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Abstract
This paper presents, in brief, the fundamentals of optimal control theory together with some notes for differential games, which is the game theoretic analogue of the optimal control. As it is recommended by literature references the main tool of analysis in open loop information structure for environmental models is the Pontryagin's Maximum Principle, while the Hamilton–Jacobi–Bellman equation is the tool of analysis for any closed loop informational structure. As applications of the above theoretic considerations we present some environmental economic models which are solved both as optimal control problems and as differential games as well.

Keywords: Optimal control; Differential games; Renewable resources; Environnemental and Resource Economies.

JEL Classification Codes: C61; C62; D43; Q0; Q2; Q20; Q50; Q52; Q53.

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1. The history of the Optimal Control Theory

Optimal control is one of many strands of control theory which uses mathematical methods to address a wide area of applications in many scientific fields. The mathematics of optimal control theory is the generalization of the ancient theory called "calculus of variations". The early applications in calculus of variations were in physics, since 1662 Fermat derived "*the law of refraction*" as a solution to a minimum time problem. Only after more than 250 years, in 1924, Evans studied a dynamic economic model for monopolists, whereas Ramsey (1928), using techniques of calculus of variations, solved the famous capital accumulation model (the well known Ramsey model). The first environmental model analyzed with the calculus of variations was the optimal exploitation of exhaustible resources, first proposed by Hotelling (1931). To begin with optimal control theory it is better to set the statement of a calculus of variations problem and then to compare with the same optimal control problem statement and solution.

The fundamental calculus of variations problem appears as an optimization problem of the form:

$$\begin{array}{lll} \text{maximize} \quad \text{or} \quad \text{minimize} & V[x] = \int_0^T F\left[t, x(t), \frac{dx(t)}{dt}\right] dt & \\ \text{subject to} & x(0) = A \quad (\text{A given}) & (1) \\ \text{and} & x(T) = Z \quad (\text{T, Z given}) & \end{array}$$

The task of the calculus of variations is to select from a set of admissible x paths the one that yields an extreme value of the integral $V[x]$. Note that the solution path is restricted to those curves that are continuous with continuous derivatives.

For the solution process of problem (1) one has to deal with the basic first order condition, also called the Euler equation, which briefly says that every small perturbation $\varepsilon \cdot p(t)$ of the optimal time path $x^*(t)$, i.e. $x(t) = x^*(t) + \varepsilon \cdot p(t)$, has no action on the integral $V[x]$, as this perturbation tends to zero, or formally

$$\left.\frac{dV}{d\varepsilon}\right|_{\varepsilon=0}=0 \tag{2}$$

so the condition $dV/d\varepsilon=0$ is a necessary condition for the extremal.

Since (2) is not operating, as many arbitrary variables are involved, the final form of the Euler equation, after the appropriate development, becomes:

$$F_x-\frac{d}{dt}F_{x'}=0 \quad \text{for all } t\in[0,T] \tag{3}$$

and the more explicit version of the Euler equation, after (3)'s expansion, is the following second order nonlinear differential equation

$$F_{x'x'}x''(t)+F_{xx'}x'(t)+F_{tx'}-F_x=0 \ \text{ for all } t\in[0,T] \tag{4}$$

That is (4) is a more familiar, since the only calculations needed are the derivatives of the objective functional F with respect to $x'x'$, xx', tx' and x.

Suppose you need to find the extremal of the functional $V[x]=\int_0^2\left(12tx+x'^2\right)dt$ with boundary conditions $x(0)=0$ and $x(2)=8$. Since $F=12tx+x'^2$, following (4) we compute $F_x=12t$, $F_{x'}=2x'$, $F_{x'x'}=2$ and $F_{xx'}=F_{tx'}=0$. The Euler equation and its solution is the following:

$$2x''(t)-12t=0 \ \Leftrightarrow \ x''(t)=6t \ \Leftrightarrow \ x'(t)=3t^2+c_1 \ \Leftrightarrow \ x^*(t)=t^3+c_1t+c_2$$

The values of the constants of integration are $c_2=c_1=0$, setting in the solution $t=0$ and $t=2$ and substituting into the boundary conditions. So the extremal, the optimal time path, is the cubic time function $x^*(t)=t^3$.

A special class of the isoperimetric problems arising in the case the constraint is substituted by an integral of the type: $\int_0^T G(t,x,x')dt=k$ with k a constant. In such a situation the problem appears in general (with m integral constraints) as

$$\text{maximize} \quad \int_0^T F\left(t, x_1, x_2, \ldots, x_n, x_1', x_2', \ldots, x_n'\right) dt$$

$$\text{subject to} \quad \int_0^T G_1\left(t, x_1, x_2, \ldots, x_n, x_1', x_2', \ldots, x_n'\right) dt = k_1$$

$$\vdots$$

$$\int_0^T G_m\left(t, x_1, x_2, \ldots, x_n, x_1', x_2', \ldots, x_n'\right) dt = k_m$$

and appropriate boundary conditions

In this case the Euler equation becomes the following Euler–Lagrange equation (it is assumed only one integral constraint)

$$\left(F_x - \lambda G_x\right) - \frac{d}{dt}\left(F_{x'} - \lambda G_{x'}\right) = 0 \tag{5}$$

where λ is the Lagrange multiplier which in the isoperimetric case is a constant.

Moreover, in the one–state–variable problem with a single integral constraint, it can be shown that the modified Lagrange integrand $\mathcal{L} = F\left(t, x, x'\right) - \lambda G\left(t, x, x'\right)$ can be used and then apply the Euler – Lagrange equation to x alone. Now the value of the (constant) λ can be determined from the isoperimetric constraint.

In the above class of the isoperimetric problems belongs the model proposed by H. Hotelling in the classic article "*The Economics of Exhaustible Resources*" (Hotelling, 1931). The major conclusion of the Hotelling model is that the pure competition can yield a socially optimal extraction path for an exhaustible resource, while the monopoly cannot. The resulting condition, after the solution[1] of the isoperimetric problem, which ensures the above conclusion, is the following

$$P(Q) - C'(Q) = \lambda e^{\rho t} \tag{6}$$

[1] For a detailed analysis of the solution process, see among others Chiang (1982).

which in turn says that, in the pure competition, the quantity $P(Q)-C'(Q)$ grows at the interest rate ρ. Note that the Lagrange multiplier λ in (6) represents the initial value of the difference price minus marginal cost ($P(Q)-C'(Q)$).

