Economy-wide substitution and Rybczynski sign pattern in a three-factor two-good model

Nakada, Yoshiaki

Kyoto University

5 October 2015

An earlier version of this paper was presented in the seminar of International Rural Development, Kyoto University, on Oct 5, 2015.

2015.11.07.

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By Yoshiaki Nakada, Kyoto University, E-mail: nakada@kais.kyoto-u.ac.jp

Equation Section 1

Abstract

In this article, we analyze how factor endowment affects production in the three-factor two-good general equilibrium trade model. These relationships determine whether 'a strong Rybczynski result' holds or not. We search for a sufficient condition for this result to hold (or not to hold). We assume factor intensity ranking is constant. We use 'the economy-wide substitution' (or EWS) and EWS-ratios to analyze Rybczynski sign pattern in a systematic manner. We also analyze Stolper-Samuelson sign pattern, which expresses the commodity price-factor price relationships.

Keywords: three-factor two-good model; general equilibrium; Rybczynski result; economy-wide substitution.

1. Introduction

Batra and Casas (1976) (hereinafter BC) wrote an article on functional relations in a three-factor two-good neoclassical model (or 3 X 2 model), and they claimed that 'a strong Rybczynski result' arises. According to Suzuki (1983, p141), BC contended in Theorem 6 (p34) that 'if commodity 1 is relatively capital intensive and commodity 2 is relatively labor intensive, an increase in the supply of labor increases the output of commodity 2 and reduces the output of commodity 1.' This is what 'a strong Rybczynski result' implies.

Suzuki (1983) contended that this could not be the case under the assumption of 'perfect complementarity'. He used the Allen-partial elasticities of substitution (hereinafter AES) for his analysis. Jones and Easton (1983) (hereinafter JE) mainly analyzed how commodity prices affect factor prices. This relationship is the dual counter-part to the factor endowment-commodity output relationships. On this duality, see JE (p67), see also BC (p36, eq. (31)-(33)). JE (p75) defined 'the economy-wide substitution' (hereinafter EWS) for their analysis. By using EWS's, JE showed some sufficient conditions for 'a strong Rybczynski result' to hold (or not to hold) in Subsection 5.2 (p86-92), for example, under the assumption of 'perfect complementarity' defined by themselves. JE suggested that 'the factor intensity ranking' and EWS are important for their analysis (On this, see JE(p67, p96)). Thompson (1985) also tried to show some sufficient conditions for 'a strong Rybczynski result' to hold (or not to hold). He used the concept of 'aggregate substitution'.

In sum, these 3 articles tried to disprove BC's claim of 'a strong Rybczynski result'. However, Suzuki's proof is not plausible (On this, see Nakada (2015a)). JE's analysis is somewhat complicated. Especially, JE's proof in subsection 5.2.4, and 5.2.5 (pp. 90-92) is questionable (On this, see Nakada (2015b) and Appendix). Thompson's analysis is not so simple. Sometimes, it is difficult to follow his logic. In any case, before Thompson (1985), it was meaningful to disprove the results derived by BC. But after that, its significance seems to have decreased.

On the other hand, Takayama (1982, Section 4, p13-21) analyzed the factor endowment-commodity output relationships and its dual counter-part in the 3 X 2 model in his survey article. For example, he analyzed when 'extreme factors' are 'aggregate complements' (On this definition, see Takayama (p18)). And he derived the result which is equivalent to 'a strong Rybczynski result'.

After Thompson (1985), what studies have been done about the 3 X 2 model? I mainly explain the articles which dealt with the factor endowment-commodity output relationships and/or its dual-counterpart.

I can classify the articles after Thompson (1985) as follows.

(1) Studies which assumed the functional form of production functions. See e.g., Thompson (1995), Ban (2007), Ban (2008), Ban (2011).

(2) Studies which assumed another assumption about production functions (e.g., normal property, separability). See e.g., Suzuki (1985), Suzuki (1987, chapter 2), Bliss (2003).

(3) Studies which modified one of basic assumptions. See e.g., Ide (2009).

(4) Studies which modified an important basic assumption. See e.g., Ban (2010).

(5) Other studies. See e.g., Suzuki (1987, chapter 1), Teramachi (1993), Easton (2008).

In sum, some of these studies after Thompson (1985) are not so simple, but somewhat complicated. I am not sure whether all of these studies are plausible or not. I do not discuss about it. It seems that some papers tried to apply before they understood the basic functions of the model. Some articles analyzed very differently from other articles. Therefore, sometimes it is not so easy to compare with others.

At least, about a sufficient condition for 'a strong Rybczynski result' to hold (or not to hold) in the 3 X 2 model of BC's original type, nobody has analyzed systematically. The purpose of this article is to analyze it in a systematic manner. We define EWS-ratios based on EWS to analyze.

In section 2, we explain about the basic structure of the model. In section 3, we assume factor intensity ranking. In section 4, we make a system of linear equations using 5 X 5 matrix to obtain the solutions. In section 5, we make a Rybczynski matrix, and analyze its component. In section 6, we rewrite it using EWS-ratios. In section 7, we derive the important relationship among EWS-ratios. And we draw the boundary line for the region for EWS-ratio vector in the figure. In section 8, we draw the border line for a Rybczynski sign pattern to change in the figure. This border line devides the region for EWS-ratio vector into 12 subregions. In section 9, we analyze Rybczynski sign patterns by using Hadamard product of matrices, and derive a sufficient condition for ‘a strong Rybczynski result’ to hold (or not to hold). In section 10, We analyze Stolper-Samuelson sign patterns which express the commodity price-factor price relationships. In section 11, we show some applications of these results. Section 12 is a conclusion. In Appendix, we derive the important relationship among EWS’s.

The studies after Thompson (1985) are as follows.

(1) Thompson (1995) assumed that production functions were trans-log type, and he estimated the values of parameters in U.S. using econometrics. Based on these, he computed ‘the aggregate elasticities’ (equivalent to EWS). Next, he assumed Cobb-Douglas type, and CES type, and computed similarly. And after that, he assumed ‘the strong degrees of complementarity’. In sum, he used all of these for his analysis of the factor endowment-commodity output relationships, and its dual-counterpart. This is an application. Ban (2007), assumed that production functions were two-stage CES type. In his model, three factors are skilled labor, capital, unskilled labor. He assumed that capital and skilled labor could be ‘[Allen-] complements’ in two sectors. And he computed the values of AES’s theoretically. He tried to analyze how commodity prices affect the relative factor prices when skilled labor and capital are (Allen-)complements in both sector. And he described those effects when he changed factor intensity ranking. But his analysis is somewhat complicated, and his results are not so clear. This is a theoretical study. Ban (2008, p4, Table1) showed a table to classify the results in Ban (2007) by factor intensity ranking. He classified the countries in the world into 14 regions in total. And, he computed the factor intensity for these areas, using GTAP vesion 6 database. And he assumed 10 kinds of values for ‘the elasticities of substitution’ (equivalent to EWS) in order to simulate how commodity prices affect the relative factor prices. This is an application. Ban (2011, chapter 4, p87-89) summarized Ban (2007) and Ban (2008), and modified them. About his results, see Ban 2011, p96-97, Table 4-1.

(2) Suzuki (1985) analyzed the factor endowment-commodity output relationships when he assumed ‘normal property’ of the factor of production. In Suzuki (1987, Chapter 2, pp27-36), he assumed that production functions are separable (p32). Bliss (2003) assumed that only one sector has a specific factor. He assumed separability and non-separability in production functions. And he assumed that capital and land are ‘Hicksian Complement’ in agriculture (p274). He analyzed the commodity price-factor price relationships. And he tried to explain the wage movement in British economic history. This is a kind of application.

(3) Ide (2009) modified one of the basic assumptions of the model, that is, he assumed the model with increasing returns to scale technology. He assumed that extreme factors are ‘aggregate complements’. And he analyzed the commodity price-factor price relationships, and its dual-counterpart. This is a theoretical study.

(4) Ban (2010) modified an important assumption of the model, that is, he assumed that commodity

prices are endogenous. He analyzed how factor endowments affected on factor prices. This is a theoretical study.

(5) In Suzuki (1987, Chapter 1, pp17-26), he assumed that ‘extreme factors’ are ‘Allen-complements’ in both of the 2 sectors (p23). Teramachi (1993) analyzed the commodity price-factor price relationships in a similar way to Suzuki (1987, Chapter 1), and Takayama (1982) in elasticity terms. For example, he commented on Thompson (1985) in his Appendix (p66-70) that Thompson’s analysis in not plausible. Easton (2008) analyzed whether the extent of substitutability and complementarity affect the commodity price-factor price relationships. He reconsidered the analysis in JE (1983). For example, he tried to extend the concept of ‘perfect complementarity’.

2. Model

We assume similarly to BC (pp22-23). That is, we assume as follows. Products and factors markets are perfectly competitive. Supply of all factors is perfectly inelastic. Production functions are homogeneous of degree one and strictly quasi-concave. All factors are not specific and perfectly mobile between sectors, and factor prices are perfectly flexible. These two ensure the full employment of all resources. The country is small and faces exogenously given world prices, or the movement in relative price of a commodity is exogenously determined. The movements in factor endowments are exogenously determined.

Full employment of factors implies

$$\sum_j a_{ij} X_j = V_i, \; i = T, K, \quad (1)$$

where X_j denotes the amount produced of good j (j=1,2); a_{ij} the requirement of input i per unit of output of good j (or the input-output coefficient); V_i the supply of factor i; T is the land, K capital, and L labor.

In a perfectly competitive economy, the unit cost of production of each good must just equal to its price. Hence,

$$\sum_i a_{ij} w_i = p_j, \; j = 1 \; , \quad (2)$$

where p_j is the price of good j; w_i is the reward of factor i.

BC (p23) stated, ‘With quasi-concave and linearly homogeneous production functions, each input-output coefficient is independent of the scale of output and is a function solely of input prices:’

$$a_{ij} = a_{ij}\left(w_i\right),\ i=T,K\ ,L\ ,\ j\ =1,2. \quad (3)$$

And they continued, 'In particular, each C_{ij} [a_{ij} in our expression] is homogeneous of degree zero in all input prices.'

