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Abstract

The main goal of this paper is to present a different perspective than the more 'traditional' approaches to study solutions for games with externalities. We provide a direct sum decomposition for the vector space of these games and use the basic representation theory of the symmetric group to study linear symmetric solutions. In our analysis we identify all irreducible subspaces that are relevant to the study of linear symmetric solutions and we then use such decomposition to derive some applications involving characterizations of classes of solutions.

Keywords: Games in partition function form; value; representation theory; symmetric group.
JEL Classification: C71, C02.

1 Introduction

Achieving cooperation and sharing the resulting benefits are central issues in any form of organization, particularly in economic environments. These issues are often difficult to resolve especially in environments with externalities, where the surplus generated by a group of agents dependes upon the organization of agents outside the group. This problem was effectively modelled in Lucas and Thrall (1963) by the concept of games in partition function form: A partition function assigns a value to each pair consisting of a coalition and a coalition structure which includes that coalition. The advantage of this model is that it takes into account both internal factors (coalition itself) and external factors (coalition structure) that may affect cooperative outcomes and allows to go deeper into cooperation problems. Thus, it is closer to real life but more complex to analyze.

There has been a surge of literature that deals with solutions for games in partition function form. The first paper that proposed a value concept for this type of games was Myerson (1977) and then Bolger (1987) derived a class of linear, symmetric and efficient values for games in partition function form. More recently, Albizuri et al. (2005), Macho-Stadler (2007), Ju (2007), Pham Do and Norde (2007) and Hu and Yang (2010) apply the axiomatic approach to characterize a value for these games.

In this work we study linear symmetric solutions for games in partition function form using the elementary representation theory of the group of permutations of the set of players. Very roughly speaking, representation theory is a general tool for studying abstract algebraic structures by representing their elements as linear transformations of vector spaces. It makes sense to use it, since every permutation may be thought of as a linear map[1] and it presents the information in a more clear and concise way.

Briefly, what we do is to compute a decomposition of the space of games in partition function form as a direct sum of three orthogonal subspaces: a subspace of 'symmetric' games, another subspace which

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[1] The precise statement will be provided in Sec. 3.

we call V_G and the common kernel of all linear symmetric solutions. Although V_G does not have a natural definition in terms of well known game theoretic considerations, it has a simple characterization in terms of vectors in $\mathbb{R}^n$ all of whose entries add up to zero. According to this decomposition, any linear symmetric solution when restricted to any such elementary piece is either zero or multiplication by a single scalar; therefore, all linear symmetric solutions may be written as a sum of trivial maps.

With a global description of all linear and symmetric solutions, it is easy to understand the restriction imposed by the efficiency axiom. We then use such decomposition to provide, in a very economical way, a characterization for the class of linear symmetric solutions and the general expression for all linear, symmetric and efficient solutions.

It is not easy to find literature related to the approach proposed in this paper to the study of topics in cooperative game theory. Kleinberg and Weiss (1985) used the representation theory of the symmetric group to construct a direct sum decomposition of the null space of the Shapley value for games in characteristic function form (TU games). In Kleinberg and Weiss (1986), the authors followed the same line of reasoning to characterize the space of linear and symmetric values for TU games. More recently, Hernández-Lamoneda et al. (2007) provide a complete analysis following the above scheme to study solutions for TU games, where they advertise representation theory as a natural tool for research in cooperative game theory. Finally, in Sánchez-Pérez (2014), it is discussed about how to use representation techniques to the characterization of solutions for games in partition function form for the particular cases with 3 and 4 players.

The paper proceeds as follows. We first recall the main basic features of games in partition function form in the next section. A decomposition for the space of games in partition function form (which establishes the main result of this work) is introduced in section 3. We then present a couple of applications of this decomposition by giving characterizations of linear symmetric solutions and section 5 concludes the paper. Long proofs are relegated to an Appendix.

We finish this introduction with a comment on the methods employed in the article. Although it is true that the characterization results could be proved without any explicit mention to the basic representation theory of the symmetric group, we feel that by doing that we would be withholding valuable information from the reader. This algebraic tool, we believe, sheds new light on the structure of the space of games in partition function form and their solutions. Part of the purpose of the present paper is to share this viewpoint with the reader.

To make the paper as self contained as possible we have included an Appendix with some facts we need regarding basic representation theory.

2 Framework and notation

In this section we give some concepts and notations related to $n-$person games in partition function form, as well as a brief subsection of preliminaries related to integer partitions, since it is a key subject in subsequent developments.

2.1 Games in partition function form

Let $N = \{1, 2, ..., n\}$ be a fixed nonempty finite set, and let the members of N be interpreted as players in some game situation. Given N, let CL be the set of all coalitions (nonempty subsets) of N, $CL = \{S \mid S \subseteq N, S \neq \varnothing\}$. Let PT be the set of partitions of N, so

$$\{S_1, S_2, ..., S_m\} \in PT \text{ iff } \bigcup_{i=1}^{m} S_i = N,\ S_j \neq \varnothing\ \forall j,\ S_j \cap S_k = \varnothing\ \forall j \neq k$$

Additionally, for $Q \in PT$ and $i \in N$, Q^i denotes the member of Q where i belongs.

Also, let $EC = \{(S, Q) \mid S \in Q \in PT\}$ be the set of *embedded coalitions*, that is the set of coalitions together with specifications as to how the other players are aligned.

Definition 1 *A mapping*

$$w : EC \to \mathbb{R}$$

that assigns a real value, $w(S,Q)$, to each embedded coalition (S,Q) is called a game in partition function form. The set of games in partition function form with player set N is denoted by G, i.e.,

$$G = G^{(n)} = \{w \mid w : EC \to \mathbb{R}\}$$

The value $w(S,Q)$ represents the payoff of coalition S, given the coalition structure Q forms. In this kind of games, the worth of some coalition depends not only on what the players of such coalition can jointly obtain, but also on the way the other players are organized. We assume that, in any game situation, the universal coalition N (embedded in $\{N\}$) will actually form, so that the players will have $w(N,\{N\})$ to divide among themselves. But we also anticipate that the actual allocation of this worth will depend on all the other potential worths $w(S,Q)$, as they influence the relative bargaining strengths of the players.

Given $w_1, w_2 \in G$ and $c \in \mathbb{R}$, we define the sum $w_1 + w_2$ and the product cw_1, in G, in the usual form, i.e.,

$$(w_1 + w_2)(S,Q) = w_1(S,Q) + w_2(S,Q) \quad \text{and} \quad (cw_1)(S,Q) = cw_1(S,Q)$$

respectively. It is easy to verify that G is a vector space with these operations.

A *solution* is a function $\varphi : G \to \mathbb{R}^n$. If φ is a solution and $w \in G$, then we can interpret $\varphi_i(w)$ as the utility payoff which player i should expect from the game w.

Now, the group of permutations of N, $S_n = \{\theta : N \to N \mid \theta \text{ is bijective}\}$, acts on CL and on EC in the natural way; i.e., for $\theta \in S_n$:

$$\theta(S) = \{\theta(i) \mid i \in S\}$$

$$\theta(S_1, \{S_1, S_2, ..., S_l\}) = (\theta(S_1), \{\theta(S_1), \theta(S_2), ..., \theta(S_l)\})$$

And also, S_n acts on the space of payoff vectors, $\mathbb{R}^n$:

$$\theta(x_1, x_2, ..., x_n) = (x_{\theta(1)}, x_{\theta(2)}, ..., x_{\theta(n)})$$

Next, we define the usual linearity, symmetry and efficiency axioms which are asked solutions to satisfy in the cooperative game theory frame-work.