In the monopoly the final solution leads to the conclusion “the difference between the marginal revenue and the marginal cost grows at the interest rate”, i.e. $R'(Q)-C'(Q)=\lambda e^{\rho t}$, which is suboptimal compared with the socially optimal extraction. After the Pontryagin's et al. (1962) book "Mathematical Theory of Optimal Processes", the Maximum Principle became the main tool of analysis in economics and management, physics, biology and so on. The absolute success of the Maximum Principle is due to the introduction of the two, instead of one, types of variables in the optimization process. The first is the *control* and the other is the *state* variable. The *control* variable is a steering mechanism which one can maneuver so that as to drive the *state* variable to various positions at any time via one or more equations of motion. That is, the Maximum Principle is this tool which sets an order in the mess of the corner solutions that may appear in the optimization process. Here the goal of the optimal control theory, is the determination of the optimal time path of the *control* variable first and then the determination of the *state* variable, unlike the calculus of variations where the main task is to find the optimal time path of the *state* variable.

Especially the simplest optimal control problem can be derived from the calculus of variations problem if the time derivative of the state variable, involved in the objective functional, is replaced by the so called *equation of motion*. Below we present a simple calculus problem together with the equivalent optimal control problem. The calculus problem is:

$$
\begin{array}{lll}
\text{maximize} & V=\int_0^T F(t,x,\dot{x})dt & \\
\text{subject to} & x(0)=A \quad \text{(A given)} & (7a) \\
\text{and} & x(T)\,\text{free} \quad \text{(T given)} &
\end{array}
$$

Now introducing the control variable u and the equation of motion $\dot{x} = u$ the same problem in optimal control fashion can be written as:

$$\begin{array}{lll} \text{maximize} & V = \int_0^T F(t,x,u)dt & \\ \text{subject to} & \dot{x} = u & (7b) \\ \text{and} & x(0) = A,\ \ x(T)\,\text{free} \quad (A, T\ \text{given}) & \end{array}$$

and the fundamental link between the two variables became apparent.

It is important to say that at the solution process, according to the Maximum Principle, except the time, state and control variables one more class of variable(s) will emerge. This is the so called *costate* variable, measuring the shadow price of the state variable, denoted by $\lambda(t)$.

Except the *maximum principle* there is another solution method for optimal control problems which is called the "dynamic programming". Starting with a wider class of similar problems which can be solved, the original problem is embodied in the larger class of problems. A policy oriented expression for the *principle of optimality* could be the following: "*An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision*". Now, it remains to set as simple as possible in rigorous mathematics the *maximum principle* and *principle of optimality*.

2. The formulation of the problem and the solution process

We discuss the class of optimal control problems that appears in the modeling of dynamic systems. Then, the state of a system at time t can be described by the following n–dimensional column vector

$$x(t) = \left(x_1(t), x_2(t), \ldots, x_n(t)\right)' \in \mathbb{R}^n, \quad t \in [0, T]$$

where the terminal time $T > 0$ in many economic applications is infinity, i.e. $T = \infty$. Moreover suppose that there is a decision maker influencing the time path of the state variable by choosing the time path of the m–dimensional control value. That is

$$u(t) = \left(u_1(t), u_2(t), \ldots, u_m(t)\right)' \in \Omega\left(x(t), t\right), \quad t \in [0, T]$$

The control variable $u(t)$ is a piecewise continuous function and $\Omega\left(x(t), t\right)$ is the given control region, i.e. $u(t) \in \Omega\left(x(t), t\right)$. Additionally it is assumed that the dynamis of the state variable is governed by the following Ordinary Differential Equation (ODE)

$$\dot{x}(t) = f\left(x(t), u(t), t\right) \quad t \in [0, T] \tag{8a}$$

$$\text{subject to } x(0) = x_0 \tag{8b}$$

with terminal constraints:

$$x_i(T) = x_i^T, \qquad i = 1, \ldots, n' \tag{8c}$$
$$x_i(T) \geq x_i^T, \qquad i = n'+1, \ldots, n'' \tag{8d}$$
$$x_i(T) \text{ free}, \qquad i = n''+1, \ldots, n \tag{8e}$$

where $n' > 0$, $n'' > 0$, $n' + n'' \leq n$, $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ is a vector valued function $f = \left(f_1, f_2, \ldots, f_n\right)'$, where for all $i = 1, \ldots, n$, $f_i(x, u, t)$ and $\partial f_i(x, u, t) / \partial x$ are continuous functions with respect to their arguments. Equation $(8a)$ is the *system dynamics* or the *equation of motion*.

Now we suppose that the decision maker has a time discounted objective in the form of the following functional

$$V\left(u(.)\right) = \int_0^T e^{-\rho t} g\left(x(t), u(t), t\right) dt + e^{-\rho T} S\left(x(T), T\right) \tag{9}$$

$g\left(x(t), u(t), t\right)$ is the instantaneous profit gained by exerting the control variable $u(t)$ at time t, $x(t)$ is the current state, while ρ is the positive discount rate. At the end horizon T the state would be $x(T)$, while the corresponding payoff is described by the term $S\left(x(T), T\right)$ also called, in the optimal control language, the *salvage* or *scrap value*. The payoff function $g\left(x(t), u(t), t\right)$ and its partial derivative $\partial g(x, u, t) / \partial x$ are assumed continuous with respect to their arguments as well as the scrap value function $S:\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ with respect to x and T. Then the task of the regulator is to choose the best policy $u(.)$ among all the admissible trajectories. As a consequence the optimal control problem is the maximization of the reward $V(u(.))$ taking into account that the state's motion is governed by equation $(8a)$.

As it is mentioned above, generally there exists two different approaches to solve an optimal control problem of the type $(8)-(9)$. One is based on the Pontryagin's Maximum Principle (Pontryagin et al., 1962; Grass et al., 2008), while the other hinges upon the Hamilton–Jacobi–Bellman (HJB) equation introduced by Bellman (1957).

The Maximum Principle

Before we proceed with the necessary first order conditions of the maximization with the Maximum Principle approach, it is important to introduce the Hamiltonian function (H), which has as arguments all the involved variables $t,\ x,\ u,\ \lambda$. The Hamiltonian function is defined as

$$H(t,x,u,\lambda) = g(t,x,u) + \lambda(t) f(t,x,u) \qquad (10)$$

Once the Hamiltonian function is defined by (10) there is the requirement to maximized with respect to the control variable u at every point of time. Pontryagin's Maximum Principle states as:

Theorem 1 (Pontryagin et al., 1962; Grass et al, 2008).

