Economy-wide substitution and Rybczynski sign pattern in a three-factor two-good model: Further analysis

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Abstract: The position of EWS-ratio vector determines Rybczynski sign pattern, and hence, Stolper-Samuelson sign pattern in a three-factor two-good general equilibrium trade model (see Nakada (2015)). EWS-ratio vector can be defined based on EWS (or economy-wide substitution). In this article, I develop a method to estimate the position of EWS-ratio vector to some extent. Especially, I derive a sufficient condition for extreme factors to be economy-wide complements. We assume factor intensity ranking is constant. The results suggest that if we have appropriate data, we can estimate the position of EWS-ratio vector to some extent. The results are useful for application.

Keywords: three-factor two-good model; general equilibrium; Rybczynski result; economy-wide substitution, Stolper-Samuelson sign pattern.

Section 1.Introduction

Batra and Casas (1976) (hereinafter BC) wrote an article on functional relations in a three-factor two-good neoclassical model (or 3 X 2 model), and they claimed that 'a strong Rybczynski result' arises. But I proved this was not the case (see Nakada (2015)). According to Suzuki (1983, p141), BC contended in Theorem 6 (p34) that 'if commodity 1 is relatively capital intensive and commodity 2 is relatively labor intensive, an increase in the supply of labor increases the output of commodity 2 and reduces the output of commodity 1.' This is what 'a strong Rybczynski result' implies.

Nakada (2015) has succeeded to derive a sufficient condition for 'a strong Rybczynski result' to hold (or not to hold) in the 3 X 2 model of BC's original type in a systematic manner by using EWS-ratio vector, which is defined based on 'economy-wide substitution' (hereinafter EWS) originally defined by Jones and Easton (1983) (hereinafter JE). Nakada concluded that the position of EWS-ratio vector determines Rybczynski sign pattern, which expresses the factor endowment-commodity output relationships, and its dual-counterpart, Stolper-Samuelson sign pattern, which expresses the commodity price-factor price relationships. And, the following result has been established (see Nakada (2015, Theorem 1)).

Theorem 1. If extreme factors are economy-wide complements, 'a strong Rybczynski result' holds necessarily.

The purpose of this paper is as follows. First, I develop a method to estimate the position of EWS-ratio vector, (S', U') to some extent, by using the framework of general equilibrium. Next, by using this method, I derive a sufficient condition for extreme factors to be economy-wide complements, which implies $(S',U')=(+,-)$. Next, we derive a sufficient condition for a certain Stolper-Samuelson sign pattern to hold. For this purpose, we will derive some important relationships among some variables. We can conclude that EWS-ratio vector exists on the line-segment.

Some papers are interested in the role of complementarity. For example, they analyzed the factor endowment-commodity output relationships, and its dual-counterpart, that is, the commodity price-factor price relationships, when extreme factors are aggregate complements, economy-wide complements, or Allen-complements in each sector.

For example, JE assumed 2 factors are 'perfect complements' by using EWS (see JE (pp. 90-92)). Suzuki (1983) ssumed 2 factors are 'perfect complements' in each sector by using Allen-partial elasticities of substitution. Takayama (1982, Section 4, p13-21) assumed that extreme factors are aggregate complements. Thompson (1985) assumed that 2 factors are aggregate complements. Suzuki (1987, Chapter 1, p17-26) assumed that extreme factors are Allen-complements in each sector. Teramachi (1993) assumed that extreme factors are aggregate complements. Other papers, also deal with complementarity, e.g., Thompson (1995), Bliss (2003), Easton (2008), Ide (2009), Ban (2007), and Ban (2008).

In sum, some of these previous studies are not so simple, but somewhat complicated. I am not sure whether all of these studies are plausible or not. I do not discuss about it. At least, to the author's knowledge, nobody has analyzed that under what condition, can extreme factors be economy-wide complements.

In section 2, we explain about the basic structure of the model. In section 3, we assume factor intensity ranking. In section 4, we define the factor-price-change ranking. In section 5, we derive the important relationship among EWS-ratios. In section 6, we show that EWS-ratio vector is on the line-segment (or EWS-ratio vector line-segment). In section 7, we derive important relationship among some variables ($H_j<0$, $H_0<0$). In section 8, we estimate the position of EWS-ratio vector in case of P>0. First, we derive a sufficient condition for extreme factors to be economy-wide complements. Next, we derive a sufficient condition for a certain Stolper-Samuelson sign pattern to hold. In section 9, we estimate the position of EWS-ratio vector in case of P<0. Section 10 is a conclusion.

In addition, section 2, 3 and 5 include the same contents as in Nakada (2015).

Section 2. Model

We assume similarly to BC (pp22-23). That is, we assume as follows. Products and factors markets are perfectly competitive. Supply of all factors is perfectly inelastic. Production functions are homogeneous of degree one and strictly quasi-concave. All factors are not specific and perfectly mobile between sectors, and factor prices are perfectly flexible. These two ensure the full employment of all resources. The country is small and faces exogenously given world prices, or the movement in relative price of a commodity is exogenously determined. The movements in factor endowments are exogenously determined.

Full employment of factors implies

$$\sum_j a_{ij} X_j = V_i,\ i = T, K\ , L\ , \quad (1)$$

where X_j denotes the amount produced of good j (j=l, 2); a_{ij} the requirement of input i per unit of output of good j (or the input-output coefficient); V_i the supply of factor i; T is the land, K capital, and L labor.

In a perfectly competitive economy, the unit cost of production of each good must just equal to its price. Hence,

$$\sum_i a_{ij} w_i = p_j,\ j = 1\ , 2, \quad (2)$$

where p_j is the price of good j; w_i is the reward of factor i.

BC (p23) stated, 'With quasi-concave and linearly homogeneous production functions, each input-output coefficient is independent of the scale of output and is a function solely of input prices:'

$$a_{ij} = a_{ij}\left(w_i\right),\ i = T, K\ , L\ , j\ = 1, 2. \quad (3)$$

And they continued, 'In particular, each C_{ij} [a_{ij} in our expression] is homogeneous of degree zero in all input prices.'

We might need some explanation about this. Samuelson (1953) suggested that a_{ij} is homogeneous of degree zero in all input prices under the assumption of cost minimization. But he did not prove it. That is, he (1953, chapter 4, p59) defined the function: $v_i = f^i(\mathrm{x}, \mathrm{w}_1, \cdots, \mathrm{w}_n), (\mathrm{i} = 1, \cdots, \mathrm{n})$. v_i is 'an optimum value for each productive factor' to derive 'the minimum total cost for each output (p58)', x is production, and w_i is 'prices of productive factors'. Samuelson (1953, chapter 4, p68) stated that v_i 'must be homogeneous of order zero in the variables $(\mathrm{w}_1, \cdots, \mathrm{w}_n)$, x being constant' (see also Samuelson (1983, chapter4, eq. (5) in p61; eq. (52) in p70)).

Eq. (1)-(3) describe the production side of the model. These are equivalent to eq. (1)-(5) in BC. The set includes 11 equations in 11 endogenous variables (X_j, a_{ij}, and w_i) and 5 exogenous variables (V_i and p_j). The small-country assumption simplifies the demand side of the economy.

Totally differentiate eq.(1):

$$\Sigma_j(\lambda_{ij}a_{ij}{*} \; + \lambda_{ij}X_j{*}) \; = V_i{*}, \; i = T, K, L \; , \quad (4)$$

where an asterisk denotes a rate of change (e.g., $X_j{*}=d\,X_j / X_j$), and where λ_{ij} is the proportion of the total supply of factor i in sector j (that is, $\lambda_{ij}=a_{ij}\,X_j / V_i$). Note that $\Sigma_j \lambda_{ij}=1$.

The minimum-unit-cost equilibrium condition in each sector implies $\Sigma_i\, w_i\, da_{ij} = 0$. Hence, we derive (see JE (p73, eq. (9)), BC (p24, n5),

$$\Sigma_i \theta_{ij} a_{ij}{*} = 0, \; j = 1, \; 2, \quad (5)$$

where θ_{ij} is the distributive share of factor i in sector j (that is, $\theta_{ij}=a_{ij}w_i/ p_j$). Note that $\Sigma_i\theta_{ij}$=l; da_{ij} is the differential of a_{ij}.

Totally differentiate eq.(2):

$$\Sigma_i \theta_{ij} w_i{*} = p_j{*}. \quad (6)$$

Subtract $p_j{*}$ from the both sides of(6):

$$\begin{aligned} &\Sigma_i \theta_{i1} w_{i1}{*} = 0, \\ &\Sigma_i \theta_{i2} w_{i1}{*} = -P, \; i = T, \; K, \; L \; , \end{aligned} \quad (7)$$

where $P=p_1{*}-p_2{*}= (p_1/p_2){*}$, $w_{i1}{*}= w_i{*}-p_1{*}$, $w_{i1} = w_i /p_1$; P is the rate of change in the relative price of a commodity; w_{i1} is the real factor price measured by the price of good 1.