Axiom 1 (Linearity) *The solution φ is linear if $\varphi(w_1 + w_2) = \varphi(w_1) + \varphi(w_2)$ and $\varphi(cw_1) = c\varphi(w_1)$, for all $w_1, w_2 \in G$ and $c \in \mathbb{R}$.*

Axiom 2 (Symmetry) *The solution φ is said to be symmetric if and only if $\varphi(\theta \cdot w) = \theta \cdot \varphi(w)$ for every $\theta \in S_n$ and $w \in G$, where the game $\theta \cdot w$ is defined as*

$$(\theta \cdot w)(S,Q) = w[\theta^{-1}(S,Q)]$$

Axiom 3 (Efficiency) *The solution φ is efficient if $\sum_{i \in N} \varphi_i(w) = w(N,\{N\})$ for all $w \in G$.*

2.2 Integer partitions

A partition of a nonnegative integer is a way of expressing it as the unordered sum of other positive integers, and it is often written in tuple notation. Formally,

Definition 2 *$\lambda = [\lambda_1, \lambda_2, ..., \lambda_l]$ is a partition of n iff $\lambda_1, \lambda_2, ..., \lambda_l$ are positive integers and $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$. Two partitions which only differ in the order of their elements are considered to be the same partition.*

The set of all partitions of n will be denoted by $\Pi(n)$, and, if $\lambda \in \Pi(n)$, $|\lambda|$ is the number of elements of λ.

For example, the partitions of $n = 4$ are $[1,1,1,1]$, $[2,1,1]$, $[2,2]$, $[3,1]$ and $[4]$. Sometimes we will abbreviate this notation by dropping the commas, so $[2,1,1]$ becomes $[211]$.

If $Q \in PT$, there is a unique partition $\lambda_Q \in \Pi(n)$, associated with Q, where the elements of λ_Q are exactly the cardinalities of the elements of Q. In other words, if $Q = \{S_1, S_2, ..., S_m\} \in PT$, then $\lambda_Q = [|S_1|, |S_2|, ..., |S_m|]$.

For a given $\lambda \in \Pi(n)$, we represent by λ° the set of numbers determined by the λ_i's and for $k \in \lambda^\circ$, we denote by m_k^λ the multiplicity of k in partition λ. So, if $\lambda = [4,2,2,1,1,1]$, then $|\lambda| = 6$, $\lambda^\circ = \{1,2,4\}$ and $m_1^\lambda = 3$, $m_2^\lambda = 2$, $m_4^\lambda = 1$.

If $[\lambda_1, \lambda_2, ..., \lambda_l] \in \Pi(n)$, for $k \geq 1$ we define $[\lambda_1, \lambda_2, ..., \lambda_l] - [\lambda_1, \lambda_2, ..., \lambda_k] = [\lambda_{k+1}, \lambda_{k+2}, ..., \lambda_l]$. For example, $[4,3,2,1,1,1] - [3,1,1] = [4,2,1]$.

For every $\lambda \in \Pi(n) \backslash \{[n]\}$ and every $k \in \lambda^\circ$, let $I_{\lambda,k}$ be a set such that

$$I_{\lambda,k} = \begin{cases} \lambda^\circ \backslash \{k\} & \text{if } m_k^\lambda = 1 \\ \lambda^\circ & \text{if } m_k^\lambda > 1 \end{cases}$$

Finally, we need to define certain sets which are used in the sequel.

Definition 3 *Let C_n and D_n be sets defined by*

$$C_n = \{(\lambda, k) \mid \lambda \in \Pi(n), k \in \lambda^\circ\}$$

and

$$D_n = \{(\lambda, k, j) \mid \lambda \in \Pi(n) \backslash \{[n]\}, k \in \lambda^\circ, j \in I_{\lambda,k}\}$$

Example 1 *If $n = 4$, then*

$$C_4 = \{([1111], 1), ([211], 1), ([211], 2), ([22], 2), ([31], 1), ([31], 3), ([4], 4)\}$$

and

$$D_4 = \{([1111], 1, 1), ([211], 1, 1), ([211], 1, 2), ([211], 2, 1), ([22], 2, 2), ([31], 1, 3), ([31], 3, 1)\}$$

3 Decomposition of G

Precise definitions and some proofs for this section may be found in the Appendix at the end of the article. Nevertheless, for the sake of easier reading we repeat here a few definitions, sometimes in a less rigorous but more accessible manner.

The group S_n acts naturally on the space of games in partition function form, G, via linear transformations (i.e., G is a representation of S_n). That is, each permutation $\theta \in S_n$ corresponds to a linear, invertible transformation, which we still call θ, of the vector space G; namely

$$(\theta \cdot w)(S, Q) = w[\theta^{-1}(S, Q)]$$

for every $\theta \in S_n$, $w \in G$ and $(S, Q) \in EC$.

Moreover, this assignmet preserves multiplication (i.e., is a group homomorphism) in the sense that the linear map corresponding to the product of the two permutations $\theta_1\theta_2$ is the product (or composition) of the maps corresponding to θ_1 and θ_2, in that order.

In this way[2], the vector space G may be considered as a module over the group algebra $\mathbb{R}S_n$. We shall exploit this point of view in the remainder of this paper.

Similarly, the space of payoff vectors, $\mathbb{R}^n$, is a representation for S_n:

$$\theta(x_1, x_2, ..., x_n) = (x_{\theta(1)}, x_{\theta(2)}, ..., x_{\theta(n)})$$

[2] As noted by Kleinberg and Weiss (1985), for the space of TU games.

Definition 4 *Let X_1 and X_2 be two representations for the group S_n. A linear map $T : X_1 \to X_2$ is said to be S_n-equivariant if $T(\theta \cdot x) = \theta \cdot T(x)$, for every $\theta \in S_n$ and every $x \in X_1$.*

Remark 1 *Notice that, in the language of representation theory, what we are calling a linear symmetric solution is a linear map $\varphi : G \to \mathbb{R}^n$ that is S_n-equivariant.*

Definition 5 *Let Y be a subspace of a vector space X.*

- *Y is invariant (for the action of S_n) if for every $y \in Y$ and every $\theta \in S_n$, we have that*

$$\theta \cdot y \in Y$$

- *Y is irreducible if Y itself has no invariant subspaces other than $\{0\}$ and Y itself.*

We begin with the decomposition of $\mathbb{R}^n$ into irreducible representations, which is easier, and then proceed to do the same thing for G; that is, we wish to write $\mathbb{R}^n$ as a direct sum of subspaces, each invariant for all permutations in S_n and in such way that the summands cannot be further decomposed (i.e., they are irreducible).

For this, set $\mathbf{1} = (1, 1, ..., 1) \in \mathbb{R}^n$ and

$$U = \langle \mathbf{1} \rangle \quad \text{and} \quad V = U^{\perp} = \{z \in \mathbb{R}^n \mid z \cdot \mathbf{1} = 0\}$$

The spaces U and V are usually called the "trivial" and "standard" representations, respectively. Notice that U is a trivial subspace in the sense that every permutation acts as the identity transformation.