Let $x^*(.),\ u^*(.)$ be *an optimal solution of the problem* $(8)-(9)$ *with free terminal state. Then there exists a continuous and piecewise continuously differentiable function* $\lambda(.)$ *with* $\lambda(t) \in \mathbb{R}^n$ *satisfying for all* $t \in [0,T]$

$$H(x^*(t),u^*(t),\lambda(t),t) = \max_{u \in \Omega(x^*(t),t)} H(x^*(t),u(t),\lambda(t),t)$$

and at every point of time t *where* $u(.)$ *is continuous*

$$\dot{\lambda}(t) = \rho\lambda(t) - H_x\left(x^*(t), u^*(t), \lambda(t), t\right) \qquad (10a)$$

Furthermore the transversality condition

$$\lambda(T) = S_x\left(x^*(T), T\right) \qquad (10b)$$

holds, where the Hamiltonian function is defined as (10*).*

Next in the lines of Forster (1980) we provide an example of a pollution abatement model solved as an optimal control problem.

Example 1

A question raised in Environmental Economics is how much of a given level of emissions should be abated (with a given abatement technology) and how much should be diffused in the environment. To focus on this problem let us assume that $P(t)$ represents the pollutants flows generated by the firms' production process and $E(P)$ are the emissions produced by these pollutants flows. These emissions can either be abated or diffused in the environment. Let A be the amount of emissions allocated for abatement, so $D = E(P) - A$ is the corresponding diffusion rate or net emissions dispersed in the environment. The stock of pollutants is raised according to the equation

$$\dot{P} = D - \delta P = E(P) - A - \delta P, \qquad P(0) = P_0$$

where δ is natural decay rate.

Furthermore let $U(A)$ be the utility which the society enjoys from the abatement at rate A and ρ is the discount factor of the society. Then the regulator has to solve the following optimal control problem

$$\max_{A(.)}\left\{\int_0^T e^{-\rho t}U(A)dt + e^{-\rho T}S\left(P(T)\right)\right\} \qquad (11a)$$

$$\text{subject to } \dot{P} = E(P) - A - \delta P \qquad (11b)$$

$$P(0) = P_0 \qquad (11c)$$

where S is the salvage function mentioned above.

The necessary assumptions on the functions U and E are the following:

$$\mathrm{U}'(\mathrm{A}) > 0, \quad \mathrm{U}'(0) = \infty, \; \mathrm{U}''(\mathrm{A}) < 0 \;\text{ for }\; \text{all}\,\mathrm{A} \geq 0 \tag{12a}$$
$$\mathrm{E}'(\mathrm{P}) > 0, \quad \mathrm{E}''(\mathrm{P}) < 0 \;\text{ for }\; \text{all }\; \mathrm{P} > 0 \tag{12b}$$
$$\mathrm{E}(0) = 0, \quad \mathrm{E}'(0) > \rho + \delta, \;\; \mathrm{E}'(\infty) < \delta \tag{12c}$$
$$\mathrm{S}''(\mathrm{P}) \leq 0 \tag{12d}$$

The properties summarized in (12) are the well known *Inada conditions*. For the solution of the optimal control problem (11), first we formulate the Hamiltonian function

$$\mathrm{H} = \mathrm{U}(\mathrm{A}) + \lambda\big(\mathrm{E}(\mathrm{P}) - \delta\mathrm{P} - \mathrm{A}\big) \tag{13}$$

The Hamiltonian function is concave with respect to A due to the assumptions (12), i.e. $\mathrm{H}_{\mathrm{AA}} = \mathrm{U}'' < 0$. Thus the maximizer A^* of the Hamiltonian $\mathrm{H}(\mathrm{P},\mathrm{A},\lambda)$ for fixed P and λ lies in the interior of $\Omega = [0,\infty)$ and satisfies the following first order condition

$$\mathrm{H}_{\mathrm{A}}\left(\mathrm{P},\mathrm{A}^*,\lambda\right) = \mathrm{U}'\left(\mathrm{A}^*\right) - \lambda = 0$$

from which the Maximum Principle yields

$$\lambda = \mathrm{U}'\left(\mathrm{A}^*\right) \tag{14}$$

Due to the concavity of the utility function U, the inverse function $\left(\mathrm{U}'\right)^{-1}$ exists and therefore A^* is a function of the adjoint variable λ given by

$$\mathrm{A}^*(\lambda) = \left(U'\right)^{-1}(\lambda) \tag{15}$$

The Hamiltonian's concavity in (P,A) is assured. This is easily seen, by using the positivity of λ, which can be deduced from (14) and (12a), which in turn implies the negative definiteness of the matrix

$$\begin{pmatrix} \mathrm{H}_{\mathrm{PP}} & \mathrm{H}_{\mathrm{PA}} \\ \mathrm{H}_{\mathrm{AP}} & \mathrm{H}_{\mathrm{AA}} \end{pmatrix} = \begin{pmatrix} \lambda\mathrm{E}''(\mathrm{P}) & 0 \\ 0 & \mathrm{U}'' \end{pmatrix}$$

and therefore the concavity of the Hamiltonian. Moreover the hypothesis that any solution that satisfies the necessary conditions is optimal is ensured (applying the maximum principle), due to the concavity of the salvage function.

Next we derive the equation of motion for the costate variable by applying (10b). For the Hamiltonian (13), (10b) yields

$$\dot{\lambda} = \rho\lambda - \mathrm{H_P} = \rho\lambda + \lambda\left(\delta - \mathrm{E}'(\mathrm{P})\right) = \left(\rho + \delta - \mathrm{E}'(\mathrm{P})\right)\lambda \tag{16}$$

Substituting (15) into the state equation (11b) establishes

$$\dot{\mathrm{P}} = \mathrm{E}(\mathrm{P}) - (U')^{-1}(\lambda) - \delta\mathrm{P} \tag{17}$$

Equations (16) an (17) is the so called *canonical system* of equations which is appropriate for further analysis.

Since the control function given by (15) is differentiable with respect to time, the time derivative of the $\mathrm{H_A}(\mathrm{P},\mathrm{A},\lambda) = 0$ is:

$$\frac{d}{d\mathrm{t}}\mathrm{H_A} = \mathrm{U}''(\mathrm{A})\dot{\mathrm{A}} - \dot{\lambda}$$

and using the adjoint equation (16) and equation (14), the time derivative of the control A can be written as:

$$\dot{\mathrm{A}} = \frac{\mathrm{U}'(\mathrm{A})}{\mathrm{U}''(\mathrm{A})}(\delta + \rho) - \frac{\mathrm{E}'(\mathrm{P})}{\mathrm{U}''(\mathrm{A})} \tag{17a}$$

Equation (17a) together with the state dynamics $\dot{\mathrm{P}} = \mathrm{E}(\mathrm{P}) - \mathrm{A} - \delta\mathrm{P}$ constitute the transformed state–control system.

