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Abstract

What if an unsustainable economy decides to switch in finite time to a sustainable path of a nonrenewable resource extraction which is optimal with respect to some criterion? We consider this problem on the example of the Dasgupta-Heal-Solow-Stiglitz model (DHSS) using constant consumption over time as a criterion. It turns out that if the criterion has no connections with the "opportunities" of the economy (initial conditions) then the resulting "optimal" path of consumption can be inferior to the one along some sub-optimal sustainable paths of extraction calibrated on the original initial conditions. In our case we have obtained under the standard Hartwick Rule bounded and unbounded growth of consumption along these sub-optimal paths.

- Keywords: Essential nonrenewable resource; Sustainable extraction; Criterion inconsistency; Hartwick Rule
- JEL classification: Q32, Q38

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1 Introduction

The Hartwick Investment Rule (Hartwick, 1977) for the Dasgupta-Heal-Solow-Stiglitz (DHSS) model (Dasgupta and Heal, 1974; Solow, 1974; Stiglitz, 1974) implies constant consumption over time. Solow (1974) used the constant-consumption criterion as a result of application of the maximin (Rawls, 1971) to the question of just intertemporal allocation of an essential nonrenewable resource. Constant consumption in this model is obtained under the Hotelling Rule as a condition of efficient extraction. This condition for our model implies that the rate of extraction $r(t)$ must be always declining including the starting point ($\dot{r}(0) < 0$). Moreover, the value of $\dot{r}(0)$ is strictly defined by the initial rate of extraction $r(0)$, amount of reserve s_0 and technological parameters of the economy.

But what if the economy's technology and (or) the initial conditions are not compatible with the requirements of the criterion which the economy decided to use?[1] For example, if the elasticity of factor substitution is less than unity then the economy with a nonrenewable resource will collapse regardless any efforts in saving (Dasgupta and Heal, 1979) and therefore this economy is not compatible with the criteria implying nondecreasing consumption. The unit-elasticity Cobb-Douglas economy can exhibit various patterns of declin-

[1] We assume here that stickiness of both the extraction and saving prevents the economy from changing the initial conditions in discontinuous way.

ing, growing, and constant per capita consumption depending on the paths of saving and extraction and on the initial conditions. Therefore it is natural to expect that some plausible criterion implying a sustainable path of consumption can be "inferior" to this economy combined with the initial conditions if this criterion is not "linked" parametrically to the potential opportunities of the economy which are expressed in technological parameters and the initial conditions.

We assume here that the economy is non-sustainable at the initial moment due to some externalities implying the modified Hotelling Rule and unsustainable pattern of extraction ($\dot{r}(0) > 0$). In general case the economic non-sustainability can be two-dimensional when an economy in addition to unsustainable extraction follows unsustainable pattern of saving. For simplicity we assume that our economy invests in the optimal with respect to our criterion way, namely it follows the Hartwick Saving Rule.[2]

In our numerical examples we have shown that the consumption along some sub-optimal sustainable paths of extraction linked to the initial condi-

[2] There is empirical evidence (e.g. Pearce and Atkinson 1993) that net investment, which takes into account natural capital, is around zero for some countries (Mexico, Philippines) and is mixed positive and negative for some others. Hamilton et al (2006) also obtained mixed result examining the satisfaction of the Hartwick Rule for 70 countries. Therefore our use of zero net investment (Hartwick Rule) has some justification. In a specific case, when investment behaviour is different from that for a sustainable path, the current pattern of saving must be also adjusted during the transition period. This is a separate complicated problem which also can not be solved immediately as was mentioned in (e.g. Kuznets 1946; Poterba 1988) because household saving behavior is slow to change despite changes of governments and government policy. We discussed in detail the question of sticky extraction and saving patterns in (Bazhanov 2007c).

tions can be superior to the one along the "optimal" path after the moment of switching with the initial conditions adjusted during the transition period. Namely, we have obtained bounded and unbounded growth of consumption (depending on the extraction path) under the standard Hartwick Rule what looks more attractive than the positive net saving (Hamilton et al, 2006) since it does not require decreasing of consumption for the present generation.

We describe the model in Section 2; discuss the nature of the extraction and saving stickiness (which imply the necessity of the transition period) in Section 3; consider different formulations of the problem of optimal adjustment of the initial conditions in finite time (Section 4); in Section 5 we describe the properties of the extraction paths which we use for the transition period; Section 6 provides theoretical result on impossibility of switching in finite time to the "optimal" path along the "optimal" transition path; Section 7 offers the opportunities of approximations for the problem of finite-time switching; Section 8 provides the numerical examples of approximate solutions for the finite-time switching; Section 9 considers an example of changing of saving rule which is necessary to compare correctly the patterns of consumption along different extraction paths; Section 10 concludes.

2 The model

We use the DHSS model with the Cobb-Douglas technology. For simplicity we consider the case with zero population growth[3], zero cost of extraction and technological progress compensating for capital depreciation. The last assumption allows to consider the basic DHSS model for the cases with a growing economy what is important for our numerical examples. Plausible patterns of technological progress compensating for capital depreciation were examined in (Bazhanov 2007b). All the paths in our economy such as output $q(t)$, consumption $c(t)$, produced capital $k(t)$ and so on are defined below in per capita units. For our case we have output $q = f(k, r) = k^\alpha r^\beta$ where r - current resource use, $r = -\dot{s}$, s - per capita resource stock ($\dot{s} = ds/dt$), α, $\beta \in (0, 1)$ are constants. Prices of capital and the resource are $f_k = \alpha q/k$ and $f_r = \beta q/r$ where $f_x = \partial f/\partial x$. Per capita consumption is $c = q - \dot{k}$. The Hartwick Saving Rule implies $c = q - rf_r$ or, substituting for f_r, we have $c = q(1-\beta)$, which means that instead of $\dot{c} = 0$ we can check $\dot{q} = 0$. Hence, the efficient path of extraction in our simple case can be derived from the standard Hotelling Rule $\dot{f}_r/f_r = f_k$ which implies $\alpha\beta q/k + \dot{r}(\beta - 1)/r = f_k =$

[3] In fact, numerous literature on sustainable development starting T. Malthus work in 1798 and some recent papers, e.g., (Brander 2007) consider the population growth as the main threat to sustainability. The debates on this problem are concentrating around the estimate of the constant which could be the limit to the population growth. Hence, we can assume that the population is already stabilized on this limit.

$\alpha q/k$ or

$$\dot{r}/r = -\alpha q/k. \tag{1}$$

Then

$$\dot{q}/q = \alpha\dot{k}/k + \beta\dot{r}/r = \beta(\alpha q/k + \dot{r}/r) = 0, \tag{2}$$

which means that we really have $\dot{q} = \dot{c} = 0$ or $q = const$. Then $rf_r = \beta q = const$ and we have $\dot{k} = \beta q = const$ for deriving $k(t)$ and (1) for deriving $r(t)$. We can find two constants of integration k_0 for $k(t) = k_0 + \beta qt$ and the constant of equation $\dot{r}/r = -1/\left(k_0/\alpha q + \beta t/\alpha\right)$ using initial conditions $r(0) = r_0$ and $s(0) = s_0$, where s_0 is the given resource stock which must be used for production over infinite time: $s_0 = \int_0^\infty r(t)dt$. Then we have

$$r(t) = r_0\left[1 + r_0\beta t/s_0(\alpha - \beta)\right]^{-\alpha/\beta}, \tag{3}$$

where $\alpha > \beta$ (Solow condition) and

$$\dot{r}(t) = -\ddot{s}(t) = -\alpha r_0^2/s_0(\alpha - \beta)\left[1 + r_0\beta t/s_0(\alpha - \beta)\right]^{-(\alpha+\beta)/\beta}. \tag{4}$$

Since we assume that our economy depends on the resource essentially, the path $r(t)$ (Hartwick curve (3)), asymptotically approaches zero (dotted line on Fig. 1 a) is $R_{Hart}(t)-$ in absolute units) and the path of extraction changes $\dot{r}(t)$ (or negative acceleration of stock $s(t)$ diminishing, dotted line on Fig. 1 b)) also approaches zero, but starting from negative value

$$\dot{r}_0 = -\alpha r_0^2/s_0(\alpha - \beta). \tag{5}$$

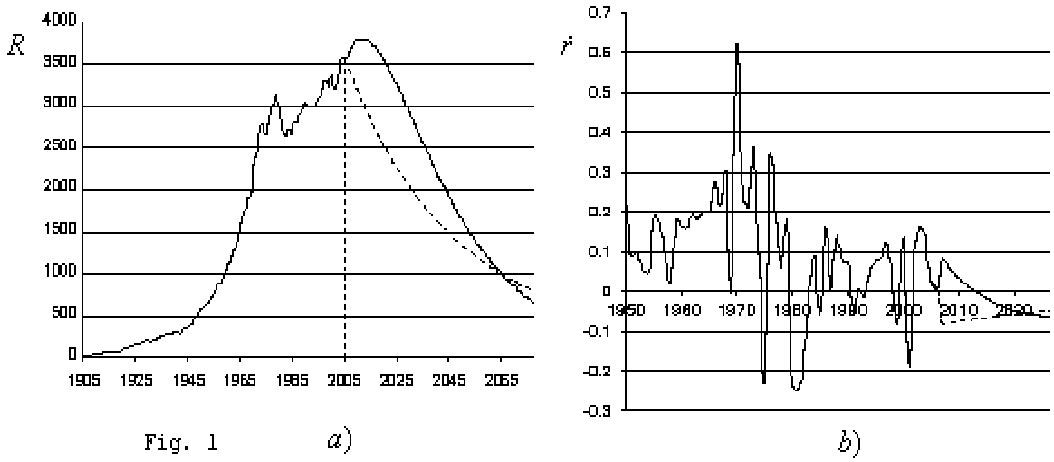

Figure 1: World oil a) extraction (mln t/year); b) per capita extraction accelerations: historical data (before 2006); Hartwick curve (dotted); transition curve (solid).