Every permutation fixes every element of U, so, in particular, it is an invariant subspace of $\mathbb{R}^n$. Being 1-dimensional, it is automatically irreducible. Its orthogonal complement, V, consists of all vectors such that the sum of their coordinates is zero. Clearly, if we permute the coordinates of any such vector, its sum will still be zero. Hence V is also an invariant subspace.

Proposition 1 *The decomposition of $\mathbb{R}^n$, under S_n, into irreducible subspaces is:*

$$\mathbb{R}^n = U \oplus V$$

Proof. First, it is clear that $U \cap V = \{\mathbf{0}\}$.[3] We now prove that $\mathbb{R}^n = U + V$:

i) If $z \in (U + V)$, then $z \in \mathbb{R}^n$ since $(U + V)$ is a subspace of $\mathbb{R}^n$.

ii) For $z \in \mathbb{R}^n$, let $\overline{z} = \frac{1}{n} \sum_{i=1}^{n} z_i$ and z can be written as $z = (\overline{z}, \overline{z}, ..., \overline{z}) + (z_1 - \overline{z}, z_2 - \overline{z}, ..., z_n - \overline{z})$; and so, $z \in (U + V)$.

Finally, since U is 1-dimensional, then it is irreducible. To check that V is also irreducible, it is an induction argument that can be found in Hernández-Lamoneda et. al. (2007). ■

Thus, this result tell us that $\mathbb{R}^n$ as a vector space with group of symmetry S_n, can be written as an orthogonal sum of the subspaces U and V, which are invariant under permutations and which can no longer be further decomposed.

The decomposition of G is carried out in three steps. For each $\lambda \in \Pi(n)$, define the subspace of games

$$G_\lambda = \{w \in G \mid w(S, Q) = 0 \text{ if } \lambda_Q \neq \lambda\}$$

Thus,

$$G = \bigoplus_{\lambda \in \Pi(n)} G_\lambda$$

Whereas, given $\lambda \in \Pi(n)$ and for $k \in \lambda^\circ$, inside G_λ define the subspace

$$G_\lambda^k = \{w \in G_\lambda \mid w(S, Q) = 0 \text{ if } |S| \neq k\}$$

[3] Here, $\mathbf{0} = (0, 0, ..., 0) \in \mathbb{R}^n$.

Then each G_λ has a decomposition $G_\lambda = \underset{k \in \lambda^\circ}{\oplus} G_\lambda^k$ and so, we have the following decomposition for G:

$$G = \underset{(\lambda,k)\in C_n}{\oplus} G_\lambda^k \tag{1}$$

Each subspace G_λ^k is invariant under S_n and the decomposition is orthogonal with respect to the invariant inner product on G given by $\langle w_1, w_2\rangle = \sum_{(S,Q)\in EC} w_1(S,Q)\cdot w_2(S,Q)$. Here, invariance of the inner product means that every permutation $\theta \in S_n$ is not only a linear map on G, but an orthogonal map with respect to this inner product. Formally, $\langle \theta \cdot w_1, \theta \cdot w_2\rangle = \langle w_1, w_2\rangle$ for every $w_1, w_2 \in G$.

Example 2 *For the case $N = \{1,2,3,4\}$, $\dim G^{(4)} = 37$ and according to (1) it decomposes:*

$$G^{(4)} = G^1_{[1,1,1,1]} \oplus G^1_{[2,1,1]} \oplus G^2_{[2,1,1]} \oplus G^1_{[3,1]} \oplus G^3_{[3,1]} \oplus G^2_{[2,2]} \oplus G^4_{[4]}$$

The next goal is to get a decomposition of each subspace of games G_λ^k into irreducible subspaces and so, we will get it for G.

The following games play an important role in describing the decomposition of the space of games. For each $(\lambda,k)\in C_n$, define $u_\lambda^k \in G_\lambda^k$ as follows

$$u_\lambda^k(S,Q) = \begin{cases} 1 & \text{if } |S| = k \text{ and } \lambda_Q = \lambda \\ 0 & \text{otherwise} \end{cases}$$

Notice that $G^n_{[n]} = \mathbb{R}u^n_{[n]}$.

Also, for each $(\lambda,k)\in C_n$ and for each $x \in \mathbb{R}^n$; define the game $x^{(\lambda,k)} \in G_\lambda^k$ as follows

$$x^{(\lambda,k)}(S,Q) = \begin{cases} \sum\limits_{i\in S} x_i & \text{if } |S| = k \text{ and } \lambda_Q = \lambda \\ 0 & \text{otherwise} \end{cases}$$

Definition 6 *Suppose X_1 and X_2 are two representations for the group S_n, i.e., we have two vector spaces X_1 and X_2 where S_n is acting by linear maps. We say that X_1 and X_2 are isomorphic if there is a linear map between them, which is $1-1$ and onto and that commutes with the respective S_n-actions. Formally, there is an invertible linear map $T : X_1 \to X_2$ such that $T(\theta\cdot x) = \theta \cdot T(x)$, for every $\theta \in S_n$ and every $x \in X_1$. We then write $X_1 \simeq X_2$.*

For our purposes, X_1 will be an irreducible subspace of G and X_2 an irreducible subspace of $\mathbb{R}^n$.

Isomorphic representations are essentially "equal"; not only are they spaces of the same dimension, but the actions are equivalent under some linear invertible map between them.

The next Theorem provides us a decomposition of the space of games, into irreducible subspaces.

Theorem 1 *For $(\lambda,k)\in C_n \backslash \{([n],n)\}$,*

$$G_\lambda^k = U_\lambda^k \oplus V_\lambda^k \oplus W_\lambda^k$$

where $U_\lambda^k = \langle u_\lambda^k \rangle \simeq U$, $V_\lambda^k = \underset{j\in I_{\lambda,k}}{\oplus} \left\{ x_j^{(\lambda,k)} \mid x_j \in V \right\}$ in which each $\left\{ x_j^{(\lambda,k)} \mid x_j \in V \right\} \simeq V$; and W_λ^k does not contain any summands isomorphic to either U nor V. The decomposition is orthogonal.
Proof. *See Appendix.* ■

Remark 2 *Theorem 1 does not quite give us a decomposition of G_λ^k into irreducible summands. The subspace U_λ^k is irreducible and V_λ^k is a direct sum of irreducible subspaces. Whereas W_λ^k may or may not be irreducible (depending on λ and k), but as we shall see the exact nature of this subspace plays no role in the study of linear symmetric solutions since it lies in the kernel of any such solution.*

Set $U_G = \bigoplus_{(\lambda,k)\in C_n} U_\lambda^k$. This is a subspace of games, whose values $w(S,Q)$ depends only on the cardinality of S and on the structure of Q.[4] According to Theorem 1, U_G is the largest subspace of G where S_n acts trivially.

Let $V_G = \bigoplus_{(\lambda,k)\in C_n\setminus\{([n],n)\}} V_\lambda^k$ and $W_G = \bigoplus_{(\lambda,k)\in C_n\setminus\{([n],n)\}} W_\lambda^k$, then:

$$G = U_G \oplus V_G \oplus W_G$$

Notice that W_G will be non-zero as soon as $n > 3$.