The infinite horizon version of the Maximum Principle was first introduced by Halkin (1974) as:

Theorem 2 (Maximum Principle for an Infinite Time Horizon)

Let the pair $\left(\mathrm{x}^*(.),\ \mathrm{u}^*(.)\right)$ *be an optimal solution of the infinite horizon problem analogue to* (8)-(9) *problem. Then there exists a continuous and piecewise continuously differentiable function* $\lambda(.)$ *with* $\lambda(\mathrm{t}) \in \mathbb{R}^n$ and a constant $\lambda_0 > 0$ *satisfying for all* $\mathrm{t} \in [0,\mathrm{T}]$

$$\left(\lambda_0, \lambda(t)\right) \neq 0$$

$$H\left(x^*(t), u^*(t), \lambda(t), t\right) = \max_{u \in \Omega\left(x^*(t), t\right)} H\left(x^*(t), u(t), \lambda(t), t\right)$$

and at every point of time t *where* $u(.)$ *is continuous*

$$\dot{\lambda}(t) = \rho\lambda(t) - H_x\left(x^*(t), u^*(t), \lambda(t), \lambda_0, t\right)$$

Note that there is no transversality condition in the sense of (10b), a result that is a consequence of the proof strategy presented in Halkin (1974).

Continuing with the pollution abatement model in infinite horizon, the basic equations are transformed below as

$$\max_{A(.)} \left\{ \int_0^\infty e^{-\rho t} U(A) dt \right\} \tag{18a}$$

$$\text{subject to } \dot{P} = E(P) - A - \delta P \tag{18b}$$

$$P(0) = P_0 \tag{18c}$$

$$0 \leq A \leq E(P) \tag{18d}$$

$$P \geq P' \tag{18e}$$

and the canonical system

$$\dot{P} = E(P) - A - \delta P \tag{19a}$$

$$\dot{A} = \frac{U'(A)}{U''(A)}\left(\delta + \rho - E'(P)\right) \tag{19b}$$

Next we draw the phase portrait for the canonical system of equations (19a)-(19b). Therefore we consider the $\dot{P}$, $\dot{A}$, isoclines, yielding

$$A = E(P) - \delta P \tag{20a}$$

$$E'(P) = \delta + \rho \tag{20b}$$

Under the assumptions (12b), (12c), the $\dot{P}$ isocline (20a) reduces to a strictly concave function. This concave function vanishes at the origin and for some $\tilde{P} > 0$, but meets its maximum at some $0 < P_m < \tilde{P}$. The other isocline $\dot{A}$ becomes a vertical line. The condition

(12c) together with (20b) now assures the existence of a unique $\dot{\mathrm{P}}$ satisfying (20b). Finally we find a unique equilibrium at $\left(\tilde{\mathrm{P}},\tilde{\mathrm{A}}\right)$ with $\tilde{\mathrm{A}}=\mathrm{E}\left(\tilde{\mathrm{P}}\right)-\delta\tilde{\mathrm{P}}$ for which the corresponding Jacobian is the following matrix:

$$\hat{\mathrm{J}}\left(\tilde{\mathrm{P}},\tilde{\mathrm{A}}\right)=\begin{pmatrix} \mathrm{E}'\left(\tilde{\mathrm{P}}\right)-\delta & -1 \\ -\dfrac{\mathrm{U}'\left(\tilde{\mathrm{A}}\right)}{\mathrm{U}''\left(\tilde{\mathrm{A}}\right)}\mathrm{E}''\left(\tilde{\mathrm{P}}\right) & 0 \end{pmatrix}$$

Since $\det\ \hat{\mathrm{J}}<0$ there exists a saddle point equilibrium, i.e., the equilibrium exhibits a stable path. Therefore, for initial values in a neighborhood of P the stable path is a possible candidate for an optimal solution.

Further phase portrait analysis includes the following two cases.

Case 1: Under the constraint (18d) there exist points $\tilde{\mathrm{P}}_1<\tilde{\mathrm{P}}_2$, with the property: for initial values between these points the resulting path is the unique optimal solution (see Figure 1). The exit point $\tilde{\mathrm{P}}_1$ is an intersection point of the state path with the axis $\mathrm{A}=0$, but the point $\tilde{\mathrm{P}}_2$ lies into the intersection of the stable path with the curve $\mathrm{A}=\mathrm{E}\left(\mathrm{P}\right)$.

Case 2: With the constraint (18e) the solution for $\tilde{\mathrm{P}}<\mathrm{P}'<\tilde{\mathrm{P}}_1$ is depicted in Figure 1.b. In this case it is optimal to control the system into the marginal equilibrium point $\left(\mathrm{P}',\mathrm{A}'\right)$. For initial values of the state into the open interval between P' and $\tilde{\mathrm{P}}_1$, the optimality of the above solution can be explicitly shown. Since $\lambda\left(\mathrm{t}\right)\geq 0$ for all t and $\lim_{\mathrm{t}\to\infty}\mathrm{P}\left(\mathrm{t}\right)\geq\tilde{\mathrm{P}}$, the limiting transversality condition is satisfied for any admissible orbit of the state. Finally, we conclude that the depicted solution in Figure 1.b is the unique optimal solution, because the adjoint and the control variables are both continuous at the point $\tilde{\mathrm{P}}_1$.

Note that Figures 1.a and 1.b are drawn for the functional forms $\mathrm{E}\left(\mathrm{P}\right)=\sqrt{\mathrm{P}}$ and $\mathrm{U}\left(\mathrm{A}\right)=\log\mathrm{A}$ and the parameter values are δ=0.5 and ρ=0.1.