Totally differentiate eq.(3) to obtain

$$a_{ij}{*} = \; \Sigma_h \varepsilon^{ij}{}_h w_h{*} \; = 0, \; i = T, \; K, \; L, \; j = \; 1, \; 2, \quad (8)$$

where

$$\varepsilon^{ij}{}_h = \partial log a_{ij} / \partial log\ w_h = \theta_{hj}\sigma^{ij}{}_h. \quad (9)$$

$\sigma^{ij}{}_h$ is the Allen-partial elasticities of substitution (hereinafter AES) between the ith and the hth factors in the jth industry. For additional definition of these symbols, see Sato and Koizumi (1973, pp47-49), and BC (p24). AES's are symmetric in the sense that

$$\sigma^{ij}{}_h = \sigma^{hj}{}_i. \quad (10)$$

And according to BC (p33), 'Given the assumption that production functions are strictly quasi-concave and linearly homogeneous,'

$$\sigma^{ij}{}_i < 0. \quad (11)$$

Since a_{ij} is homogeneous of degree zero in input prices, we have

$$\Sigma_h \varepsilon^{ij}{}_h = \Sigma_h \theta_{hj}\sigma^{ij}{}_h = 0,\ i = T,\ K,\ L,\ j = 1,\ 2. \quad (12)$$

Eq. (8) and (12) are equivalent to the expressions in BC (p24, n. 6). See also JE (p74, eq. (12)-(13)). From these:

$$a_{ij}{}^* = \Sigma_h \varepsilon^{ij}{}_h w_{h1}{}^*. \quad (13)$$

Substitute eq. (13) in eq.(4):

$$\Sigma_j(\lambda_{ij}\Sigma_h \varepsilon^{ij}{}_h w_{h1}{}^* + \lambda_{ij} X_j{}^*) = \Sigma_h g_{ih} w_{h1}{}^* + \Sigma_j \lambda_{ij} X_j{}^* = V_i{}^*,\ i = T,\ K,\ L, \quad (14)$$

where

$$g_{ih} = \Sigma_j \lambda_{ij} \varepsilon^{ij}{}_{h,} i,\ h = T,\ K,\ L. \quad (15)$$

This is the EWS (or 'the economy-wide substitution') between factors i and h defined by JE (p75). They stated, 'Clearly, the substitution terms in the two industries are always averaged together. With this in mind we define the term $\sigma^i{}_k$ to denote the economy-wide substitution towards or away from the use of factor i when the kth factor becomes more expensive, under the assumption that each industry's output is kept constant:…'.

Note that

$$\Sigma_h g_{ih} = 0, i = T,\ K,\ L, \quad (16)$$
$$g_{ih} = (\theta_h / \theta_i) g_{hi}\ , i,\ h = T,\ K,\ L. \quad (17)$$

g_{ih} is not symmetric. Namely, $g_{ih} \neq g_{hi}$, $i \neq h$ in general. On eq.(17), see also JE (p85). From(17), for example, we obtain

$$g_{KK} = -\left(g_{KT} + g_{KL}\right), g_{TT} = -\left(g_{TK} + g_{TL}\right), g_{LL} = -(g_{LK} + g_{LT}). \quad (18)$$

Recall(11), that is, $\sigma^{ij}{}_i<0$. From this equation and(9), we derive $\varepsilon^{ij}{}_i<0$. Further, from this equation and(15), we derive

$g_{ii}<0$. (19)

Section 3. Factor intensity ranking

In this article, we assume

$$\frac{\theta_{T1}}{\theta_{T2}} > \frac{\theta_{L1}}{\theta_{L2}} > \frac{\theta_{K1}}{\theta_{K2}}, \quad (20)$$
$$\theta_{L1} > \theta_{L2}. \quad (21)$$

Eq.(20) is, what you call, 'the factor intensity ranking' (see JE (p69), see also BC (pp26-27), Suzuki (1983, p142),). This implies that sector 1 is relatively land intensive, and sector 2 is relatively capital intensive, and that labor is the middle factor, and land and capital are extreme factors (see also Ruffin (1981, p180)). Eq.(21) is 'the factor intensity ranking for middle factor'. It implies that the middle factor is used relatively intensively in sector 1.

In the following sections, I show the analysis under these assumptions. Nakada (2015) also assumed (20) and (21) hold. Of course, even if we assume $\theta_{L1} < \theta_{L2}$, we can analyze similarly.

Section 4. Factor-price-change ranking segment

Subsection 4.1. In case of P>0

For example, we assume

P>0. (22)

Recall (7), that is, $P = p_1* - p_2*$. We can draw the line which expresses the change in real-factor-price in the figure, and can compare them. Recall(7):

$$\begin{aligned} &\Sigma_i \theta_{i1} w_{i1}* = 0, \\ &\Sigma_i \theta_{i2} w_{i1}* = -P, \ i = T, \ K, \ L. \end{aligned} \quad (23)$$

We define that, for ease of notation:

$$(X, Y, Z) = (w_{T1}*, w_{K1}*, w_{L1}*) = (w_T* - p_1*, w_K* - p_1*, w_L* - p_1*). \quad (24)$$

This is a change in the real factor price. Transform (23) using(24):

$$\begin{bmatrix} \theta_{T1} & \theta_{K1} \\ \theta_{T2} & \theta_{K2} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} -\theta_{L1} Z \\ -P - \theta_{L2} Z \end{bmatrix}. \quad (25)$$

Solve eq.(25):

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{D_1} \begin{bmatrix} \theta_{K2} & -\theta_{K1} \\ \theta_{T2} & \theta_{T1} \end{bmatrix} \begin{bmatrix} -\theta_{L1} Z \\ -P - \theta_{L2} Z \end{bmatrix}, \quad (26)$$

where $D_1 = \theta_{T1}\theta_{K2} - \theta_{K1}\theta_{T2}$. Hence, we have

$$X = \frac{1}{D_1} \theta_{K1} P - \frac{D_2}{D_1} Z,$$

$$Y = \frac{1}{D_1} (-\theta_{T1} P) - \frac{D_3}{D_1} Z,$$

$$Z = Z, \quad (27)$$

where $D_2 = \theta_{K2}\theta_{L1} - \theta_{K1}\theta_{L2},\ D_3 = \theta_{T1}\theta_{L2} - \theta_{T2}\theta_{L1}$. Eq.(20) implies

$$(D_1, D_2, D_3) = (+,+,+). \quad (28)$$

We can treat as if X, Y were dependent variables, and Z was the independent variable. Eq.(27) express the straight lines in 2 dimensions. We call these, respectively, as line-X, -Y, -Z. Because we assume (22)(P>0), the sign of gradient and intercept of line-X, line-Y is, respectively:

$$-\frac{D_2}{D_1} = (-),\ \frac{1}{D_1}\theta_{K1}P = (+), \text{for line-X;}$$
$$-\frac{D_3}{D_1} = (-),\ \frac{1}{D_1}(-\theta_{T1}P) = (-), \text{for line-Y.} \quad (29)$$

Hence, we can draw line-X, -Y, -Z in the figure (see Fig. 1). We can show that line-X and line-Y has an intersection point in quadrant IV, because we assume eq. (21) ($\theta_{L1} > \theta_{L2}$).

Only 4 rankings are possible, that is,

X>Y>Z, X>Z>Y, Z>X>Y, Z>Y>X. (30)

Either of the 4 patterns is possible. We call this as the factor-price-change-ranking. For example, we can assume as follows.

X>Z>Y $\leftrightarrow$ $w_T{*} > w_L{*} > w_K{*}$. (31)

If (31) holds, the change in real reward for labor is intermediate (or middle), and the change in real reward for land and capital are extreme.

Subsection 4.2.In case of P<0

On the other hand, if we assume

P<0, (32)

we can draw line-X,-Y,-Z in the figure similarly. The sign of intercept of line-X,-Y is opposite to(29).

But the sign of gradient is similar to(29). This implies that only 4 rankings are possible, that is,

X>Y>Z, Y>X>Z, Y>Z>X, Z>Y>X. (33)

Either of the 4 patterns is possible.