We might need some explanation about this. Samuelson (1953, chapter 4, p59) defined the function, $v_i = f^{i}(\mathrm{x}, \mathrm{w}_1, \cdots, \mathrm{w}_n), (\mathrm{i}=1,\cdots,\mathrm{n})$. v_i is 'an optimum value for each productive factor' to derive 'the minimum total cost for each output (p58)', x is production, and w_i is 'prices of productive factors'. Samuelson (1953, chapter 4, p68) stated that v_i 'must be homogeneous of order zero in the variables $(\mathrm{w}_1,\cdots,\mathrm{w}_n)$, x being constant' (see also Samuelson (1986, chapter4, eq. (5) in p61; eq. (52) in p70)). This implies that a_{ij} is homogeneous of degree zero in all input prices.

Eq. (1)-(3) describe the production side of the model. These are equivalent to eq. (1)-(5) in BC. The set includes 11 equations in 11 endogenous variables (X_j, a_{ij}, and w_i) and 5 exogenous variables (V_i and p_j). The small-country assumption simplifies the demand side of the economy.

Totally differentiate eq. (1):

$$\Sigma_j\ (\lambda_{ij}a_{ij}^* + \lambda_{ij} X_j^*) = V_i^*,\ i=T,K,L, \quad (4)$$

where an asterisk denotes a rate of change (e.g., $X_j^*=dX_j / X_j$), and where λ_{ij} is the proportion of the total supply of factor i in sector j (that is, $\lambda_{ij}=a_{ij} X_j / V_i$). Note that $\Sigma_j \lambda_{ij}=1$.

The minimum-unit-cost equilibrium condition in each sector implies $\Sigma_i w_i da_{ij}=0$, hence we derive (see JE (p73, eq. (9)), BC (p24, n5),

$$\Sigma_i\ \theta_{ij}a_{ij}^*=0,\ j=1,\ 2, \quad (5)$$

where θ_{ij} is the distributive share of factor i in sector j (that is, $\theta_{ij}=a_{ij}w_i/ p_j$). Note that $\Sigma_i\theta_{ij}=l$; da_{ij} is the differential of a_{ij}.

Totally differentiate eq. (2):

$$\Sigma_i\theta_{ij}w_i^*=p_j^*. \quad (6)$$

Subtract p_j^* from the both sides of (6):

$$\Sigma_i\theta_{i1}\ w_{i1}^*=0,$$
$$\Sigma_i\theta_{i2}\ w_{i1}^*=-P,\ i=T,\ K,\ L, \quad (7)$$

where $P=p_1^*-p_2^*$, $w_{i1}^*= w_i^*-p_1^*$, $w_{i1} = w_i /p_1$; P is the change in the relative price of a commodity; w_{i1} is the real factor price measured by the price of good 1.

Totally differentiate eq. (3) to obtain

$$a_{ij}^*= \Sigma_h \varepsilon^{ij}_h w_h^* =0, i=T, K , L , j= 1, 2, \quad (8)$$

where

$$\varepsilon^{ij}_h = \partial \log a_{ij} /\partial \log w_h =\theta_{hj}\sigma^{ij}_h ; \quad (9)$$

σ^{ij}_h is the AES (or the Allen-partial elasticities of substitution) between the ith and the hth factors in the jth industry. For additional definition of these symbols, see Sato and Koizumi (1973, pp47-49), BC (p24). Since a_{ij} is homogeneous of degree zero in input prices, we have

$$\Sigma_h \varepsilon^{ij}_h =\Sigma_h \theta_{hj}\sigma^{ij}_h =0, i=T, K, L, j=1, 2. \quad (10)$$

Eq. (8)-(10) are equivalent to the expressions in BC (p24, n. 6). See also JE (p74, eq. (12)-(13)). From these:

$$a_{ij}^*=\Sigma_h \varepsilon^{ij}_h w_{h1}^*. \quad (11)$$

Substitute eq. (11) in eq. (4):

$$\Sigma_j(\lambda_{ij} \Sigma_h \varepsilon^{ij}_h w_{h1}^*+\lambda_{ij} X_j ^*) =\Sigma_h g_{ih} w_{h1}^*+\Sigma_j \lambda_{ij} X_j ^* = V_i ^*, i =T, K, L. \quad (12)$$

where

$$g_{ih} = \Sigma_j \lambda_{ij}\varepsilon^{ij}_{h}, \quad i, h=T, K, L. \quad (13)$$

This is the EWS (or ‘the economy-wide substitution’) between factors i and h defined by JE (p75). They stated, ‘Clearly, the substitution terms in the two industries are always averaged together. With this in mind we define the term $\sigma^i{}_k$ to denote the economy-wide substitution towards or away from the use of factor i when the kth factor becomes more expensive, under the assumption that each industry's output is kept constant’.

Note that

$$\Sigma_h g_{ih} =0, \quad i=T, K, L.(14)$$

$$g_{ih} = (\theta h / \theta i) g_{hi} , \quad i, h=T, K, L. \quad (15)$$

g_{ih} is not symmetric. Namely, $g_{ih} \neq g_{hi}$, $i \neq h$ in general. On eq.(15), see also JE (p85).
From (14), we obtain

$$g_{KK} = -(g_{KT} + g_{KL}), \text{ and } g_{TT} = -(g_{TK} + g_{TL}). \quad (16)$$

Combine eq. (12) and (7) to make a system of linear equations. Use 5 X 5 matrix, we obtain

$$\boldsymbol{AX}=\boldsymbol{P}, \quad (17)$$

where $$\boldsymbol{A}=\begin{bmatrix} \theta_{T1} & \theta_{K1} & \theta_{L1} & 0 & 0 \\ \theta_{T2} & \theta_{K2} & \theta_{L2} & 0 & 0 \\ g_{TT} & g_{TK} & g_{TL} & \lambda_{T1} & \lambda_{T2} \\ g_{KT} & g_{KK} & g_{KL} & \lambda_{K1} & \lambda_{K2} \\ g_{LT} & g_{LK} & g_{LL} & \lambda_{L1} & \lambda_{L2} \end{bmatrix}, \boldsymbol{X}=\begin{bmatrix} w_{T1}* \\ w_{K1}* \\ w_{L1}* \\ X_1* \\ X_2* \end{bmatrix}, \boldsymbol{P}=\begin{bmatrix} 0 \\ -P \\ V_T* \\ V_K* \\ V_L* \end{bmatrix}.$$

A is a 5 X 5 coefficient matrix, and **X, P** are column vectors.

3. Factor intensity ranking

In this article, we assume:

$$\frac{a_{T1}}{a_{T2}} > \frac{a_{L1}}{a_{L2}} > \frac{a_{K1}}{a_{K2}}. \quad (18)$$

This implies:

$$\frac{\theta_{T1}}{\theta_{T2}} > \frac{\theta_{L1}}{\theta_{L2}} > \frac{\theta_{K1}}{\theta_{K2}}. \quad (19)$$

This is, what you call, 'the factor intensity ranking' (see JE (p69), see also BC (pp26-27), Suzuki (1983, p142),). This implies that sector 1 is relatively land intensive, and sector 2 is relatively capital intensive, and that labor is the middle factor, and land and capital are extreme factors (see also Ruffin (1981, p180)).

If eq.(19) holds, we have

$$\frac{\theta_{L1}}{\theta_{L2}} > 1 \leftrightarrow \theta_{L1} > \theta_{L2}, \quad (20)$$

$$\text{or} \frac{\theta_{L1}}{\theta_{L2}} < 1 \leftrightarrow \theta_{L1} < \theta_{L2} \quad (21)$$

Note that we do not assume that $\theta_{L1} = \theta_{L2}$ holds. JE (p70) called eq.(20),(21), as 'the factor intensity ranking for middle factor'. It implies that the middle factor is used relatively intensively in sector 1.

Define that

$$(A,\ B, E) = \ (\theta_{T1} - \theta_{T2},\ \theta_{K1} - \theta_{K2}, \theta_{L1} - \theta_{L2}). \quad (22)$$

This is the inter-sectoral difference in distributional share. The equation of $\Sigma_i \theta_{ij}$=l (see eq.(5)) implies

$$\mathrm{A} + \mathrm{B} + E = 0. \quad (23)$$

From (23):

$$(\mathrm{A}, \mathrm{B}, E) = (-,+,-),(-,+,+),(+,+,-),(-,-,+),(+,-,+),(+,-,-) \quad (24)$$

But, eq.(19) implies

(A, B, E)=(+, -, ?). (25)

That is,

$$(\mathrm{A}, \mathrm{B}, E) = (+,-,+),(+,-,-) \quad (26)$$

From (23), e.g.,

$$E = -(\mathrm{A} + \mathrm{B}).$$
$$\mathrm{B} = -(\mathrm{A} + E). \quad (27)$$

If we assume eq. (20) holds, we derive

(A, B, E)=(+, -, +). (28)

On the other hand, if we assume eq.(21) holds, we derive

$(A, B, E)=(+, -, -)$. (29)

4. Solution

Use Cramer's rule to solve eq. (17) for X_2*:

$$X_2^*=\Delta_5/\Delta, \quad (30)$$

where Δ=det (**A**), Δ_5= det (**A**$_5$)= $\begin{vmatrix} \theta_{T1} & \theta_{K1} & \theta_{L1} & 0 & 0 \\ \theta_{T2} & \theta_{K2} & \theta_{L2} & 0 & -P \\ g_{TT} & g_{TK} & g_{TL} & \lambda_{T1} & V_T^* \\ g_{KT} & g_{KK} & g_{KL} & \lambda_{K1} & V_K^* \\ g_{LT} & g_{LK} & g_{LL} & \lambda_{L1} & V_L^* \end{vmatrix}$.