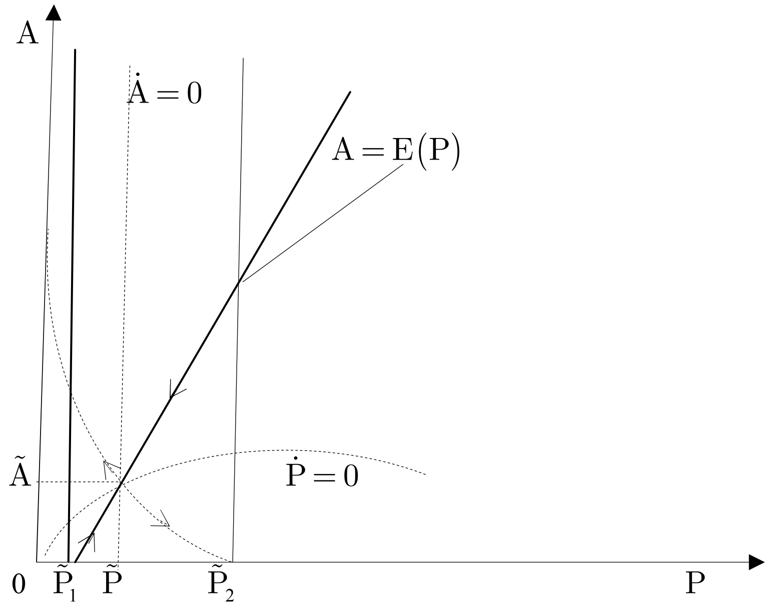

Figure 1.a. The black dotted curve is the optimal solution path for the pollution abatement model. Starting between the states $\tilde{P}_1$ and $\tilde{P}_2$ the path which converges to the saddle point $(\tilde{A},\tilde{P})$ is the optimal solution. For all other initial values except the previously noted the control trajectory under consideration is on its boundary until the exit point $\tilde{P}_1$ or $\tilde{P}_2$ is met.

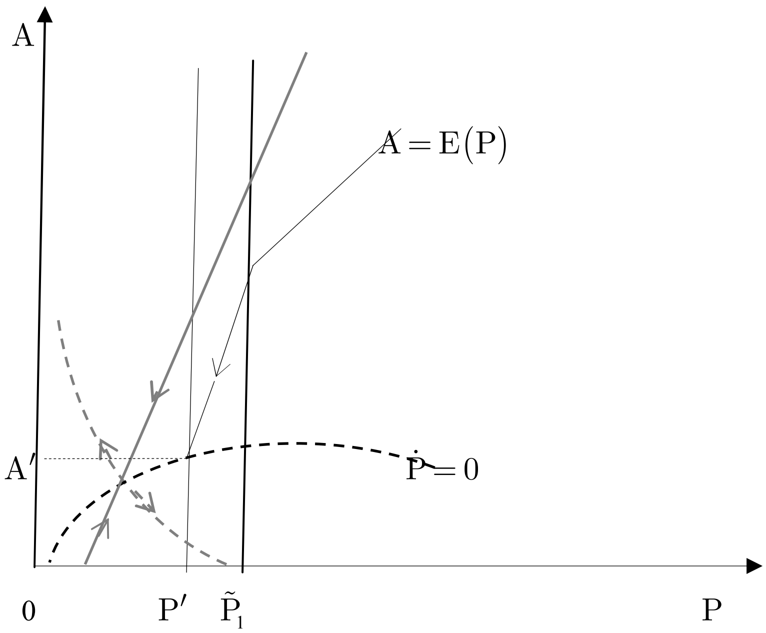

Figure 1.b. Here we consider $P \geq P'$, i.e., for state values into the interval between P' and $\tilde{P}_1$, the optimal control line lies in the interior of the control region and the optimal path leads to the boundary equilibrium (A',P'). For states $P \geq \tilde{P}_1$ the control values are chosen from the upper boundary of the control region, until the exit point $\tilde{P}_1$ is reached.

The Principle of Optimality

As it is mentioned above the other approach to solve optimal control problems is the principle of Optimality and is based on the HJB equation. According to that principle, the wider class of these problems, in which an optimal control problem belongs, is sated as follows:

$$\max_{u(.)} \int_t^T g\left(x(s),u(s),s\right)ds + S\left(x(T),T\right) \tag{21a}$$

Subject to $\dot{x}(s) = f\left(x(s),u(s),s\right) \quad s \in [t,T]$ (21b)

$$x(t) = \xi$$

As it is assumed above the optimal control problem under consideration has an optimal solution for any pair (ξ,t) . The Bellman equation with the pair (ξ,t), $V(\xi,t)$ as arguments, is defined as

$$V(\xi,t) = \max_{u(.)} \int_t^T g\left(x(s),u(s),s\right)ds + S\left(x(T),T\right) \tag{22}$$

Now in order to produce the HJB equation the following Principle of Optimality must be used.

Theorem 3 (Principle of Optimality)

We suppose that there exists a solution $\left(x^*(.),u^*(.)\right)$ *of the problem* (21) *and this solution exists for each pair* (ξ,t) *with* $t \in [0,T]$, $\xi \in \mathbb{R}^n$. *Then* $\left(x^*(.),u^*(.)\right)$ *is an optimal solution for the problem of class* (21) *with* $x(t) = \xi$ *if and only if*

$$V(\xi,t) = \int_t^s g\left(x^*(\tau),u^*(\tau),\tau\right)d\tau + V\left(x^*(s),s\right) \tag{23a}$$

and
$$V\left(x^*(T),T\right) = S\left(x^*(T),T\right) \tag{23b}$$

Note that, the information which records the relative change of $W(\xi,t)$ with respect to ξ, when s tends to t is given by relation (23a). The resulting HJB equation formally is defined as follows.

Theorem 4 (HJB equation).

Let there exist an admissible control $u^*(.)$ *and their corresponding trajectory* $x^*(.)$ *for the state. Moreover the Bellman function* $V(\xi,t)$ *is continuously differentiable with respect to* ξ *and* t. *Then* $\left(x^*(.),u^*(.)\right)$ *is an optimal solution of the problem* (21) *if and only if the Bellman function* $V(\xi,t)$ *satisfies the HJB equation:*

$$-V_t(\xi,t)=\max_u\left\{g(\xi,u,t)+V_x(\xi,t)f(\xi,u,t)\right\} \tag{24a}$$

and

$$V(\xi,T)=S(\xi,T) \tag{24b}$$

for all $(\xi,t)\in\mathbb{R}^n\times[0,T]$ *for which* $u^*(.)$ *is continuous.*

Note that for the problems which the discount factor is entered into the objective functional, equation (24a) is not operative in the solution process. Therefore another condition, for the HJB equation provided by Dockner et al (2000), satisfies the following partial differential equation:

$$\rho V(x,t)-V_t(x,t)=\max_u\left\{g(x,u,t)+V_x(x,t)f(x,u,t)\right\} \tag{25}$$

and (25) is the HJB function for discounted problems, which is very useful for our economic problems under consideration.

Next we present an example of a very simple environmental model for which the HJB equation is used in order to extract feedback strategies and the optimal value function.