However, according to our assumptions about stickiness of extraction and saving, we are not able to realize the efficient Hartwick's curve at $t = 0$ and we must switch to the optimal path along some "smooth continuation" (solid line on Fig. 1 a) after 2006).[4] Our definitions in the next section reflect these restrictions.

[4] In the current paper we assume that there exists a path of tax (or policy which can be expressed in terms of tax) which can influence the rate of extraction in a corresponding way. So we will concentrate on some normative and technical problems which can arise during the switching in finite time to the path with desirable properties. There is literature on the design of government interventions for realizing sustainable resource use via price changes. For example, Karp and Livernois (1992) obtained the tax which brings the monopolist extraction to an efficient path . A review on regulation under asymmetric information is in (Caillaud et al 1988) and a recent review of sustainability and environmental policies can be found in (Pezzey 2002). Government can influence the extraction activity directly (using regulations as in (Davis and Cairns 1999) or affecting the households' demand with environmental policy as in (Grimaud and Rouge 2005; Pezzey 2002).

3 Feasibility, efficiency, and restrictions on $\dot{r}$

The constant per capita consumption over time in our case is the result of

1) total investment of oil rent in capital ($\dot{k} = rf_r$) and

2) fulfillment of the standard Hotelling Rule ($\dot{f}_r/f_r = f_k$).

In our case the Hotelling Rule is modified by some externalities at the initial point, namely $\dot{f}_r/f_r = f_k + \tau(t)$, where $\tau(0) \neq 0$. These externalities cause inefficient extraction in terms of the standard Hotelling Rule[5] or we will say that the path of extraction is "$\tau-$*zero inefficient*". Technical and political restrictions prevent us from starting the extraction using the "$\tau-$zero efficient" path and so we must find the optimal path in the first (transition) period among $\tau-$zero inefficient curves.[6] We set down these assumptions below in the definitions 1 and 2 and the Propositions 1 and 2.

Definition 1 An *intertemporal program* $\langle f(t), c(t), k(t), r(t)\rangle_{t=0}^{\infty}$ is a set of paths $f(t)$, $c(t)$, $k(t)$, $r(t)$, $t \geq 0$ such that $f(t) = f[k(t), r(t)]$ and $c(t) = f(t) - \dot{k}(t)$.

We use below the notation $(x_1, \ldots x_n) \gg 0$ if $x_i > 0$ for all $i = \overline{1,n}$.

Definition 2 For positive initial stock of capital and resource $(k_0, s_0) \gg 0$ the set of the programs $F = \{\langle f(t), c(t), k(t), r(t)\rangle_{t=0}^{\infty}\}$ is *a feasible sheaf* at

[5] The most recent analysis of the reasons of distortion in the Hotelling Rule in its original form and alternative formulations of the Rule, reconciling it with the patterns of price and extraction, can be found in (Gaudet 2007).

[6] For simplicity we will omit below the expression "$\tau-$zero" assuming that "efficiency" means satisfaction of the standard Hotelling Rule.

$t = 0$ and each of the paths $f(t)$, $c(t)$, $k(t)$, $r(t)$ is *a feasible path* if any program $\langle f(t), c(t), k(t), r(t)\rangle_{t=0}^{\infty}$ from F for all $t \geq 0$ satisfies the conditions:

1) $(f(t), c(t), k(t), r(t)) \gg 0$;

2) $r(t), k(t), c(t)$ are continuously differentiable and $\sup_t |\dot{r}(t)| \leq \dot{r}_{\max} < \infty$;

3) $f(t)$ is twice continuously differentiable;

4) $\int_t^{\infty} r(t)dt \leq s(t)$;

5) $k(0) = k_0, c(0) = c_0, r(0) = r_0, \dot{r}(0) = \dot{r}_0 \leq \dot{r}_{\max}$.

Definitions 1 and 2 are based on the definition of the interior feasible path in (Asheim et al, 2007). The differences reflect our assumptions: a) population is constant; b) the speed of change of the extraction rate $\dot{r}$ is limited and continuous for all t including $t = 0$. Henceforth, a "program" and a "path" will refer to a feasible program and a feasible path.

We use below the notions of intertemporally inefficient and efficient programs which are introduced in (Dasgupta and Heal 1979 p 214). We denote the set of the efficient programs as E.

Proposition 1 *If* $\dot{f}_r(0)/f_r(0) \neq f_k(0)$ *then* $F \cap E = \emptyset$ *or all the feasible paths are inefficient.*

Proof is in (Bazhanov 2007d).

Now we will show that in our assumptions (zero extraction cost) all the growing paths of extraction are inefficient.

Proposition 2 *For an economy with technology $q = k^{\alpha}r^{\beta}$ where $\alpha, \beta \in (0,1)$; $k(t), r(t) > 0$ and $\dot{k}(t) < q(t)$ for all t, the path of extraction is inefficient if there is $\bar{t} \geq 0$ such that $\dot{r}(\bar{t}) > 0$.*

Proof is in (Bazhanov 2007d).

According to our formulation of the problem and the definition of the feasible paths, we have the restriction on changes in extraction: $\sup_t |\dot{r}(t)| \leq \dot{r}_{\max} < \infty$. This condition means that the extraction can be reduced without losing consumption only with the rate not exceeding $\dot{r}_{\max}$ which is defined by the rate of introducing the substitute technology. It is interesting to examine the behavior of the model depending on the specific functions introducing substitute technology. However we think that this problem needs special careful consideration in a separate paper.

For the purpose of the current paper we assume that the feasible dynamics of introducing substitute technology affects the path of extraction only at the point with maximum $|\dot{r}|$. Therefore for the numerical examples below it is enough to estimate $\dot{r}_{\max}$ from historical data (Fig. 1 b)).

The methodology of estimation of $\dot{r}(t)$ for historical data is described in (Bazhanov 2006b). It is shown, that there is empirical evidence that the Hamilton variation principle holds in economics of nonrenewable resources. Then the changes of the rates of extraction can be estimated as follows: $\dot{r}_i = 2[s_i - r_i(t_{i+1} - t_i) - s_{i+1}]/(t_{i+1} - t_i)^2$ where $s_i, s_{i+1}-$ reserves at t_i and

t_{i+1}; r_i- rate of extraction at t_i; $[t_i, t_{i+1})$ - the period when the sum of all the reasons influencing the resource extraction can be considered as a constant. The reserve at initial point of extraction was considered as the sum of the final historical reserve estimate and the sum of all historical extractions. Since acceleration is proportional to the generalized force (reason of changes), the values of $\dot{r}_i$ can be considered as the indices of the resource market. We can add to the results in (Bazhanov 2006b) that a coefficient of proportionality (inertia coefficient or coefficient of stickiness) between force and acceleration can be obtained from the Hotelling Rule. In our case with zero extraction cost (resource price equals to per unit rent) it has the form of $\dot{p}(t) = p(t)\rho$ where $p(t)$ - current price and ρ - interest rate. It can be rewritten as $[dp/dr] \cdot [dr/dt] = -[dp/dr]\,\ddot{s} = p(t)\rho$. The reason of extraction (force) here is rent $p(t)$ which is expressed via the acceleration of the resource extraction $\ddot{s}$ as follows: $p(t) = m(t)\ddot{s}$. This expression is equivalent to the Newton's second law which is the corollary of the Hamilton variation principle. The stickiness or inertia coefficient is $m(t) = [-dp/dr]/\rho$ or it can be also expressed via the price elasticity: $m(t) = p(t)/[\varepsilon_r(p)r(t)\rho]$. The less is the marginal price $-dp/dr$ (with ρ fixed) the less must be an effort (e.g., tax imposed on p) in order to obtain the unity change in $\ddot{s}$.

We turn to estimation of $\dot{r}_{\max}$ for our numerical examples. Note that $\dot{r}$ oscillated around 0.2 before 1980 (Fig. 1 b)). As a result of energy crises in 1973 and 1979-80 it was a period of introducing new technologies. Then after 1980 per capita accelerations oscillated already around zero. But these energy crises followed by declines in output and consumption. Hence, since we consider the problem of switching to sustainable path without losing consumption we can take as a reasonable estimate for our simplified economy $\dot{r}_{\max} = 0.1$.