The next result provides a good example of how the decomposition of G can be used to gain information about linear symmetric solutions.

Corollary 1 *If $\varphi : G \to \mathbb{R}^n$ is a linear symmetric solution, then $\varphi(w) = 0$ for every $w \in W_G$.*
Proof. *Let $\varphi : G = U_G \oplus V_G \oplus W_G \to \mathbb{R}^n = U \oplus V$ be a linear symmetric solution. Suppose $Z \subset W_G$ is an irreducible summand in the decomposition of W_G (even while we do not know the decomposition of W_G as a sum of irreducible subspaces, it is known that such a decomposition exists). Let p_1 and p_2 denote orthogonal projection of $\mathbb{R}^n$ onto U and V, respectively. Now, $\varphi : G \to \mathbb{R}^n = U \oplus V$ may be written as $\varphi = (p_1 \circ \varphi, p_2 \circ \varphi)$. Denote by $\iota : Z \to G$ the inclusion, then, the restriction of φ to Z may be expressed as $\varphi_{|_Z} = \varphi \circ \iota = (p_1 \circ \varphi \circ \iota, p_2 \circ \varphi \circ \iota)$.*

Now, $p_1 \circ \varphi \circ \iota : Z \to U$ and $p_2 \circ \varphi \circ \iota : X \to V$ are linear symmetric maps; since Z is not isomorphic to either of these two spaces, thus Schur's Lemma (see Appendix for the statement) says that $p_1 \circ \varphi \circ \iota$ and $p_2 \circ \varphi \circ \iota$ must be zero. Since this is true for every irreducible summand Z of W_G, φ is zero on all of W_G. ■

Remark 3 *In other words, Corollary 1 is the statement that the common kernel of all linear symmetric solutions is W.*

Remark 4 *According to Theorem 1 and Corollary 1, in order to study linear symmetric solutions, one needs to look only at those games inside $U_G \oplus V_G$. Also, from Theorem 1 we know that for every $(\lambda, k) \in C_n\setminus\{([n], n)\}$, G_λ^k contains exactly 1 copy of U and $|I_{\lambda,k}|$ copies of V.*

Example 3 *For $n = 5$, we compute the number of copies of V inside of each G_λ^k:*

G_λ^k	*# of copies of V*
$G^1_{[11111]}$	1
$G^1_{[2111]}$	2
$G^2_{[2111]}$	1
$G^1_{[311]}$	2
$G^3_{[311]}$	1
$G^1_{[41]}$	1

G_λ^k	*# of copies of V*
$G^4_{[41]}$	1
$G^1_{[221]}$	1
$G^2_{[221]}$	2
$G^2_{[32]}$	1
$G^3_{[32]}$	1
$G^5_{[5]}$	0

And for $n = 7$, the number of copies of V inside of G_λ^k for some values of λ and k:

G_λ^k	*# of copies of V*
$G^2_{[211111]}$	1
$G^2_{[2221]}$	2
$G^1_{[3211]}$	3
$G^2_{[322]}$	2
$G^3_{[322]}$	1
$G^4_{[421]}$	2

[4] Such type of games may be thought of as its counterpart for symmetric games in TU games.

From the decomposition of G, given a game $w \in G$ we may decompose it relative to the above as $w = u + v + r$, where in turn $u = \sum a_{\lambda,k} u_\lambda^k$ and $v = \sum z_{\lambda,k,j}^{(\lambda,k)}$. This decomposition is very well suited to study the image of w under any linear symmetric solution. The reason being the following version of the well known Schur's Lemma[5].

Theorem 2 (Schur's Lemma) *Any linear symmetric solution*

$$\varphi : G = U_G \oplus V_G \oplus W_G \to \mathbb{R}^n = U \oplus V$$

satisfies

a) $\varphi(U_G) \subset U$

b) $\varphi(V_G) \subset V$

Moreover,

- *for each* $(\lambda, k) \in C_n$*, there is a constant* $\alpha(\lambda, k) \in \mathbb{R}$ *such that, for every* $u \in U_\lambda^k$*,*

$$\varphi(u) = \alpha(\lambda, k) \cdot \mathbf{1} \in U$$

- *for each* $(\lambda, k, j) \in D_n$*, there is a constant* $\beta(\lambda, k, j) \in \mathbb{R}$ *such that, for every* $z_{\lambda,k,j}^{(\lambda,k)} \in V_\lambda^k$*,*

$$\varphi\left(z_{\lambda,k,j}^{(\lambda,k)}\right) = \beta(\lambda, k, j) \cdot z_{\lambda,k,j} \in V$$

For many purposes it suffices to use merely the existence of the decomposition of the game $w \in G$, without having to worry about the precise value of each component. Nevertheless it will be useful to have it. Thus we give a formula for computing it.

Proposition 2 *Let* $w \in G$*. Then*

$$w = \sum_{(\lambda,k)\in C_n} a_{\lambda,k} u_\lambda^k + \sum_{(\lambda,k,j)\in D_n} z_{\lambda,k,j}^{(\lambda,k)} + r \tag{2}$$

where,

1. $a_{\lambda,k}$ *is the average of the values* $w(S,Q)$ *with* S *containing* k *players and* Q *with structure according to* λ*:*

$$a_{\lambda,k} = \frac{\sum_{\substack{(S,Q)\in EC \\ |S|=k, \lambda_Q=\lambda}} w(S,Q)}{|(S,Q) \in EC \mid |S| = k, \lambda_Q = \lambda|}$$

2. *For every* $(\lambda, k, j) \in D_n$*,* $z_{\lambda,k,j} \in V$ *is given by:*

$$(z_{\lambda,k,j})_i = \sum_{\substack{(S,Q)\in EC \\ S\ni i, |S|=k \\ \lambda_Q=\lambda}} \sum_{\substack{T\in Q\setminus\{S\} \\ |T|=j}} \frac{j}{n} w(S,Q) - \sum_{\substack{(S,Q)\in EC \\ S\not\ni i, |S|=k \\ \lambda_Q=\lambda, |Q^i|=j}} \frac{k}{n} w(S,Q)$$

3. *r may be computed as "the rest", i.e.,*

$$r = w - \sum_{(\lambda,k)\in C_n} a_{\lambda,k} u_\lambda^k - \sum_{(\lambda,k,j)\in D_n} z_{\lambda,k,j}^{(\lambda,k)}$$

[5] See the Appendix for a precise statement.

Proof. *See Appendix.* ■

Example 4 *Let $N = \{1,2,3\}$ and a game w described by*

(S,Q)			$w(S,Q)$		
$\{1\}$	$\{2\}$	$\{3\}$	5	10	2
$\{1,2\}$	$\{3\}$		18	0	
$\{1,3\}$	$\{2\}$		12	6	
$\{2,3\}$	$\{1\}$		20	3	
$\{1,2,3\}$			30		

One then computes with the above formula the decomposition of such a game w as

$$w = \frac{17}{3}u^1_{[111]} + 3u^1_{[21]} + \frac{50}{3}u^2_{[21]} + 30u^3_{[3]} + z^{([111],1)}_{[111],1,1} + z^{([21],1)}_{[21],1,2} + z^{([21],2)}_{[21],2,1}$$

where

$$z_{[111],1,1} = \left(-\frac{2}{3}, \frac{13}{3}, -\frac{11}{3}\right), \quad z_{[21],1,2} = (0,3,-3) \quad \textit{and} \quad z_{[21],2,1} = \left(-\frac{10}{3}, \frac{14}{3}, -\frac{4}{3}\right)$$

For the case $n = 3$, it turns out that $r = 0$.