Section 5. EWS-ratio vector boundary

Each a_{ij} is homogeneous of degree zero in all input prices (see(3)). Recall(11), $\sigma^{ij}_i<0$. These implies (see Nakada (2015), Appendix, eq. (A14)):

$$g_{KK}g_{TT}- g_{TK}g_{KT} >0. \quad (34)$$

Recall eq.(18). That is, $g_{KK} = -\left(g_{KT} + g_{KL}\right), and\ g_{TT} = -\left(g_{TK} + g_{TL}\right)$. Substitute these equations to eliminate g_{KK}, g_{TT} from L.H.S. of eq.(34). Next, recall eq.(17), that is, $g_{ih} = (\theta_h / \theta_i) g_{hi}$. Use this equation to eliminate g_{KL}, g_{TL}, g_{TK}. That is, express using only 3 EWS's, namely, g_{LK}, g_{LT}, g_{KT} :

$$\text{L.H.S. of } (34)= g_{KT}g_{TL} + g_{KL}g_{TK} + g_{KL}g_{TL} = \frac{\theta_L}{\theta_T}[g_{KT}(g_{LT} + g_{LK}) + \frac{\theta_L}{\theta_K} g_{LK}g_{LT}](> 0) .(35)$$

From(35), we derive

$$g_{KT}(g_{LT} + g_{LK}) < -\frac{\theta_L}{\theta_K} g_{LK}g_{LT} .(36)$$

From(18) and(19), we derive $g_{LK} + g_{LT} = -g_{LL} > 0$. Divide(36) by $(g_{LK} + g_{LT})$, we have

$$g_{KT} > -\frac{\theta_L}{\theta_K}\frac{g_{LK}g_{LT}}{g_{LK} + g_{LT}} . \quad (37)$$

If we define, for ease of notation,

$$(S,\ T,\ U) = (g_{LK}, g_{LT}, g_{KT}) , \ (38)$$

we derive

$$U > -\frac{\theta_L}{\theta_K}\frac{ST}{S+T}. \quad (39)$$

Divide the both sides by T:

$$U' > -\frac{\theta_L}{\theta_K}\frac{S'}{S'+1}, \text{ if T>0;} U' < -\frac{\theta_L}{\theta_K}\frac{S'}{S'+1}, \text{ if T<0,} \quad (40)$$

where

$$(S', U') = (S/T, U/T) = (g_{LK}/g_{LT}, g_{KT}/g_{LT}). \quad (41)$$

We call this as EWS-ratio vector. Transform that

$$U' = -\frac{\theta_L}{\theta_K}\frac{S'}{S'+1} = -\frac{\theta_L}{\theta_K} + \frac{\theta_L}{\theta_K}\frac{1}{S'+1}, \quad (42)$$

which expresses the rectangular hyperbola. We call it as the equation for EWS-ratio vector boundary. It passes on the origin of O (0, 0). The asymptotic lines are

$$S' = -1, \quad U' = -\theta_L/\theta_K. \quad (43)$$

We can draw this boundary in the figure (see Fig. 2). S' is written along the horizontal axis, and U' along the vertical axis. EWS-ratio vector boundary demarcates the boundary of the region for EWS-ratio vector. This implies that EWS-ratio vector is not so arbitrary, but exists within this bounds.

Note that:

EWS-ratio vector exists in the upper right region of EWS-ratio vector boundary, if T>0,
EWS-ratio vector exists in the lower left region of EWS-ratio vector boundary, if T<0.(44)

The sign patterns of EWS-ratio vector are, in each quadrant:

quad. I: (S', U')=(+, +)↔(S, T, U) = (+, +, +);
quad. II: (S', U')=(-, +)↔(S, T, U) =(-, +, +);
quad. III: (S', U')= (-, -)↔(S, T, U) =(+, -, +);
quad. IV: (S', U')= (+, -)↔(S, T, U) =(+, +, -). (45)

Hence, one of EWS's can be negative at most. Note that

T>0, if (S', U') exists in quadrant I, II, or IV;
T<0, if (S', U') exists in quadrant III. (46)

Recall (38) and (41), that is, (S', U')=(S/T, U/T)=(g_{LK} /g_{LT}, g_{KT} /g_{LT}), (S, T, U) = (g_{LK}, g_{LT}, g_{KT}). We may define (for i≠h),

Factors i and h are economy-wide substitutes, if g_{ih} >0;
Factors i and h are economy-wide complements, if g_{ih} <0. (47)

Section 6. EWS-ratio vector line-segment

Eliminate $\varepsilon^{ij}{}_{i}$ from (13) using(12) to obtain (on this, see BC (p33, note6 in p24)):

$$a_{Tj}* = \sigma_{TKj}\theta_{Kj}(\mathrm{w_K}* - \mathrm{w_T}*) + \sigma_{TLj}\theta_{Lj}(\mathrm{w_L}* - \mathrm{w_T}*)\ ,$$
$$a_{Kj}* = \sigma_{KLj}\theta_{Lj}(\mathrm{w_L}* - \mathrm{w_K}*) + \sigma_{KTj}\theta_{Tj}(\mathrm{w_T}* - \mathrm{w_K}*)\ ,$$
$$a_{Lj}* = \sigma_{LTj}\theta_{Tj}(\mathrm{w_T}* - \mathrm{w_L}*) + \sigma_{LKj}\theta_{Kj}(\mathrm{w_K}* - \mathrm{w_L}*)\ , \quad (48)$$

where $\sigma_{ihj} = \sigma^{ij}{}_{h}$. By using(9), that is, $\varepsilon^{ij}{}_{h} = \partial log a_{ij} / \partial log\ w_h = \theta_{hj}\sigma^{ij}{}_{h}$, transform(48):

$$a_{Tj}* = \varepsilon_{TKj}(\mathrm{w_K}* - \mathrm{w_T}*) + \varepsilon_{TLj}(\mathrm{w_L}* - \mathrm{w_T}*)\ ,$$
$$a_{Kj}* = \varepsilon_{KLj}(\mathrm{w_L}* - \mathrm{w_K}*) + \varepsilon_{KTj}(\mathrm{w_T}* - \mathrm{w_K}*)\ ,$$
$$a_{Lj}* = \varepsilon_{LTj}(\mathrm{w_T}* - \mathrm{w_L}*) + \varepsilon_{LKj}(\mathrm{w_K}* - \mathrm{w_L}*)\ , \quad (49)$$

where $\varepsilon_{ihj} = \varepsilon^{ij}{}_{h}$.

We define:

$$a_{i0}' = \sum_{j} \lambda_{ij} a_{ij}*, \mathrm{i} = T, K, L. \quad (50)$$

This is the aggregate of input-output-coefficient-change $(a_{ij}*)$. Substitute (49) in eq.(50), we derive, for example,

$$a_{T0}' = \sum_j \lambda_{Tj} a_{Tj}* = \sum_j \lambda_{Tj} \{ \varepsilon^{Tj}{}_K (w_K* - w_T*) + \varepsilon^{Tj}{}_L (w_L* - w_T*) \},$$

$$a_{K0}' = \sum_j \lambda_{Kj} a_{Kj}* = \sum_j \lambda_{Kj} \{ \varepsilon^{Kj}{}_L (w_L* - w_K*) + \varepsilon^{Kj}{}_T (w_T* - w_K*) \},$$

$$a_{L0}' = \sum_j \lambda_{Lj} a_{Lj}* = \sum_j \lambda_{Lj} \{ \varepsilon^{Lj}{}_T (w_T* - w_L*) + \varepsilon^{Lj}{}_K (w_K* - w_L*) \}. \quad (51)$$

Rewrite (51) using eq.(15) ($g_{ih} = \Sigma_j \lambda_{ij} \varepsilon^{ij}{}_{h,} i,\ h = T,\ K,\ L.$):

$$a_{T0}' = g_{TK}(w_K* - w_T*) + g_{TL}(w_L* - w_T*)\}, \quad (52)$$
$$a_{K0}' = g_{KL}(w_L* - w_K*) + g_{KT}(w_T* - w_K*), \quad (53)$$
$$a_{L0}' = g_{LT}(w_T* - w_L*) + g_{LK}(w_K* - w_L*)\}. \quad (54)$$

Recall eq.(17), that is, $g_{ih} = (\theta_h / \theta_i) g_{hi}$. Using this, eliminate g_{TK}, g_{TL}, g_{KL}, from (52),(53):

$$a_{T0}' = \theta_{KT} g_{KT}(w_K* - w_T*) + \theta_{LT} g_{LT}(w_L* - w_T*), \quad (55)$$
$$a_{K0}' = \theta_{LK} g_{LK}(w_L* - w_K*) + g_{KT}(w_T* - w_K*), \quad (56)$$

where

$$\theta_{ih} = \theta_i / \theta_h, i \neq h. \quad (57)$$

In sum, from(55),(56), and(54), we have

$$a_{T0}' = \theta_{KT} g_{KT}(w_K* - w_T*) + \theta_{LT} g_{LT}(w_L* - w_T*), \quad (58)$$
$$a_{K0}' = \theta_{LK} g_{LK}(w_L* - w_K*) + g_{KT}(w_T* - w_K*), \quad (59)$$
$$a_{L0}' = g_{LT}(w_T* - w_L*) + g_{LK}(w_K* - w_L*). \quad (60)$$

Multiply(58),(59), and(60) by $g_{LK}\theta_T, g_{LT}\theta_K, g_{KT}\theta_K$, respectively, and take the difference:

$$a_{T0}' g_{LK}\theta_T - a_{K0}' g_{LT}\theta_K = (w_K* - w_T*) G_0,$$
$$a_{K0}' g_{LT}\theta_K - a_{L0}' g_{KT}\theta_K = (w_L* - w_K*) G_0,$$
$$a_{L0}' g_{KT}\theta_K - a_{T0}' g_{LK}\theta_T = (w_T* - w_L*) G_0, \quad (61),$$

where

$$G_0 = g_{KT}\theta_K(g_{LK} + g_{LT}) + g_{LT}g_{LK}\theta_L(>0)\ . \quad (62)$$