4 Formulations of the transition problem

For the economy $q = k^{\alpha} r^{\beta}$ given the initial reserve $s(0) = s_0$ and the initial conditions r_0, $\dot{r}_0$, $\dot{q}_0/q_0$ (which imply the expressions for k_0 (Section 8), q_0, and $\dot{k}_0 = \beta q_0$) we are going to find among the feasible paths (definition 2, Section 3) such a path of extraction $r(t)$ and such a finite moment of time $\widetilde{t}$ that $r(0) = r_0$, $\dot{r}(0) = \dot{r}_0$, and

$$\dot{r}(\widetilde{t}) = -\alpha r^2(\widetilde{t}) / \left[s(\widetilde{t})(\alpha - \beta) \right]$$

In other words a finite moment of time $\widetilde{t}$ must be such that the change of the rate of extraction along the transition path coincides with the initial change of the rate of extraction for the Hartwick's curve given the current state as the initial. Besides these conditions we can require the optimality of the transition path $r(t)$ with respect to some criterion consistent with our main goal - constant consumption over time.

For example, we could formulate the problem of optimality of the transition path in the following way: find $\widetilde{t}$ and $r(t)$ such that

$$\int_0^{\widetilde{t}} U\left[c(r(t))\right] dt \to \max_{\widetilde{t}, r(t)}$$

where $U(\cdot)$ is monotonically nondecreasing. However this approach implies $\widetilde{t} \to \infty$. If we consider the problem with a fixed $\widetilde{t}$ then this criterion will be a "dictatorship of the present" (Chichilnisky, 1996) and it will imply that almost all the resource reserve must be extracted in period $\left[0, \widetilde{t}\right]$ in order to maximize consumption in this period.

We can reformulate this criterion making use of the fact that our goal is the constant consumption over time. Then we can require that the path of consumption during the transition period must be as close to a constant as possible. Using the least-square approach it can be formulated as follows

$$\int_0^{\widetilde{t}} \left[c(r(t)) - \overline{c}(\widetilde{t}, r(t))\right]^2 dt \to \min_{\widetilde{t}, r(t)}$$

where $\overline{c}(\widetilde{t}, r(t))$ is the constant path of consumption after the moment of switching $\widetilde{t}$. The level $\overline{c}$ must depend on the rest of reserve $s(\widetilde{t})$ and so it depends on the pattern of extraction in previous period $r(t)$. However this problem has no solution in continuous functions because the value of the criterion will be infinitely approaching zero with the path of consumption (and corresponding path of extraction) approaching discontinuous function

$$c(t) = \begin{cases} c_0, & t = 0, \\ \bar{c}, & t > 0. \end{cases}$$

which is not feasible in our problem.

We will obtain the same result if we remember that the reason of the "wrong" behaviour of extraction and consumption in our case is the distortion $\tau(t)$ in the Hotelling Rule by some externality. In this case the Hotelling Rule is $\dot{f}_r/f_r = f_k + \tau(t)$. Requiring

$$\int_0^{\widetilde{t}} \left[\tau(t, r(t))\right]^2 dt \longrightarrow \min_{\widetilde{t}, r(t)}$$

we will also obtain the solution as a discontinuous (unfeasible) function

$$\tau(t) = \begin{cases} \tau_0, & t = 0, \\ 0, & t > 0. \end{cases}$$

Another approach is to use our stickiness argument in order to justify the restriction on the change of accelerations of the extraction, namely, require that $|\ddot{r}| \le \ddot{r}_{max}$. Then the problem of transition to the path with desirable properties in minimal time is

$$\begin{aligned} \widetilde{t}(r) &\longrightarrow \min_{r(t)} \\ s.t.\ r(0) &= r_0;\ \dot{r}(0) = \dot{r}_0; |\ddot{r}| \le \ddot{r}_{max}, \end{aligned}$$

where $\widetilde{t}$ is the minimal positive solution of equation $\dot{r}(\widetilde{t}) = -\alpha r^2(\widetilde{t}) / \left[s(\widetilde{t})(\alpha - \beta)\right]$. In this case the problems which have no solutions in continuous functions will have corner solutions. For example, in the last problem we will have the solution with $\ddot{r}(t) \equiv -\ddot{r}_{max}$ for all $t \in [0, \widetilde{t}]$, which implies $r(t) = -0.5\ddot{r}_{max}t^2 + \dot{r}_0 t + r_0$. However this path is not feasible ($r(t)$ must be positive for all $t \ge 0$) and so it can be used only for the transition period.

Despite these technical difficulties in formulation of the problem of optimal transition path, which can be compared with the optimal path after switching, we can find the answer on our main question[7] using some particular class of functions. These functions must be such that they can be used to describe the path of extraction with the specific initial conditions, they must be feasible, and they must allow for sustainable paths of consumption. Then if we show that for some functions from this particular class, calibrated on the initial conditions, the path of consumption can be superior to the one, which is optimal with respect to our criterion, then it will mean that our criterion is "inefficient" for our economy (the reverse in this case is not true because we use specific functions as sub-optimal solutions). As a particular class of functions satisfying these conditions we will consider the transition paths offered in (Bazhanov, 2007c).

5 Transition curves

For the transition period we can use the constant-consumption curve offered in (Bazhanov 2007c). This path is optimal among the transition curves with respect to our criterion. The transition path belongs to the same class of rational functions as the Hartwick curve (3). The difference is in the numerator, which in the expression for the changes of extraction rate $\dot{r}$ depends on t

[7] Namely, if the criterion which we use can be "inefficient" for our economy.

with a negative coefficient to control "smooth breaking" in the neighborhood of $t = 0$. Namely, $\dot{r}(t)$ has the form of

$$\dot{r}(t, b, c, d) = (\dot{r}_0 + bt)/(1 + ct)^d, \tag{6}$$

where $b < 0, c > 0, d > 1$ (for convergence $\dot{r}(t) \to -0$ with $t \to \infty$). We have $r_0 = r(0)$ to express b and then $r(t)$ has a dependence on c and d in

$$r(t) = r_0 \left(1 + b_r t\right) / (1 + ct)^{d-1} \tag{7}$$

where $b_r = c(d-1) + \dot{r}_0/r_0$. Coefficient c is expressed from the condition that resource is finite $s_0 = \int_0^\infty r(t)dt$:

$$c(d) = \left[r_0/(d-3) + \left\{r_0^2/(d-3)^2 + s_0\dot{r}_0/[(d-3)(d-2)]\right\}^{0.5}\right]/s_0. \tag{8}$$

Hence, we have a single independent parameter d which defines the shape of the curve (including its peak) and we can use this parameter as a control variable in some selected optimization problem

$$F\left[r(t, d)\right] \to \max_d$$

which can be connected with the short- or long-run policy in output or in consumption. The dependence of the long-run consumption on parameter d and technological parameters of the economy α and β is formulated in the following proposition.

Proposition 3 *If an economy with technology* $q = k^{\alpha}r^{\beta}$ *is such that* $\alpha, \beta \in (0,1)$; $\beta < \alpha$ *and*

1) resource rent is completely invested in capital;

2) there is no time lag between the moment of investment and the corresponding increase in capital;

3) rate of extraction $r(t)$ *is such that*

$$\dot{r}(t) = (\dot{r}_0 + bt)/(1 + ct)^d, \ b < 0, \ c > 0, \ d > 3,$$

where $b = b(r_0), c = c(s_0)$, *then consumption* $c(t)$ *is*

1) asymptotically decreasing if $d > \alpha/\beta + 2$;

2) asymptotically constant if $d = \alpha/\beta + 2$;

3) asymptotically growing if $3 < d < \alpha/\beta + 2$.

Proof of the Proposition is in (Bazhanov 2006a & 2007c). According to this proposition the transition path with $d = \alpha/\beta + 2$ is optimal among the transition paths with respect to the constant-consumption welfare criterion.

6 Switching to the "optimal" path

We define the moment of shifting to the second period $\widetilde{t}_0$ (the period of "efficient and optimal" extraction) as a solution of the "smooth switching" problem. Namely, the economy enters the optimal path when the change of rate of extraction (acceleration) $\dot{r}$ along the transition curve is equal to the initial acceleration of the optimal path which is being constructed at the each current moment. In our case the optimal curve (3) is being dynamically constructed with the use of "floating" initial conditions $\widetilde{r}_0(t), \widetilde{\dot{r}}_0(t), \widetilde{s}_0(t)$ which are being calculated along the transition path. Equations (5) and (6) for the accelerations imply that $\widetilde{t}_0$ must be a solution of the equation

$$(\dot{r}_0 + b\widetilde{t}_0)/(1 + c\widetilde{t}_0)^d = -\alpha r(\widetilde{t}_0)^2 / \left[\widetilde{s}_0(\widetilde{t}_0)(\alpha - \beta)\right] \tag{9}$$

where $r(\widetilde{t}_0) = \widetilde{r}_0(\widetilde{t}_0)$ is defined by equation (7) and the rest of resource $\widetilde{s}_0$ at $\widetilde{t}_0$ is $\widetilde{s}_0(\widetilde{t}_0) = s_0 - \int_0^{\widetilde{t}_0} r(t)dt$. Since our efficient curve (3) with the Hartwick Investment Rule gives us constant consumption over time, it is natural to construct the transition path (7) which is consistent with the same welfare criterion. Namely, according to Proposition 3, the curve (7) with $d = \alpha/\beta + 2$ implies asymptotically constant consumption. This path is optimal in the class of rational functions (7), e.g., with respect to the following criterion, consistent with constant consumption over time: $F(d) = \min_d \max_t |c_{\max} - c(t)|$, where $c_{\max}-$ asymptote for the path with asymptotically constant consumption. Indeed, for any $d_1$and d_3 such that $d_1 < d_2 = \alpha/\beta + 2 < d_3$ Proposition 3 implies that $F(d_1) = \infty > F(d_3) = c_{\max} \geq F(d_2) = c_{\max} - c_0$.