Example 2

Assume we have a nonrenewable resource extraction monopolistic firm that sells the extracted product at a fixed price $p>0$. We denote by $u(t)$ the resource's extraction rate and we suppose that this rate equals to the sales rate, thus preventing the resource's stock up. Moreover we denote by $x(t)$ the remainder resource stock at time t. The system dynamics is described as "*the rate of reduction of the resource stock equals to the extraction rate*". Thus the equation of motion is the following:

$$\dot{x}(t) = -u(t) \qquad (26)$$

and with boundary conditions $x(t) \geq 0, \quad u(t) > 0$

Extraction cost is an increasing function with respect to the extraction rate $u(t)$ and decreasing with respect to the remainder stock $x(t)$.

The monopolistic firm maximizes its discounted profits, given by the objective functional:

$$J(u(.)) = \int_0^\infty e^{-\rho t}\left[pu(t) - c(u(t), x(t))\right]dt \qquad (27)$$

And the optimal control problem is:

$$\max \quad J(u(.)) = \int_0^\infty e^{-\rho t}\left[pu(t) - c(u(t), x(t))\right]dt \qquad (28)$$

Subject to $\dot{x}(t) = -u(t)$

With the boundary conditions $x(t) \geq 0, \quad u(t) > 0$

Specifying the cost function as:

$$c(u, x) = \frac{\gamma u^2}{2x} \qquad (29)$$

we have the following result.

Proposition 1

"*An optimal feedback extraction strategy* $u(x,t)$ *of the problem* (28) under the constraint (26) *is the following:*

$$u(x(t), t) = \frac{x\left[p - A(t)\right]}{\gamma}$$

where $A(t)$ *is the unique solution of the following Riccati differential equation:*

$$\dot{A}(t) = \rho A(t) - \frac{\left[p - A(t)\right]^2}{2\gamma} \text{,,}[2]$$

Proof

The HJB equation of the above problem is:

$$\rho V(x,t) - V_t(x,t) = \max_u \left\{ g(x,u,t) + V_x(x,t) f(x,u,t) \,|\, u \in U(x,t) \right\}$$

with $g(x,u,t) = pu(t) - \frac{\gamma u^2}{2x}$, $f(x,u,t) = -u(t)$

Taking the first order conditions of the above HJB function we have:

$$\frac{\partial \left\{ g(x,u,t) + V_x(x,t) f(x,u,t) \right\}}{\partial u} = 0 \Rightarrow \frac{\partial \left\{ pu - \frac{\gamma u^2}{2x} - V_x(x,t) u \right\}}{\partial u} = 0 \Rightarrow$$

$$\Rightarrow p - \frac{\gamma u}{x} - V_x(x,t) = 0 \Rightarrow u(x,t) = \frac{x\left(p - V_x(x,t)\right)}{\gamma} \qquad (30)$$

Making use of the well informed guess for the value function

$$V(x,t) = A(t)x$$

thus giving the following derivatives:

$$V_x(x,t) = A(t)$$
$$V_t(x,t) = \dot{A}(t)x$$

Now substituting the value function derivative (with respect to state) into the strategy (30) we have the final strategy

$$u(x(t),t) = \frac{x\left[p - A(t)\right]}{\gamma}$$

Now it remains to verify that this strategy satisfies the initial HJB equation for the conjectured linear value function $V(x,t) = A(t)x$.

First, substituting the strategy into the right hand side of the HJB equation gives:

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[2] The solution of the differential equation $\dot{A}(t) = \rho A(t) - \frac{\left(p - A(t)\right)^2}{2\gamma}$ is

$$A(t) = \gamma\rho + p + \tanh\left(\frac{\sqrt{\gamma^2\rho + 2\gamma\rho p}\,(t + C_1)}{2\gamma} \right) \sqrt{\gamma^2 r^2 + 2\gamma\rho p}$$, where $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$$\text{RHS(HJB)} = p\left[\frac{x\left(p-A(t)\right)}{\gamma}\right] - \frac{\gamma\frac{\left[x^2\left(p-A(t)\right)^2\right]}{\gamma^2}}{2x} + A(t)\left[-\frac{x\left(p-A(t)\right)}{\gamma}\right] =$$

$$= x\frac{\left(p-A(t)\right)^2}{2\gamma}$$

Second, the left hand side of the same equation becomes:

$$\text{LHS}\left(\text{HJB}\right) = rV(x,t) - V_t(x,t) = x\left[\rho A(t) - \dot{A}(t)\right]$$

Equating both sides, i.e. $\text{LHS}\left(\text{HJB}\right) = \text{RHS(HJB)}$ the result is the differential equation

$$\dot{A}(t) = \rho A(t) - \frac{\left[p-A(t)\right]^2}{2\gamma}$$

for which the solution must be $A(t)$ in order to satisfy the HJB equation.

3. Differential Games

Game theory is intended to be a useful tool for modeling situations in which there are many (rational) decision makers and for guessing the outcome of decision makers' competition or cooperation. Here we deal only with differential games. Differential games involved in dynamic conflict situations, for which an arbitrary number of decision makers (such as renewable or nonrenewable resources extractors, pollution regulators etc) interact in an environment e.g., a fishery place, a mine, a factory or a society) .

In fact, differential games are those dynamic games for which the maximization of each player's objective is subject to some limitations. All those constraints which are subject to the payoffs of each player are included in one or more differential equations describing the state's evolution of the game.

Since every player involved in a differential game has its own objective functional to maximize (or minimize), optimal control theoretic methods can be used. Considering the game’s solution, we seek for the Nash equilibrium which is the appropriate, but not the only, concept of solution. Under the Nash equilibrium concept there no incentive for none of the involved players to deviate from his/her own Nash equilibrium strategy.

Before we continue with the (brief) description of the solution it is necessary to give some definitions of the type of the available strategies depending on information patterns. An **open loop** strategy is only a time dependent rule of decision, i.e., the resulting controls are functions of time as:

$$u_i(t) = \phi_i(t)$$

An open loop strategy is used only if the players commit at the start of the game to follow a fixed time path. This strategy is applied only if it is impossible for every one player to observe the current state variable involved.

