Subgame Perfect Equilibrium in a Bargaining Model with Deterministic Procedures

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Abstract

Two players, A and B, bargain to divide a perfectly divisible pie. In a bargaining model with constant discount factors, δ_A and δ_B, we extend Rubinstein (1982)'s alternating offers procedures to more general deterministic procedures so that any player in any period can be the proposer. We show that each bargaining game with a deterministic procedure has a unique subgame perfect equilibrium (SPE) payoff outcome, which is efficient. Conversely, each efficient division of the pie can be supported as an SPE outcome by some procedure if $\delta_A + \delta_B \geq 1$, while almost no division can ever be supported in SPE if $\delta_A + \delta_B < 1$.

Keywords: noncooperative bargaining, subgame perfect equilibrium, bargaining procedure

JEL Classification: C78

1 Introduction

In two-player noncooperative bargaining theory, the most often used bargaining procedure is the alternating offers procedure discussed by Ståhl (1972), Rubinstein (1982) and many followers. In this procedure, a player (proposer) suggests a division of a pie in each period, and the rejection of this offer will lead the game to the next period when the other player becomes the proposer. This bargaining

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procedure has many obvious advantages over other procedures. It is quite natural to model everyday bargaining activities with this procedure. In addition, the stationary structure[1] of an infinite-horizon alternating offers procedure makes it relatively easy to analyze. In particular, Rubinstein (1982)'s seminal paper shows that a bargaining game with such a procedure has a unique subgame perfect equilibrium outcome. Shaked and Sutton (1984), Fudenberg and Levine (1983) provide some alternative proofs of Rubinstein's conclusion in the special case of common discount factor.

Although the alternating offers procedure is important, sometimes it is necessary to study more general bargaining procedures, mainly because this procedure is overly simple and lack the flexibility to handle more complex situations. We provide two examples below.

First, sometimes it is important to understand how bargaining outcomes are related to procedures. In other words, one may wonder to what extent different bargaining procedures could result in different bargaining outcomes. An example is the noncooperative implementation of cooperative solutions, which is often referred to as the Nash program[2]. It is natural to ask which cooperative outcomes in a bilateral bargaining situation can be sustained as noncooperative equilibria, using appropriate bargaining procedures.

Second, in reality, bargaining procedures are sometimes not exogenously given, but are endogenously determined. For example, the players in a bargaining game will first negotiate the bargaining protocol before the actual bargaining takes place. They will do so to achieve a better bargaining position[3]. A person (called the designer) who has the authority to design the bargaining procedure will choose a procedure to influence the bargaining outcome in her best interest. In such cases, the players (or the designer) would like to know the outcomes that different procedures may lead to before bargaining for (or designing) the bargaining procedure.

The literature has emphasized the important role that procedures play in bilateral bargaining models, and has discussed many specific procedures. Here I only list a few of them. Osborne and Rubinstein (1990, 1994), and Muthoo (1999)

[1]For any t, the subgame that starts from period $t+2$ has exactly the same structure as that which starts from period t.

[2]See, among others, Nash (1953), Binmore et al. (1986), Ju and Wettstein (2009).

[3]See, for example, Fershtman (1990), Anesi and Seidmann (2014).

mention some specific procedures such as repeated offers, simultaneous offers, and random procedures. Muthoo (1990) allows the proposer to withdraw an offer even after his opponent has already accepted it.

Following the above literature, this paper tries to generalize the alternating offers procedure in a specific way. For a two-player (A and B) noncooperative bargaining games with constant discount factors δ_A and δ_B, we allow a player to successively make offers for several periods according to an exogenously given protocol before the other player begins to propose. In other words, one player is assigned as the proposer in each period of the game according to a deterministic (not random) procedure, which is common knowledge to both players. No further assumptions are placed on the procedure; it can be finite, infinite, stationary, or non-stationary.

Note that these procedures are an extension of the alternating offers procedure, and are fully general within the class of deterministic protocols. However, we do not consider random procedures where the proposer in some period is assigned stochastically, since in reality randomization mechanisms are not always available in determining bargaining procedure. In addition, our generalization is essentially different from changing the factors involving players' time preference, such as discount factors or the time interval between two adjacent periods[4]. This is because players' time preference is by nature exogenously given, while bargaining procedures at least have the potential to be endogenously determined, although in this paper we simply assume them to be exogenously given.

A natural question is whether bargaining games with deterministic general procedures still possess some good theoretical properties. After introducing the basic model and some notations in Sections 2 and 3, we prove in Section 4 that the existence, uniqueness and efficiency of the subgame perfect equilibrium (SPE) outcome are retained for general deterministic procedures (Lemma 6 and Theorem 1). The alternating offers procedure can thus be generalized without losing its theoretical attraction.

Furthermore, in Section 5 we try to determine the influence of the bargaining procedure on the bargaining outcome. More specifically, we examine which payoff

[4]For example, see Binmore et al. (1986) and Binmore (1987) for some discussions on how the subgame perfect equilibrium outcomes can be affected by players' time preference.

outcomes could be supported in SPE by some general procedures. It turns out that all efficient payoff outcomes can be supported in SPE if $\delta_A+\delta_B \geq 1$ (Theorem 2); while almost no outcomes can be supported in SPE if $\delta_A+\delta_B < 1$, in the sense that the measure of the set of SPE outcomes is zero (Theorem 3).

2 Bargaining with deterministic procedures

Two players, A and B, bargain to divide a pie that is perfectly divisible. Time is discrete, and can be denoted by period $t = 1, 2, \cdots$. The constant discount factor of player i is $\delta_i \in [0, 1)$, $i = A, B$.

We define a deterministic procedure of the bargaining game, denoted by

$$\omega = (\omega_1, \omega_2, \ldots, \omega_k, \ldots),$$

to be a sequence of A's and B's, where $\omega_k = A$ or B denotes the kth element of the sequence. A procedure ω can be finite or infinite. Let $T(\omega)$ denote the number of elements ω contains. That is, if ω is finite, then it can be written as $\omega = (\omega_1, \omega_2, \ldots, \omega_{T(\omega)})$; if ω is infinite, then $T(\omega) = \infty$.