The following propositions show that the finite solution of equation (9) does not exist.

Proposition 4 *Equation (9) has real roots if and only if the value of d in*

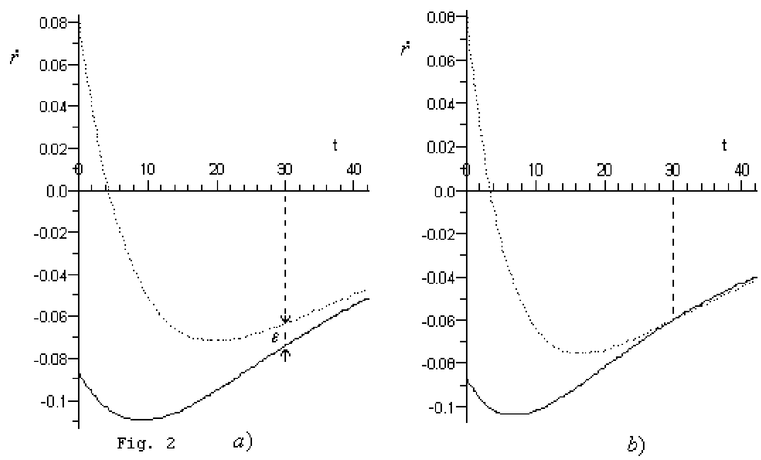

Figure 2: Changes in extraction rates for the transition curve (left hand side of equation (9), dotted line) with a) $d = \alpha/\beta + 2 = 8$; b) $\varepsilon-$optimal transition curve with $d = 5.875$; the solid line for both cases is a plot of the initial accelerations for the Hartwick's curve (right hand side of equation (9)), constructed along the transition path.

(7) is such that

$$d \leq \alpha/\beta + 2. \tag{10}$$

There are two real roots if inequality (10) is strict and one real root if it holds as an equality.

Proof (Appendix 1).

Proposition 5 *Equation (9) has only one real finite positive root if and only if* $d < \alpha/\beta + 2$.

Proof (Appendix 2).

Proposition 5 implies that the transition path with $d = \alpha/\beta + 2$, which is optimal with respect to our criterion, does not give us the opportunity to switch to the efficient path in finite time. As an illustration of Proposition 5 we consider an example with world oil extraction data. The accelerations of the transition path (7) with $d = \alpha/\beta + 2$ (left hand side of equation (9)) and dynamically constructed initial accelerations of the efficient curve (right hand side of equation (9)) are shown on Fig. 2 a). It can be seen that the residual of equation (9) approaches zero only asymptotically which means that our problem of "smooth switching" in finite time has no solution in this framework

7 Approximations of the "smooth switching" problem

In our numerical examples below we use $\dot{r}_0 = 0.08$ and as world oil reserves and extraction on January 1, 2007 (Oil & Gas J 2006, 104(47): 20-23.): $R_0 = 72,486.5$ [1,000 bbl/day] $\times 365 = 26,457,572$ [1,000 bbl/year] (or 3.6243 bln t/year); $S_0 = 1,317,447,415$ [1,000 bbl] (or 180.47 bln t). We use coefficient 1 ton of crude oil = 7.3 barrel.

We will consider two approaches to the approximate solution of the problem (9).

(a) "*$\varepsilon-$smooth switching*" which means that the economy will enter the efficient curve of extraction in a "regime shifting" way when the residual of equation (9) is small enough. For example, using our historical data estimate for $\dot{r}_{\max}$ we can define this moment as $\bar{t}_0$ such that

$$\begin{aligned} |\dot{r}_{trans} - \dot{r}_{Hart}| &= \left|(\dot{r}_0 + b\bar{t}_0)/(1+c\bar{t}_0)^d + \alpha r(\bar{t}_0)^2 / \left[\widetilde{s}_0(\bar{t}_0)(\alpha-\beta)\right]\right| \\ &\leq \varepsilon = 0.1\dot{r}_{\max} = 0.01. \end{aligned}$$

As an approximate solution of this problem we can take $\bar{t}_0 = 30$ (see Fig. 2 a)).

(b) *"$\varepsilon-$optimal transition curve"* which means that using the result of Proposition 5 the economy will follow some $\varepsilon-$optimal (with respect to constant consumption over time) transition curve with $d < \alpha/\beta + 2$ for which equation (9) has a single finite positive root. For the comparison between consumption paths in cases (a) and (b) we will take $d = 5.875$ (given $\alpha = 0.3$ and $\beta = 0.05$). For this value of d we have the same moment of switching $\bar{t}_0 = 30$ (see Fig. 2 b)). The difference between two cases is that in case (a) we must apply some "additional efforts" at the moment $\bar{t}_0$ to make discontinuous switch to the efficient path while in case (b) realization of the transition path with $d < \alpha/\beta + 2$ needs more efforts during all transition period (substitute technologies must be introduced faster).

Note that for $\alpha = 0.2$ and $\beta = 0.05$ estimated in (Nordhaus and Tobin 1972) the prospects for growth along the rational paths are less optimistic. The peak of oil extraction for the "borderline"-transition path with $d =$

$\alpha/\beta + 2$ must be closer which implies that the substitute technologies must be introduced faster, and the level of asymptote for consumption in the case with $\alpha = 0.2$ is less than for $\alpha = 0.3$ (see Bazhanov 2007a).

8 Consumption along the "approximate switching" scenarios of extraction

The Hartwick saving rule implies that the consumption path is $c = q - \dot{k}$ $= (1-\beta)q = (1-\beta)k^{\alpha}r^{\beta}$ where $r(t)$ is a known combination of the transition and efficient paths and $k(t)$ is an unknown path of capital. We can calculate $k(t)$ from the equation for the saving rule $\dot{k} = \beta k^{\alpha}r^{\beta}$ assuming that we have estimation of k_0. From (2) we have $\dot{q}/q = \beta(\alpha q/k + \dot{r}/r)$ which implies the expression for k_0, given $r_0, \dot{r}_0$, and output percent change $(\dot{q}/q)_0$:

$$k_0 = \left\{ \left[\left(\dot{q}/q \right)_0 / \beta - \dot{r}_0/r_0 \right] / \left(\alpha r_0^{\beta} \right) \right\}^{1/(\alpha-1)}. \tag{11}$$

Using $\left(\dot{q}/q\right)_0 = 0.04$ and estimates of $\dot{r}_0$, r_0 for world oil extraction we have $k_0 = 0.2810456$ and $c_0 = 0.692337$ which gives us the paths of consumption along the transition curves. In order to construct the consumption path along the Hartwick's curve (3) we must assume that we manage not only to change instantly acceleration of the extraction at the moment of switching $\bar{t}_0$ but also to stop the growth of our economy. The last requirement connected with condition $\dot{q} = 0$ along the Hartwick's curve including the initial point $\bar{t}_0$.

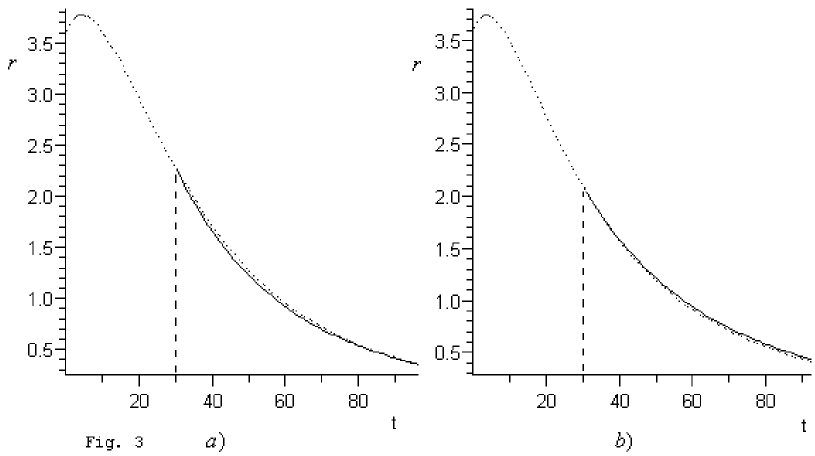

Figure 3: Switching the extraction from the transition curve (dotted line) to the efficient curve (solid line). Case (a): "$\varepsilon-$smooth switching"; case (b): switching from the "$\varepsilon-$optimal transition curve".

Note that at the moment of switching to the Hartwick's curve ($\bar{t}_0 = 30$) we have output growth at rate $(\dot{q}/q)_{\bar{t}_0} = 0.00886$ for the case (a) and $(\dot{q}/q)_{\bar{t}_0} = 0.00881$ for the case (b). Substituting for $\dot{r}_{0Hart} = -\alpha r_0(\bar{t}_0)^2/\left[\widetilde{s}_0(\bar{t}_0)(\alpha-\beta)\right]$ and $\dot{q} = 0$ into (11) we have the expression of capital in "different units": $k_0 = 30.47656$. Since physical capital is the same at this moment,[8] we must adjust its value using scale factor in order to obtain paths of consumption expressed in the "same units".