4 Some basic applications

Now, we show how to get characterizations of solutions easily by using the decomposition of a game given by (2) in conjunction with Schur's Lemma. We start by providing a characterization of all linear symmetric solutions $\varphi : G \to \mathbb{R}^n$ in the following

Proposition 3 *The linear symmetric solutions $\varphi : G \to \mathbb{R}^n$ are precisely those of the form*

$$\varphi_i(w) = \sum_{(\lambda,k)\in C_n} \sum_{\substack{(S,Q)\in EC \\ S\ni i, |S|=k \\ \lambda_Q=\lambda}} \gamma(\lambda,k)\cdot w(S,Q) + \sum_{(\lambda,k,j)\in D_n} \sum_{\substack{(S,Q)\in EC \\ S\not\ni i, |S|=k \\ \lambda_Q=\lambda, |Q^i|=j}} \delta(\lambda,k,j)\cdot w(S,Q) \tag{3}$$

for arbitrary real numbers $\{\gamma(\lambda,k) \mid (\lambda,k)\in C_n\} \cup \{\delta(\lambda,k,j) \mid (\lambda,k,j)\in D_n\}$.
Proof. *Let $\varphi : G \to \mathbb{R}^n$ be a linear symmetric solution. From Proposition 2, $w \in G$ decomposes as*

$$w = \sum_{(\lambda,k)\in C_n} a_{\lambda,k} u^k_\lambda + \sum_{(\lambda,k,j)\in D_n} z^{(\lambda,k)}_{\lambda,k,j} + r$$

where by linearity,

$$\varphi_i(w) = \sum_{(\lambda,k)\in C_n} a_{\lambda,k}\varphi_i\left(u^k_\lambda\right) + \sum_{(\lambda,k,j)\in D_n} \varphi_i\left(z^{(\lambda,k)}_{\lambda,k,j}\right) + \varphi_i(r)$$

Now, $\varphi_i(r) = 0$ by Corollary 1 and Schur's Lemma implies

$$\varphi_i(w) = \sum_{(\lambda,k)\in C_n} a_{\lambda,k}\alpha(\lambda,k) + \sum_{(\lambda,k,j)\in D_n} \beta(\lambda,k,j)\cdot (z_{\lambda,k,j})_i$$

for some constants $\{\alpha(\lambda,k) \mid (\lambda,k)\in C_n\} \cup \{\beta(\lambda,k,j) \mid (\lambda,k,j)\in D_n\}$.

Set $|(S,Q)_{\lambda,k}| = |(S,Q) \in EC \mid |S| = k, \lambda_Q = \lambda|$, *then*

$$\begin{aligned}
\varphi_i(w) &= \sum_{(\lambda,k)\in C_n} \frac{\alpha(\lambda,k)}{|(S,Q)_{\lambda,k}|} \sum_{\substack{(S,Q)\in EC \\ |S|=k,\lambda_Q=\lambda}} w(S,Q) \\
&\quad + \sum_{(\lambda,k,j)\in D_n} \beta(\lambda,k,j) \left[\sum_{\substack{(S,Q)\in EC \\ S\ni i,|S|=k \\ \lambda_Q=\lambda}} \sum_{\substack{T\in Q\setminus\{S\} \\ |T|=j}} j w(S,Q) - \sum_{\substack{(S,Q)\in EC \\ S\not\ni i,|S|=k \\ \lambda_Q=\lambda,|Q^i|=j}} k w(S,Q) \right] \\
&= \sum_{(\lambda,k)\in C_n} \sum_{\substack{(S,Q)\in EC \\ S\ni i,|S|=k \\ \lambda_Q=\lambda}} \frac{\alpha(\lambda,k)}{|(S,Q)_{\lambda,k}|} w(S,Q) + \sum_{(\lambda,k)\in C_n} \sum_{\substack{(S,Q)\in EC \\ S\not\ni i,|S|=k \\ \lambda_Q=\lambda}} \frac{\alpha(\lambda,k)}{|(S,Q)_{\lambda,k}|} w(S,Q) \\
&\quad + \sum_{(\lambda,k,j)\in D_n} \sum_{\substack{(S,Q)\in EC \\ S\ni i,|S|=k \\ \lambda_Q=\lambda}} \sum_{\substack{T\in Q\setminus\{S\} \\ |T|=j}} \frac{j}{n} \beta(\lambda,k,j) w(S,Q) - \sum_{(\lambda,k,j)\in D_n} \sum_{\substack{(S,Q)\in EC \\ S\not\ni i,|S|=k \\ \lambda_Q=\lambda,|Q^i|=j}} \frac{k}{n} \beta(\lambda,k,j) w(S,Q)
\end{aligned}$$

Observe that

$$\sum_{(\lambda,k)\in C_n} \sum_{\substack{(S,Q)\in EC \\ S\not\ni i,|S|=k \\ \lambda_Q=\lambda}} w(S,Q) = \sum_{(\lambda,k,j)\in D_n} \sum_{\substack{(S,Q)\in EC \\ S\not\ni i,|S|=k \\ \lambda_Q=\lambda,|Q^i|=j}} w(S,Q)$$

and

$$\sum_{(\lambda,k,j)\in D_n} \sum_{\substack{(S,Q)\in EC \\ S\ni i,|S|=k \\ \lambda_Q=\lambda}} \sum_{\substack{T\in Q\setminus\{S\} \\ |T|=j}} w(S,Q) = \sum_{(\lambda,k)\in C_n} \sum_{\substack{(S,Q)\in EC \\ S\ni i,|S|=k \\ \lambda_Q=\lambda}} \sum_{j\in I_{\lambda,k}} m_j^{\lambda-[|S|]} w(S,Q)$$

Thus

$$\begin{aligned}
\varphi_i(w) &= \sum_{(\lambda,k)\in C_n} \sum_{\substack{(S,Q)\in EC \\ S\ni i,|S|=k \\ \lambda_Q=\lambda}} \frac{\alpha(\lambda,k)}{|(S,Q)_{\lambda,k}|} w(S,Q) + \sum_{(\lambda,k,j)\in D_n} \sum_{\substack{(S,Q)\in EC \\ S\not\ni i,|S|=k \\ \lambda_Q=\lambda,|Q^i|=j}} \frac{\alpha(\lambda,k)}{|(S,Q)_{\lambda,k}|} w(S,Q) \\
&\quad + \sum_{(\lambda,k)\in C_n} \sum_{\substack{(S,Q)\in EC \\ S\ni i,|S|=k \\ \lambda_Q=\lambda}} \sum_{j\in I_{\lambda,k}} \frac{j}{n} m_j^{\lambda-[|S|]} \beta(\lambda,k,j) w(S,Q) - \sum_{(\lambda,k,j)\in D_n} \sum_{\substack{(S,Q)\in EC \\ S\not\ni i,|S|=k \\ \lambda_Q=\lambda,|Q^i|=j}} \frac{k}{n} \beta(\lambda,k,j) w(S,Q)
\end{aligned}$$

The result follows by setting $\gamma(\lambda,k) = \frac{\alpha(\lambda,k)}{|(S,Q)_{\lambda,k}|} + \sum_{j\in I_{\lambda,k}} \frac{j}{n} m_j^{\lambda-[|S|]} \beta(\lambda,k,j)$ *and* $\delta(\lambda,k,j) = \frac{\alpha(\lambda,k)}{|(S,Q)_{\lambda,k}|} - \frac{k}{n}\beta(\lambda,k,j)$. ■

A similar expression of linear and symmetric solutions for games in partition function form has been obtained by Hernández-Lamoneda et al. (2009).