The bargaining takes place according to the procedure, which is exogenously given and is common knowledge to both players. Given the procedure ω, in period $t = 1$ the initial proposer ω_1 makes an offer from the agreement set

$$D = \{(d_A, d_B) \mid d_A, d_B \geq 0,\ d_A + d_B = 1\},$$

where d_A and d_B are the respective shares of the pie A and B get in the agreement. The other player decides whether to accept or reject this offer. By induction, if an offer is rejected in some period $t = k \leq T(\omega) - 1$, then the game proceeds to the next period $t = k+1$ when proposer ω_{k+1} makes an offer from D and her opponent responds. Once an offer $d = (d_A, d_B)$ is accepted in period t, the game ends and the accepted agreement is enforced. Each player i's payoff $u_i(d_i, t) = \delta_i^{t-1} d_i$ is her share of the pie in this agreement discounted to $t = 1$. If no agreement is ever accepted in all periods $t \leq T(\omega)$, both players receive a payoff of zero.

For notational simplicity, a procedure

$$\omega = (\underbrace{j_1, \cdots, j_1}_{n_1}, \underbrace{j_2, \cdots, j_2}_{n_2}, \underbrace{j_3, \cdots, j_3}_{n_3}, \cdots)$$

can be written as $\omega = j_1^{n_1} j_2^{n_2} j_3^{n_3} \cdots$, where $j_k = A, B$, $j_k \neq j_{k-1}$, and n_k is the length of the kth group of one-player successive offering periods. This procedure can also be denoted by $\omega = j_1[n_1, n_2, n_3, \ldots]$, where $j_1 = \omega_1$ is the initial proposer. For example, $\omega = A^3 B^2 = A[3, 2]$ is a procedure of five periods, in the first three of which A is the proposer ($\omega_1 = \omega_2 = \omega_3 = A$), while B proposes in the next two periods ($\omega_4 = \omega_5 = B$). In particular, if ω is an infinite procedure and $n_k = 1$, $\forall k$, then ω is the alternating offers procedure discussed by Rubinstein (1982).

Let Ω_i denote the set of all procedures whose initial proposers are $\omega_1 = i$ ($i = A, B$), and let $\Omega = \Omega_A \cup \Omega_B$ be the set of all procedures. Given procedure $\omega \in \Omega$ and discount factors δ_A, δ_B, the bargaining game defined above is denoted by $G(\omega, \delta_A, \delta_B)$.

Given any $\omega \in \Omega$ and any $k \leq T(\omega)$, let $\omega(k)$ denote the subprocedure of ω starting from ω_k. In other words, suppose now that the bargaining has come to period $t = k$, then $\omega(k)$ is the part of ω that starts at period k. For example, suppose $\omega = A^2 B^3 A$, then $\omega(3) = B^3 A$. An infinite procedure ω is said to be stationary, if there exists an integer $k \geq 2$ such that $\omega(nk) = \omega$ for all integers $n \geq 1$. It is obvious that an infinite horizon alternating offers procedure is stationary.

Each player i's strategy in a game $G(\omega, \delta_A, \delta_B)$, denoted by S_i, specifies the action i will take in any time of the game, given the history by that time. More specifically, when it is i's turn to propose, S_i specifies which agreement she will offer; when j is the proposer, S_i specifies which response (accept or reject) i will make to j's offer, where $i, j = A, B$, $i \neq j$. Following Rubinstein (1982) and many others, this paper uses subgame perfect equilibrium (SPE for short) to predict the outcome of the game $G(\omega, \delta_A, \delta_B)$. An SPE of $G(\omega, \delta_A, \delta_B)$ is a strategy pair $S = (S_A, S_B)$ that induces a Nash equilibrium in each subgame.

3 Technical preparation

Note that if the procedure $\omega = j_1^{n_1} j_2^{n_2} \cdots j_k^{n_k}$ has finite groups of one-player successive offering periods, then n_k (the length of the last group's offering periods) is irrelevant to SPE payoffs. This is straightforward since during the last group of offering periods, any SPE always involves the proposer suggesting the agreement in which her own share is one, and the other player accepting this offer. Thus, we may regard two procedures that only differ in the length of the last group's offering periods as essentially identical. For simplicity, hereafter we suppose that the last group of one-player offering periods of each finite procedure contains only one period. That is, if ω is a finite procedure, then it can be written as $\omega = j_1^{n_1} j_2^{n_2} \cdots j_k^{n_k} j_{k+1} = j_1[n_1, n_2, \ldots, n_k, 1]$. Similarly, suppose each infinite procedure contains infinitely many groups of one-player offering periods; in other words, we preclude the case $\omega = j_1^{n_1} j_2^{n_2} \cdots j_k^{n_k} j_{k+1}^{\infty}$, since it is essentially identical to $\omega = j_1^{n_1} j_2^{n_2} \cdots j_k^{n_k} j_{k+1}$.

We introduce the following notations for future convenience. Given a finite or infinite procedure $\omega = j_1^{n_1} j_2^{n_2} \cdots$, let

$$r(\omega) = \begin{cases} 0, & \text{if } \omega = A \text{ or } \omega = B \\ m, & \text{if } \omega = j_1^{n_1} j_2^{n_2} \cdots j_m^{n_m} j_{m+1} \\ \infty, & \text{if } \omega \text{ is infinite} \end{cases}.$$

That is, the identity of the proposer changes $r(\omega)$ times throughout ω. In other words, there are $r(\omega) + 1$ groups of one-player successive offering periods in ω. Furthermore, given procedure $\omega = j_1^{n_1} j_2^{n_2} \cdots$ and nonnegative integer $k \leq r(\omega)$, define

$$p(\omega, k) = \begin{cases} 1, & \text{if } k = 0 \\ \sigma_1^{n_1} \sigma_2^{n_2} \cdots \sigma_k^{n_k}, & \text{if } k = 1, 2, \ldots, r(\omega) \end{cases},$$

where $\sigma_i = \delta_A$, if $j_i = B$; $\sigma_i = \delta_B$, if $j_i = A$. Finally, write

$$\theta(\omega) = \sum_{k=0}^{r(\omega)} (-1)^k p(\omega, k). \tag{1}$$

As an example, suppose $\omega^1 = B[n_1, n_2, n_3, n_4, 1]$, then $r(\omega^1) = 4$, and

$$\theta(\omega^1) = 1 - \delta_A^{n_1} + \delta_A^{n_1}\delta_B^{n_2} - \delta_A^{n_1+n_3}\delta_B^{n_2} + \delta_A^{n_1+n_3}\delta_B^{n_2+n_4}.$$