For our numerical example the process of switching from the transition path with $d = \alpha/\beta + 2$ to the efficient curve (case (a)) is depicted on the

[8] By the time $\bar{t}_0 = 30$ for our example the value of capital along the transition curve with $d = 5.875$ is $k(\bar{t}_0) = 1.8206$ and for the path with $d = 8$ it is 1.8251.

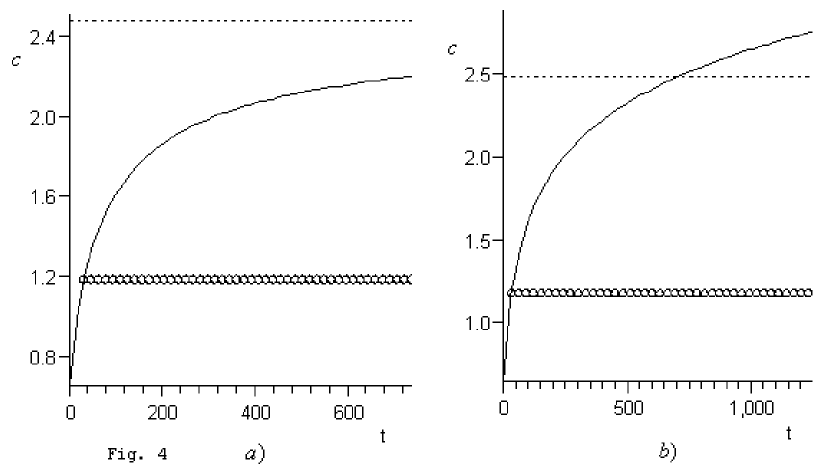

Figure 4: Consumption along the transition curve (solid) and the Hartwick's curve (circles) for switching a) in case (a); b) in case (b).

Fig. 3 a). Consumption paths are on the Fig. 4 a). The dash line is the limit ($c_{\max} = 2.480$) for the growth of consumption along the transition path.The process of switching from the $\varepsilon-$optimal transition path with $d < \alpha/\beta + 2$ to the efficient curve (case (b)) is on Fig. 3 b) and the corresponding consumption paths are on Fig. 4 b). Note that according to Proposition 3 consumption along the rational curve with $d < \alpha/\beta + 2$ grows with no limit which looks more attractive than opportunity of unbounded consumption obtained with positive net saving (Hamilton et al, 2006) because positive net saving requires sacrificing of consumption from the present generation while in our case the economy follows the standard Hartwick Rule. We can see on Fig. 4 b) that consumption along this path exceeds the limit for the path with $d = \alpha/\beta + 2$ (dash line).

Hence, in both cases (a) and (b) our attempts to switch to the efficient sustainable path of extraction gave us unexpected and seemingly paradoxical results. Consumption along the efficient path of extraction (circled lines on Fig. 4 a) and Fig. 4 b)) is inferior to the consumption along the inefficient transition path at all moments of time except the point of switching $\bar{t}_0 = 30$ where they are equal. At first glance the example contradicts the definition of inefficient curve (Dasgupta and Heal 1979 p 214) according to which everything must be exactly vice versa. However this definition works only for the feasible paths $r(t), k(t), c(t)$ which according to definition 2 must be continuously differentiable and $f(t)$ must be twice continuously differentiable. This implies the continuity of the output percent change $\dot{q}/q$ but in our "approximate solutions" we violated this requirement assuming that we manage to stop the growth of economy at the moment of switching to the efficient path. This violation explains also the big differences in consumption along the transition and the efficient paths (Fig. 4 a) and Fig. 4 b)) despite very small residual in extractions (Fig. 3 a) and Fig. 3 b)). So, if it is really possible to change the economy in a "regime shifting" manner as a result of some political actions or natural disaster, then we can not be sure that the continuation of the inefficient program from the "previous life" would be inferior to our efficient program which we have managed to realize.

For our economy with technology $q = k^{\alpha} r^{\beta}$ and the Hartwick Rule output can be only growing ($\dot{q} > 0$) for all t when $\dot{r} > 0$. This implies that for our model consumption must exhibit an infinite growth along the sustainable (in the weak sense) patterns of the resource use (limited growth as in case (a) along the transition curve or unlimited as in case (b)). Otherwise, if we discontinuously switch our economy into "different world" which is inferior with respect to future levels of consumption, the comparison of consumption along the paths from these "different worlds" will be incorrect.

In order to estimate the amount of consumption which we lose due to the inefficient extraction, we must compare correctly the consumption behavior along the transition and the efficient paths. To draw this comparison we will construct a saving rule for the transition path which implies $\dot{q}(\bar{t}) = 0$ at the moment of switching to the Hartwick's curve and which also is "close" asymptotically to the Hartwick rule. We will use this saving rule in the second period only as an artificial tool for correct comparison of the consumption along our paths. This means that the efficient Hartwick's curve will be used in the second period with another saving rule which will lead to the consumption behavior different from the constant over time. We consider this case in the following section.

9 Constant output at the moment of switching

Technical and political restrictions (definition 2) imply, that given $\dot{q}(0) > 0$, there is no saving rule which will give us $\dot{q}(t) \equiv 0$ for all t in the transition period including the moment of switching $\bar{t}$. Then we will construct a saving rule for which $\dot{q}(\bar{t}) = 0$ and $\dot{q}(t)$ has arbitrary (feasible) values at all other moments t in the transition period $t \in [0, \bar{t})$. Another requirement for this saving rule is that it must have a feasible continuation for the second period of efficient extraction ($t \in [\bar{t}, \infty)$) in order to draw the correct comparison of the consumption paths for this rule. Note that in our formulation we can not find this saving rule in the class of the rules with constant saving rates $\dot{k} = \delta q$ because we will obtain qualitatively the same behavior of consumption which will vary only in parameters. For example, for the transition path with $d = \alpha/\beta + 2$, which implies asymptotically constant consumption we will have different levels of asymptote $c_{\max}$ for different δ with monotonically growing consumption c and output q. So we will construct a feasible function $\delta(t)$ which gives us $\dot{q}(\bar{t}) = 0$. Since $\delta(t)$ has some level of arbitrariness, we can construct it in such a way that $q(t)$ is nonmonotonic in transition period and $\delta(t)$ asymptotically approaches β. Then our saving rule asymptotically approaches the Hartwick rule and the consumption paths will have to be asymptotically constant. Thus, for our numerical example we can find $\delta(t)$,

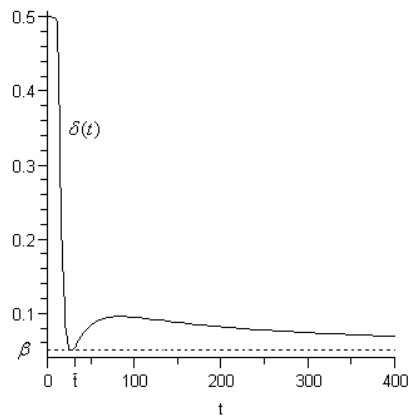

Figure 5: An example of saving rate $\delta(t)$ in the transition period (solid line) and the Hartwick saving rate (dotted line).

for example, in the following form (Fig. 5):

$$\delta(t) = \delta_0 - (\delta_0 - \beta) \exp\left[-\nu(t - t^*)^2/(1 + t)^3\right]$$

with parameters $\delta_0 = 0.5$, $\nu = 20$, and t^* defined from the condition $\dot{q}(\bar{t}) = 0$ using an iterative numerical procedure. The condition $\dot{q}(\bar{t}) = 0$ implies that the expression $\alpha k^{\alpha-1} r^{\beta} \dot{k} + \beta r^{\beta-1} k^{\alpha} \dot{r}$ or (substituting for $\dot{k} = \delta q$ and expressing k)

$$k(\bar{t}) - \left[-\beta \dot{r}(\bar{t}) / \left\{\alpha \delta(\bar{t}) r(\bar{t})^{\beta+1}\right\}\right]^{1/(\alpha-1)} \tag{12}$$

must be equal to zero. Then for defining t^* we can use the following procedure:

(1) set t_0^*, iterative parameter $i = 0$, and define $\delta^i(t) = \delta_0 - (\delta_0 -$

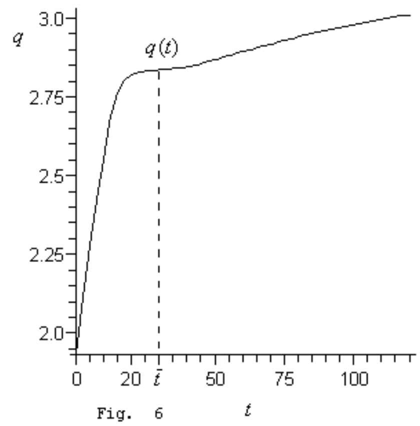

Figure 6: Output $q(t)$ along the transition curve (for $d = \alpha/\beta + 2$) with the saving rule $\dot{k} = \delta(t)q$; $\bar{t}$- the moment of switching to the efficient path.

$\beta) \exp\left[-\nu(t - t_i^*)^2/(1+t)^3\right]$;

(2) calculate (from equation 2)

$$k_0^i = \left\{\left[(\dot{q}/q)_0 - \beta\dot{r}_0/r_0\right] / \left[\delta^i(t)\alpha r_0^\beta\right]\right\}^{1/\alpha - 1}.$$

Note, that for $\delta^i(t) \equiv \beta$ this formula coincides with (11).