Corollary 2 *The space of all linear and symmetric solutions on* G *has dimension* $|C_n| + |D_n|$.

Once we have such a global description of all linear symmetric solutions, we can understand restrictions imposed by other conditions or axioms, for example, the efficiency axiom.

Proposition 4 *Let $\varphi : G \to \mathbb{R}^n$ be a linear symmetric solution. Then φ is efficient if and only if*

i) $\varphi_i(u^k_\lambda) = 0$*, for all* $(\lambda, k) \in C_n \setminus \{([n], n)\}$*; and*

ii) $\varphi_i(u^n_{[n]}) = \frac{1}{n}$

Proof. First of all, $\left(U^n_{[n]}\right)^\perp$ is exactly the subspace of games w where $w(N, \{N\}) = 0$. Of these, those in V_G trivially satisfy $\sum_{i \in N} \varphi_i(w) = 0$, since (by Schur's Lemma) $\varphi(V_G) \subset V$.

Thus, efficiency need only be checked in U_G. Since u^k_λ is fixed by every permutation in S_n, we have

$$\sum_{i \in N} \varphi_i(u^k_\lambda) = n\varphi_i(u^k_\lambda)$$

and so, φ is efficient if and only if for $\lambda \neq [n]$, $n\varphi_i(u^k_\lambda) = u^k_\lambda(N, \{N\}) = 0$ and $n\varphi_i(u^n_{[n]}) = u^n_{[n]}(N, \{N\}) = 1$. ■

Recall that U_G is a subspace of games whose value on a given embedded coalition (S, Q), depends only on the cardinality of S and on the structure of Q. The next Corollary characterizes the solutions to these games in terms of linearity, symmetry and efficiency. It turns out that among all linear symmetric solutions, the egalitarian solution is characterized as the unique efficient solution on U_G. Formally,

Corollary 3 *Let $\varphi : G \to \mathbb{R}^n$ be a linear, symmetric and efficient solution. Then for all $w \in U_G$:*

$$\varphi_i(w) = \frac{w(N, \{N\})}{n}$$

In other words, all linear symmetric and efficient solutions (e.g. Myerson's value) coincide with the egalitarian solution when restricted to these type of games, U_G.

Now, another immediate application is to provide a characterization of all linear, symmetric and efficient solutions.[6]

Theorem 3 *The solution $\varphi : G \to \mathbb{R}^n$ satisfies linearity, symmetry and efficiency axioms if and only if it is of the form*

$$\varphi_i(w) = \frac{w(N, \{N\})}{n} + \sum_{(\lambda,k,j) \in D_n} \delta(\lambda, k, j) \left[\sum_{\substack{(S,Q) \in EC \\ S \ni i, |S| = k \\ \lambda_Q = \lambda}} \sum_{\substack{T \in Q \setminus \{S\} \\ |T| = j}} j w(S, Q) - \sum_{\substack{(S,Q) \in EC \\ S \not\ni i, |S| = k \\ \lambda_Q = \lambda, |Q^i| = j}} k w(S, Q) \right] \quad (4)$$

for some real numbers $\{\delta(\lambda, k, j) \mid (\lambda, k, j) \in D_n\}$.

Proof. *Let $\varphi : G \to \mathbb{R}^3$ be a linear, symmetric and efficient solution; and $w \in G$. Then, applying*

[6] An equivalent expression to (4) can be found in Hernández-Lamoneda et al. (2009).

Proposition 2 and 4, Corollary 1 and Schur's Lemma:

$$\begin{aligned}
\varphi_i(w) &= \sum_{(\lambda,k)\in C_n} a_{\lambda,k}\varphi_i\left(u_\lambda^k\right) + \sum_{(\lambda,k,j)\in D_n} \varphi_i\left(z_{\lambda,k,j}^{(\lambda,k)}\right) + \varphi_i(r) \\
&= a_{[n],n}\varphi_i\left(u_{[n]}^n\right) + \sum_{(\lambda,k,j)\in D_n} \beta(\lambda,k,j)\cdot (z_{\lambda,k,j})_i \\
&= \frac{w(N,\{N\})}{n} + \sum_{(\lambda,k,j)\in D_n} \beta(\lambda,k,j) \left[\sum_{\substack{(S,Q)\in EC \\ S\ni i,|S|=k \\ \lambda_Q=\lambda}} \sum_{\substack{T\in Q\setminus\{S\} \\ |T|=j}} \frac{j}{n} w(S,Q) - \sum_{\substack{(S,Q)\in EC \\ S\not\ni i,|S|=k \\ \lambda_Q=\lambda,|Q^i|=j}} \frac{k}{n} w(S,Q) \right]
\end{aligned}$$

The result follows by setting $\delta(\lambda,k,j) = \frac{\beta(\lambda,k,j)}{n}$. ■

Corollary 4 *The space of all linear, symmetric and efficient solutions on* G *has dimension* $|D_n|$.

We denote by $LS(G)$ the vector space of all linear symmetric solutions on G and by $LSE(G)$ the vector space of all linear symmetric and efficient solutions on G.

The following result shows that there is a relationship between those vector spaces. It states that there are as many linear symmetric solutions in n players and linear symmetric and efficient solutions in $n+1$ players.

Proposition 5 *The relation*

$$\dim LS(G^{(n)}) = \dim LSE(G^{(n+1)})$$

holds for every n.

Proof. We have to prove that, for $n \geq 1$

$$|C_n| + |D_n| = |D_{n+1}|$$

Define $f : C_n \cup D_n \to D_{n+1}$as follows. Given $\lambda = [\lambda_1, \lambda_2, ..., \lambda_p] \in \Pi(n)$, denote $\lambda \cup [1] := [\lambda_1, \lambda_2, ..., \lambda_p, 1] \in \Pi(n+1)$. For $(\lambda, s) \in C_n$ define

$$f(\lambda, s) := (\lambda \cup [1], s, 1)$$

For $(\lambda, s, t) \in D_n$ we pick j such that $\lambda_j = t$ and let j be the smallest k such that $\lambda_k = t$. Let $\widetilde{\lambda} := [\lambda_1, \lambda_2, ..., \lambda_{j-1}, \lambda_j + 1, \lambda_{j+1}, ..., \lambda_p] \in \Pi(n+1)$ and define

$$f(\lambda, s, t) := (\widetilde{\lambda}, s, t+1)$$

Then f is a bijection. Indeed,

- To show that f is bijective, take $a, b \in C_n \cup D_n$ such that $f(a) = f(b) = (\mu, x, y)$.

i) If $y = 1$. $a, b \in C_n$ and clearly $a = b$.

ii) If $y > 1$. $a, b \in D_n$ and it is easy to check that $a = b$.