From(35), we can derive(62). From(61), we have:

$$\mathrm{G}_0 = \frac{a_{T0}' g_{LK}\theta_T - a_{K0}' g_{LT}\theta_K}{(\mathrm{w_K}* - \mathrm{w_T}*)} = \frac{a_{K0}' g_{LT}\theta_K - a_{L0}' g_{KT}\theta_K}{(\mathrm{w_L}* - \mathrm{w_K}*)} = \frac{a_{L0}' g_{KT}\theta_K - a_{T0}' g_{LK}\theta_T}{(\mathrm{w_T}* - \mathrm{w_L}*)} \quad .$$

(63)

From(63), we have:

$$(a_{T0}' g_{LK}\theta_T - a_{K0}' g_{LT}\theta_K)(\mathrm{w_L}* - \mathrm{w_K}*) = (a_{K0}' g_{LT}\theta_K - a_{L0}' g_{KT}\theta_K)(\mathrm{w_K}* - \mathrm{w_T}*)\ . \quad (64)$$

Divide the both sides of (64) by g_{LT}. Recall (38) and(41), that is, (S, T, U) = (g_{LK}, g_{LT}, g_{KT}), (S', U') =(S/T, U/T) = (g_{LK}/g_{LT}, g_{KT}/g_{LT}). Use these symbols, we have

$$(a_{T0}' S' \theta_T - a_{K0}' \theta_K)(\mathrm{w_L}* - \mathrm{w_K}*) = (a_{K0}' \theta_K - a_{L0}' U' \theta_K)(\mathrm{w_K}* - \mathrm{w_T}*)\ . \quad (65)$$

From(65), we derive

$$U' = -a_1 S' + \mathrm{b}_1, \quad (66)$$

where

$$a_1 = \frac{a_{T0}' \theta_T W_{LK}}{a_{L0}' \theta_K W_{KT}},\ b_1 = \frac{a_{K0}' W_{LT}}{a_{L0}' W_{KT}},\ W_{ih} = w_i * - w_h * = (w_i / w_h)*, \mathrm{i} = \mathrm{T,K,L,i} \neq \mathrm{h}\ . \quad (67)$$

W_{ih} is the change in relative factor price between factors i and h. Eq.(66) expresses the straight line. EWS-ratio vector (S', U') exists on this line. We call it as EWS-ratio vector line. Hence, U' has a linear relationship with S'.

From (66) and(42), we can make a system of equations:

$$U' = -a_1 S' + \mathrm{b}_1, \quad (68)$$

$$U' = -\frac{\theta_L}{\theta_K}\frac{S'}{S'+1}\ . \quad (69)$$

From (68) and(69), we obtain a quadratic equation in S'. Solve this to derive two solutions. Each solution denotes the S’ coordinate value of intersection point of EWS-ratio vector line and EWS-ratio vector boundary. The solutions are:

$$S' = \frac{-W_{TL}}{W_{KL}}, \frac{\alpha_{K0}'}{\alpha_{T0}'}\theta_{KT}. \quad (70)$$

Hence, the Cartesian coordinates of intersection point is,

$$(S', \mathrm{U}') = (\frac{-W_{TL}}{W_{KL}}, \frac{\theta_L}{\theta_K}\frac{-W_{LT}}{W_{KT}}), (\alpha_0\theta_{KT}, \beta_0), \quad (71)$$

where

$$\alpha_0 = \frac{\alpha_{K0}'}{\alpha_{T0}'}, \beta_0 = \frac{\alpha_{K0}'}{\alpha_{L0}'}. \quad (72)$$

We call these points as point A and B, respectively.

In general, EWS-ratio vector, (S', U') exists on the line-segment AB. We can call it as EWS-ratio vector line-segment.

Section 7. Derivation of important relationship among some variables (H_j<0, H_0<0)

Unit cost of production of good j equals the price of good j (see eq.(2)):

$$\sum_i a_{ij} w_i = p_j, j = 1, 2. \quad (73)$$

Eq.(73) expresses the isocost surface (or IC) (see Fig. 3). IQ is the isoquant surface. We define as follows.

$$\vec{w} = w_i, \vec{w}' = w_i + \Delta w_i, \overrightarrow{OB} = a_{ij} + \Delta a_{ij}, \overrightarrow{OA} = a_{ij},$$
$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (a_{ij} + \Delta a_{ij}) - a_{ij} = \Delta a_{ij}, i = T, K, L, j = 1, 2., \quad (74)$$

where Δ denotes the small variation. Vector w_i is vertical to the isocost surface (IC). Because production functions are homogeneous of degree one and strictly quasi-concave, isoquant surface is convex to the origin. Isocost surface has a point of contact on point A with isoquant surface. That is the equilibrium point. The input-output coefficient (a_{ij}) is determined on this point. I draw this figure by analogy from the figure of isocost line and isoquant curve in case of 2 factor case. If isocost surface changes its position and becomes IC', the equilibrium point move to new point, point B. Angles θ_A and θ_B are the angles between vector $\vec{w}$ and $\overrightarrow{AB}$, $\vec{w}'$ and $\overrightarrow{BA}$, respectively.

We obtain for angles θ_A, θ_B :

$$0 < \theta_A < \pi/2, 0 < \theta_B < \pi/2. \quad (75)$$

Hence, the inner product of vectors satisfies:

$$\vec{w} \cdot \overrightarrow{AB} = \sum_i w_i \Delta a_{ij} = |\vec{w}| \left|\overrightarrow{AB}\right| \cos\theta_A > 0, \quad (76)$$

$$\vec{w}' \cdot \overrightarrow{BA} = \sum_i (w_i + \Delta w_i)(-\Delta a_{ij}) = |\vec{w}'| \left|\overrightarrow{BA}\right| \cos\theta_B > 0. \quad (77)$$

By summing up(76) and(77), we have:

$$-\sum_i \Delta w_i \Delta a_{ij} > 0, \ j = 1,\ 2. \quad (78)$$

From(78):

$$H_j < 0, \ j = 1,\ 2, \quad (79)$$

where

$$H_j = \sum_i \Delta w_i \Delta a_{ij} \quad . \quad (80)$$

H_j is the inner product of 2 vectors, $\overrightarrow{\Delta w}\ (= \Delta w_i)$, and $\overrightarrow{AB} (= \Delta a_{ij})$.

Eq.(79) is very similar to the equation which Samuelson derived (see Samuelson (1983, chapter 4, p78, eq. (82)), that is,

$$\sum_1^n \Delta w_i \Delta v_i \leq 0, \quad (81)$$

where w_i is the price of factor i; v_i is the combination of factors which minimize the total cost. But he derived (81) in another way.

Transform (80):

$$H_j = \sum_i w_i * a_{ij} * \mathrm{p_j} \theta_{ij},\ j = 1,\ 2. \quad (82)$$

Of course, from eq.(79), we derive

$$\frac{1}{p_j} H_j < 0,\ j = 1,\ 2. (83)$$

We define

$$H_0 = \sum_j \theta_j \frac{1}{p_j} H_j. \quad (84)$$

We call this as the aggregate of H_j /p_j. From (84) and(83), we derive

$$H_0 < 0. (85)$$

Substitute (82) in(84):

$$H_0 = \sum_j \theta_j \frac{1}{p_j} \sum_i w_i * a_{ij} * \mathrm{p_j} \theta_{ij} = \sum_i \sum_j \lambda_{ij} a_{ij} * \theta_i w_i *. \quad (86)$$

Recall(50), that is, $a_{i0}' = \sum_j \lambda_{ij} a_{ij} *, \mathrm{i} = T, K, L.$ Rewrite (86) using(50), and from(85), we have:

$$H_0 = \sum_i a_{i0}' \theta_i w_i * < 0. \quad (87)$$

Recall eq.(5)($\Sigma_i\ \theta_{ij} a_{ij}*=0, j=1, 2,$). From(50), and(5), we can show that

$$\sum_i a_{i0}'\theta_i = 0\,. \quad (88)$$

From(88), we derive:

$$a_{L0}'\theta_L = -(a_{T0}'\theta_T + a_{K0}'\theta_K)\,. \quad (89)$$

Substitute (89) in (87) to derive:

$$H_0 = (\mathrm{w_T}^* - \mathrm{w_L}^*)a_{T0}'\theta_T + (\mathrm{w_K}^* - \mathrm{w_L}^*)a_{K0}'\theta_K < 0. \quad (90)$$

Similarly, we derive

$$H_0 = (\mathrm{w_T}^* - \mathrm{w_K}^*)a_{T0}'\theta_T + (\mathrm{w_L}^* - \mathrm{w_K}^*)a_{L0}'\theta_L < 0, \quad (91)$$
$$H_0 = (\mathrm{w_K}^* - \mathrm{w_T}^*)a_{K0}'\theta_K + (\mathrm{w_L}^* - \mathrm{w_T}^*)a_{L0}'\theta_L < 0. \quad (92)$$

Eq.(88) implies that

$$(a_{T0}', a_{K0}', a_{L0}') \neq (+,+,+)(-,-,-)\,,$$
$$(a_{T0}', a_{K0}', a_{L0}') = (-,+,-),(-,+,+),(+,+,-),(-,-,+),(+,-,+),(+,-,-)\,. \quad (93)$$

One or two of a_{i0}' can be negative. We call these sign patterns as sign A, B, C, D, E, F, respectively.