In addition, suppose $\omega^2 = A[1, 1, \ldots]$ is an infinite alternating offers procedure, then

$$\theta(\omega^2) = 1 - \delta_B + \delta_B\delta_A - \delta_B^2\delta_A + \delta_B^2\delta_A^2 - \delta_B^3\delta_A^2 + \cdots = \frac{1-\delta_B}{1-\delta_A\delta_B}. \tag{2}$$

We list some properties of $p(\omega, k)$ and $\theta(\omega)$ below. These will be useful for future analysis. Roughly speaking, these properties suggest that $\theta(\omega)$ is a well-defined partition of the pie (Lemma 1, 4); that this partition changes if one player offers for an additional period (Lemma 5); and that the elements in the sequence $\{z_r : r = 1, \ldots, r(\omega)\}$ where $z_r = \sum_{k=0}^{r}(-1)^k p(\omega, k)$ are alternately larger and smaller than $\theta(\omega)$ (Lemma 2, 3).

Lemma 1. *If $r(\omega) = \infty$, then $\sum_{k=t}^{\infty}(-1)^k p(\omega, k)$ is absolutely convergent for all $t \geq 0$.*

Proof. Let $\delta = \max\{\delta_A, \delta_B\} < 1$. For all $t \geq 0$, the series $\sum_{k=t}^{\infty}\delta^k$ is convergent. Since $0 < p(\omega, k) \leq \delta^{\sum_{i=1}^{k} n_i} \leq \delta^k$, we know by the comparison test that $\sum_{k=t}^{\infty} p(\omega, k)$ is also convergent. Thus $\sum_{k=t}^{\infty}(-1)^k p(\omega, k)$ is absolutely convergent. □

Lemma 2. *Suppose $t < r \leq r(\omega)$. If t is odd, then $-p(\omega, t) < \sum_{k=t}^{r}(-1)^k p(\omega, k) < 0$; if t is even, then $0 < \sum_{k=t}^{r}(-1)^k p(\omega, k) < p(\omega, t)$.*

Proof. We shall only prove the case $r = \infty$, since the proof of the case $r < \infty$ is similar. According to Lemma 1, we can rearrange the elements of $\sum_{k=t}^{\infty}(-1)^k p(\omega, k)$ without affecting the sum of this series. Assume without loss of generality that t is even. Then

$$\begin{aligned} 0 &< [p(\omega,t) - p(\omega,t+1)] + [p(\omega,t+2) - p(\omega,t+3)] + \cdots = \sum_{k=t}^{\infty}(-1)^k p(\omega,k) \\ &= p(\omega,t) - [p(\omega,t+1) - p(\omega,t+2)] - [p(\omega,t+3) - p(\omega,t+4)] - \cdots < p(\omega,t), \end{aligned}$$

which is exactly what we want to prove. □

Lemma 3. *Given $r \le r(\omega)$, let $z_r = \sum_{k=0}^{r}(-1)^k p(\omega, k)$. For any $t < s \le r(\omega)$, if t is odd, then $z_t < z_s$; if t is even, then $z_t > z_s$.*

Proof. Since $z_t = \sum_{k=0}^{t}(-1)^k p(\omega, k) = z_s - \sum_{k=t+1}^{s}(-1)^k p(\omega, k)$, we can prove the conclusion directly from Lemma 2. □

Lemma 4. *For any $\omega \in \Omega$, $0 \le \theta(\omega) \le 1$.*

Proof. Suppose without loss of generality that $\omega = A^{n_1}B^{n_2}\cdots \in \Omega_A$. If $r(\omega) = 0$, then $\theta(\omega) = 1$. If $r(\omega) = 1$, then $\theta(\omega) = 1 - \delta_B^{n_1}$. If $r(\omega) \ge 2$, then according to Lemma 3, we have $0 < 1 - \delta_B^{n_1} = z_1 < \theta(\omega) = z_{r(\omega)} < z_0 = 1$. □

Lemma 5. *Suppose $\omega^1, \omega^2 \in \Omega$, $r(\omega^2) = r(\omega^1) + 1$; there exists an integer $t \ge 2$ such that $\omega_k^2 = \omega_k^1$, for all $k = 1, \ldots, t-1$; $\omega_k^2 = \omega_{k-1}^1$, for all $k = t+1, \ldots, r(\omega^2)$. Then $\theta(\omega^2) > \theta(\omega^1)$ if $\omega_t^2 = \omega_1^1$, while $\theta(\omega^2) < \theta(\omega^1)$ if $\omega_t^2 \ne \omega_1^1$.*

Proof. Without loss of generality, suppose $\omega^1 = A^{n_1}\cdots B^{n_{s-1}}A^{n_s}B^{n_{s+1}}\cdots$, and $\omega^2 = A^{n_1}\cdots B^{n_{s-1}}A^{n_s+1}B^{n_{s+1}}\cdots$. According to (1), $\theta(\omega^2) = \sum_{k=0}^{r(\omega^2)}(-1)^k p(\omega^2, k)$ $= \sum_{k=0}^{s-1}(-1)^k p(\omega^1, k) + \delta_B \sum_{k=s}^{r(\omega^1)}(-1)^k p(\omega^1, k)$, whereas $\sum_{k=s}^{r(\omega^1)}(-1)^k p(\omega^1, k) < 0$ due to Lemma 2. Hence, $\theta(\omega^2) > \sum_{k=0}^{s-1}(-1)^k p(\omega^1, k) + \sum_{k=s}^{r(\omega^1)}(-1)^k p(\omega^1, k) = \sum_{k=0}^{r(\omega^1)}(-1)^k p(\omega^1, k) = \theta(\omega^1)$. □

4 Subgame perfect equilibrium

In this section, we examine the subgame perfect equilibria of bargaining games with deterministic procedures. We first establish that in any SPE the bargaining in $G(\omega, \delta_A, \delta_B)$ will end by its first period. Therefore, there is no delay in SPE, and SPE outcomes, if they exist, must be efficient[5].