This shows that f is injective.

- To show that f is surjective, take $(\mu, x, y) \in D_n$.

i) If $y = 1$. There is at least a part equal to 1. Let λ obtained from μ by removing the last zero in μ. Then $\lambda \in \Pi(n)$, $(\lambda, x) \in C_n$ and $f(\lambda, x) = (\mu, x, y)$.

ii) If $y > 1$. Let j the greatest value of the k's such that $\mu_k = y$. Define $\lambda = [\mu_1, \mu_2, ..., \mu_{j-1}, \mu_j - 1, \mu_{j+1}, ...]$ and so, $\lambda \in \Pi(n)$. We can easily check that $f(\lambda, x, y-1) = (\mu, x, y)$.

Therefore, f is surjective. This proves that $|C_n| + |D_n| = |D_{n+1}|$ ■

Example 5 *We compute some cases for the dimension of families of solutions:*

n	$\dim G^{(n)}$	$\dim LS(G^{(n)})$	$\dim LSE(G^{(n)})$
2	3	3	1
3	10	7	3
4	37	14	7
5	151	26	14
6	674	45	26

5 Concluding remarks

We have noticed that the point of view we take in this article depends heavily on a decomposition of the space of games as a direct sum of 'special' subspaces and characterizations of solutions follow from such decomposition in an very economical way. The space of games was decomposed as a direct sum of three orthogonal subspaces: $G = U_G \oplus V_G \oplus W_G$. U_G is a subspace of games whose values $w(S, Q)$ depends only on the cardinality of S and on the structure of Q. V_G does not have a natural definition in terms of well known game theoretic considerations, but it has a simple characterization in terms of vectors all of whose entries add up to zero. And W_G which plays only the role of the common kernel of every linear symmetric solution.

We showed that in order to study linear symmetric solutions, one needs to look only at those games inside $U_G \oplus V_G$ and we presented a global description of all such solutions. Besides linearity and symmetry, we studied the efficiency axiom and provided the restrictions that this property imposed in the components of the decomposition of G. Once we understood those restrictions, we obtained characterizations of classes of solutions.

The analysis throughout the paper proceeded under the consideration of the axioms of linearity, symmetry and efficiency. The consideration of other axioms or properties (e.g. a nullity axiom) following our approach, remains an interesting topic for further research.

Although it is true that the characterization results could be proved without any explicit mention to the representation theory of the symmetric group, we feel that by doing that we would be withholding valuable information from the reader. This algebraic tool, we believe, sheds new light on the structure of the space of games in partition function form and their solutions. Part of the purpose of the present paper is to share this viewpoint with the reader.

Appendix

A reference for basic representation theory is Fulton and Harris (1991). Nevertheless, we recall all basic facts that we need.

The symmetric group S_n acts on G via linear transformations (i.e., G is a representation of S_n). That is, there is a group homomorphism $\rho : S_n \to GL(G)$, where $GL(G)$ is the group of invertible linear maps in G. This action is given by:

$$(\theta \cdot w)(S, Q) := [\rho(\theta)(w)](S, Q) = w[\theta^{-1}(S, Q)]$$

for every $\theta \in S_n$, $w \in G$ and $(S, Q) \in EC$.

Definition 7 *Let H be an arbitrary group. A representation for H is a homomorphism $\rho : H \to GL(X)$, where X is a vector space and $GL(X) = \{T : X \to X \mid T$ linear and invertible$\}$.*

In other words, a representation of H is a map assigning to each element $h \in H$ a linear map $\rho(h) : X \to X$ that respects multiplication:

$$\rho(h_1 h_2) = \rho(h_1)\rho(h_2)$$

for all $h_1, h_2 \in H$.

One usually abuses notation and talks about the representation X without explicitly mentioning the homomorphism ρ. Thus, when applying the linear transformation corresponding to $h \in H$ on the element $x \in X$, we write $h \cdot x$ rather than $(\rho(h))(x)$.

The space of payoff vectors, $\mathbb{R}^n$, is also a S_n-representation:

$$\theta(x_1, x_2, ..., x_n) := [\rho(\theta)](x_1, x_2, ..., x_n) = (x_{\theta(1)}, x_{\theta(2)}, ..., x_{\theta(n)})$$

Definition 8 *Let X_1 and X_2 be two representations for the group H.*

- *A linear map $T : X_1 \to X_2$ is said to be H-equivariant if $T(h \cdot x) = h \cdot T(x)$, for every $h \in H$ and every $x \in X_1$.*
- *X_1 and X_2 are said to be isomorphic H-representations, $X_1 \simeq X_2$, if there exists an H-equivariant isomorphism between them.*

Thus, two representations that are isomorphic are, as far as all problems dealing with linear algebra with a group of symmetries, the same. They are vector spaces of the same dimension where the actions are seen to correspond under a linear isomorphism.

Definition 9 *A representation X is irreducible if it does not contain a nontrivial invariant subspace. That is, if $Y \subset X$ is also a representation for H (meaning that $h \cdot y \in Y \ \forall h \in H$), then Y is either $\{0\}$ or all of X.*

Proposition 6 *For any representation X of a finite group H, there is a decomposition*

$$X = X_1^{\oplus a_1} \oplus X_2^{\oplus a_2} \oplus \cdots \oplus X_j^{\oplus a_j}$$

where the X_i are distinct irreducible representations. The decomposition is unique, as are the X_i that occur and their multiplicities a_i.

This property is called "complete reducibility" and the extent to which the decomposition of an arbitrary representation into a direct sum of irreducible ones is unique is one of the consequences of the following:

Theorem 4 (Schur's Lemma) *Let X_1,X_2 be irreducible representations of a group H. If $T : X_1 \to X_2$ is H-equivariant, then $T = 0$ or T is an isomorphism.*

Moreover, if X_1 and X_2 are complex vector spaces, then T is unique up to multiplication by a scalar $\lambda \in \mathbb{C}$.

The previous theorem is one of the reasons why it is worth carrying around the group action when there is one. Its simplicity hides the fact that it is a very powerful tool.

There is a remarkably effective technique for decomposing any given finite dimensional representation into its irreducible components. The secret is *character theory.*

Definition 10 *Let $\rho : H \to GL(X)$ be a representation. The character of X is the complex-valued function $\chi_X : H \to \mathbb{C}$, defined as:*

$$\chi_X(h) = Tr\left(\rho(h)\right)$$

The character of a representation is easy to compute. If H acts on an n-dimensional space X , we write each element h as an $n \times n$ matrix according to its action expressed in some convenient basis, then sum up the diagonal elements of the matrix for h to get $\chi_X(h)$. For example, the trace of the identity map of an n-dimensional vector space is the trace of the $n \times n$ identity matrix, or n. In fact, $\chi_X(e) = \dim X$ for any finite dimensional representation X of any group.