Section 8. Estimating the position of EWS-ratio vector in case of P>0

In this section, I estimate the position of EWS-ratio vector. We assume P>0.

Subsection 8.1. A sufficient condition for extreme factors to be economy-wide complements

For example, we assume:

$$X > Z > Y \leftrightarrow \mathrm{w_T}^* > \mathrm{w_L}^* > \mathrm{w_K}^*\,. \quad (94)$$

This assumption is plausible (see eq.(30)). Hence,

$$(\mathrm{w_T}^* - \mathrm{w_L}^*, \mathrm{w_K}^* - \mathrm{w_L}^*) = (+,-)\,, (\mathrm{w_T}^* - \mathrm{w_K}^*, \mathrm{w_L}^* - \mathrm{w_K}^*) = (+,+)\,,$$

$$(w_K^* - w_T^*, w_L^* - w_T^*) = (-,-)\,. \quad (95)$$

Substitute (95) in eq.(90),(91), and(92), we obtain

$$(a_{T0}', a_{K0}') \neq (+,-)\,, \quad (a_{T0}', a_{L0}') \neq (+,+)\,, \quad (a_{K0}', a_{L0}') \neq (-,-)\,. \quad (96)$$

From(96), we have

$$(a_{T0}', a_{K0}', a_{L0}') \neq (+,-,+), (+,-,-)\,. \quad (97)$$

Hence, sign E and F are impossible. The sign A, B, C, and D are possible (see(93)). That is,

$$(a_{T0}', a_{K0}', a_{L0}') = (-,+,-), (-,+,+), (+,+,-), (-,-,+)\,. \quad (98)$$

Because we assume(94), we derive

$$\left(\frac{-W_{TL}}{W_{KL}}, \frac{\theta_L}{\theta_K}\frac{-W_{LT}}{W_{KT}}\right) = \left(\frac{-(w_T^* - w_L^*)}{(w_K^* - w_L^*)}, \frac{\theta_L}{\theta_K}\frac{-(w_L^* - w_T^*)}{(w_K^* - w_T^*)}\right) = (+,-)\,. \quad (99)$$

Hence, point A is in quad IV. And we derive the results as follows.

(i)If sign A holds, we derive $(\alpha_0\theta_{KT}, \beta_0) = (-,-)$, point B is in quad. III. (100)
(ii)If sign B holds, we derive $(\alpha_0\theta_{KT}, \beta_0) = (-,+)$, point B is in quad. II. (101)
(iii)If sign C holds, we derive $(\alpha_0\theta_{KT}, \beta_0) = (+,-)$, point B is in quad. IV. (102)
(iv)If sign D holds, we derive $(\alpha_0\theta_{KT}, \beta_0) = (+,-)$, point B is in quad. IV.(103)

Hence, we can plot point A and B in the figure. If sign C holds, both point A and B is in quad. IV. Similarly, if sign D holds, both point A and B is in quad. IV. Therefore, if sign D or C holds, we have to show that which point is on the L.H.S., point A or B. We can analyze the distribution of point A and B by using eq. (90) as follows.

For example, we assume sign C holds, that is,

$$(a_{T0}', a_{K0}', a_{L0}') = (+,+,-)\,. \quad (104)$$

Recall(90), that is,

$$H_0 = (w_T* - w_L*)a_{T0}'\theta_T + (w_K* - w_L*)a_{K0}'\theta_K < 0. \quad (105)$$

Transform(105):

$$H_0 = (w_K* - w_L*)a_{T0}'\theta_T\{\frac{(w_T* - w_L*)}{(w_K* - w_L*)} + \frac{a_{K0}'\theta_K}{a_{T0}'\theta_T}\}$$

$$= W_{KL}\, a_{T0}'\theta_T\{\frac{W_{TL}}{W_{KL}} + \alpha_0\theta_{KT}\} < 0. \quad (106)$$

We have $W_{KL} = (-)$, $a_{T0}' = (+)$. Hence, we derive

$$\frac{W_{TL}}{W_{KL}} + \alpha_0\theta_{KT} > 0. \quad (107)$$

This implies

$$-\frac{W_{TL}}{W_{KL}} < \alpha_0\theta_{KT}. \quad (108)$$

Hence, point A is on the L.H.S. of point B (see(71)). From (99) and(102), both of point A and B exist in quadrant IV. Hence, the point of (S', U') exists on the line-segment AB, which exists in quad. IV (see Fig. 2).

From the above, if we assume sign C holds, the point of (S', U') satisfies as follows:

$$\frac{-W_{TL}}{W_{KL}} < S' < \alpha_0\theta_{KT}, \frac{\theta_L}{\theta_K}\frac{-W_{LT}}{W_{KT}} > U' > \beta_0. \quad (109)$$

Hence, we derive

$$(S', U') = (+,-) \leftrightarrow (S, T, U) = (g_{LK}, g_{LT}, g_{KT}) = (+,+,-). \quad (110)$$

This implies that factors T and K are economy-wide complements. In sum, the following result has been established.

Theorem 2. If we assume

$$P>0, X > Z > Y \leftrightarrow w_T* > w_L* > w_K*, (a_{T0}', a_{K0}', a_{L0}') = (+,+,-), \quad (111)$$

we derive

$$\frac{-W_{TL}}{W_{KL}} < S' < \alpha_0 \theta_{KT}, \frac{\theta_L}{\theta_K}\frac{-W_{LT}}{W_{KT}} > \mathrm{U}' > \beta_0, \quad (112)$$

$$(S', U') = (+, -) \leftrightarrow \ (S, \mathrm{T}, U) = (g_{LK}, g_{LT}, g_{KT}) = (+, +, -). \quad (113)$$

This implies that extreme factors are economy-wide complements.

On the other hand, we can show that if sign D holds, point A is on the R.H.S. of point B. We can do the similar analysis. But I omit.

Subsection 8.2. A sufficient condition for a certain Stolper-Samuelson sign pattern to hold

We assume (111) holds, hence, (112)and(113) hold. On the other hand, the following result has been established already (see Nakada (2015, section 10)).

Theorem 1. If extreme factors are economy-wide complements, 'a strong Rybczynski result' holds necessarily. In this case, Stolper-Samuelson sign patterns for subregion P1-P3 are:

$$\mathrm{sign}[\frac{w_i * - p_j *}{P}] = \begin{array}{ccc} \mathrm{P1} & \mathrm{P2} & \mathrm{P3} \\ \begin{bmatrix} + & - & - \\ + & - & - \end{bmatrix} & \begin{bmatrix} + & - & - \\ + & - & + \end{bmatrix} & \begin{bmatrix} + & - & + \\ + & - & + \end{bmatrix} \end{array}, \quad (114)$$

where

$$P = p_1 * - p_2 *, [\frac{w_i * - p_j *}{P}] = \begin{bmatrix} w_T * - p_1 * & w_K * - p_1 * & w_L * - p_1 * \\ w_T * - p_2 * & w_K * - p_2 * & w_L * - p_2 * \end{bmatrix} \frac{1}{P}. (115)$$

I use this theorem for the analysis. I show some examples of a sufficient condition for a certain Stolper-Samuelson sign pattern to hold. For the analysis shown below, compare (112) with the results in Nakada (2015, example 2 in section 11).

(i) Example 1

If we assume:

$$\frac{\theta_{K2}}{\theta_{T2}} < \frac{-W_{TL}}{W_{KL}} < S' < \alpha_0\theta_{KT}, \quad (116)$$

EWS-ratio vector exists in the subregion P1. Hence, we derive

$$\text{sign}[\frac{w_i*-p_j*}{P}] = \begin{bmatrix} + & - & - \\ + & - & - \end{bmatrix}. \quad (117)$$

What is the sufficient condition for(116)? If (116) holds, we have

$$\frac{\theta_{K2}}{\theta_{T2}} < \frac{-W_{TL}}{W_{KL}}. \quad (118)$$

Multiply by W_{KL} (<0), we have:

$$\theta_{K2}W_{KL} > -W_{TL}\theta_{T2} \leftrightarrow \theta_{K2}(\mathrm{w_K}*-\mathrm{w_L}*) > -(\mathrm{w_T}*-\mathrm{w_L}*)\theta_{T2}$$

$$\leftrightarrow \sum_i \theta_{i2}\,\mathrm{w_i}*-\mathrm{w_L}* > 0. \quad (119)$$

Recall (6)($\Sigma_i\theta_{ij}w_i* = p_j*.$). Use (6) to transform(119), we have

$$\mathrm{p_2}*-\mathrm{w_L}* > 0 \leftrightarrow \mathrm{w_L}*-\mathrm{p_2}* = (-) < 0. \quad (120)$$

In sum, if (120) holds, (116) holds. Hence, (117) holds.