Lemma 6. *Suppose a bargaining game $G(\omega, \delta_A, \delta_B)$ has come to some period, then in any SPE, the offer proposed in that period must be accepted.*

Proof. First note that if ω is finite and the game has come to the last period $T(\omega)$, then in SPE the offer proposed in this period must be accepted. Now for any finite or infinite ω, suppose the bargaining has come to period $k_1 \le T(\omega) - 1$. We

[5]In this paper, a payoff outcome (u_A, u_B) where $u_i \ge 0$ is said to be efficient if $u_A + u_B = 1$.

assume for a contradiction that in some SPE, $S = (S_A, S_B)$, the offer in period k_1 is rejected. Then either some agreement is reached in some period $t > k_1$, or no agreement is ever reached in finite time (when ω is infinite).

In the first case, there is an integer k, $1 \leq k \leq T(\omega) - k_1$, such that according to S the offers in periods $k_1, k_1 + 1, \ldots, k_1 + k - 1$ are all rejected, but the offer d^* proposed in period $k_1 + k$ is accepted. Suppose the proposer i in period $k_1 + k - 1$ proposes an agreement d' in which j gets $\delta_j x + \varepsilon$ and i gets $1 - \delta_j x - \varepsilon$, where x is j's share in d^*, and $0 < \varepsilon < \min\{1 - \delta_A, 1 - \delta_B\}$. It is obvious that j will accept d' in period $k_1 + k - 1$ according to S_j, since otherwise he is worse off by getting x in period $k_1 + k$. Since the offer in period $k_1 + k - 1$ is rejected, player i would not propose d' according to S_i. This implies $1 - \delta_j x - \varepsilon \leq \delta_i(1 - x)$, and hence $\varepsilon \geq 1 - \delta_i + (\delta_i - \delta_j)x$, which contradicts the assumption $\varepsilon < \min\{1 - \delta_A, 1 - \delta_B\}$.

In the second case, we can similarly prove that the proposer i in period k_1 has an incentive to deviate from S_i and to propose an agreement (for example, j gets ρ and i gets $1 - \rho$, where $\delta_j < \rho < 1$) which will be accepted by j. This contradicts the assumption that S is an SPE, and completes the proof. □

The following theorem shows that for each bargaining game with a deterministic procedure $G(\omega, \delta_A, \delta_B)$, there is a unique SPE payoff outcome, in which the initial proposer ω_1's payoff is $\theta(\omega)$.

Theorem 1. *There exists a unique pair $(x, 1-x)$ that can be supported as an SPE outcome of $G(\omega, \delta_A, \delta_B)$, where $x = \theta(\omega)$ if $\omega \in \Omega_A$, and $x = 1 - \theta(\omega)$ if $\omega \in \Omega_B$.*

Proof. See the appendix. □

Note that if $\omega = A[1, 1, \ldots]$ is an infinite-horizon alternating offers procedure, then it follows from (2) that $\theta(\omega) = \frac{1-\delta_B}{1-\delta_A\delta_B}$. Hence, Theorem 1 can be regarded as a generalization of Rubinstein (1982)'s main theorem for the special case of constant discount factors. In fact, one contribution of Theorem 1 is that it suggests that the existence and uniqueness of the SPE outcome does not depend on the stationary structure of the alternating offers procedure, and can be extended to non-stationary bargaining procedures as well.

Shaked and Sutton (1984) suggests that when $\delta_A = \delta_B = \delta$, each player's SPE payoff in an alternating offers bargaining game is basically the discounted sum of all those pieces of pie that shrink (at a speed of δ) while that player is the proposer. It follows from Theorem 1 that this intuitive statement remains true for all deterministic bargaining procedures. As an example, consider $\omega = A^2B^3A$, then the SPE partition is $(1-\delta^2+\delta^5, \delta^2-\delta^5)$. Suppose that the size of the pie is one in period 1; in each period k, $1 < k \leq 6$, the pie is δ times as large as that in period $k-1$; in period $k > 6$, the size is zero. If B rejects A's offers in periods 1 and 2, the size of the pie will decrease by $1-\delta^2$; if A's offer is rejected in period 6, the remaining pie (of size δ^5) will disappear. Hence, $1-\delta^2+\delta^5$ is the total size of the pie that shrinks while A is the proposer.

Moreover, Theorem 1 may serve as a basis for analyzing the influence of bargaining procedure on the bargaining outcome. This is the main task of the next section. For example, Theorem 1 and Lemma 5 together imply that a player's SPE payoff increases as she gains one additional offering period in a new procedure.

5 The measure of SPE payoff outcomes

In this section, we mainly focus on the converse of Theorem 1. That is, given δ_A and δ_B, can an efficient partition of the pie $(x, 1-x)$ be supported as an SPE outcome? The following theorem suggests that when $\delta_A + \delta_B \geq 1$, any efficient partition can be supported as an SPE outcome by some procedure. The proof in appendix illustrates how to explicitly construct such a procedure.

Theorem 2. *Suppose $\delta_A + \delta_B \geq 1$, then for any $x \in [0, 1]$, there exists $\omega \in \Omega$ such that the SPE payoff pair supported by ω is $(x, 1-x)$.*

Proof. See the appendix. □

The condition $\delta_A + \delta_B \geq 1$ in Theorem 2 is essential. In fact, if $\delta_A + \delta_B < 1$, then there are some $(x, 1-x)$ that cannot be supported in SPE by any procedure. For example, each $\omega \in \Omega_A$ will result in the SPE payoff pair $(x, 1-x)$ such that $x \geq 1-\delta_B$,[6] while each $\omega \in \Omega_B$ will lead to $(x, 1-x)$ such that $x \leq \delta_A$. Hence, if $\delta_A + \delta_B < 1$ and $x \in (\delta_A, 1-\delta_B)$, then $(x, 1-x)$ cannot be supported in SPE by any procedure $\omega \in \Omega$.