Δ is the determinant of matrix **A**. We can show that $\Delta<0$, but we do not need to prove this. It is because Δ is equivalent to the 3 X 3 determinant D in BC, and they proved that D<0 (on this, see BC (pp25-26)). Replace column 5 of matrix **A** with column vector **P**, we derive matrix **A**$_5$. Δ_5 is the determinant of matrix **A**$_5$. Sum columns 1 and 2 in column 3, and subtract row 2 from row 1:

$$\Delta_5=\begin{vmatrix} A & B & 0 & 0 & P \\ \theta_{T2} & \theta_{K2} & 1 & 0 & -P \\ g_{TT} & g_{TK} & 0 & \lambda_{T1} & V_T^* \\ g_{KT} & g_{KK} & 0 & \lambda_{K1} & V_K^* \\ g_{LT} & g_{LK} & 0 & \lambda_{L1} & V_L^* \end{vmatrix}.$$

Recall that $(A, B)=(\theta_{T1}-\theta_{T2}, \theta_{K1}-\theta_{K2})$ (see eq. (22)). Express the above as a cofactor expansion along the 3rd column:

$$\Delta_5= (1)(-1)^{2+3}\begin{vmatrix} A & B & 0 & P \\ g_{TT} & g_{TK} & \lambda_{T1} & V_T^* \\ g_{KT} & g_{KK} & \lambda_{K1} & V_K^* \\ g_{LT} & g_{LK} & \lambda_{L1} & V_L^* \end{vmatrix}.$$

Express the above as a cofactor expansion along the 4th column:

$$\Delta_5 = (-1)^{2+3}\,[P\,(-1)^{1+4}\,C_{P2} + V_T*(-1)^{2+4}C_{T2} + V_K*(-1)^{3+4}C_{K2} + V_L*(-1)^{4+4}\,C_{L2}\,],$$

where

$$C_{P2}=\begin{vmatrix} g_{TT} & g_{TK} & \lambda_{T1} \\ g_{KT} & g_{KK} & \lambda_{K1} \\ g_{LT} & g_{LK} & \lambda_{L1} \end{vmatrix},\; C_{T2}=\begin{vmatrix} A & B & 0 \\ g_{KT} & g_{KK} & \lambda_{K1} \\ g_{LT} & g_{LK} & \lambda_{L1} \end{vmatrix},\; C_{K2}=\begin{vmatrix} A & B & 0 \\ g_{TT} & g_{TK} & \lambda_{T1} \\ g_{LT} & g_{LK} & \lambda_{L1} \end{vmatrix},\; C_{L2}=\begin{vmatrix} A & B & 0 \\ g_{TT} & g_{TK} & \lambda_{T1} \\ g_{KT} & g_{KK} & \lambda_{K1} \end{vmatrix}.$$

Hence,

$$X_2*=\Delta_5/\Delta=1/\Delta\,(-1)\,[\,P(-C_{P2})+V_T*C_{T2}+V_K*(-C_{K2})+V_L*C_{L2}\,]. \quad (31)$$

On the other hand, solve eq. (17) for X_1*:

$$X_1*=\Delta_4/\Delta, \quad (32)$$

where $\Delta_4=\det(\mathbf{A}_4)=\begin{vmatrix} \theta_{T1} & \theta_{K1} & \theta_{L1} & 0 & 0 \\ \theta_{T2} & \theta_{K2} & \theta_{L2} & -P & 0 \\ g_{TT} & g_{TK} & g_{TL} & V_T* & \lambda_{T2} \\ g_{KT} & g_{KK} & g_{KL} & V_K* & \lambda_{K2} \\ g_{LT} & g_{LK} & g_{LL} & V_L* & \lambda_{L2} \end{vmatrix}.$

Replace column 4 of matrix **A** with column-vector **P**, we derive the matrix $\mathbf{A}_4$. Δ_4 is the determinant of matrix $\mathbf{A}_4$. Sum columns 1 and 2 in column 3, and subtract row 2 from row 1:

$$\Delta_4=\begin{vmatrix} A & B & 0 & P & 0 \\ \theta_{T2} & \theta_{K2} & 1 & -P & 0 \\ g_{TT} & g_{TK} & 0 & V_T* & \lambda_{T2} \\ g_{KT} & g_{KK} & 0 & V_K* & \lambda_{K2} \\ g_{LT} & g_{LK} & 0 & V_L* & \lambda_{L2} \end{vmatrix}.$$

Express the above as a cofactor expansion along the 3rd column:

$$\Delta_4 = (1)(-1)^{2+3}\begin{vmatrix} A & B & P & 0 \\ g_{TT} & g_{TK} & V_T{*} & \lambda_{T2} \\ g_{KT} & g_{KK} & V_K{*} & \lambda_{K2} \\ g_{LT} & g_{LK} & V_L{*} & \lambda_{L2} \end{vmatrix}.$$

Express the above as a cofactor expansion along the 3rd column:

$$\Delta_4 = (-1)^{2+3}[P(-1)^{1+3}C_{P1} + V_T{*}(-1)^{2+3}C_{T1} + V_K{*}(-1)^{3+3}C_{K1} + V_L{*}(-1)^{4+3}C_{L1}],$$

where

$$C_{P1} = \begin{vmatrix} g_{TT} & g_{TK} & \lambda_{T2} \\ g_{KT} & g_{KK} & \lambda_{K2} \\ g_{LT} & g_{LK} & \lambda_{L2} \end{vmatrix},\ C_{T1} = \begin{vmatrix} A & B & 0 \\ g_{KT} & g_{KK} & \lambda_{K2} \\ g_{LT} & g_{LK} & \lambda_{L2} \end{vmatrix},\ C_{K1} = \begin{vmatrix} A & B & 0 \\ g_{TT} & g_{TK} & \lambda_{T2} \\ g_{LT} & g_{LK} & \lambda_{L2} \end{vmatrix},\ C_{L1} = \begin{vmatrix} A & B & 0 \\ g_{TT} & g_{TK} & \lambda_{T2} \\ g_{KT} & g_{KK} & \lambda_{K2} \end{vmatrix}.$$

Hence,

$$X_1{*} = \Delta_4/\Delta = 1/\Delta\ (-1)\ [PC_{P1} + V_T{*}(-C_{T1}) + V_K{*}C_{K1} + V_L{*}(-C_{L1})]. \quad (33)$$

In sum, from eq. (31), (33), we obtain:

$$X_2{*} = 1/\Delta\ [PC_{P2} + V_T{*}(-C_{T2}) + V_K{*}C_{K2} + V_L{*}(-C_{L2})]. \quad (34)$$

$$X_1{*} = 1/\Delta [P(-C_{P1}) + V_T{*}C_{T1} + V_K{*}(-C_{K1}) + V_L{*}C_{L1}]. \quad (35)$$

5. Rybczynski matrix

From the above, Rybczynski matrix $\left[X_j{*}/V_i{*}\right]$(to use Thompson's terminology (1985, p619)) in elasticity terms is:

$$\begin{bmatrix} X_1{*}/V_T{*} & X_1{*}/V_K{*} & X_1{*}/V_L{*} \\ X_2{*}/V_T{*} & X_2{*}/V_K{*} & X_2{*}/V_L{*} \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} C_{T1} & -C_{K1} & C_{L1} \\ -C_{T2} & C_{K2} & -C_{L2} \end{bmatrix}. \quad (36)$$

Express in general:

$X_j*/V_i*=1/\Delta(-1)^{i+j}C_{ij}$, i=T, K, L, j=1, 2. (37)

Substitute 1, 2, 3 instead of T, K, L, respectively, when we compute $(-1)^{i+j}$. Sign patterns are of interest. We can show that X_1*/V_L*, X_2*/V_L*, are respectively equivalent to eq. (26), (27) in BC (p32). BC only obtained these 2 equations. It seems that BC's method of derivation is somewhat complicated. The method shown here is simpler.

Use Saruss's rule to expand the determinant of C_{ij}, i=T, K, L, j=1, 2:

$$C_{T1}=A\,g_{KK}\lambda_{L2}+B\,\lambda_{K2}g_{LT}-(A\,g_{LK}\lambda_{K2}+Bg_{KT}\lambda_{L2}),$$
$$C_{K1}=Ag_{TK}\lambda_{L2}+B\,\lambda_{T2}g_{LT}-(A\,g_{LK}\lambda_{T2}+Bg_{TT}\lambda_{L2}),$$
$$C_{L1}=Ag_{TK}\lambda_{K2}+B\,\lambda_{T2}g_{KT}-(A\,g_{KK}\lambda_{T2}+Bg_{TT}\lambda_{K2});$$
$$C_{T2}=A\,g_{KK}\lambda_{L1}+B\,\lambda_{K1}g_{LT}-(A\,g_{LK}\lambda_{K1}+Bg_{KT}\lambda_{L1}),$$
$$C_{K2}=Ag_{TK}\lambda_{L1}+B\,\lambda_{T1}g_{LT}-(A\,g_{LK}\lambda_{T1}+Bg_{TT}\lambda_{L1}),$$
$$C_{L2}=Ag_{TK}\lambda_{K1}+B\,\lambda_{T1}g_{KT}-(A\,g_{KK}\lambda_{T1}+Bg_{TT}\lambda_{K1}). \quad (38)$$

Recall eq.(16), that is, $g_{KK}=-(g_{KT}+g_{KL})$, and $g_{TT}=-(g_{TK}+g_{TL})$. Substitute these equations into eq. (38) to eliminate g_{KK}, g_{TT}. Next, recall $g_{ih}=(\theta h/\theta i)g_{hi}$ (see eq.(15)). Use this equation to eliminate g_{KL}, g_{TL}, g_{TK}. Define that

(S, T, U)=(g_{LK}, g_{LT}, g_{KT}). (39)