(ii)Example 2

If we assume:

$$\frac{\theta_{K1}}{\theta_{T1}} < \frac{-W_{TL}}{W_{KL}} < S' < \alpha_0\theta_{KT} < \frac{\theta_{K2}}{\theta_{T2}}, \quad (121)$$

EWS-ratio vector exists in the subregion P2. Hence, we derive

$$\text{sign}[\frac{w_i*-p_j*}{P}] = \begin{bmatrix} + & - & - \\ + & - & + \end{bmatrix}. \quad (122)$$

What is the sufficient condition for(121)? If (121) holds, because (57)($\theta_{ih} = \theta_i / \theta_h, i \neq h$) implies $\theta_{KT} = \theta_K / \theta_T$, we derive

$$\frac{\theta_{K1}}{\theta_{T1}} < \frac{-W_{TL}}{W_{KL}} \leftrightarrow \mathrm{w_L}^* - \mathrm{p_1}^* = (-) < 0\,; \quad (123)$$

$$\alpha_0 \theta_{KT} < \frac{\theta_{K2}}{\theta_{T2}} \leftrightarrow \frac{\alpha_{K0}'}{\alpha_{T0}'} \frac{\theta_K}{\theta_T} < \frac{\theta_{K2}}{\theta_{T2}}, \quad (124)$$

Recall(88). From this, we have $\alpha_{K0}'\theta_K = -(\alpha_{L0}'\theta_L + \alpha_{T0}'\theta_T)$. Substitute this in L.H.S. of(124):

$$\text{L.H.S. of(124)} = \frac{\alpha_{K0}'}{\alpha_{T0}'} \frac{\theta_K}{\theta_T} = \frac{-\alpha_{L0}'\theta_L - \alpha_{T0}'\theta_T}{\alpha_{T0}'\theta_T} = \frac{-\alpha_{L0}'}{\alpha_{T0}'} \frac{\theta_L}{\theta_T} - 1 = \frac{-\sum_j \lambda_{Lj} a_{Lj}^*}{\sum_j \lambda_{Tj} a_{Tj}^*} \frac{\theta_L}{\theta_T} - 1. \quad (125)$$

Substitute (125) in(124):

$$\frac{-(\lambda_{L1} a_{L1}^* + \lambda_{L2} a_{L2}^*)}{(\lambda_{T1} a_{T1}^* + \lambda_{T2} a_{T2}^*)} \frac{\theta_L}{\theta_T} < 1 + \frac{\theta_{K2}}{\theta_{T2}} \leftrightarrow \frac{-(\theta_1 \theta_{L1} a_{L1}^* + \theta_2 \theta_{L2} a_{L2}^*)}{(\theta_1 \theta_{T1} a_{T1}^* + \theta_2 \theta_{T2} a_{T2}^*)} < 1 + \frac{\theta_{K2}}{\theta_{T2}}. \quad (126)$$

In sum, if (126) and (123) hold, (121) holds. Hence, (122) holds. Eq.(126) seems useful for application.

(iii)Example 3

If we assume

$$0 < \frac{-W_{TL}}{W_{KL}} < S' < \alpha_0 \theta_{KT} < \frac{\theta_{K1}}{\theta_{T1}}, \quad (127)$$

EWS-ratio vector exists in the subregion P3. Hence, we derive

$$\text{sign}[\frac{w_i^* - p_j^*}{P}] = \begin{bmatrix} + & - & + \\ + & - & + \end{bmatrix}. \quad (128)$$

What is the sufficient condition for(127)? If (127) holds, we derive

$$0<\frac{-W_{TL}}{W_{KL}};\ (129)$$

$$\alpha_0\theta_{KT}<\frac{\theta_{K1}}{\theta_{T1}}\ \leftrightarrow\ \frac{\alpha_{K0}'}{\alpha_{T0}'}\frac{\theta_K}{\theta_T}<\frac{\theta_{K1}}{\theta_{T1}}.\ (130)$$

(129) holds, because we assume (94). Use (125) to transform(130):

$$\frac{-(\lambda_{L1}a_{L1}*+\lambda_{L2}a_{L2}*)}{(\lambda_{T1}a_{T1}*+\lambda_{T2}a_{T2}*)}\frac{\theta_L}{\theta_T}<1+\frac{\theta_{K1}}{\theta_{T1}}\leftrightarrow\frac{-(\theta_1\theta_{L1}a_{L1}*+\theta_2\theta_{L2}a_{L2}*)}{(\theta_1\theta_{T1}a_{T1}*+\theta_2\theta_{T2}a_{T2}*)}<1+\frac{\theta_{K1}}{\theta_{T1}}.\ (131)$$

In sum, (131) and (129) hold, (127) holds. Hence, (128) holds. Eq. (131) seems useful for application.

Section 9. Estimating the position of EWS-ratio vector in case of P<0

In this section, I estimate the position of EWS-ratio vector. We assume P<0.

Subsection 9. 1. A sufficient condition for extreme factors to be economy-wide complements

We assume:

$$X<Z<Y\leftrightarrow w_T*<w_L*<w_K*.\ (132)$$

This assumption is plausible (see(33)). Hence,

$$(w_T*-w_L*, w_K*-w_L*)=(-,+),(w_T*-w_K*, w_L*-w_K*)=(-,-),$$
$$(w_K*-w_T*, w_L*-w_T*)=(+,+).\quad (133)$$

Substitute (133) in eq.(90),(91), and(92), we obtain

$$(a_{T0}', a_{K0}')\neq(-,+),(a_{T0}', a_{L0}')\neq(-,-),(a_{K0}', a_{L0}')\neq(+,+).\ (134)$$

From the above,

$$(a_{T0}', a_{K0}', a_{L0}')\neq(-,+,+),(-,+,-).\ (135)$$

Hence, sign B and A are impossible. The sign C, D, E, and F are possible (see(93)). That is,

$$(a_{T0}', a_{K0}', a_{L0}') = (+,+,-),(-,-,+),(+,-,+),(+,-,-)\,. \quad (136)$$

Because we assume(132), we derive

$$(\frac{-W_{TL}}{W_{KL}}, \frac{\theta_L}{\theta_K}\frac{-W_{LT}}{W_{KT}}) = (\frac{-(\mathrm{w_T}^* - \mathrm{w_L}^*)}{(\mathrm{w_K}^* - \mathrm{w_L}^*)}, \frac{\theta_L}{\theta_K}\frac{-(\mathrm{w_L}^* - \mathrm{w_T}^*)}{(\mathrm{w_K}^* - \mathrm{w_T}^*)}) = (+,-)\,.(137)$$

Hence, point A is in quad. IV. And we derive the results as follows.

(i)If sign E holds, we derive $(\alpha_0\theta_{KT}, \beta_0) = (-,-)$, point B is in quad. III. (138)
(ii)If sign F holds, we derive $(\alpha_0\theta_{KT}, \beta_0) = (-,+)$, point B is in quad. II. (139)
(iii)If sign C holds, we derive $(\alpha_0\theta_{KT}, \beta_0) = (+,-)$, point B is in quad. IV. (140)
(iv)If sign D holds, we derive $(\alpha_0\theta_{KT}, \beta_0) = (+,-)$, point B is in quad. IV.(141)

Hence, we can plot point A and B in the figure. If sign D or C holds, we have to show that which point is on the L.H.S., point A or B. We can analyze the distribution of point A and B by using eq.(90) as follows.

For example, we assume sign C holds, that is,

$$(a_{T0}', a_{K0}', a_{L0}') = (+,+,-)\,. \quad (142)$$

Recall(90), that is,

$$H_0 = (\mathrm{w_T}^* - \mathrm{w_L}^*)a_{T0}'\theta_T + (\mathrm{w_K}^* - \mathrm{w_L}^*)a_{K0}'\theta_K < 0. \quad (143)$$

Transform this:

$$H_0 = (\mathrm{w_K}^* - \mathrm{w_L}^*)a_{T0}'\theta_T\{\frac{(\mathrm{w_T}^* - \mathrm{w_L}^*)}{(\mathrm{w_K}^* - \mathrm{w_L}^*)} + \frac{a_{K0}'\theta_K}{a_{T0}'\theta_T}\}$$

$$= \mathrm{W_{KL}}\, a_{T0}'\theta_T\{\frac{\mathrm{W_{TL}}}{\mathrm{W_{KL}}} + \alpha_0\theta_{KT}\} < 0\,. \quad (144)$$

We have $\mathrm{W_{KL}} = (+), \mathrm{a_{T0}}' = (+)$. Hence, we derive

$$\alpha_0\theta_{KT} < \frac{-W_{TL}}{W_{KL}}. \quad (145)$$

Hence, point A is on the R.H.S. of point B (see(71)). From (137) and(140), both of point A and B exist in quadrant IV. Hence, the point of (S',U') exists on the line-segment AB, which exists in quad. IV.

From the above, if we assume sign C holds, the point of (S',U') satisfies as follows:

$$\alpha_0\theta_{KT} < S' < \frac{-W_{TL}}{W_{KL}}, \beta_0 > \mathrm{U'} > \frac{\theta_L}{\theta_K}\frac{-W_{LT}}{W_{KT}}. \quad (146)$$

Hence, we derive

$$(S',U') = (+,-) \leftrightarrow \ (S,\mathrm{T},U) = (g_{LK}, g_{LT}, g_{KT}) = (+,+,-). \quad (147)$$

This implies that factors T and K are economy-wide complements. In sum, the following result has been established.