[6] Suppose $\omega = A^{n_1}B^{n_2}\cdots$, then according to Lemma 3 and Theorem 1, $x = \theta(\omega) = z_{r(\omega)} \geq z_1 = 1-\delta_B^{n_1} \geq 1-\delta_B$ if $r(\omega) \geq 1$; $x = \theta(\omega) = 1 > 1-\delta_B$ if $r(\omega) = 0$.

Furthermore, we are interested in "how many" payoff divisions can ever be supported as SPE outcomes. Given δ_A and δ_B, let

$$\Gamma(\delta_A, \delta_B) = \big\{x \in [0,1] \,|\, \exists\, \omega \in \Omega,\ s.t. \text{ SPE outcome is } (x, 1-x)\big\}$$

be the set of player A's share of the pie in all SPE payoff divisions. Let $m[Q]$ denote the (Lebesgue) measure of a measurable set $Q \subset \mathbb{R}$. If $\Gamma(\delta_A, \delta_B)$ is measurable, then $m[\Gamma(\delta_A, \delta_B)]$ measures the scale of SPE payoff divisions. Since $\Gamma(\delta_A, \delta_B) \subset [0,1]$, we have $0 \leq m[\Gamma(\delta_A, \delta_B)] \leq 1$ if $\Gamma(\delta_A, \delta_B)$ is measurable.

It follows from Theorem 2 that if $\delta_A + \delta_B \geq 1$, then $\Gamma(\delta_A, \delta_B) = [0,1]$, and thus $m[\Gamma(\delta_A, \delta_B)] = 1$. On the contrary, the next theorem establishes that if $\delta_A + \delta_B < 1$, then $\Gamma(\delta_A, \delta_B)$ is also measurable, but its measure is zero. It suggests that almost no partitions of the pie can ever be supported in SPE if $\delta_A + \delta_B < 1$.

Theorem 3. *Suppose* $\delta_A + \delta_B < 1$*, then* $m[\Gamma(\delta_A, \delta_B)] = 0$*.*

Proof. See the appendix. □

The contrast between Theorem 2 and Theorem 3 has an intuitive explanation. When players are relatively impatient ($\delta_A + \delta_B < 1$), different procedures are less capable of yielding different bargaining outcomes than in the situations with sufficiently patient players ($\delta_A + \delta_B \geq 1$), since impatient players care less about the arrangement of future procedures and are less sensitive to a change in procedure.

As a possible application of these conclusions, suppose there is a designer who tries to achieve his ideal payoff division $(x^*, 1-x^*)$ by designing a deterministic bargaining procedure. To an outside observer, x^* is an ex ante random variable, which is uniformly distributed over $[0,1]$. It follows from Theorem 2 and Theorem 3 that from the outside observer's view, with probability one the designer can "implement" his ideal payoff division if $\delta_A + \delta_B \geq 1$, while the corresponding probability is zero if $\delta_A + \delta_B < 1$. Hence, $m[\Gamma(\delta_A, \delta_B)]$ can be regarded as the outside observer's estimation of the designer's control power over the bargaining outcome. In particular, when $\delta_A + \delta_B$ is close to 1, a designer who has full control over all bargaining outcomes may, after a slight perturbation of discount factors, suddenly lose almost all his power in determining any specific outcome.

6 Conclusion

This paper focuses on the role bargaining procedures play in a bilateral noncooperative bargaining model. We consider natural extensions and generalizations of the alternating offers procedure. We explore the potential of these generalized procedures to achieve a particular bargaining outcome, and also investigate the limitations of these procedures.

The theoretical contribution of this paper is two-fold. First, we show that a bargaining game with any deterministic procedure has a unique subgame perfect equilibrium outcome, which is efficient. Second, all efficient partitions of the pie can be supported as SPE outcomes by some procedures if players are sufficiently patient; while only a degenerate set of partitions can be supported in SPE if players are impatient.

Finally, we list here some extensions that might be worth exploring in future research. First, it might be helpful to discuss the endogenously determination of the bargaining procedure. Second, the general bargaining game under incomplete information might be very complex but also interesting. Third, the analysis can be extended to bargaining games with more than two players.

Appendix

Proof of Theorem 1.

By symmetry, we only consider $\omega \in \Omega_A$. The theorem can be easily proved if $\omega = A$, so we only consider procedures ω such that $r(w) \geq 1$.

First define a strategy pair $S^* = (S^*_A, S^*_B)$ as follows. When the game comes to period s, for any history, the proposer $\omega_s = i$ proposes $\theta(\omega(s))$ as the share for herself and $1 - \theta(\omega(s))$ as the share for her opponent $j \neq i$ according to S^*_i; the other player j accepts any offer in which the proposer's share is no more than $\theta(\omega(s))$ and rejects all other offers according to S^*_j. It is obvious that if players follow S^*, then the game will end by period $t = 1$ with agreement $\big(\theta(\omega), 1-\theta(\omega)\big)$. We shall prove that S^* is an SPE of $G(\omega, \delta_A, \delta_B)$, and $\big(\theta(\omega), 1-\theta(\omega)\big)$ is the only payoff pair that could ever be supported in SPE.