Use these symbols for ease of notation:

$$C_{T1}=E\,\lambda_{L2}U-A\frac{\theta_2}{\theta_K}(1-\theta_{T2})S+B\,\lambda_{K2}T,$$

$$C_{K1}=(-E)\frac{\theta_K}{\theta_T}\lambda_{L2}U-A\,\lambda_{T2}S+B\frac{\theta_2}{\theta_T}(1-\theta_{K2})T,$$

$$C_{L1}=(-E)\frac{\theta_2}{\theta_T}(1-\theta_{L2})U+A\frac{\theta_L}{\theta_K}\lambda_{T2}S+B\frac{\theta_L}{\theta_T}\lambda_{K2}T;$$

$$C_{T2}=E\,\lambda_{L1}U-A\frac{\theta_1}{\theta_K}(1-\theta_{T1})S+B\,\lambda_{K1}T,$$

$$C_{K2}=(-E)\frac{\theta_K}{\theta_T}\lambda_{L1}U-A\,\lambda_{T1}S+B\frac{\theta_1}{\theta_T}(1-\theta_{K1})T,$$

$$C_{L2}=(-\mathrm{E})\frac{\theta_1}{\theta_T}(1-\theta_{L1})\mathrm{U}+\mathrm{A}\frac{\theta_L}{\theta_K}\lambda_{T1}S+B\frac{\theta_L}{\theta_T}\lambda_{k1}T. \quad (40)$$

C_{ij} is a linear function in S, T, and U. Recall that $E = -(A+B)$ (see eq. (27)) ; θ_i and θ_j are, respectively, the share of factor i and good j in total income. That is, $\theta_j = p_j x_j / I$, $\theta_i = w_i V_i / I$, where $I = \Sigma_j p_j X_j = \Sigma_i w_i V_i$. On this, see BC (p25, eq. (16)). Hence, we obtain $\lambda_{ij} = (\theta_j / \theta_i)\theta_{ij}$ (see JE (p72, n. 9)). Note that $\Sigma_j \theta_j = 1,\ \Sigma_i \theta_i = 1$.

6. Transforming the Rybczynski matrix by using EWS-ratios

Define that

$$(S', U') = (S/T, U/T) = (g_{LK} / g_{LT},\ g_{KT} / g_{LT}), \qquad (41)$$

which we call as EWS-ratio vector. S' and U' denote, respectively, the relative magnitude of EWS's between factors L and K, factors K and T, compared to EWS between factors L and T. Using these, transform the above:

$$C_{T1} = E\lambda_{L2}\, T\,[U'\text{-}f_{T1}(S')]],\ C_{K1} = (-E)\frac{\theta_K}{\theta_T}\,\lambda_{L2}\, T[U'\text{-}f_{K1}(S')]\ ,$$

$$C_{L1} = (-E)\frac{\theta_2}{\theta_T}(1\text{-}\theta_{L2})\, T[U'\text{-}f_{L1}(S')];$$

$$C_{T2} = E\lambda_{L1}\, T\,[U'\text{-}f_{T2}(S')]],\ C_{K2} = (-E)\frac{\theta_K}{\theta_T}\,\lambda_{L1}\, T[U'\text{-}f_{K2}(S')]\ ,$$

$$C_{L2} = (-E)\frac{\theta_1}{\theta_T}(1\text{-}\theta_{L1})T[U'\text{-}f_{L2}(S')], \qquad (42),$$

where

$$f_{T1}(S') = [A\frac{\theta_2}{\theta_K}(1\text{-}\theta_{T2})S'\text{-}B\lambda_{K2}]\,(E\lambda_{L2})^{-1},$$

$$f_{K1}(S') = [A\lambda_{T2}S'\text{-}\,B\frac{\theta_2}{\theta_T}(1\text{-}\theta_{K2})][(-E)\frac{\theta_K}{\theta_T}\,\lambda_{L2}]^{-1},$$

$$f_{L1}(S') = [\text{-}\,A\frac{\theta_L}{\theta_K}\,\lambda_{T2}\, S'\text{-}B\frac{\theta_L}{\theta_T}\,\lambda_{K2}][(-E)\frac{\theta_2}{\theta_T}(1\text{-}\theta_{L2})]^{-1}\ ;$$

$$f_{T2}(S') = [A\frac{\theta_1}{\theta_K}(1\text{-}\theta_{T1})S'\text{-}B\lambda_{K1}]\,(E\lambda_{L1})^{-1},$$

$$f_{K2}(S') = [A\lambda_{T1}S'\text{-}\,B\frac{\theta_1}{\theta_T}\,(1\text{-}\theta_{K1})][(-E)\frac{\theta_K}{\theta_T}\,\lambda_{L1}]^{-1},$$

$$f_{L2}(S') = [\text{-}\,A\frac{\theta_L}{\theta_K}\,\lambda_{T1}\, S'\text{-}B\frac{\theta_L}{\theta_T}\,\lambda_{K1}][(-E)\frac{\theta_1}{\theta_T}(1\text{-}\theta_{L1})]^{-1}.$$

Define that

$$f_{ij}\left(S'\right)=[\mathrm{A_{ij}S'+B_{ij}}]\mathrm{E_{ij}}^{-1}, \text{ and } C_{ij}'=U'-f_{ij}\left(S'\right), i=T,K,\mathrm{L}, \mathrm{j}=1,2. \quad (43)$$

C_{ij}' is a linear function in S' and U'. In these expressions, A_{ij}, B_{ij}, E_{ij} are the parameters respectively related to A, B, E. That is,

$$(A_{ij},\quad B_{ij},\quad E_{ij}) = (A\frac{\theta_2}{\theta_K}(1-\theta_{T2}),\ -B\lambda_{K2},\ E\lambda_{L2}),\ for\ ij=T1,$$

$$= (A\lambda_{T2},\ -B\frac{\theta_2}{\theta_T}(1-\theta_{K2}),\ (-E)\frac{\theta_K}{\theta_T}\lambda_{L2}),\ for\ ij=K1,$$

$$= (-A\frac{\theta_L}{\theta_K}\lambda_{T2},\ -B\frac{\theta_L}{\theta_T}\lambda_{K2},\ (-E)\frac{\theta_2}{\theta_T}(1-\theta_{L2})),\ for\ ij=L1;$$

$$= (A\frac{\theta_1}{\theta_K}(1-\theta_{T1}),\ -B\lambda_{K1},\ E\lambda_{L1}),\ for\ ij=T2,$$

$$= (A\lambda_{T1},\ -B\frac{\theta_1}{\theta_T}(1-\theta_{K1}),\ (-E)\frac{\theta_K}{\theta_T}\lambda_{L1}),\ for\ ij=K2,$$

$$= (-A\frac{\theta_L}{\theta_K}\lambda_{T1},\ -B\frac{\theta_L}{\theta_T}\lambda_{K1},\ (-E)\frac{\theta_1}{\theta_T}(1-\theta_{L1})),\ for\ ij=L2. \quad (44)$$

Express in general:

$$C_{ij}=E_{ij}T\ C_{ij}', \mathrm{i=T,K,L}, \mathrm{j}=1,2. \quad (45)$$

7. EWS-ratio boundary and the region for EWS-ratio vector

According to BC (p33), 'Given the assumption that production functions are strictly quasi-concave and linearly homogeneous,' $\sigma^{ij}_i<0$. This implies (see Appendix , eq. (A5)):

$$g_{KK}g_{TT}-g_{TK}g_{KT}>0. \quad (46)$$

Recall eq. (16). That is, $g_{KK}=-(g_{KT}+g_{KL})$, *and* $g_{TT}=-(g_{TK}+g_{TL})$. Substitute these equations to eliminate g_{KK}, g_{TT} from L.H.S. of eq. (46). Next, recall $g_{ih}=(\theta h/\theta i)g_{hi}$ (see eq. (14)). Use this equation to eliminate g_{KL}, g_{TL}, g_{TK}. That is, express using only 3 EWS's, namely, g_{LK}, g_{LT}, g_{KT}:

$$\text{L.H.S.} = g_{KT}g_{TL} + g_{KL}g_{TK} + g_{KL}g_{TL} \;= \frac{\theta_L}{\theta_T}[g_{KT}(g_{LT} + g_{LK}) + \frac{\theta_L}{\theta_K}g_{LK}g_{LT}].$$

From $\sigma^{ij}_i<0$, we derive $\varepsilon^{ij}_i<0$, hence, $g_{ii}<0$. Racall eq.(14), that is, $\Sigma_h g_{ih}=0$, i=T, K, L. This implies $g_{LK} + g_{LT} = -g_{LL} > 0$. Using this, transform eq.(46):

$$g_{KT} > -\frac{\theta_L}{\theta_K}\frac{g_{LK}g_{LT}}{g_{LK} + g_{LT}}. \quad (47)$$

Replace g_{LK}, g_{LT}, g_{KT}, respectively with S, T, U for ease of notation:

$$U > -\frac{\theta_L}{\theta_K}\frac{ST}{S+T}. \quad (48)$$

Devide the both sides by T:

$$U' > -\frac{\theta_L}{\theta_K}\frac{S'}{S'+1}, \text{ if } T>0; \quad U' < -\frac{\theta_L}{\theta_K}\frac{S'}{S'+1}, \text{ if } T<0. \quad (49)$$

Recall that (S', U')=(S/T, U/T) (see eq. (41)). This is EWS-ratio vector. Transform that

$$U' = -\frac{\theta_L}{\theta_K}\frac{S'}{S'+1} = -\frac{\theta_L}{\theta_K} + \frac{\theta_L}{\theta_K}\frac{1}{S'+1}, \quad (50)$$

which expresses the rectangular hyperbola. We call it as the equation for EWS-ratio vector boundary. It passes on the origin of O (0, 0). The asymptotic lines are S'=-1, $U' = -\theta_L / \theta_K$. We can draw this boundary in the figure (see Fig. 1). S' is written along the horizontal axis, and U' along the vertical axis. EWS-ratio boundary demarcates the boundary of the region for EWS-ratio vector. This implies that the EWS-ratio vector is not so arbitrary, but exists within this bounds.