Theorem 3. If we assume

$$\mathrm{P}<0, X < Z < Y \leftrightarrow \mathrm{w_T}^* < \mathrm{w_L}^* < \mathrm{w_K}^*, (a_{T0}', a_{K0}', a_{L0}') = (+,+,-), \quad (148)$$

we derive

$$\alpha_0\theta_{KT} < S' < \frac{-W_{TL}}{W_{KL}}, \beta_0 > \mathrm{U'} > \frac{\theta_L}{\theta_K}\frac{-W_{LT}}{W_{KT}}, \quad (149)$$

$$(S',U') = (+,-) \leftrightarrow \ (S,\mathrm{T},U) = (g_{LK}, g_{LT}, g_{KT}) = (+,+,-). \quad (150)$$

This implies that extreme factors are economy-wide complements.

On the other hand, we can show that if sign D holds, point A is on the L.H.S. of point B. We can do the similar analysis. But I omit.

Subsection 9.2. A sufficient condition for a certain Stolper-Samuelson sign pattern to hold

We assume (148) holds, hence, (149)and(150) hold. We can analyze similarly to Subsection 8.2. I show only one example for a certain Stolper-Samuelson sign pattern to hold. For the analysis shown below, compare (149) with the results in Nakada (2015, example 2 in section 11).

(i)Example 1.

If we assume

$$0 < \alpha_0\theta_{KT} < S' < \frac{-W_{TL}}{W_{KL}} < \frac{\theta_{K1}}{\theta_{T1}}, \quad (151)$$

EWS-ratio vector exists in the subregion P3. Hence, we derive

$$\text{sign}[\frac{w_i^* - p_j^*}{P}] = \begin{bmatrix} + & - & + \\ + & - & + \end{bmatrix}. \quad (152)$$

What is the sufficient condition for(151)? If (151) holds, we have

$$0 < \alpha_0\theta_{KT}; \quad (153)$$

$$\frac{-W_{TL}}{W_{KL}} < \frac{\theta_{K1}}{\theta_{T1}} \leftrightarrow w_L^* - p_1^* = (-) < 0. \quad (154)$$

(153) holds, because we assume (142). In sum, if (154) holds, (151) holds. Hence, (152) holds.

Section 10. Conclusion

I assumed a certain pattern of factor intensity ranking, including that of middle factor. In general, I can estimate the position of EWS-ratio vector, (S', U') to some extent. It is because EWS-ratio vector exists on the line-segment AB (or EWS-ratio vector line-segment). Therefore, if we know the position of point A and B, we can estimate the position of EWS-ratio vector. Point A and B are the intersection points of EWS-ratio vector line and EWS-ratio vector boundary. Hence, we can know which Stolper-Samuelson sign pattern holds to some extent.

In other words, if the data available satisfies a certain condition, we can estimate the position of EWS-ratio vector to some extent. I have deepened the analysis in Nakada (2015).

Especially, I have derived a sufficient condition for extreme factors to be economy-wide complements, which implies $(S', U') = (+, -)$. In this case, we can derive 3 patterns of Stolper-Samuelson sign patterns. Next, I have derived a sufficient condition for a certain Stolper-Samuelson sign pattern to hold. These results suggest that if we have the appropriate data available, we can specify only one Stolper-Samuelson sign pattern.

Of course, we can apply these results. In order to apply, we need the data about the change in some variables, which requires the data on 2 time-points. That is, the change in relative price of a commodity, in real factor price measured by both goods, and in input-output coefficient. On the other hand, normal CGE (or computable general equilibrium) analysis only needs the data only on 1 time-point in order to estimate the value of basic parameters.

This article suggests as follows. Sometimes it is not plausible to assume the functional form of production functions, such as Cobb-Douglas, or all-constant CES in each sector. It is because they do not allow any 2 factors to be Allen-complements. It is because extreme factors can be economy-wide complements.

References:

Ban, H., 2007. Capital-skill complementarity, factor intensity and relative factor prices: a model with three factors and two goods. Kobe-gakuin economic papers 39, 101-122 (in Japanese).

Ban, H., 2008. Capital-skill complementarity and the Stolper-Samuelson effect: an analysis by a three-factor two-commodity model. Kobe-gakuin economic papers 40, 1-17(in Japanese).

Batra, R.N. and Casas F.R., 1976. A synthesis of the Heckscher-Ohlin and the neoclassical models of international trade. Journal of International Economics 6, 21-38.

Bliss, C, 2003. A Specific-Factors Model with Historical Application. Review of International Economics 11, 268–278.

Easton, S, T., 2008. Something new for something old: Reflections on a model with three factors and two goods, in: S. Marjit, E.S.H. Yu, eds., Contemporary and emerging issues in trade theory and policy, Emerald group Ltd., Bingley.

Ide, T., 2009. The two commodities and three factors model with increasing returns to scale technology: another interpretation of the Leontief paradox. Fukuoka University Review of Economics 53, 165-179.

Jones, R.W. and Easton S.T., 1983. Factor intensities and factor substitution in general equilibrium. Journal of International Economics 15, 65-99.

Nakada, Y., 2015. Economy-wide substitution and Rybczynski sign pattern in a three-factor two-good model. Obtainable on ResearchGate.

Ruffin, R.J., 1981. Trade and factor movements with three factors and two goods. Economics Letters, 7, pp. 177–182.

Samuelson, P.A., 1953. Foundations of economic analysis. Harvard University Press.

Samuelson, P.A., 1983. Foundations of economic analysis, enlarged edition. Harvard University Press. [Keizai Bunseki no Kiso, Zoho ban. Keiso-shobo, Tokyo, (translated by Sato Ryuzo in 1986 in Japanese)]

Sato, R. and Koizumi, T., 1973. On the Elasticities of Substitution and Complementarity. Oxford

Economic Papers, 25, 44-56.

Suzuki, K., 1983. A synthesis of the Heckscher-Ohlin and the neoclassical models of international trade: a comment. Journal of International Economics 14, 141-4.

Suzuki, K., 1987. Boeki to Shigen Haibun [Trade and Resource Allocation]. Yuhikaku, Tokyo (in Japanese).

Takayama, A., 1982. On Theorems of General Competitive Equilibrium of Production and Trade—A Survey of Some Recent Developments in the Theory of International Trade. *Keio Economic Studies* **19,** 1–37.

Teramachi, N., 1993. The Structure of Production with Two Goods and Three Factors. The bulletin of the Institute for World Affairs and Cultures, Kyoto Sangyo University 13, 37-71 (in Japanese).

Thompson, H., 1985. Complementarity in a simple general equilibrium production model. Canadian Journal of Economics 17, 616-621.

Thompson, H., 1995. Factor Intensity versus Factor Substitution in a Specified General Equilibrium Model. Journal of economic integration 10, 283-297.

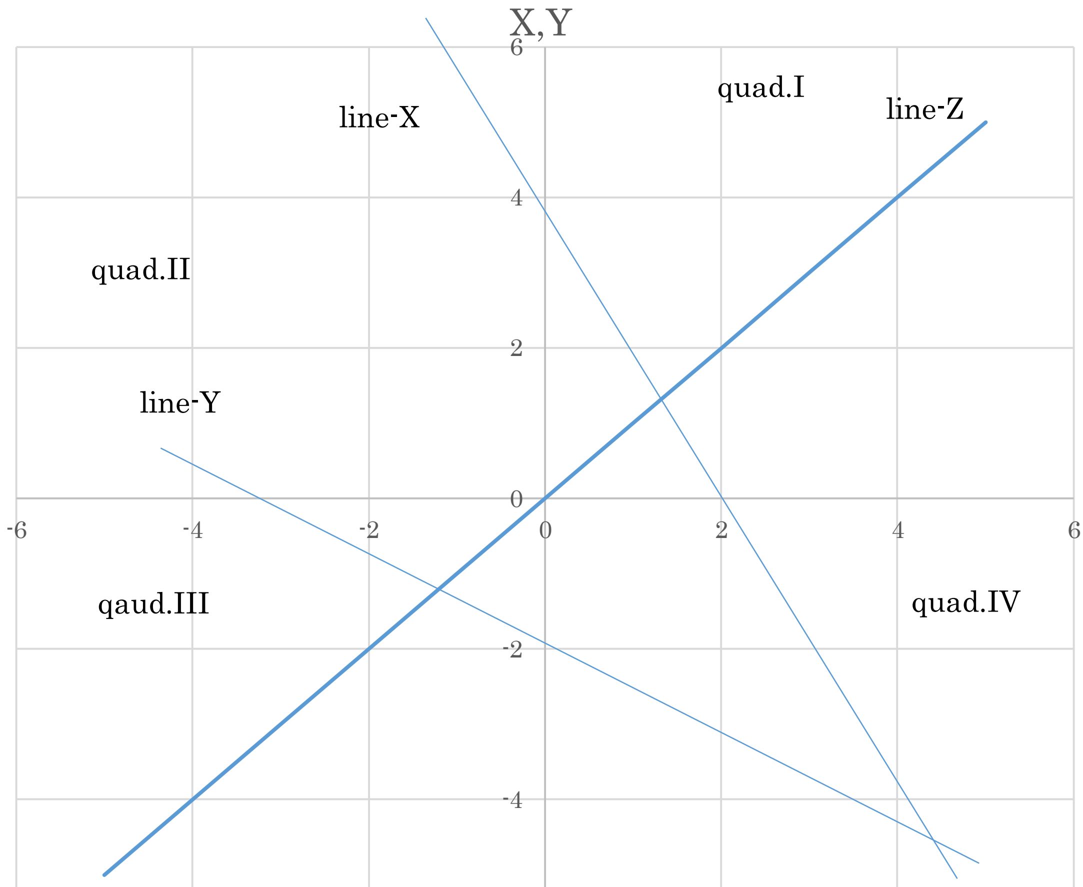

Fig.1. Distribution of line-X, -Y, -Z (the change in real factor price)