(3) given k_0^i solve differential equation $\dot{k} = \delta^i k^\alpha r^\beta$ for $k(t)$;

(4) if the expression in (12) is close enough to zero then $t^* = t_i^*$ and our saving rule is constructed; else change t_i^* to reduce the residual in (12), $i := i + 1$, and go to (2).

Since $\delta(t)$ can be chosen in such a way that output q is nonmonotonic along $\delta(t)$ (see Fig. 6) and since points with $\dot{q}(t) = 0$ depend on parameter t^*, it can be shown that the procedure converges. For our numerical example we

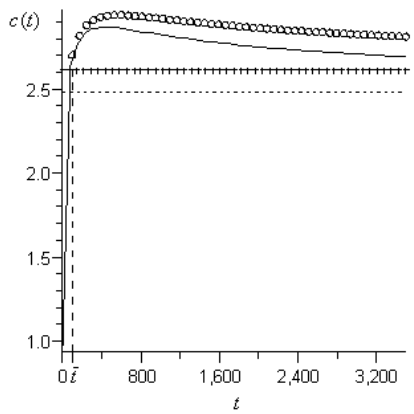

Fig. 7

Figure 7: Consumption with the saving rate $\delta(t)$ along the Hartwick's curve (circled) and the transition curve with $d = \alpha/\beta + 2$ (solid); the line in crosses is the asymptote for the Hartwick's path, dotted line - asymptote for the transition path.

obtained $t^* = \bar{t} - 3.48946$ which gave us the difference (12) equal to $7.8 \cdot 10^{-7}$.

Now, given the saving rate $\delta(t)$ which implies $\dot{q}(\bar{t}) = 0$, we can correctly switch at $\bar{t}$ to the efficient Hartwick's curve and compare the levels of consumption (Fig. 7). Note that when $\dot{q}(\bar{t}) = 0$, the estimates for the capital value at the moment of switching coincide for the transition and the efficient paths and we have no scaling problem for the correct comparison. The Hotelling Rule modifier $\tau(t)$ asymptotically approaches zero along the transition path (Fig. 8a) and in combination with the modified saving rule $\dot{k} = \delta(t)q(t)$ its convergence to zero is much faster (Fig. 8b).

We can see from Fig. 7 that consumption along the efficient curve is

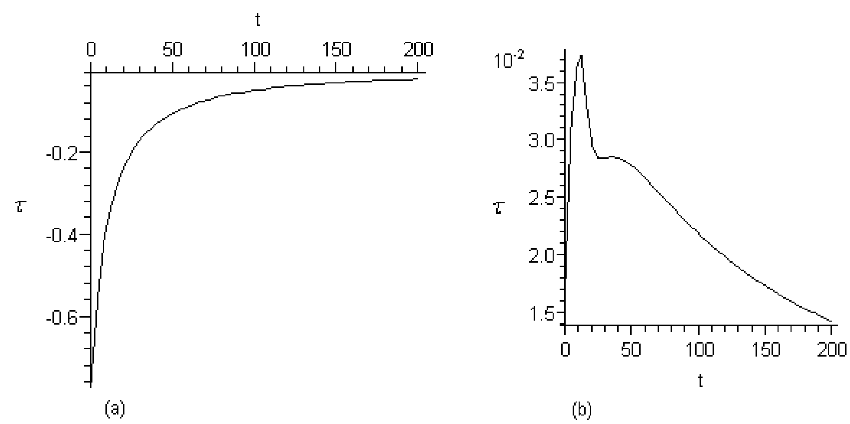

Figure 8: The paths of additive modifier for the Hotelling Rule $\tau(t)$: (a) for the transition extraction path with the standard Hartwick Rule; (b) for the transition path with the modified saving rule $\dot{k} = \delta(t)q(t)$.

always superior to the consumption along the transition path except the point of switching $\bar{t}$ where they are the same. The asymptote for the efficient path (crosses) $c_{\max Hart} = 2.6145$ is also higher then the one for the transition path (dotted) $c_{\max trans} = 2.4802$. Hence, we can conclude that it makes sense to control the efficiency of the extraction path because, as we can see, the economy in our example is losing more than 5% of consumption at each moment of time in the long run along the sustainable but inefficient path of extraction.

An interesting source for contemplation is the example for "correct switching" in case (b), when we use "$\varepsilon-$optimal" transition path with $d = 5.875 < \alpha/\beta + 2$. Using the described above procedure we obtained that in this case the consumption path along the efficient curve is also superior but only in the

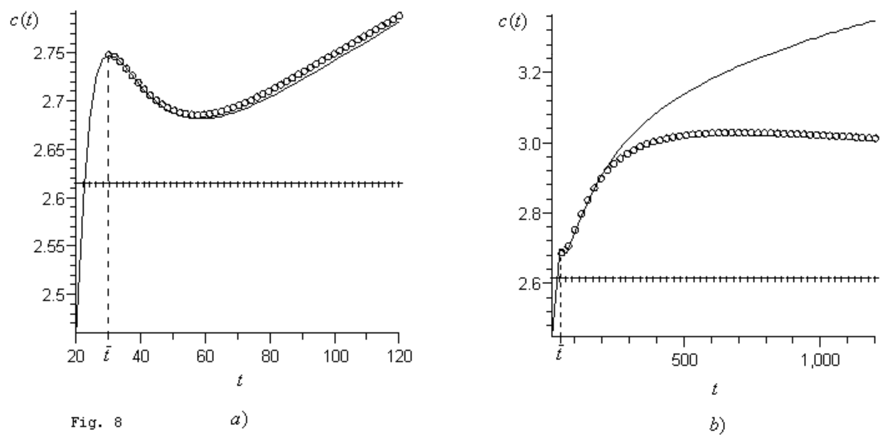

Figure 9: Consumption with the saving rate $\delta(t)$ along the Hartwick's curve (circled) and the transition curve with $d = 5.875$ (solid); the line in crosses is the asymptote for the Hartwick's path (a - short run; b - long run).

short run (Fig. 9 a). Then consumption is growing along the transition path with no limit (Proposition 3) while along the efficient curve it is decreasing to the same asymptote depicted with crosses (Fig. 9 b).

Of course, our comparison of the satisfactoriness of the extraction paths in this case is problematic because the transition curve is optimal with respect to a different welfare criterion, one which implies unlimited growth of consumption. But the example is interesting from the point of view of selecting a criterion. We can see how small sacrifices of consumption in the short run yield large future benefits even for the case when an "almost superior" but inefficient path of consumption is constructed.

10 Concluding remarks

An economy can enter the “inferior” path of consumption if the economy is controlled by the criterion which is not linked to the economy’s “abilities to grow”. These abilities or the “potential for the sustainable growth” are connected with the initial conditions of the economy and with the properties of the production function.

We have obtained this result for the Dasgupta-Heal-Solow-Stiglitz model (DHSS) with constant consumption over time as an example of plausible criterion for sustainable development. We assumed that the initial conditions of the economy did not satisfy the necessary implications of our criterion, namely, that the output and the rates of extraction were growing at the initial moment. We also assumed stickiness of extraction and saving paths or, in other words, a restricted rate of substitution between the resource and man-made capital. We think that this restriction is plausible when the man-made capital is represented by new technologies (e.g. solar plants), rather than financial capital in some fund. The restriction implies the necessity of a transition period in order to adjust the initial conditions in accord with the criterion requirements. We have constructed sub-optimal transition paths of extraction in a specific class of functions which allowed the economy to enter the “optimal” path in finite time.

However the level of consumption along these sub-optimal paths of extraction was always superior to the one along the "optimal" path after the moment of switching.[9] Namely, we have obtained bounded and unbounded growth of consumption under the standard Hartwick Rule what looks more attractive than the positive net saving (Hamilton et al, 2006) since it does not require an additional sacrificing of consumption from the present generation. This means that an economy with growing extraction and output can follow some more attractive sustainable program than the one implying constant per capita consumption. It raises a question of construction of a criterion which is consistent with the initial conditions of the economy and implies the corresponding pattern of sustainable growth. For example we can use a variant of the generalized maximin (Bazhanov 2006a) in a form of $\dot{c}^{\gamma} c^{1-\gamma} = \overline{U} = const$ which implies quasi-arithmetic growth $c(t) = c_0(1+\varphi t)^{\gamma}$ where $\varphi = \left(\overline{U}/c_0\right)^{1/\gamma}/\gamma$. There is also a question about technical possibility for an unsustainable economy to adjust its initial conditions in order to "catch-up" a "superior" criterion which the economy can not "afford" at the current moment. We think that this problems deserve a separate investigation.

[9] Except the case when we adjusted the saving rule during the transition period in order to obtain zero growth at the moment of switching ($\dot{q} = 0$). This artificial step brought different patterns of extraction under the same efficiency conditions and made it possible to compare correctly the level of consumption along the efficient and transition paths.