Note that:

EWS-ratio vector (S', U') exists in the upper right region of the EWS-ratio boundary, if T>0 ,

EWS-ratio vector (S', U') exists in the lower left region of the EWS-ratio boundary, if T<0.(51)

The sign patterns of the EWS-ratio vector are, in each quadrant:

quad. I: (S', U')=(+, +)$\leftrightarrow$(S, T, U) =(+, +, +);
quad. II: (S', U')=(-, +)$\leftrightarrow$(S, T, U) =(-, +, +);
quad. III: (S', U')= (-, -)$\leftrightarrow$(S,T, U) =(+, -, +);
quad. IV: (S', U')= (+, -)$\leftrightarrow$(S, T, U) =(+, +, -). (52)

Hence, one of EWS's can be negative at most. Note that

T>0, if (S', U') exists in quadrant I, II, or IV,
T<0, if (S', U') exists in quadrant III. (53)

Recall that (S', U')=(S/T, U/T)=(g_{LK} /g_{LT}, g_{KT}/g_{LT}), (S, T, U) =(g_{LK}, g_{LT}, g_{KT}) . On this, see eq.(41) (39). We may define (for i$\neq$h),

Factors i and h are economy-wide substitutes, if g_{ih} >0,
Factors i and h are economy-wide complements, if g_{ih} <0. (54)

In addition, we may define (for i$\neq$h)(on this, see e.g., Takayama (1982, p17, eq. (35)),

Factors i and h are Allen-substitutes, if $\sigma^{ij}{}_{h} > 0$,

Factors i and h are Allen-complements, if $\sigma^{ij}{}_{h} < 0$. (55)

8. Drawing the border line for a Rybczynski sign pattern to change

We derive:

$$X_j*/V_i* = 0 \leftrightarrow C_{ij} = 0 \leftrightarrow C_{ij}' = 0$$
$$\leftrightarrow U' = f_{ij}(S') = [\mathrm{A_{ij}S'} + \mathrm{B_{ij}}]\mathrm{E_{ij}}^{-1}, \mathrm{i} = \mathrm{T,K,L}, \mathrm{j} = 1,2. \quad (56)$$

This equation expresses the straight line in two dimensions. We call it as the equation for line ij, which expresses the border line for a Rybczynski sign pattern to change. The gradient and intercept of line ij are, respectively, $\mathrm{A_{ij}E_{ij}}^{-1}$, and $\mathrm{B_{ij}E_{ij}}^{-1}$.

Make a system of equations using eq.(56), and eq.(50):

$$U' = f_{ij}(S') = [A_{ij}S' + B_{ij}]E_{ij}^{-1}, i = T, K, L, j = 1, 2, \quad (57)$$

$$U' = -\frac{\theta_L}{\theta_K}\frac{S'}{S'+1}. \quad (58)$$

From these, we obtain a quadratic equation in S' for each i, j. Solve this to derive two solutions. Each solution denotes the S' coordinate value of intersection point of line ij and EWS-ratio boundary. The solutions are, for line-T1, K1, L1; T2, K2, L2, respectively:

$$S' = \frac{B}{A}, \frac{-\theta_{K2}}{1-\theta_{T2}}, \; S' = \frac{B}{A}, \frac{-(1-\theta_{K2})}{\theta_{T2}}, \; S' = \frac{B}{A}, \frac{\theta_{K2}}{\theta_{T2}};$$

$$S' = \frac{B}{A}, \frac{-\theta_{K1}}{1-\theta_{T1}}, \; S' = \frac{B}{A}, \frac{-(1-\theta_{K1})}{\theta_{T1}}, \; S' = \frac{B}{A}, \frac{\theta_{K1}}{\theta_{T1}}. \quad (59)$$

In sum, there are 7 intersection points. Each line ij passes through the same point, which we call as the point Q. The Caretesian coordinates of the point Q is

$$(S', U') = (B/A, (B/E)(\theta_L/\theta_K)). \quad (60)$$

We call the 6 intersection points other than point Q as the point R_{ij}, i=T, K, L, j=1, 2. The Cartesian coordinates of these points are, for line-T1, K1, L1; T2, K2, L2, respectively:

$$(S', U') = \left(\frac{-\theta_{K2}}{1-\theta_{T2}}, \frac{\theta_{K2}}{\theta_{L2}}\frac{\theta_L}{\theta_K}\right), \left(\frac{-(1-\theta_{K2})}{\theta_{T2}}, \frac{1-\theta_{K2}}{-\theta_{L2}}\frac{\theta_L}{\theta_K}\right), \left(\frac{\theta_{K2}}{\theta_{T2}}, \frac{-\theta_{K2}}{1-\theta_{L2}}\frac{\theta_L}{\theta_K}\right);$$

$$\left(\frac{-\theta_{K1}}{1-\theta_{T1}}, \frac{\theta_{K1}}{\theta_{L1}}\frac{\theta_L}{\theta_K}\right), \left(\frac{-(1-\theta_{K1})}{\theta_{T1}}, \frac{1-\theta_{K1}}{-\theta_{L1}}\frac{\theta_L}{\theta_K}\right), \left(\frac{\theta_{K1}}{\theta_{T1}}, \frac{-\theta_{K1}}{1-\theta_{L1}}\frac{\theta_L}{\theta_K}\right). \quad (61)$$

The sign patterns of point R_{ij} are, respectively,

sign (S', U') = (-, +), (-, -), (+, -); (-, +), (-, -), (+, -).

Hence, point R_{T1}, R_{T2} is in quadrant II; point R_{K1}, R_{K2} is in quadrant III; point R_{L1}, R_{L2} is in quadrant IV.

From the factor intensity ranking, (A, B, E)=(+, -, +)(see (28)). Hence, the sign pattern of point Q is (-, -)(see (60)). This implies that point Q belongs to quadrant III.

Next, we investigate the relative position of point R_{ij}, and Q. From eq. (17), we can prove for S' values of point R_{K1}, R_{K2}:

$$\frac{-(1-\theta_{K2})}{\theta_{T2}} < \frac{-(1-\theta_{K1})}{\theta_{T1}} \quad (62)$$

Eq.(62) tells us about the relative position of 2 points (R_{K1}, R_{K2}). Similarly, from (17), we can prove for S' values of point R_{T1}, R_{T2}, the origin of O, R_{L2}, R_{L1}:

$$\frac{-\theta_{K2}}{1-\theta_{T2}} < \frac{-\theta_{K1}}{1-\theta_{T1}} < 0 < \frac{\theta_{K1}}{\theta_{T1}} < \frac{\theta_{K2}}{\theta_{T2}}. \quad (63)$$

Eq.(63) tells us about the relative position of these 5 points.

We can prove for S' values of point R_{K2}, Q:

$$\frac{-(1-\theta_{K1})}{\theta_{T1}} < \frac{B}{A}. \quad (64)$$

The derivation of (64) is as follows. Because we assume (A,B,E)=(+, -, +)((see (28)), we have A=(+), Hence,

$$-(1-\theta_{K1})\mathrm{A} < \theta_{T1}B \quad (65)$$

Substitute B=-(A+E) (see eq. (27)?), and multiply the both sides by (-1), we have

$$\theta_{L1}\mathrm{A} - E\theta_{T1} > 0 \quad (66)$$

We can show that

$$L.H.S. = \theta_{L1}(\theta_{T1} - \theta_{T2}) - (\theta_{L1} - \theta_{L1})\theta_{T1}$$

From eq. (17), we have

$$L.H.S. > 0$$

From the above, we can draw point Q, R_{ij}, and hence, line ij in the figure. Each line ij divides the region for EWS-ratio vector into 12 subregions, that is, the subregion P1-5 (upper right region) and M1-7 (lower left region)(see Fig. 1).

9. Rybczynski sign patterns

Define the 2 X 3 matrices:

$$\mathbf{F}=[\mathrm{F_{ij}}]=\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix},\ \mathbf{C}=[\mathrm{C_{ij}}]=\begin{bmatrix} C_{T1} & C_{K1} & C_{L1} \\ C_{T2} & C_{K2} & C_{L2} \end{bmatrix},$$

$$\mathbf{E}=\left[E_{ij}\right]=\begin{bmatrix} E_{T1} & E_{K1} & E_{L1} \\ E_{T2} & E_{K2} & E_{L2} \end{bmatrix}=\begin{bmatrix} E\lambda_{L2} & (-E)\dfrac{\theta_K}{\theta_T}\lambda_{L2} & (-E)\dfrac{\theta_2}{\theta_T}(1-\theta_{L2}) \\ E\lambda_{L1} & (-E)\dfrac{\theta_K}{\theta_T}\lambda_{L1} & (-E)\dfrac{\theta_1}{\theta_T}(1-\theta_{L1}) \end{bmatrix},$$

$$\mathbf{C'}=[\mathrm{C_{ij}'}]=\begin{bmatrix} C_{T1}' & C_{K1}' & C_{L1}' \\ C_{T2}' & C_{K2}' & C_{L2}' \end{bmatrix}=\begin{bmatrix} U'-f_{T1}(S') & U'-f_{K1}(S') & U'-f_{L1}(S') \\ U'-f_{T2}(S') & U'-f_{K2}(S') & U'-f_{L2}(S') \end{bmatrix}. \quad (67)$$

Using the Hadamard product of these matrices, the Rybczynski matrix is expressed as (see (36)):

$$\left[X_j*/V_i*\right]=\frac{1}{\Delta}\mathbf{F}\circ\mathbf{C}, \quad (68)$$

where (see (45))

$$\mathbf{C}=\mathbf{E}\circ\mathbf{C'}T. \quad (69)$$

In general, if **A**=[a_{ij}] and **B**=[b_{ij}] are each m X n matrices, their Hadamard product is the matrix of element-wise products , that is, $\mathbf{A}\circ\mathbf{B}=[a_{ij}b_{ij}]$. On this definition, see, e.g. Styan (1973, p217-218). Hadamard product is known, for example, in the literature of statistics.