A **closed loop** of feedback or Markovian strategy is that for which each player observes the system's current state i.e., according to the state – time pair (x,t) and decides about her action according to the rule:

$$u_i(t) = \phi_i(x(t),t)$$

while the stationary closed loop strategy is defined independently of the time as:

$$u_i(t) = \phi_i(x(t))$$

The major question raised in differential games is how we can compute the Nash equilibrium. Supposing that all the other N–1 rivals of player i use closed loop strategies $u_j(t) = \phi_j(x(t),t),\ j \neq i$, then player i has to solve an optimal control type problem, which is of the following form:

$$\max_{u_i(.)\in\Omega_i} \int_0^T e^{-\rho_i t} g_i(x, u_i, \phi_i(x,t), t)\,dt \; + \; e^{-\rho_i T} S_i(x(T))$$

$$\text{subject to} \quad \dot{x} = f(x, u_i, \phi_{-i}(x,t), t,) \quad x(0) = x_0$$

where $\quad \phi_{-i}(x,t) = (\phi_1(x,t), \phi_2(x,t), \ldots, \phi_{i-1}(x,t), \phi_{i+1}(x,t), \ldots, \phi_N(x,t))$

Since one differential game is faced as N optimal control games the above theorems 2 and 4 for the Maximum Principle and for the Principle of Optimality are in use.

Next we present an example of a differential game model.

Example 3

As a differential game example we deal with the basic renewable resource model, but we modify its growth function to be a Gompertz type. The Gompertz growth function is given by the expression (see for instance Schafer, 1967)

$$g(x) = x(t)\left[1 - \ln\left(x(t)\right)\right]$$

Concerning the properties of the Gompertz growth this function first of all fulfills the conditions:

$$g'(x) = -\ln(x) \qquad g''(x) = -\frac{1}{x} < 0 \qquad g(0) = 0$$

Second, it is a concave function and therefore it has "the pure compensation property" as it is defined by Clark (1984).

Third, it is right–skewed and has the same properties as the logistic growth function, while the upper stationary solution of $\dot{x} = g(x)$, i.e. the solution $x = e$, is asymptotically stable.

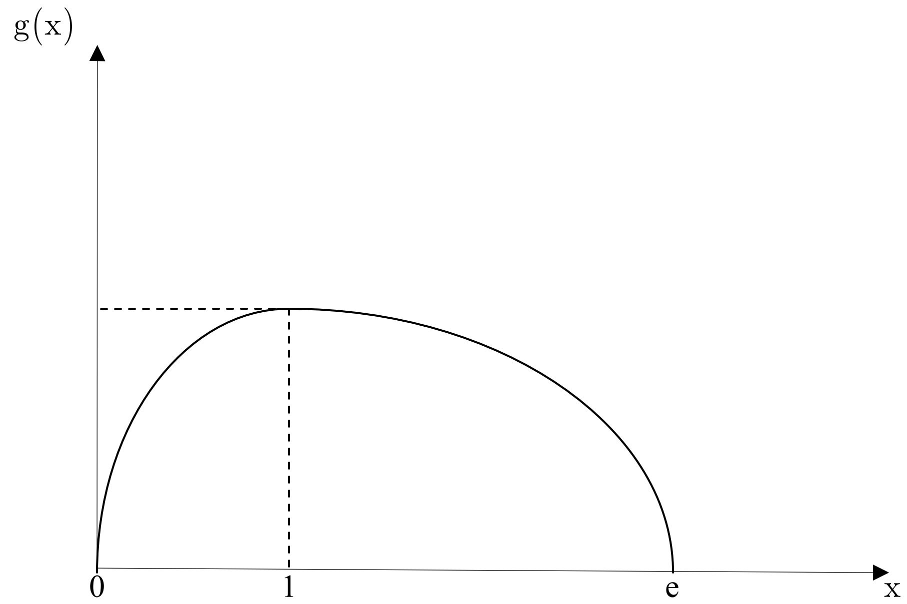

Figure 2: The shape of the Gompertz growth function $g(x) = x(t)\left[1 - \ln\left(x(t)\right)\right]$

According to that growth function the stock of the resource obeys to the following differential equation law of motion:

$$\dot{x}(t)=x(t)\left[1-\ln\left(x(t)\right)\right]-\phi_1-\phi_2$$

where $\phi_i,\ i=1,2$ is the harvesting function for the two players of the model. If we define the fishing effort for the i player as $a_i(t)=\phi_i(t)/x(t)$, then the game is a non-cooperative one for which every agent chooses a time path of his own fishing effort $a_i(t)$ that maximizes the discounted utility. We transform the utility in the form of an additive separable function, i.e. dependent on the fish stock $x(t)$ and on utility that every player enjoys from harvesting $\phi_i(t)$ as well.

We specify the utility functions to be in logarithmic form arising from the following utility function specification often used in growth models

$$U(x)=\begin{cases}\dfrac{x^{\beta}-1}{\beta} & \beta\in(0,1)\\ \ln(x) & \beta=0\end{cases}$$

for which the elasticity of intertemporal substitution is given by $1/(1-\beta)$. Moreover, we define $y(t)=\ln x(t)$ in the case $\beta=0$.

A number of calculations are performed in order to set up the problem. The calculations are the following:

$$y(t)=\ln x(t)\Rightarrow x(t)=e^{y(t)}\Rightarrow\frac{dx(t)}{dt}=\dot{x}(t)=e^{y(t)}\dot{y}(t)\Rightarrow\dot{x}(t)=x(t)\dot{y}(t)$$

Now, the transformed evolution equation becomes:

$$\dot{x}(t)=x(t)\left[1-\ln\left(x(t)\right)\right]-\phi_1-\phi_2\Rightarrow\frac{\dot{x}(t)}{x(t)}=1-\ln\left(x(t)\right)-\frac{\phi_1}{x(t)}-\frac{\phi_2}{x(t)}\Rightarrow$$

$$\Rightarrow\dot{y}(t)=1-y(t)-a_1(t)-a_2(t)$$

This is the transformed stock evolution equation that depends on the logarithm of the resource stock as well as on the players' fishing effort.

The utility function that is maximized is depending on the resource stock and on effort as well. It is assumed that original present value maximized utility is dependent on the harvesting function, i.e.: $\max \int_0^\infty e^{-\rho t} \ln(\phi_i(t))dt$, but the latter can be transformed as follows:

$$\begin{aligned}
&\max_{\phi_i} \int_0^\infty e^{-\rho t} \ln(\phi_i(t))dt = \max_{\phi_i} \int_0^\infty e^{-\rho t} \left[\ln(\phi_i(t)) - \ln(x(t)) + \ln(x(t))\right]dt \\
&= \max_{\phi_i} \int_0^\infty e^{-\rho t} \left[\ln\left(\frac{\phi_i(t)}{x(t)}\right) + \ln(x(t))\right]dt = \max_{a_i} \int_0^\infty e^{-\rho t} \left[\ln(a_i(t)) + y(t)\right]dt
\end{aligned}$$

The differential game now becomes:

$$\max_{a_i} \int_0^\infty e^{-\rho t} \left[\ln(a_i(t)) + y(t)\right]dt \qquad (31)$$

subject to $\quad \dot{y}(t) = 1 - y(t) - a_1(t) - a_2(t) \qquad (32)$

In what follows we explore the Nash equilibria of the game which may be a time consistent one in the sense of subgame perfectness.