11 Acknowledgments

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12 Appendix 1

In order to proof the main statement of Proposition 4 we will state some auxiliary facts which we will formulate as

Lemma 1 *The rational curve of extraction (7) is such that*

a) $s_0 = \frac{r_0 p_0}{c(d-2)}$;

b) the rest of the resource $s(\bar{t})$ along this curve *at* $\bar{t} \geq 0$ *is*

$$s(\bar{t}) = s_0 - \int_0^{\bar{t}} r(t)dt = \frac{r_0}{c(d-2)} \frac{p_0 + p_1 \bar{t}}{\left(1 + c\bar{t}\right)^{d-2}} = s_0 \frac{1 + \frac{p_1}{p_0}\bar{t}}{\left(1 + c\bar{t}\right)^{d-2}}$$

where $p_0 = 1 + \frac{b_r}{c(d-3)}$, $p_1 = \frac{b_r(d-2)}{d-3}$;

$r_0 = r(0)-$ *initial rate of extraction,* s_0- *initial stock;*

$b_r = b_r(d)$, $c = c(d)$, *and* d *are the parameters of the curve (7).*

Proof. a) By the construction of $r(t)$ and since $d > 3$ we have

$$\begin{aligned} \frac{s_0}{r_0} &= \int_0^{\infty} (1+ct)^{1-d} dt + b_r \int_0^{\infty} t(1+ct)^{1-d} dt = \frac{1}{c(d-2)} \left[1 + \frac{b_r}{c(d-3)}\right] \\ &= \frac{p_0}{c(d-2)}. \end{aligned}$$

b) By direct calculations we have

$$s(\bar{t}) = s_0 - \int_0^{\bar{t}} r(t)dt = s(\bar{t}) = s_0 - r_0 \left[\frac{1}{c(d-2)} \left\{1 - \left(1 + c\bar{t}\right)^{2-d}\right\} + b_r I(\bar{t})\right] \tag{13}$$

where

$$I(\bar{t}) = \frac{1}{c^2} \left\{\frac{1}{d-3}\left[1 - \left(1 + c\bar{t}\right)^{3-d}\right] - \frac{1}{d-2}\left[1 - \left(1 + c\bar{t}\right)^{2-d}\right]\right\}$$

$$
\begin{aligned}
&= \frac{1}{c^2\,(d-2)\,(d-3)} \\
&\quad \times \left\{ (d-2)\left[1-\left(1+c\bar{t}\right)\left(1+c\bar{t}\right)^{2-d}\right]-(d-3)\left[1-\left(1+c\bar{t}\right)^{2-d}\right]\right\} \\
&= \frac{1}{c^2\,(d-2)\,(d-3)} \\
&\quad \times \left\{ \left(1+c\bar{t}\right)^{2-d}\left[(d-3)-(d-2)\left(1+c\bar{t}\right)\right]+(d-2)-(d-3)\right\} \\
&= \frac{1}{c^2\,(d-2)\,(d-3)}\left\{1-\frac{1+(d-2)\,c\bar{t}}{\left(1+c\bar{t}\right)^{d-2}}\right\}.
\end{aligned}
$$

Then the bracket $[\cdot]$ in (13) is

$$
\begin{aligned}
[\cdot] &= \frac{1}{c(d-2)} \\
&\quad \times \left\{\frac{\left(1+c\bar{t}\right)^{d-2}-1}{\left(1+c\bar{t}\right)^{d-2}}\right\}+\frac{b_r}{c^2\,(d-2)\,(d-3)}\left\{\frac{\left(1+c\bar{t}\right)^{d-2}-1-(d-2)\,c\bar{t}}{\left(1+c\bar{t}\right)^{d-2}}\right\} \\
&= \frac{1}{c\,(d-2)\left(1+c\bar{t}\right)^{d-2}} \\
&\quad \times \left\{\left(1+\frac{b_r}{c\,(d-3)}\right)\left(1+c\bar{t}\right)^{d-2}-\left(1+\frac{b_r}{c\,(d-3)}\right)-\frac{b_r\,(d-2)}{(d-3)}\bar{t}\right\} \\
&= \frac{1}{c\,(d-2)}\left\{p_0-\frac{p_0+p_1\bar{t}}{\left(1+c\bar{t}\right)^{d-2}}\right\}.
\end{aligned}
$$

Then (13) can be rewritten as follows

$$
s(\bar{t}) = s_0 - \frac{r_0}{c\,(d-2)}\left\{p_0-\frac{p_0+p_1\bar{t}}{\left(1+c\bar{t}\right)^{d-2}}\right\}.
$$

Using the result of the case a) we have

$$
s(\bar{t}) = \frac{r_0}{c\,(d-2)}\frac{p_0+p_1\bar{t}}{\left(1+c\bar{t}\right)^{d-2}} = s_0\frac{1+\frac{p_1}{p_0}\bar{t}}{\left(1+c\bar{t}\right)^{d-2}}
$$

or the assertion of the case b)■

Proof of Proposition 4. We will show that the equation defining the moment $\widetilde{t}_0$ of "smooth switching" to the efficient curve

$$\frac{\dot{r}_0 + b\widetilde{t}_0}{(1 + c\widetilde{t}_0)^d} = -\frac{\alpha r^2(\widetilde{t}_0)}{\widetilde{s}_0(\widetilde{t}_0)(\alpha - \beta)} \tag{14}$$

has real roots if and only if parameter d of the rational curve (7) is such that

$$d \leq \frac{\alpha}{\beta} + 2$$

and that there are two real roots if the last inequality is strict and one real root if it holds as an equality.

Substituting for $r(\widetilde{t}_0)$ and multiplying both sides of (14) by $\left(1 + c\widetilde{t}_0\right)^d$ we have

$$\dot{r}_0 + b\widetilde{t}_0 = -\frac{\alpha r_0^2}{\widetilde{s}_0(\widetilde{t}_0)(\alpha - \beta)} \frac{\left(1 + b_r\widetilde{t}_0\right)^2}{\left(1 + c\widetilde{t}_0\right)^{d-2}}.$$

Applying assertion b) of Lemma 1 it can be written as

$$\dot{r}_0 + b\widetilde{t}_0 = -\frac{\alpha r_0^2}{s_0(\alpha - \beta)} \frac{\left(1 + b_r\widetilde{t}_0\right)^2}{\left(1 + \frac{p_1}{p_0}\widetilde{t}_0\right)}$$

which means that the moment of "smooth switching" $\widetilde{t}_0$ is a solution of quadratic equation

$$\left(\dot{r}_0 + b\widetilde{t}_0\right)\left(1 + \frac{p_1}{p_0}\widetilde{t}_0\right) + \frac{\alpha r_0^2}{s_0(\alpha - \beta)}\left(1 + b_r\widetilde{t}_0\right)^2 = 0$$

or

$$\lambda_2\widetilde{t}_0^2 + \lambda_1\widetilde{t}_0 + \lambda_0 = 0 \tag{15}$$

where $\lambda_2 = b\frac{p_1}{p_0} + \frac{b_r^2 \alpha r_0^2}{s_0(\alpha-\beta)}$, $\lambda_1 = \frac{p_1}{p_0}\dot{r}_0 + b + \frac{2b_r \alpha r_0^2}{s_0(\alpha-\beta)}$, $\lambda_0 = \dot{r}_0 + \frac{\alpha r_0^2}{s_0(\alpha-\beta)}$. This equation has at least one real root (two if inequality is strict) if and only if $D = \lambda_1^2 - 4\lambda_2\lambda_0 \geq 0$ where

$$\begin{aligned}
\lambda_1^2 &= \frac{1}{s_0^2(\alpha-\beta)^2} \\
&\quad \times \left\{ \left(\frac{p_1}{p_0}\dot{r}_0 + b \right)^2 s_0^2(\alpha-\beta)^2 + 4b_r \alpha r_0^2 \left(\frac{p_1}{p_0}\dot{r}_0 + b \right) s_0(\alpha-\beta) + 4b_r^2 \alpha^2 r_0^4 \right\} \\
\lambda_2\lambda_0 &= \frac{1}{s_0^2(\alpha-\beta)^2} \left\{ b\frac{p_1}{p_0}\dot{r}_0 s_0^2(\alpha-\beta)^2 + s_0(\alpha-\beta)\left[b_r^2 \alpha r_0^2 \dot{r}_0 + b\frac{p_1}{p_0}\alpha r_0^2 \right] + b_r^2 \alpha^2 r_0^4 \right\}
\end{aligned}$$

Cancelling like terms and multiplying by $s_0(\alpha-\beta) > 0$ we can write our condition as $\widetilde{D} \geq 0$ where

$$\begin{aligned}
\widetilde{D} &= s_0(\alpha-\beta)\left[\left(\frac{p_1}{p_0}\dot{r}_0 + b \right)^2 - 4b\frac{p_1}{p_0}\dot{r}_0 \right] \\
&\quad +4\left[b_r \alpha r_0^2 \left(\frac{p_1}{p_0}\dot{r}_0 + b \right) - b_r^2 \alpha r_0^2 a_0 - b\frac{p_1}{p_0}\alpha r_0^2 \right]
\end{aligned}$$

Note that the first bracket $[\cdot]$ in this expression is

$$\left(\frac{p_1}{p_0}\dot{r}_0 + b \right)^2 - 4b\frac{p_1}{p_0}\dot{r}_0 = \left(\frac{p_1}{p_0}\dot{r}_0 - b \right)^2$$

and the second bracket is

$$\begin{aligned}
\left[b_r \alpha r_0^2 \left(\frac{p_1}{p_0}\dot{r}_0 + b \right) - b_r^2 \alpha r_0^2 \dot{r}_0 - b\frac{p_1}{p_0}\alpha r_0^2 \right] &= \alpha r_0^2 \left[b_r \dot{r}_0 \left(\frac{p_1}{p_0} - b_r \right) + b\left(b_r - \frac{p_1}{p_0} \right) \right] \\
&= \alpha r_0^2 \left(\frac{p_1}{p_0} - b_r \right)(b_r \dot{r}_0 - b).
\end{aligned}$$