Hence, Rybczynski sign patterns are:

$$sign\left[X_j*/V_i*\right]=sign\frac{1}{\Delta}\mathbf{F}\circ\mathbf{C}=sign\frac{1}{\Delta}\mathbf{F}\circ sign\mathbf{C}, \quad (70)$$

where

$$sign\mathbf{C}=sign(\mathbf{E}\circ\mathbf{C'}T)=sign\mathbf{E}\circ sign\mathbf{C'}T. \quad (71)$$

Recall that $\Delta<0$ (see eq.(30)). Hence,

$$\operatorname{sign}\frac{1}{\Delta}\mathbf{F}=\operatorname{sign}\frac{1}{\Delta}\ sign\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & -1\end{bmatrix}=(-)\begin{bmatrix}+ & - & +\\ - & + & -\end{bmatrix}=\begin{bmatrix}- & + & -\\ + & - & +\end{bmatrix}.(72)$$

Recall that we assume (A, B, E)=(+, -, +)(see eq.(28)). Hence,

$$\operatorname{sign}\mathbf{E}=\begin{bmatrix}+ & - & -\\ + & - & -\end{bmatrix}.\quad (73)$$

In general, if EWS-ratio vector (S', U') exists in the subregion above line ij (resp. below line ij), we derive

$$C_{ij}'=U'-f_{ij}\left(S'\right)=(+)>0 \text{ (resp. } C_{ij}'=U'-f_{ij}\left(S'\right)=(-)<0). \quad (74)$$

For example, if EWS-ratio vector exists in the subregion P2, that is, below line T1, T2, L2, and above line L1, K1, K2, the sign pattern of matrix **C'** is:

$$\operatorname{sign}\mathbf{C'}=[C_{ij}']=\begin{bmatrix}- & + & +\\ - & + & -\end{bmatrix}.$$

Sign patterns of the matrix **C'** are, respectively, for each subregion:

$$\operatorname{sign}\mathbf{C'}=\begin{array}{ccccc}\text{P1} & \text{P2} & \text{P3} & \text{P4} & \text{P5}\\ \begin{bmatrix}- & + & -\\ - & + & -\end{bmatrix} & \begin{bmatrix}- & + & +\\ - & + & -\end{bmatrix} & \begin{bmatrix}- & + & +\\ - & + & +\end{bmatrix} & \begin{bmatrix}- & + & +\\ + & + & +\end{bmatrix} & \begin{bmatrix}+ & + & +\\ + & + & +\end{bmatrix}\end{array}$$

$$\begin{array}{ccccccc}\text{M1} & \text{M2} & \text{M3} & \text{M4} & \text{M5} & \text{M6} & \text{M7}\\ \begin{bmatrix}+ & + & +\\ + & + & +\end{bmatrix} & \begin{bmatrix}+ & + & +\\ + & - & +\end{bmatrix} & \begin{bmatrix}+ & - & +\\ + & - & +\end{bmatrix} & \begin{bmatrix}+ & - & -\\ + & - & +\end{bmatrix} & \begin{bmatrix}+ & - & -\\ + & - & -\end{bmatrix} & \begin{bmatrix}+ & - & -\\ - & - & -\end{bmatrix} & \begin{bmatrix}- & - & -\\ - & - & -\end{bmatrix}\end{array}(75)$$

In sum, the position of EWS-ratio vector determines the sign pattern of the matrix **C'**.

Of course, we derive

T>0, if EWS-ratio vector exists in either of the subregion P1-P5,

T<0, if EWS-ratio vector exists in either of the subregion M1-M7. (76)

From eq.(75),(76), sign patterns of the matrix **C'T** are, for each subregion:

P1 P2 P3 P4 P5

$$
sign\mathbf{C}' = \begin{bmatrix} - & + & - \\ - & + & - \end{bmatrix} \begin{bmatrix} - & + & + \\ - & + & - \end{bmatrix} \begin{bmatrix} - & + & + \\ - & + & + \end{bmatrix} \begin{bmatrix} - & + & + \\ + & + & + \end{bmatrix} \begin{bmatrix} + & + & + \\ + & + & + \end{bmatrix}
$$

$$
\begin{array}{ccccccc} \text{M1} & \text{M2} & \text{M3} & \text{M4} & \text{M5} & \text{M6} & \text{M7} \\ \begin{bmatrix} - & - & - \\ - & - & - \end{bmatrix} & \begin{bmatrix} - & - & - \\ - & + & - \end{bmatrix} & \begin{bmatrix} - & + & - \\ - & + & - \end{bmatrix} & \begin{bmatrix} - & + & + \\ - & + & - \end{bmatrix} & \begin{bmatrix} - & + & + \\ - & + & + \end{bmatrix} & \begin{bmatrix} - & + & + \\ + & + & + \end{bmatrix} & \begin{bmatrix} + & + & + \\ + & + & + \end{bmatrix} \end{array} \quad (77)
$$

Note that the sign patterns for P1-P5 are, respectively, the same as those for M3-M7.

Recall that (see eq.(70))

$$
sign\mathbf{C} = sign(\mathbf{E} \circ \mathbf{C}'T) = sign\mathbf{E} \circ sign\mathbf{C}'T . \quad (78)
$$

Substitute eq. (77), (73) in eq.(78):

$$
sign\mathbf{C} = \begin{array}{ccccc} \text{P1} & \text{P2} & \text{P3} & \text{P4} & \text{P5} \\ \begin{bmatrix} - & - & + \\ - & - & + \end{bmatrix} & \begin{bmatrix} - & - & - \\ - & - & + \end{bmatrix} & \begin{bmatrix} - & - & - \\ - & - & - \end{bmatrix} & \begin{bmatrix} - & - & - \\ + & - & - \end{bmatrix} & \begin{bmatrix} + & - & - \\ + & - & - \end{bmatrix} \end{array}
$$

$$
\begin{array}{ccccccc} \text{M1} & \text{M2} & \text{M3} & \text{M4} & \text{M5} & \text{M6} & \text{M7} \\ \begin{bmatrix} - & + & + \\ - & + & + \end{bmatrix} & \begin{bmatrix} - & + & + \\ - & - & + \end{bmatrix} & \begin{bmatrix} - & - & + \\ - & - & + \end{bmatrix} & \begin{bmatrix} - & - & - \\ - & - & + \end{bmatrix} & \begin{bmatrix} - & - & - \\ - & - & - \end{bmatrix} & \begin{bmatrix} - & - & - \\ + & - & - \end{bmatrix} & \begin{bmatrix} + & - & - \\ + & - & - \end{bmatrix} \end{array} \quad (79)
$$

Recall that (see eq.(70)) :

$$
\text{sign}\left[X_j * / V_i *\right] = \text{sign}\frac{1}{\Delta}\mathbf{F} \circ sign\mathbf{C} . \quad (80)
$$

Substitute eq. (79), (72) in eq.(80), we derive Rybczynski sign patterns. They are, for each subregion:

$$
\text{sign}\left[X_j * / V_i *\right] = \begin{array}{ccccc} \text{P1} & \text{P2} & \text{P3} & \text{P4} & \text{P5} \\ \begin{bmatrix} + & - & - \\ - & + & + \end{bmatrix} & \begin{bmatrix} + & - & + \\ - & + & + \end{bmatrix} & \begin{bmatrix} + & - & + \\ - & + & - \end{bmatrix} & \begin{bmatrix} + & - & + \\ + & + & - \end{bmatrix} & \begin{bmatrix} - & - & + \\ + & + & - \end{bmatrix} \end{array}
$$

$$
\begin{array}{ccccccc} \text{M1} & \text{M2} & \text{M3} & \text{M4} & \text{M5} & \text{M6} & \text{M7} \\ \begin{bmatrix} + & + & - \\ - & - & + \end{bmatrix} & \begin{bmatrix} + & + & - \\ - & + & + \end{bmatrix} & \begin{bmatrix} + & - & - \\ - & + & + \end{bmatrix} & \begin{bmatrix} + & - & + \\ - & + & + \end{bmatrix} & \begin{bmatrix} + & - & + \\ - & + & - \end{bmatrix} & \begin{bmatrix} + & - & + \\ + & + & - \end{bmatrix} & \begin{bmatrix} - & - & + \\ + & + & - \end{bmatrix} \end{array} \quad (81)
$$

In sum, the position of EWS-ratio vector determines Rybczynski sign pattern. There are 12 patterns in total. Note that the sign patterns for P1-P5 are, respectively, the same as those for M3-M7. If we do not count the duplication, there are 7 patterns in total.

We can state as follows.

'A strong Rybczynski result' holds, if EWS-ratio vector exists in the subregion P1, P2, P3; M3, M4, or M5.

'A strong Rybczynski result' does not hold, if EWS-ratio vector exists in the subregion P4, P5; M1, M2, M6, or M7. (82)

10. The commodity price-factor price relationship

From the reciprocity relations derived by Samuelson, BC (p36, eq. (31)-(33)) derived:

$$\frac{w_i^* - p_1^*}{P} = \frac{X_2^*}{V_i^*}\frac{-\theta_2}{\theta_i},$$
$$\frac{w_i^* - p_2^*}{P} = \frac{X_1^*}{V_i^*}\frac{\theta_1}{\theta_i}, \text{ i=T, K, L.} \quad (83)$$

Recall that P=p_1*-p_2* (see eq. (7)). Define the Stolper-Samuelson matrix in elasticity terms:

$$[\frac{w_i^* - p_j^*}{P}] = \begin{bmatrix} w_T^* - p_1^* & w_K^* - p_1^* & w_L^* - p_1^* \\ w_T^* - p_2^* & w_K^* - p_2^* & w_L^* - p_2^* \end{bmatrix}\frac{1}{P}. \quad (84)$$

This matrix shows how the relative price of a commodity affects the real factor prices. Sign patterns are of interest. Multiply row 2 of Rybczynski sign pattern (eq.(81)) by (-1), and interchange row 1 and row 2, we derive Stolper-Samuelson sign pattern as follows.