In order to show that S^* is an SPE, it is sufficient to verify that S^* satisfies the one-shot deviation principle, which asserts that a strategy profile S is an SPE of an extensive game with constant discount factors, if and only if no player can ever become strictly better off by deviating from S for just one period and then reverting to S[7]. Suppose the game has come to period $t = s$. If the proposer ω_s, without loss of generality assumed to be A, deviates from S_A^* and suggests an offer in which her own share is $y_{1,s} > x_{1,s}$, then player B will reject this offer according to S_B^* and let the game enter period $s+1$. If $\omega_{s+1} = A$, then A will get $v(s+1) = \theta(\omega(s+1))$ in period $s+1$ according to S^*; if $\omega_{s+1} = B$, then A will get $v(s+1) = 1 - \theta(\omega(s+1))$ in period $s+1$. In either case, if $\delta_B > 0$, we have

$$\begin{aligned} \theta(\omega(s)) &= 1 + \sum_{k=1}^{r(\omega(s))} (-1)^k p(\omega(s), k) \\ &> 1 + \frac{1}{\delta_B} \sum_{k=1}^{r(\omega(s))} (-1)^k p(\omega(s), k) \\ &= v(s+1) > \delta_A v(s+1), \end{aligned}$$

where $\sum_{k=1}^{r(\omega(s))} (-1)^k p(\omega_s, k) < 0$ due to Lemma 2; if $\delta_B = 0$, then $\theta(\omega(s)) = 1$, thus we also have $\theta(\omega(s)) > \delta_A v(s+1)$ due to Lemma 4. This implies that A has no incentive to increase her own share in the offer in period s, since otherwise her payoff will decline from $u_A(\theta(\omega(s)), s)$ to $u_A(v(s+1), s+1)$. On the other hand, it is obvious that A will also not decrease her own share in the offer in period s. Similarly, we can show that B will neither increase nor decrease her reservation share $1 - \theta(\omega(s))$ for accepting an offer proposed by A in period s. Thus we have proved S^* satisfies the one-shot deviation principle, and hence is an SPE.

Now we turn to the uniqueness part of the theorem, using a technique introduced by Shaked and Sutton (1984). Given $i = 0, 1, \ldots, r(\omega)$, let Λ_i be a subset of $[0, 1]$ so that for each $x \in \Lambda_i$, there exists an SPE of $G(\omega, \delta_A, \delta_B)$ such that in this SPE, when the game has come to period $t_i = \sum_{j=0}^{i} n_j + 1$ (where $n_0 = 0$) the proposer suggests an agreement in which his own share is x and his opponent's share is $1 - x$. Let M_i and m_i denote the supremum and infimum of Λ_i, respectively.

In period $n_1 + 1$ player B can get no more than M_1 in any SPE. Hence, A will

[7] See, for example, Fudenberg and Tirole (1991, section 4.2).

offer B no more than $\delta_B M_1$ in period n_1, no more than $\delta_B^2 M_1$ in period $n_1 - 1$, ..., no more than $\delta_B^{n_1} M_1$ in period 1. Thus $m_0 \geq 1 - \delta_B^{n_1} M_1$. Similarly, we have

$$m_{j-1} \geq 1 - \delta_j^{n_j} M_j, \quad j = 1, \ldots, r(\omega), \tag{3}$$

where $\delta_j = \delta_A$ if j is even, and $\delta_j = \delta_B$ if j is odd.

If player B rejects A's offers in period $t = 1, \ldots, n_1$, he will get no less than m_1 in period $n_1 + 1$ in any SPE. Thus due to Lemma 6, A will offer B no less than $\delta_B^{n_1} m_1$ in period $t = 1$. Hence $M_0 \leq 1 - \delta_B^{n_1} m_1$. Similarly, we have

$$M_{j-1} \leq 1 - \delta_j^{n_j} m_j, \quad j = 1, \ldots, r(\omega), \tag{4}$$

where $\delta_j = \delta_A$ if j is even, and $\delta_j = \delta_B$ if j is odd.

If ω is finite, by repeatedly using (3) and (4), we have

$$\begin{aligned} m_0 &\geq 1 - \delta_B^{n_1} M_1 \geq 1 - \delta_B^{n_1} + \delta_B^{n_1} \delta_A^{n_2} m_2 \geq 1 - \delta_B^{n_1} + \delta_B^{n_1} \delta_A^{n_2} - \delta_B^{n_1+n_3} \delta_A^{n_2} M_3 \\ &\geq \cdots \geq \sum_{k=0}^{r(\omega)-1} (-1)^k p(\omega, k) + (-1)^{r(\omega)} p(\omega, r(\omega)) \overline{m}_{r(\omega)}, \\ M_0 &\leq 1 - \delta_B^{n_1} m_1 \leq 1 - \delta_B^{n_1} + \delta_B^{n_1} \delta_A^{n_2} M_2 \leq 1 - \delta_B^{n_1} + \delta_B^{n_1} \delta_A^{n_2} - \delta_B^{n_1+n_3} \delta_A^{n_2} m_3 \\ &\leq \cdots \leq \sum_{k=0}^{r(\omega)-1} (-1)^k p(\omega, k) + (-1)^{r(\omega)} p(\omega, r(\omega)) \underline{m}_{r(\omega)}, \end{aligned}$$

where $\overline{m}_{r(\omega)} = m_{r(\omega)}$, $\underline{m}_{r(\omega)} = M_{r(\omega)}$ if $r(\omega)$ is even, and $\overline{m}_{r(\omega)} = M_{r(\omega)}$, $\underline{m}_{r(\omega)} = m_{r(\omega)}$ if $r(\omega)$ is odd. However, in the last period of the game $t_{r(\omega)} = T(\omega)$, in any SPE the proposer always suggests the agreement in which her own share is one. Hence $m_{r(\omega)} = M_{r(\omega)} = 1$. Since $m_0 \leq M_0$, we have $m_0 = M_0 = \sum_{k=0}^{r(\omega)} (-1)^k p(\omega, k) = \theta(\omega)$.

If ω is infinite, we can similarly get

$$\sum_{k=0}^{j-1} (-1)^k p(\omega, k) + (-1)^j p(\omega, j) \overline{m}_j \leq m_0 \leq M_0 \leq \sum_{k=0}^{j-1} (-1)^k p(\omega, k) + (-1)^j p(\omega, j) \underline{m}_j$$

for each $j \geq 1$, where $\overline{m}_j = m_j$, $\underline{m}_j = M_j$ if j is even, and $\overline{m}_j = M_j$, $\underline{m}_j = m_j$ if j is odd. Note that $\lim_{j \to \infty} (-1)^j p(\omega, j) \overline{m}_j = \lim_{j \to \infty} (-1)^j p(\omega, j) \underline{m}_j = 0$. Due to

Lemma 1, let $j \to \infty$, we have $m_0 = M_0 = \sum_{k=0}^{\infty}(-1)^k p(\omega, k) = \theta(\omega)$. Hence we have proved the theorem. □

Proof of Theorem 2.

