{\mathcal{D}}_{u_i}(\cdot, s_{-i})$ is not nonnegative) for all $s_{-i} \in S_{-i}$. Then:*

- (i) *player i's best reply function b_i is greater than ω_i (resp. smaller than α_i);*
- (ii) *player i's best reply function b_i is concave (resp. convex) if and only if $\mathcal{D}_{u_i}$ (resp. $\widetilde{\mathcal{D}}_{u_i}$) has a convex upper (resp. lower) level set at height 0.*

The variant of Corollary E3 which establishes the convexity of the set $\Omega_+ = \{s \in S : b_i(s) > \omega_i\}$ (resp. of $\Omega^+ = \{s \in S : b_i(s) < \alpha_i\}$) and the concavity (resp. the convexity) of b_i on Ω_+ (resp. Ω^+) is left to the reader.

Remark E1 *Note that Corollary E3 implies a result for "linear" best reply functions: just impose that b_i is both concave and convex.*

Introducing some definitions, we obtain a simple extension of Corollary E1 for the case of continuous differentiability of u_i in the i-th argument on $\mathrm{int}(S_i)$.

Definition E1 *We say that a game* Γ *is a **smooth**$^{\#}$ **compact nice**$^{\#}$ **game** if, for all* $i \in N$*:*

- S_i *is a proper compact real interval;*
- u_i *is upper semicontinuous in the* i*-th argument;*
- u_i *is strictly quasiconcave in the* i*-th argument;*
- u_i *is continuously differentiable in the* i*-th argument on* $\mathrm{int}\,(S_i)$*.*

Notation ($\mathcal{M}_i^{\#}$) *Given a smooth*$^{\#}$ *compact nice*$^{\#}$ *game* Γ*, we denote player* i*'s "**marginal**$^{\#}$ **utility function**" by* $\mathcal{M}_i^{\#} : \mathrm{int}\,(S_i) \times S_{-i} \to \mathbb{R} : s \mapsto \frac{\partial u_i}{\partial s_i}(s)$*.*

Corollary E4 *Let* Γ *be a smooth*$^{\#}$ *compact nice*$^{\#}$ *game* Γ *and* $i \in N$*. Player* i*'s best reply function is isotone (resp. antitone) if and only if player* i*'s marginal*$^{\#}$ *utility function* $\mathcal{M}_i^{\#}$ *is quasiincreasing (resp. quasidecreasing) in the* j*-th argument, for all* $j \in N \setminus \{i\}$*.*

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