Note: (X, Y, Z)=(wT1*, wK1*, wL1*)

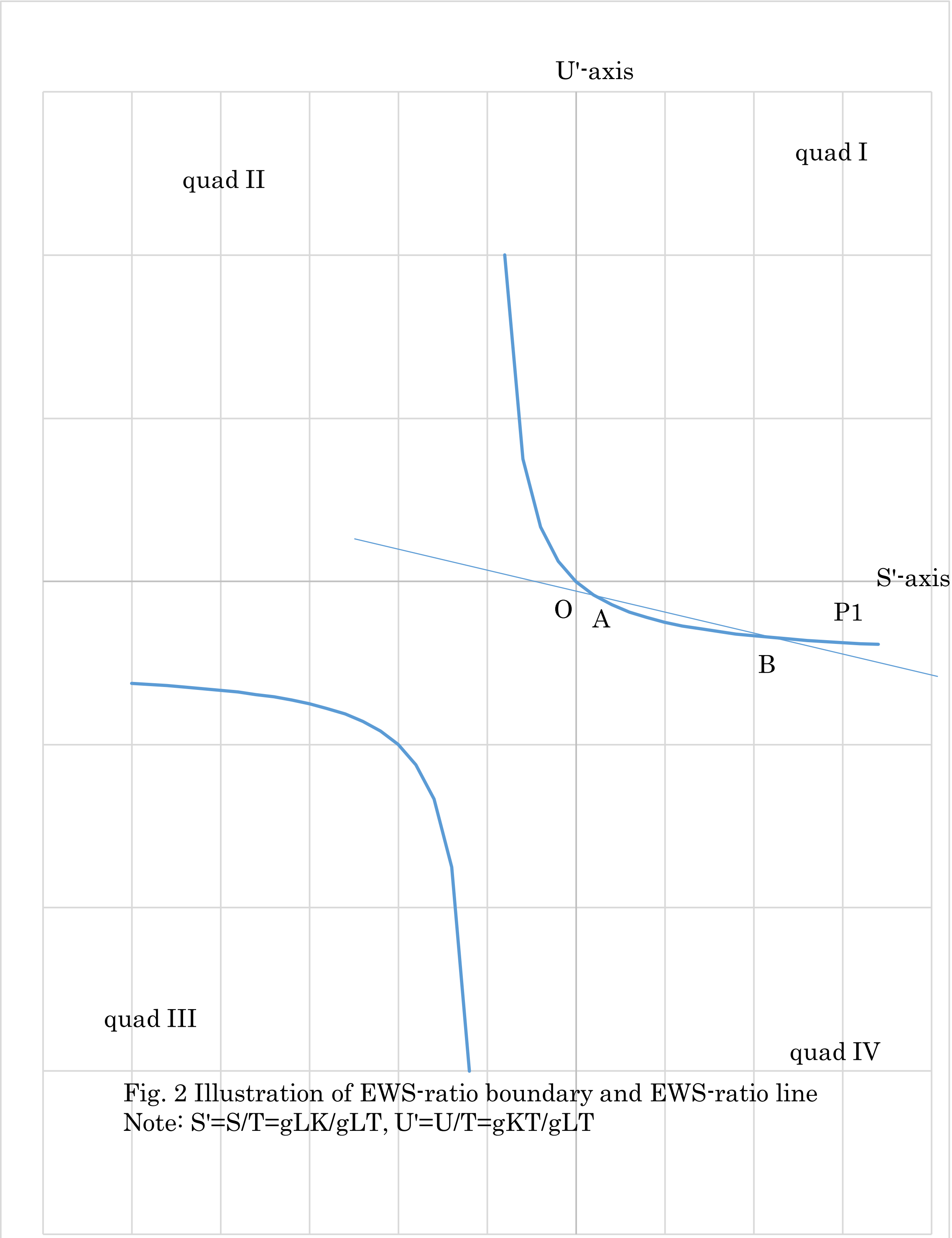

Fig. 2 Illustration of EWS-ratio boundary and EWS-ratio line
Note: S'=S/T=gLK/gLT, U'=U/T=gKT/gLT

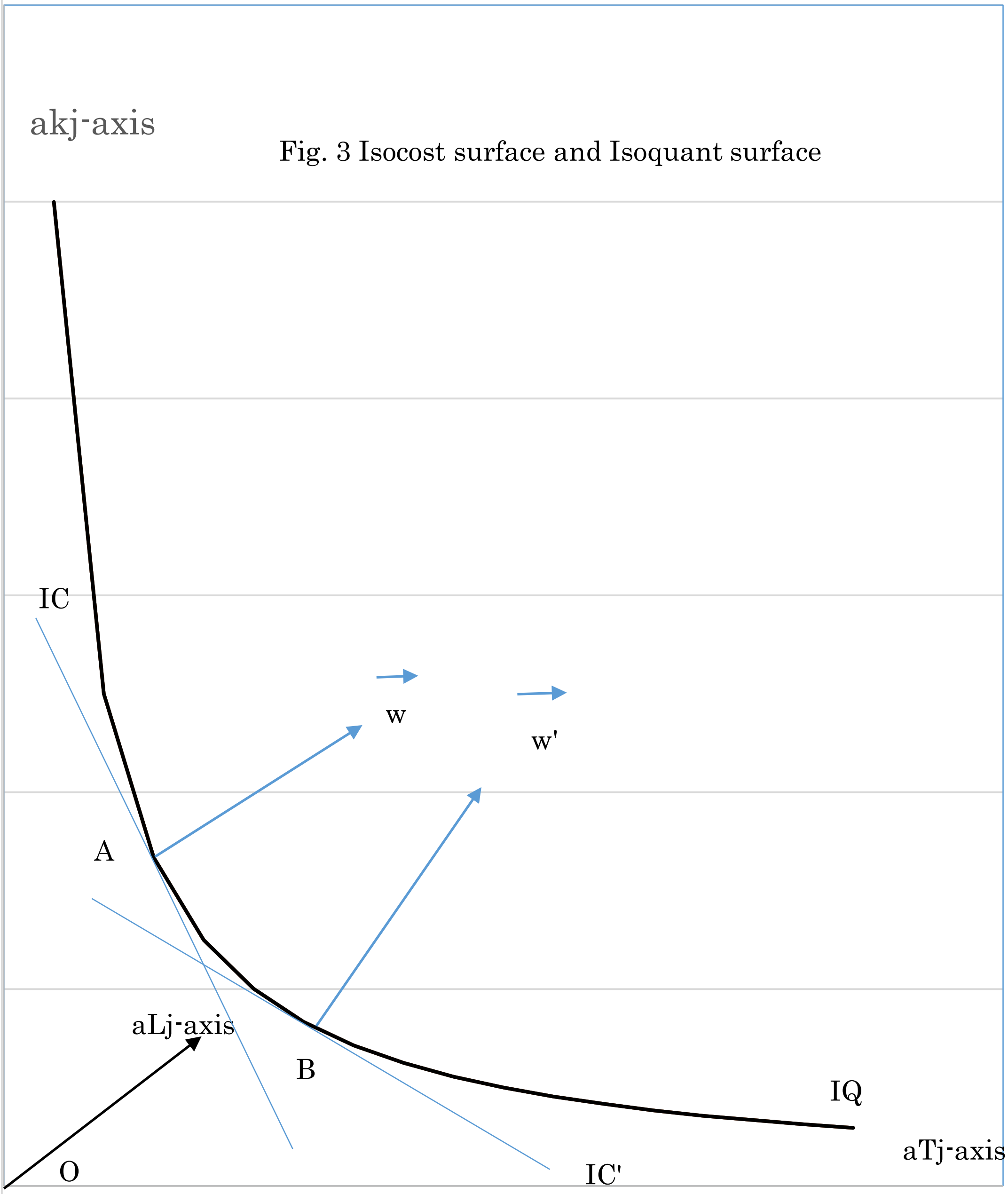

Fig. 3 Isocost surface and Isoquant surface