I shall only prove that for any $x \in [1-\delta_B, 1]$, there exists $\omega \in \Omega_A$ such that the payoff pair supported in SPE by ω is $(x, 1-x)$. For any $x \in [0, 1-\delta_B) \subset [0, \delta_A)$, a similar proof applies for $\omega \in \Omega_B$.

We can construct a procedure $\omega \in \Omega_A$ inductively as follows. If $x = 1$, then $\omega = A$ suffices for the proof. Otherwise, we have $1 - \delta_B \leq x < 1$. Let n_1 be the maximal integer such that $1 - \delta_B^{n_1}$ is not greater than x. That is,

$$1 - \delta_B^{n_1} \leq x < 1 - \delta_B^{n_1+1}. \tag{5}$$

If the equality in (5) holds, that is, $x = 1 - \delta_B^{n_1}$, we can implement $(x, 1-x)$ as an SPE outcome by $\omega = A[n_1, 1]$. Otherwise, we have $1 - \delta_B^{n_1} < x < 1 - \delta_B^{n_1+1}$. In this case, let n_2 be the maximal integer such that $1 - \delta_B^{n_1} + \delta_A^{n_2}\delta_B^{n_1}$ is not less than x. That is,

$$1 - \delta_B^{n_1} + \delta_A^{n_2+1}\delta_B^{n_1} < x \leq 1 - \delta_B^{n_1} + \delta_A^{n_2}\delta_B^{n_1}. \tag{6}$$

If the equality in (6) holds, that is, $x = 1 - \delta_B^{n_1} + \delta_A^{n_2}\delta_B^{n_1}$, we can implement $(x, 1-x)$ as an SPE outcome by $\omega = A[n_1, n_2, 1]$. Otherwise, we have $1 - \delta_B^{n_1} + \delta_A^{n_2+1}\delta_B^{n_1} < x < 1 - \delta_B^{n_1} + \delta_A^{n_2}\delta_B^{n_1}$, and should continue to construct ω.

Now, suppose by induction that $n_1, n_2, \ldots, n_r$ have already been defined, where $r \geq 2$. If $x = \sum_{k=0}^{r}(-1)^k p(\omega, k)$, $(x, 1-x)$ can be supported in SPE by $\omega = A[n_1, n_2, \ldots, n_r, 1]$. Otherwise, if r is odd, then

$$\sum_{k=0}^{r}(-1)^k p(\omega, k) < x < \sum_{k=0}^{r-1}(-1)^k p(\omega, k) - \delta_B p(\omega, r), \tag{7}$$

while if r is even, then

$$\sum_{k=0}^{r-1}(-1)^k p(\omega, k) + \delta_A p(\omega, r) < x < \sum_{k=0}^{r}(-1)^k p(\omega, k).$$

In either case, we need to define n_{r+1}. If r is odd, let n_{r+1} be the integer such that

$$\sum_{k=0}^{r}(-1)^k p(\omega,k) + \delta_A p(\omega, r+1) < x \le \sum_{k=0}^{r+1}(-1)^k p(\omega,k). \tag{8}$$

If r is even, let n_{r+1} be the integer such that

$$\sum_{k=0}^{r+1}(-1)^k p(\omega,k) \le x < \sum_{k=0}^{r}(-1)^k p(\omega,k) - \delta_B p(\omega, r+1). \tag{9}$$

We need to show that n_{r+1} is well defined, that is, there exists exactly one integer n_{r+1} such that (8) or (9) holds. If r is odd, $\sum_{k=0}^{r+1}(-1)^k p(\omega,k)$ is decreasing in n_{r+1}. When $n_{r+1}=1$, according to (7) we have

$$\begin{aligned}\sum_{k=0}^{r+1}(-1)^k p(\omega,k) &= \sum_{k=0}^{r}(-1)^k p(\omega,k) + \delta_A p(\omega,r)\\ &\ge \sum_{k=0}^{r}(-1)^k p(\omega,k) + (1-\delta_B) p(\omega,r)\\ &= \sum_{k=0}^{r-1}(-1)^k p(\omega,k) - \delta_B p(\omega,r) > x,\end{aligned}$$

while as $n_{r+1}\to\infty$, $\sum_{k=0}^{r+1}(-1)^k p(\omega,k) \to \sum_{k=0}^{r}(-1)^k p(\omega,k) < x$. Thus, there exists a unique integer n_{r+1} that satisfies (8), that is, n_{r+1} is well defined. Similarly, we can prove n_{r+1} is well defined if r is even. We have thus finished defining n_{r+1}.

Given $x \in [1-\delta_B, 1]$, if there exists an integer h such that for $n_1, n_2, \ldots, n_h$ defined above, $x = \sum_{k=0}^{h}(-1)^k p(\omega,k)$, then from Theorem 1, the payoff pair $(x, 1-x)$ can be supported as an SPE outcome by $\omega = A[n_1, n_2, \ldots, n_h, 1]$. If such a finite sequence of integers cannot be found, we will eventually get an infinite sequence $n_1, n_2, \ldots$ and a corresponding infinite procedure $\omega = A[n_1, n_2, \ldots]$. Let

$$z_r = \sum_{k=0}^{r}(-1)^k p(\omega,k), \quad r = 1, 2, \ldots.$$

The elements in the sequence $\{z_1, z_2, \ldots\}$ are alternately larger and smaller than

x. According to Lemma 1, this sequence converges, and thus its limit is x. That is, $x = \sum_{k=0}^{\infty}(-1)^k p(\omega, k)$. It follows form Theorem 1 that $(x, 1-x)$ can be supported as an SPE outcome by ω. Thus, we have proved the theorem. $\square$

Proof of Theorem 3.

The proof of the theorem proceeds in steps.