Then the condition of the root existence is

$$\widetilde{D} = s_0(\alpha-\beta)\left(\frac{p_1}{p_0}\dot{r}_0 - b \right)^2 + 4\alpha r_0^2 \left(\frac{p_1}{p_0} - b_r \right)(b_r \dot{r}_0 - b) \geq 0 \qquad (16)$$

where

$$\begin{aligned}
\frac{p_1}{p_0}\dot{r}_0 - b &= c(d-2)r_0 b_r + \dot{r}_0 \frac{b_r(d-2)}{d-3}\frac{c(d-3)}{c(d-3)+b_r} \\
&= b_r c(d-2)\left[r_0 + \frac{\dot{r}_0}{c(d-3)+b_r}\right] \\
&= b_r c(d-2) r_0 \left[\frac{c(d-3)+b_r+\frac{\dot{r}_0}{r_0}}{c(d-3)+b_r}\right] \\
&= b_r c(d-2) r_0 \left[\frac{c(d-3)+c(d-1)+\frac{\dot{r}_0}{r_0}+\frac{\dot{r}_0}{r_0}}{c(d-3)+c(d-1)+\frac{\dot{r}_0}{r_0}}\right] \\
&= 2b_r c(d-2) r_0 \left[\frac{c(d-2)+\frac{\dot{r}_0}{r_0}}{2c(d-2)+\frac{\dot{r}_0}{r_0}}\right],
\end{aligned}$$

$$\begin{aligned}
\frac{p_1}{p_0} - b_r &= \frac{b_r(d-2)}{d-3}\frac{c(d-3)}{c(d-3)+b_r} - b_r = b_r\left[\frac{c(d-2)}{c(d-3)+b_r} - 1\right] \\
&= b_r\left[\frac{c(d-2)-c(d-3)-b_r}{2c(d-2)+\frac{\dot{r}_0}{r_0}}\right] = -b_r\left[\frac{c(d-2)+\frac{\dot{r}_0}{r_0}}{2c(d-2)+\frac{\dot{r}_0}{r_0}}\right],
\end{aligned}$$

$$b_r\dot{r}_0 - b = b_r\dot{r}_0 + b_r c(d-2)r_0 = b_r r_0\left[c(d-2)+\frac{\dot{r}_0}{r_0}\right].$$

Substituting for these expressions in (16) we obtain

$$\begin{aligned}
\widetilde{D} &= s_0(\alpha-\beta)4b_r^2c^2(d-2)^2r_0^2\left[\frac{c(d-2)+\frac{\dot{r}_0}{r_0}}{2c(d-2)+\frac{\dot{r}_0}{r_0}}\right]^2 \\
&\geq 4\alpha r_0^3 b_r^2\left[\frac{c(d-2)+\frac{\dot{r}_0}{r_0}}{2c(d-2)+\frac{\dot{r}_0}{r_0}}\right]\left[c(d-2)+\frac{\dot{r}_0}{r_0}\right]
\end{aligned}$$

or

$$\frac{s_0(\alpha-\beta)c^2(d-2)^2}{2c(d-2)+\frac{\dot{r}_0}{r_0}} \geq \alpha r_0.$$

Substituting for s_0 (Lemma 1, a)) into the LHS we have

$$\frac{p_0 c(d-2)}{2c(d-2)+\frac{\dot{r}_0}{r_0}} \geq \frac{\alpha}{\alpha-\beta}$$

and substituting for p_0 we obtain

$$\frac{(d-2)}{(d-3)}\frac{2c(d-2)+\frac{\dot{r}_0}{r_0}}{2c(d-2)+\frac{\dot{r}_0}{r_0}} \geq \frac{\alpha}{\alpha-\beta}$$

or

$$1-\frac{\beta}{\alpha} \geq 1-\frac{1}{d-2}.$$

The last expression gives us $\frac{1}{d-2} \geq \frac{\beta}{\alpha}$ or $d \leq \frac{\alpha}{\beta}+2$■

13 Appendix 2

Proof of Proposition 5. We will show that the equation defining the moment $\widetilde{t}_0$ of "smooth switching" to the optimal curve

$$\frac{\dot{r}_0 + b\widetilde{t}_0}{(1 + c\widetilde{t}_0)^d} = -\frac{\alpha r^2(\widetilde{t}_0)}{\widetilde{s}_0(\widetilde{t}_0)(\alpha - \beta)} \tag{17}$$

has only one real finite positive root if and only if $d < \frac{\alpha}{\beta} + 2$.

It was shown in the Appendix 1 that equation (17) is equivalent to the quadratic equation (15) which using lemma 1 is equivalent to equation

$$\mu_2 \widetilde{t}_0^2 + \mu_1 \widetilde{t}_0 + \mu_0 = 0 \tag{18}$$

where

$$\mu_2 = bp_1 + \frac{b_r^2 r_0 \alpha c(d-2)}{\alpha - \beta},$$

$$\mu_1 = p_1 \dot{r}_0 + bp_0 + \frac{2b_r r_0 \alpha c(d-2)}{\alpha - \beta},$$

$$\mu_0 = \dot{r}_0 p_0 + \frac{r_0 \alpha c(d-2)}{\alpha - \beta}.$$

Substituting for b, p_0, p_1, and reorganizing we have

$$\begin{aligned}
\mu_2 &= -b_r c(d-2) r_0 \frac{b_r(d-2)}{d-3} + \frac{b_r^2 r_0 \alpha c(d-2)}{\alpha - \beta} \\
&= b_r^2 r_0 c(d-2) \left[\frac{\alpha}{\alpha - \beta} - \frac{d-2}{d-3} \right] \\
&= \frac{b_r^2 r_0 c(d-2)}{(\alpha - \beta)(d-3)} \left[\beta(d-2) - \alpha \right].
\end{aligned}$$

Note that in our formulation of the problem the multiplier $\frac{b_r^2 r_0 c(d-2)}{(\alpha-\beta)(d-3)}$ in the last formula is always positive since $d > 3$, $\alpha > \beta$, $r_0 > 0$, $\dot{r}_0 > 0$ and it follows $c > 0$. Then the sign of μ_2 is defined by the sign of $\beta(d-2)-\alpha$. Namely, μ_2 is negative when $d < \frac{\alpha}{\beta}+2$, positive when $d > \frac{\alpha}{\beta}+2$, and zero when $d = \frac{\alpha}{\beta}+2$.

Coefficient μ_1 is

$$\mu_1 = \frac{b_r(d-2)}{d-3}\dot{r}_0 - b_r c(d-2) r_0 \left(1 + \frac{b_r}{c(d-3)}\right) + \frac{2b_r r_0 \alpha c(d-2)}{\alpha-\beta}$$

$$= \quad b_r(d-2)\left[\frac{\dot{r}_0}{d-3} - r_0 \frac{2c(d-2)+\frac{\dot{r}_0}{r_0}}{d-3} + \frac{2r_0\alpha c}{\alpha-\beta}\right]$$

$$= \quad b_r(d-2)\left[\frac{\dot{r}_0 - 2r_0 c(d-2) - \dot{r}_0}{d-3} + \frac{2r_0\alpha c}{\alpha-\beta}\right].$$

Finally we have

$$\mu_1 = 2b_r r_0 c(d-2)\left[\frac{\alpha}{\alpha-\beta} - \frac{d-2}{d-3}\right].$$

Note that b_r is also positive in our formulation (because of the growing rate of extraction in the neighborhood of $t=0$). Then the sign of μ_1 like the sign of μ_2 is completely defined by the same expression $\beta(d-2)-\alpha$. It can be shown that $\mu_0 > 0$ for $a_0 > 0$. The peak of parabola (18) is defined by equation

$$t^* = -\frac{\mu_1}{2\mu_2} = -\frac{2b_r r_0 c(d-2)\left[\beta(d-2)-\alpha\right]}{2b_r^2 r_0 c(d-2)\left[\beta(d-2)-\alpha\right]} = -\frac{1}{b_r} < 0.$$

Hence our parabola is convex for $d < \frac{\alpha}{\beta}+2$ and has only one positive finite root (Fig. 10). With $d \to \frac{\alpha}{\beta}+2-0$ parabola degenerates into a positive constant and the root goes to infinity$\blacksquare$

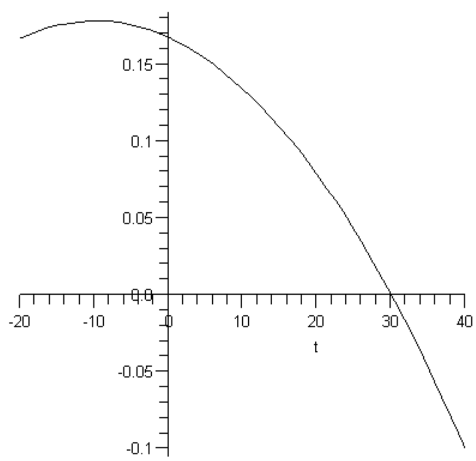

Figure 10: The root of equation (9) for $d = 5.875$.