They are, for each subregion:

$$\text{sign}[\frac{w_i^* - p_j^*}{P}] = \overset{P1}{\begin{bmatrix} + & - & - \\ + & - & - \end{bmatrix}} \overset{P2}{\begin{bmatrix} + & - & - \\ + & - & + \end{bmatrix}} \overset{P3}{\begin{bmatrix} + & - & + \\ + & - & + \end{bmatrix}} \overset{P4}{\begin{bmatrix} - & - & + \\ + & - & + \end{bmatrix}} \overset{P5}{\begin{bmatrix} - & - & + \\ - & - & + \end{bmatrix}}$$

$$\overset{M1}{\begin{bmatrix} + & + & - \\ + & + & - \end{bmatrix}} \overset{M2}{\begin{bmatrix} + & - & - \\ + & + & - \end{bmatrix}} \overset{M3}{\begin{bmatrix} + & - & - \\ + & - & - \end{bmatrix}} \overset{M4}{\begin{bmatrix} + & - & - \\ + & - & + \end{bmatrix}} \overset{M5}{\begin{bmatrix} + & - & + \\ + & - & + \end{bmatrix}} \overset{M6}{\begin{bmatrix} - & - & + \\ + & - & + \end{bmatrix}} \overset{M7}{\begin{bmatrix} - & - & + \\ - & - & + \end{bmatrix}}$$

(85)

In sum, the position of EWS-ratio vector determines the Stolper-Samuelson sign pattern.

Note that

The sign patterns of matrix [$w_i^* - p_j^*$] are similar to eq.(85), if P=(+)>0, The sign patterns of matrix [w_i*-p_j*] are opposite to eq.(85), if P=(-)<0, (86)

11. Some applications

Example 1: We assume (S', U')=(+, -)↔(S, T, U)=(g_{LK}, g_{LT}, g_{KT})=(+, +, -). This implies that factors K and T, extreme factors, are economy-wide complements. Hence, EWS-ratio vector exists in quadrant IV, that is, in the subregion P1, P2, or P3. Rybczynski sign patterns for P1-P3 hold. Hence, in this case, 'a strong Rybczynski result' holds necessarily. The following result has been established.

Theorem 1. If extreme factors are economy-wide complements, 'a strong Rybczynski result' holds necessarily. Stolper-Samuelson sign patterns for subregion P1-P3 are:

$$\begin{array}{cccc} & \text{P1} & \text{P2} & \text{P3} \\ \text{sign}[\dfrac{w_i*-p_j*}{P}] = & \begin{bmatrix} + & - & - \\ + & - & - \end{bmatrix} & \begin{bmatrix} + & - & - \\ + & - & + \end{bmatrix} & \begin{bmatrix} + & - & + \\ + & - & + \end{bmatrix} \end{array}. \quad (87)$$

For example, if we assume that P=(+)>0, the sign patterns of the matrix [w_i*-p_j*] are similar to the above. That is, both the real factor prices of land measured by good 1 and 2 increase, and both the real factor prices of capital decrease.

(i) If EWS-ratio vector exists in the subregion P1, both the real factor prices of labor measured by good 1 and 2 decrease. This is not favorable to the owner of labor.

(ii) If EWS-ratio vector exists in the subregion P2, the real factor price of labor measured by good 1 decreases, the real factor price of labor measured by good 2 increases. It is indeterminate whether this is favorable to the owner of labor or not.

(iii) If EWS-ratio vector exists in the subregion P3, both the real factor prices of labor measured by good 1 and 2 increase. This is favorable to the owner of labor..

On the other hand, if we assume P=(-)<0, the sign patterns of the matrix [w_i*-p_j*] are opposite to the above.

For example, we can apply these results to U.S. trade problem in the 1980's, as Takayama (1982, p20) did. He did not analyze in elasticity terms, but analyzed in differential forms. If we replace factors T, K, L in our analysis, respectively, with factors 1, 2, 3, his analysis is very similar to ours.

He called factors 1, 2, and 3, respectively, as skilled labor, (physical) capital, and unskilled labor (or raw labor). And he called industries 1 and 2, respectively, as exportables and importables.

And he stated, 'there seems to be strong evidence that the current U. S. commodity structure of trade is such that her exports are relatively skilled labor (or R&D) intensive vis a vis unskilled labor, and that her imports are relatively capital intensive vis a vis unskilled labor (e.g., Baldwin, 1971, 1979)'. This implies (see Takayama (p20, p14))

$$\frac{a_{11}}{a_{12}} > \frac{a_{31}}{a_{32}} > \frac{a_{21}}{a_{22}}.$$

This is the factor intensity ranking. And he continued, 'there is some evidence that skilled labor and capital are (aggregate) complements (e, g., Branson-Monoyios, 1977). This indicates that our assumption of [aggregate] complements for extreme factors are satisfied.'

This implies that s_{12}<0 (on this, see Takayama (p18)). This implies g_{12}<0, if we use EWS. It is because $s_{ih}= g_{ih}V_i/w_h$, i, h=1, 2, 3 (On this, see eq. (A-4) in Appendix). Hence, EWS-ratio vector exists in quadrant IV, that is, in either of the subregion P1, P2, or P3.

He derived the sign pattern of 'the Stolper-Samuelson matrix' (see p20, eq. (40)). It is, if we use our symbols:

$$\left[\partial X_j / \partial V_i\right] = [\partial w_i / \partial p_j*]^t = \begin{bmatrix} + & - & ? \\ - & + & ? \end{bmatrix},$$

where t denotes the transpose. And he concluded, 'we may conclude that an import restriction which raises the domestic price of importables (say, automobiles from Japan) in the U.S. increases the return on capital and lowers the return on skilled labor (or R&D) in the U.S.' Similarly, he analyzed the effect of reduction of import restrictions. It is opposite to the above.

He did not analyze how the strengthening (or reduction) of import restrictions affected the price of middle factor (factor 3, or unskilled labor). But, he only analyzed how this affected the price of extreme factors (factors 1 and 2). In our analysis, the strengthening implies that $P=p_1*-p_2*=(-)$, and the reduction implies that P=(+).

Our results suggest that it is possible to analyze how the trade-policy-change affected middle factor in U.S., if we have other two information. That is, the information about the factor intensity ranking of middle factor (that is, which equation holds, $\theta_{31} > \theta_{32}$, or $\theta_{31} < \theta_{32}$), and the information about the position of EWS-ratio vector, that is, the subregion P1, P2, or P3. Using these information, we can decide Stolper-Samuelson sign pattern and, hence, the sign pattern of matrix $[w_i*-p_j*]$.

If we assume $\theta_{31} > \theta_{32}$, we have known that 3 patterns of Stolper-Samuelson sign patterns hold as shown in eq. (87). On the other hand, if we assume $\theta_{31} < \theta_{32}$, we can analyze similarly.

Of course, if we use econometrics, we can estimate the value of each coefficient in equation for the solutions for X_1* and X_2*, that is, eq. (25) and (26). That is, we can derive the value of each element of the Rybczynski matrix. Therefore, we can derive Rybczynski sign pattern, and hence, Stolper-Samuelson sign pattern. This will be useful for us.

Example 2: Next, we show some examples of a sufficient condition for 'a strong Rybczynski result' to hold. We assume (S', U')=(+, -)↔(S, T, U)=(+, +,-). We analyze using the coordinate values of point R_{L2}, R_{L1}.

(i) If (S', U') satisfy

$$\frac{\theta_{K2}}{\theta_{T2}} < S',\quad \frac{-\theta_{K2}}{1-\theta_{L2}}\frac{\theta_L}{\theta_K} > U',$$

EWS-ratio vector (S', U') exists in the lower right of point R_{L1} . Hence, it exists in the subregion P1.

(ii) If (S', U') satisfy

$$0<\frac{\theta_{K1}}{\theta_{T1}} < S' < \frac{\theta_{K2}}{\theta_{T2}},\ \text{and}\ 0>\frac{-\theta_{K1}}{1-\theta_{L1}}\frac{\theta_L}{\theta_K} > U' > \frac{-\theta_{K2}}{1-\theta_{L2}}\frac{\theta_L}{\theta_K},$$

EWS-ratio vector (S', U') exists in the lower right of point R_{L2}, and in the upper left of point R_{L1} . Hence, it exists in the subregion P2.

(iii) If (S', U') satisfy

$$0<S'<\frac{\theta_{K1}}{\theta_{T1}},\ \text{and}\ 0>U'>\frac{-\theta_{K1}}{1-\theta_{L1}}\frac{\theta_L}{\theta_K},$$

EWS-ratio vector (S', U') exists in the lower right of the origin of O, and in the upper left of point R_{L2} . Hence, it exists in the subregion P3.

In all 3 cases, 'a strong Rybczynski result' holds.

Example 3: On the other hand, for example:

(i) If (S', U')=(+, +), EWS-ratio vector exists in quadrant I, i.e., in the subregion P1-P5.

(ii) If (S', U')= (-, +), EWS-ratio vector exists in quadrant II, i.e., in the subregion P3, P4, or P5.

(iii) If (S', U')= (-, -), EWS-ratio vector exists in quadrant III, i.e., in the subregion M1-M7.

In all 3 cases, it is indeterminate whether 'a strong Rybczynski result' holds or not.

11. Conclusion

We assumed a certain pattern of factor intensity ranking, including that of middle factor. And we analyzed the Rybczynski matrix and its sign pattern, by using EWS-ratio vector. This matrix expresses the factor endowment-commodity production relationships. The EWS-ratio boundary demarcates the boundary of the region where EWS-ratio vector can exist. line ij divides this region into 12 subregions. The position of EWS-ratio vector determines Rybczynski sign pattern. 'A strong Rybczynski result' holds for some subregions. We have succeeded to derive a sufficient condition for 'a strong Rybczynski result' to hold (or not to hold) in a systematic manner. We also analyzed the Stolper-Samuelson matrix and its sign pattern, which expresses the commodity price-factor price relationships. I showed some applications.

Equation Section 1

Appendix: Derivation of important relationship among EWS

On this, see Nakada (2015b). Here, I show only the essence of it. Thompson (1985, p618) stated, 'Aggregate substitution between factors h and k is expressed by the substitution term

$s_{kh}=\Sigma_j\ x_j\ \partial a_{kj}/\partial w_h$ [, k, h=1, 2, 3]. (A1)

The 3 X 3 matrix of substitution terms is symmetric and negative semidefinite. A result of cost minimizing behavior is

$\Sigma_i\ s_{hi}w_i=0$, for every factor h [, h=1, 2, 3].' (A2)

For definition of these symbols, it is similar to that in this paper.