Step 1: We shall show $(\delta_A, 1-\delta_B) \cap \Gamma(\delta_A, \delta_B) = \emptyset$, that is, if $\delta_A < x < 1-\delta_B$, then there does not exist $\omega \in \Omega$ such that $(x, 1-x)$ can be supported as an SPE outcome by ω. Suppose, on the contrary, that we can find such a procedure ω, and assume without loss of generality that $\omega \in \Omega_A$. According to Theorem 1 and Lemma 3, $x = \theta(\omega) = \sum_{k=0}^{r(\omega)}(-1)^k p(\omega, k) \geq 1-\delta_B^{n_1} \geq 1-\delta_B$, which contradicts $x < 1-\delta_B$. Thus $(\delta_A, 1-\delta_B) \cap \Gamma(\delta_A, \delta_B) = \emptyset$. Therefore $m[\Gamma(\delta_A, \delta_B)] \leq 1 - m[(\delta_A, 1-\delta_B)] = \delta_A + \delta_B$.

Step 2: We shall first prove that $(1-\delta_B+\delta_B\delta_A, 1-\delta_B^2) \cap \Gamma(\delta_A, \delta_B) = \emptyset$. Suppose, on the contrary, that we can find $x \in (1-\delta_B+\delta_B\delta_A, 1-\delta_B^2)$ and $\omega \in \Omega$ such that $(x, 1-x)$ can be supported as an SPE outcome by ω. Since $x > 1-\delta_B$, we have $\omega \in \Omega_A$, that is, $\omega_1 = 1$. Again, due to Theorem 1 and Lemma 3, if $\omega_2 = 1$, then $x \geq 1-\delta_B^{n_1} \geq 1-\delta_B^2$; if $\omega_2 = 2$, then $x \leq 1-\delta_B+\delta_B\delta_A^{n_2} \leq 1-\delta_B+\delta_B\delta_A$. This contradicts $x \in (1-\delta_B+\delta_B\delta_A, 1-\delta_B^2)$. Thus $(1-\delta_B+\delta_B\delta_A, 1-\delta_B^2) \cap \Gamma(\delta_A, \delta_B) = \emptyset$. Similarly, we can prove $(\delta_A^2, \delta_A-\delta_B\delta_A) \cap \Gamma(\delta_A, \delta_B) = \emptyset$. Hence, $m[\Gamma(\delta_A, \delta_B)] \leq \delta_A+\delta_B - m[(1-\delta_B+\delta_B\delta_A, 1-\delta_B^2)] - m[(\delta_A^2, \delta_A-\delta_B\delta_A)] = (\delta_A+\delta_B)^2$.

Assume by induction that by the first $k \geq 1$ steps, we have deleted the following $2^k - 1$ disjoint intervals from $\Gamma(\delta_A, \delta_B)$: $A_1^1 = (\delta_A, 1-\delta_B)$, $A_1^2 = (\delta_A^2, \delta_A-\delta_B\delta_A)$, $A_2^2 = (1-\delta_B+\delta_B\delta_A, 1-\delta_B^2)$, $A_1^k = (\delta_A^k, \delta_A^{k-1}(1-\delta_B))$, $\cdots$, $A_{2^{k-1}}^k = (1-\delta_B^{k-1}(1-\delta_A), 1-\delta_B^k)$. There remain 2^k intervals, whose total measure is $1 - \sum_{s=1}^{k}\sum_{i=1}^{2^{k-1}} m[A_i^s] = (\delta_A+\delta_B)^k$. Thus $m[\Gamma(\delta_A, \delta_B)] \leq (\delta_A+\delta_B)^k$.

Now at step $k+1$, we further delete 2^k disjoint intervals A_i^{k+1}, $i = 1, \ldots, 2^k$, from $\Gamma(\delta_A, \delta_B)$. More specifically, in each remaining interval $[P_i, Q_i]$ after step k, we delete $A_i^{k+1} = \big(P_i + \delta_A(Q_i - P_i), Q_i - \delta_B(Q_i - P_i)\big)$. There remain two disjoint intervals in $[P_i, Q_i]$: $[P_i, P_i + \delta_A(Q_i - P_i)]$ and $[Q_i - \delta_B(Q_i - P_i), Q_i]$, whose total measure is $(\delta_A+\delta_B)(Q_i - P_i)$. Therefore, after step $k+1$, there remain 2^{k+1} intervals, whose total measure is $(\delta_A+\delta_B)^{k+1}$.

To show $m[\Gamma(\delta_A, \delta_B)] \leq (\delta_A+\delta_B)^{k+1}$, we still have to prove that $A_i^{k+1} \cap$

$\Gamma(\delta_A, \delta_B) = \emptyset$, $i = 1, \dots, 2^k$. Without loss of generality, we only consider $A_1^{k+1} = \big(\delta_A^{k+1}, \delta_A^k(1-\delta_B)\big) \subset [P_1, Q_1] = [0, \delta_A^k]$. Suppose, on the contrary, that we can find $x \in A_1^{k+1}$ and $\omega \in \Omega$ such that $(x, 1-x)$ can be supported as an SPE outcome by ω. Since $0 < x < \delta_A < 1 - \delta_B$, we may write $\omega = B[n_1, n_2, \dots]$, then $x = \delta_A^{n_1} - \delta_A^{n_1}\delta_B^{n_2} + \cdots$. If $n_1 \geq k+1$, then $x < \delta_A^{n_1} \leq \delta_A^{k+1}$, which contradicts $x > \delta_A^{k+1}$. If $n_1 \leq k$, then $x \geq \delta_A^{n_1}(1 - \delta_B^{n_2}) \geq \delta_A^k(1-\delta_B)$, which contradicts $x < \delta_A^k(1-\delta_B)$. Thus $A_1^{k+1} \cap \Gamma(\delta_A, \delta_B) = \emptyset$.

Hence, we have proved $m[\Gamma(\delta_A, \delta_B)] \leq (\delta_A + \delta_B)^k$, $\forall k \geq 1$. Since $\delta_A + \delta_B < 1$, we have $(\delta_A + \delta_B)^k \to 0$ as $k \to \infty$. Thus $m[\Gamma(\delta_A, \delta_B)] = 0$. □

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