Time consistency could be seen as a minimal requirement for the credibility of an equilibrium strategy. If player i (i=1,2) had an incentive to deviate from his strategy ψ_i during the time interval $[0, T)$, the other player $j, \quad j = 1,2$ would not believe his announcement of ψ_i in the first place. Consequently, player j computes his own strategy taking into account the expected future deviation of player i which, in general, would lead to strategies different from $\psi_j, \; j \neq i$. Open-loop informational structure strategies are not in general time consistent; while closed-loop or Markovian strategies are certainly time consistent (Dockner *et al.*, 2000).

On the other hand subgame perfectness is the concept for which an equilibrium strategy remains unchanged regardless the starting period the game begins. So, subgame perfectness is a sole requirement for the credibility of an equilibrium strategy that is time consistency for that strategy. We conclude if we can found an equilibrium strategy for the game, independently of the initial state and regardless of the informational structure employed, this strategy has the subgame perfectness property and can be a time consistent strategy.

Equilibrium analysis

Proposition 2

The game with the Gompertz as the resource growth function, admits an equilibrium strategy of the form $a_i = \rho + 1$ *, which is time consistent.*

Proof

The Hamiltonian of the above problem for the player i (i=1,2) is

$$\mathrm{H}_i = \mathrm{y}(\mathrm{t}) + \ln a_i(t) + \lambda(\mathrm{t})\left[1 - \mathrm{y}(\mathrm{t}) - a_1(\mathrm{t}) - a_2(\mathrm{t})\right]$$

and the conditions for an interior solution are

$$\frac{\partial \mathrm{H}_i}{\partial a_i} = \frac{1}{a_i(\mathrm{t})} - \lambda(\mathrm{t}) = 0 \Rightarrow a_i(\mathrm{t}) = \frac{1}{\lambda(\mathrm{t})}$$

The costate's variable equation of motion becomes:

$$\dot{\lambda}(\mathrm{t}) = -\frac{\partial \mathrm{H}_i}{\partial y} + \rho\lambda(\mathrm{t}) \Rightarrow \dot{\lambda}(\mathrm{t}) = -1 + (\rho+1)\lambda(\mathrm{t})$$

with solution $$\lambda(\mathrm{t}) = \frac{1}{\rho+1} + e^{(\rho+1)\mathrm{t}}\Omega$$

along with the transversality condition $\lim_{\mathrm{t}\to\infty} \lambda(\mathrm{t})\mathrm{y}(\mathrm{t}) = 0$,

which must be satisfied, so it is reasonable to set $\Omega = 0$ and the costate variable becomes $\lambda(\mathrm{t}) = \frac{1}{\rho+1}$. Substituting the value of the costate variable into the strategy, the resulting strategy becomes $a_i = \rho + 1$ which is independent of the initial state, and therefore it is time consistent.

Proposition 3

In the case the players cooperate the joint cooperative time consistent equilibrium harvesting strategy is given by the expression $a(t) = \dfrac{\rho + 1}{2}$.

Proof

The evolution equation in the cooperative case becomes

$\dot{y}(t) = 1 - y(t) - 2a(t)$ where $a(t) = a_1(t) + a_2(t)$ is the joint fishing effort of the two players. The Hamiltonian for the cooperative case is,

$$H_c = y(t) + \ln a(t) + \lambda(t)\left[1 - y(t) - 2a(t)\right]$$

and the rest of algebraic manipulations for maximization reveals the cooperative equilibrium strategy $a = \dfrac{\rho + 1}{2}$ which is again time consistent.

The payoff (Value) function

Proposition 4

In the case the players do not cooperate the payoff function for each player is

$$V_i = \frac{y}{1+\rho} + \frac{1}{\rho}\left[\ln(1+\rho) + \frac{1}{1+\rho} - 2\right].$$

Proof

We check whether the equilibrium strategies given by proposition 2 are verified by the above value function. The Hamilton–Jacobi–Bellman (HJB) of the differential game (31)-(32) becomes:

$$\rho V_i = \max\left\{y(t) + \ln a_i(t) + \frac{\partial V_i}{\partial y}\left[1 - y(t) - a_i(t) - a_j(t)\right]\right\} \quad (i \neq j,\ \ i = 1,2\ \ j = 1,2)$$

and the maximization of the RHS of the HJB equation yields:

$$\frac{\partial\left[y(t)+\ln a_i(t)+\frac{\partial V_i}{\partial y}\left[1-y(t)-a_i(t)-a_j t(t)\right]\right]}{\partial a_i}=0 \Rightarrow \frac{\partial V_i}{\partial y}=\frac{1}{a_i(t)} \quad .$$

Differentiation of the proposition's 4 value function with respect to the state variable y, yields $\frac{\partial V_i}{\partial y}=\frac{1}{1+\rho}$. Now equating the derivatives $\frac{\partial V_i}{\partial y}$, the final result is $a_i=\rho+1$.

Main Points

In this paper we first discus the dynamical methods as they applied in environmental and resource economics, given in a rigorous mathematical language; and second, as a contribution, we introduce and solve two environmental and resource models. The first model is an optimal control one, touching the classical monopolistic extraction of a depletable resource, disposed after the extraction in a market. One of the first model's crucial characteristic is that the extraction cost is dependent not only from the monopolist's utility but also from the remaining stock of the resource. At the solution process, under the closed loop informational structure, we found the analytic expression of the optimal monopolistic strategy, which also is time consistent and therefore an objective for further research and policy instrument, as well.

In the game theory part of the paper we tackle with a renewable resource model for which as the growth function of the resource is set the well known (from biology) Gompertz growth function. In the equilibrium analysis that follows, pointing out the closed loop solutions of the game, we found the analytic expressions of the cooperative and non cooperative strategies. All the above strategies are independent the state's variable as well as the control's variable, but only hinges upon the discount factor. Therefore, these strategies have the important properties of time consistency, thus they constitute economically acceptant policies. Regarding the players' payoffs, we also found the analytic expressions of the value functions which are functions of the state variable and functions of the common discount rate as well.

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