But his explanation seems too short. The 'cost minimizing behavior' implies that each a_{ij} function is homogeneous-of-degree-zero (see eq.(3) in this paper). From this, we can derive the Thompson's result,(A2). Probably, we should prove it below.

Recall (9), $\varepsilon^{ij}{}_h = \partial\log a_{ij} /\partial\log w_h =\theta_{hj}\sigma^{ij}{}_h$. From this equation, we obtain

$$\partial a_{ij} / \partial w_h = \varepsilon^{ij}{}_h a_{ij} / w_h,\ i,\ h = T,\ K,\ L,\ j = 1,\ 2. \quad \text{(A3)}$$

Replace s_{kh} in (A1) with s_{ih}, we derive

$s_{ih}=\Sigma_j\ x_j\ \partial a_{ij}/\partial w_h$, i, h=T, K, L. (A4)

Substitute (A3) in (A4), we obtain:

$$s_{ih} = \Sigma_j x_j \varepsilon^{ij}{}_h a_{ij} / w_{h,} i,\ h = T,\ K,\ L. \quad \text{(A5)}$$

Because each a_{ij} function is homogeneous-of-degree-zero (see eq.(3)):

$$\Sigma_h \varepsilon^{ij}{}_h = \Sigma_h \theta_{hj} \sigma^{ij}{}_h = 0,\ i = T,\ K,\ L,\ j = 1,\ 2. \quad \text{(A6)}$$

From (A5),(A6), we derive:

$$\Sigma_h s_{ih} w_h = 0,\ i = T,\ K,\ L. \quad \text{(A7)}$$

This is equivalent to Thompson's result,(A2).

AES's are symmetric in the sense that

$$\sigma^{ij}{}_h = \sigma^{hj}{}_i. \quad \text{(A8)}$$

And according to BC (p33), 'Given the assumption that production functions are strictly quasi-concave and linearly homogeneous,'

$$\sigma^{ij}{}_i < 0. \quad \text{(A9)}$$

By using (A8) and (A6), we can show that $s_{ih}=s_{hi}$, namely, aggregate substitutions are symmetric. And (A9) implies that $\varepsilon^{ij}{}_i<0$, hence $s_{ii}<0$.

Next, we analyze s_{LL} in a way similar to that which BC (p33) used in analyzing AES ($\sigma^{Lj}{}_L$). Eliminate s_{TL}, s_{KL} from eq.(A7):

$$s_{LL} = \frac{1}{(w_L)^2}\left\{w_T\left(w_T s_{TT} + w_K s_{TK}\right) + w_K\left(w_T s_{KT} + w_K s_{KK}\right)\right\}. \text{(A10)}$$

Transform (A10):

$$s_{LL} = x \cdot Ax \quad , \text{(A11)}$$

where x is a vector, A a matrix, and $x \cdot Ax$ the inner product of vectors; $x = \begin{bmatrix} w_K \\ w_T \end{bmatrix}, A = \begin{bmatrix} s_{KK} & s_{KT} \\ s_{TK} & s_{TT} \end{bmatrix}$.

Quote a passage from BC (p33): 'the quadratic form on the right-hand side of the expression above must be negative definite. This in turn implies that'

$|A| = s_{KK}s_{TT} - s_{TK}s_{KT} > 0$, (A12)

where $|A|$ is the 2 X 2 determinant. Transform eq.(A5):

$s_{ih} = \Sigma_j \lambda_{ij} \varepsilon^{ij}_h V_i/w_h = g_{ih}V_i/w_h$, i, h=T, K, L. (A13)

This equation shows how aggregate substitution and EWS is related. From (A13), g_{ih} is not symmetric. Namely, $g_{ih} \neq g_{hi}$, $i \neq h$ in general. Substitute eq. (A13) in (A12):

$g_{KK}g_{TT} - g_{TK}g_{KT} > 0$. (A14)

It may be noted that this equation is useful to show that JE's proof is impossible. I show the disproof. It might be useful to readers. JE (p75) defined σ_i^k ,i, k=1, 2, 3, as EWS. In subsection 5.2.4 (p90), JE stated, 'First we assume that the two extreme factors [factors 1 and 2] are perfect complements in the sense that any factor price change does not alter the ratio of the intensities of their use ($\sigma_1^k = \sigma_2^k$ [, k=1, 2, 3]).'

Here, for them, 'perfect complementarity' implies $\sigma_1^k = \sigma_2^k$. If we replace σ_i^k with g_{ih}, this implies that

$g_{Th} = g_{Kh}$, h=T, K, L $\leftrightarrow$ $g_{TT} = g_{KT}$, $g_{TK} = g_{KK}$, $g_{TL} = g_{KL}$. (A15)

In other words, they found that the set of three equations holds for EWS under the assumption of 'perfect complementarity'. Next, they used this set to prove how commodity prices affect factor prices.

If we compare eq.(A15) with eq.(A14), we find that the latter is not consistent with the former. That is, if eq. (A15) holds, L.H.S. of eq. (A14) equals zero. Hence, JE's result is impossible. Specifically, they failed to explain what 'perfect complementarity' implies. In sum, their proof is not plausible.

In subsection 5.2.5 (p91), JE analyzed similarly to subsection 5.2.4. They assumed that extreme factor (factor 2) is a perfect complement with middle factor (factor 3). They stated that they derived $\sigma_3^1 = \sigma_2^1$. Apparently, in their context, this implies $\sigma_3^k = \sigma_2^k$, k=1, 2, 3. We can prove that it is impossible, similarly.

In addition, it may be noted that Takayama (1982) analyzed the general m X n model, and he stated that since 'substitution matrix' S is negative semidefinite and R(S)=m-1, the (m-1) x (m-1) matrix is negative definite, from which $s_{ii} < 0$, i=1, 2, …, m (p5, Theorem 1, note5). R(S) denotes the rank of a particular matrix, and S=[s_{ih}].

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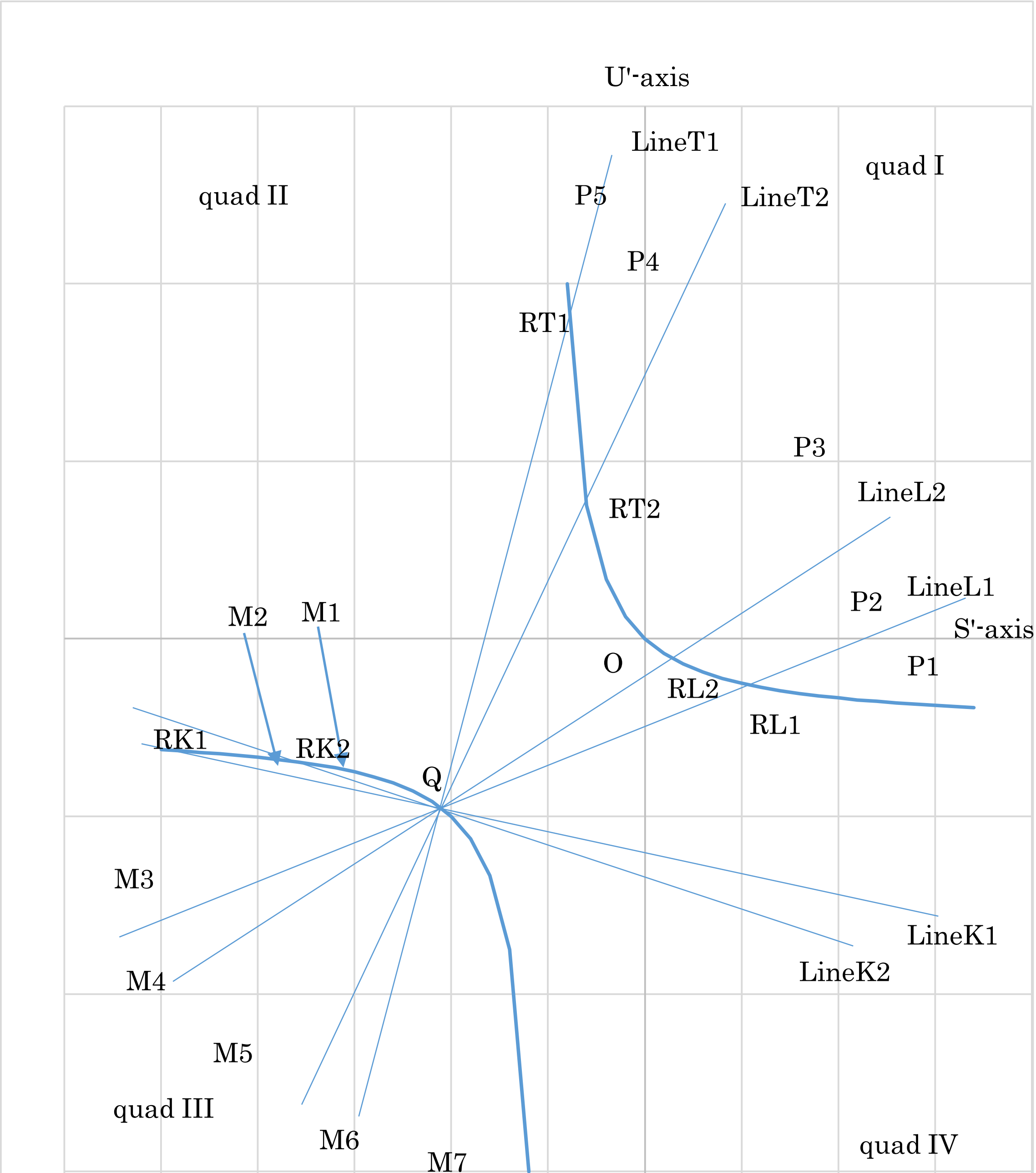

Fig. 1 Illustration of EWS-ratio boundary and Line-ij (border line for a Rybczynski sign pattern to change)
Note: S'=S/T=gLK/gLT, U'=U/T=gKT/gLT