Notice that in particular, we have $\chi_X(h) = \chi_X(ghg^{-1})$ for $g, h \in H$. So that χ_X is constant on the conjugacy classes of H; such a function is called a *class function.*

Definition 11 *Let $\mathbb{C}_{class}(H) = \{f : H \to \mathbb{C} \mid f$ is a class function on $H\}$. If $\chi_1, \chi_2 \in \mathbb{C}_{class}(H)$, we define an Hermitian inner product on $\mathbb{C}_{class}(H)$ by*

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|H|} \sum_{h \in H} \overline{\chi_1(h)} \cdot \chi_2(h) \tag{5}$$

Multiplicities of irreducible subspaces in a representation can be calculated via

Proposition 7 *If $Z = Z_1^{\oplus a_1} \oplus Z_2^{\oplus a_2} \oplus \cdots \oplus Z_j^{\oplus a_j}$, then the multiplicity Z_i (irreducible representation) in Z, is:*

$$a_i = \langle \chi_Z, \chi_{Z_i} \rangle$$

where $\langle , \rangle$ is the inner product given by (5).

Theorem 5 *For $(\lambda, k) \in C_n \backslash \{([n], n)\}$,*

$$G_\lambda^k = U_\lambda^k \oplus V_\lambda^k \oplus W_\lambda^k$$

where $U_\lambda^k = \langle u_\lambda^k \rangle \simeq U$, $V_\lambda^k = \bigoplus_{j \in I_{\lambda,k}} \left\{ x_j^{(\lambda,k)} \mid x_j \in V \right\}$ in which each $\left\{ x_j^{(\lambda,k)} \mid x_j \in V \right\} \simeq V$; and W_λ^k does not contain any summands isomorphic to either U nor V. The decomposition is orthogonal.

Proof. First, $\left\langle \chi_{G_\lambda^k}, \chi_U \right\rangle$ and $\left\langle \chi_{G_\lambda^k}, \chi_V \right\rangle$ are the number of subspaces isomorphic to the trivial (U) and standard representation (V) within G_λ^k, respectively.

We start by computing the number of copies of U in G_λ^k:

$$\left\langle \chi_{G_\lambda^k}, \chi_U \right\rangle = \frac{1}{n!} \sum_{\theta \in S_n} \chi_{G_\lambda^k}(\theta) \chi_U(\theta) = \frac{1}{n!} \sum_{\theta \in S_n} \chi_{G_\lambda^k}(\theta)$$

Notice that, $\chi_{G_\lambda^k}(\theta)$ is just the number of pairs $(S, Q) \in EC$ with $|S| = k$ and $\lambda_Q = \lambda$, that are fixed under $\theta \in S_n$.

Define

$$\{\theta\}_{(S,Q)} = \begin{cases} 1 & \text{if } \theta(S, Q) = (S, Q) \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$\chi_{G_\lambda^k}(\theta) = \sum_{\substack{(S,Q) \in EC \\ |S| = k, \lambda_Q = \lambda}} \{\theta\}_{(S,Q)}$$

and so,

$$\left\langle \chi_{G_\lambda^k}, \chi_U \right\rangle = \frac{1}{n!} \sum_{\theta \in S_n} \sum_{\substack{(S,Q) \in EC \\ |S| = k, \lambda_Q = \lambda}} \{\theta\}_{(S,Q)} = \frac{1}{n!} \sum_{\substack{(S,Q) \in EC \\ |S| = k, \lambda_Q = \lambda}} \sum_{\theta \in S_n} \{\theta\}_{(S,Q)}$$

where,

$$\sum_{\theta \in S_n} \{\theta\}_{(S,Q)} = |\{\theta \in S_n \mid \theta(Q) = Q,\ \theta(S) = S\}|$$

Now, S_n acts on the set Q_λ and take $Q \in Q_\lambda$. The orbit of Q under S_n is

$$S_n Q = \{\theta(Q) \mid \theta \in S_n\} = Q_\lambda$$

and the isotropy subgroup of Q is

$$(S_n)_Q = \{\theta \in S_n \mid \theta(Q) = Q\}$$

By Lagrange theorem, we get:

$$|S_n Q| = \frac{|S_n|}{|(S_n)_Q|} = |Q_\lambda| \quad \Rightarrow \quad |(S_n)_Q| = \frac{n!}{|Q_\lambda|}$$

Notice that $H = (S_n)_Q$ acts on Q and take $S \in Q$ such that $|S| = k$. The orbit of S under H is

$$HS = \{hS \mid h \in H\} = \{T \in Q \mid |T| = k\}$$

Observe that $|HS| = m_k^\lambda$ and the isotropy subgroup of S is

$$H_S = \{h \in H \mid h(S) = S\} = \{\theta \in S_n \mid \theta(Q) = Q,\ \theta(S) = S\}$$

Again, by Lagrange theorem, we get

$$|HS| = \frac{|H|}{|H_S|} = \frac{|(S_n)_Q|}{|H_S|} = m_k^\lambda \quad \Rightarrow \quad |H_S| = \frac{|(S_n)_Q|}{m_k^\lambda} = \frac{n!}{|Q_\lambda|\, m_k^\lambda}$$

And therefore,

$$\left\langle \chi_{G_\lambda^k}, \chi_U \right\rangle = \frac{1}{n!} \sum_{\substack{(S,Q)\in EC \\ |S|=k, \lambda_Q=\lambda}} \frac{n!}{|Q_\lambda|\, m_k^\lambda} = \frac{1}{n!} |Q_\lambda|\, m_k \frac{n!}{|Q_\lambda|\, m_k^\lambda} = 1$$

Now, we compute the multiplicity of V in G_λ^k. Since $\mathbb{R}^n = U \oplus V$, then $\chi_{\mathbb{R}^n} = \chi_U + \chi_V \Rightarrow \left\langle \chi_{G_\lambda^k}, \chi_{\mathbb{R}^n} \right\rangle = \left\langle \chi_{G_\lambda^k}, \chi_U \right\rangle + \left\langle \chi_{G_\lambda^k}, \chi_V \right\rangle \Rightarrow \left\langle \chi_{G_\lambda^k}, \chi_V \right\rangle = \left\langle \chi_{G_\lambda^k}, \chi_{\mathbb{R}^n} \right\rangle - 1.$

Notice that $G^1_{[1,1,\ldots,1]} \simeq \mathbb{R}^n$ (as a representation for S_n). Let us compute

$$\begin{aligned}
\left\langle \chi_{G_\lambda^k}, \chi_{\mathbb{R}^n} \right\rangle &= \left\langle \chi_{G_\lambda^k}, \chi_{G^1_{[1,1,\ldots,1]}} \right\rangle = \frac{1}{n!} \sum_{\theta \in S_n} \chi_{G_\lambda^k}(\theta) \chi_{G^1_{[1,1,\ldots,1]}}(\theta) \\
&= \frac{1}{n!} \sum_{\theta \in S_n} \sum_{\substack{(S,Q)\in EC \\ |S|=k, \lambda_Q=\lambda}} \{\theta\}_{(S,Q)} \sum_{\substack{(S',Q')\in EC \\ |S'|=1 \\ Q'=Q_{[1,1,\ldots,1]}}} \{\theta\}_{(S',Q')} \\
&= \frac{1}{n!} \sum_{\substack{(S,Q)\in EC \\ |S|=k, \lambda_Q=\lambda}} \sum_{\substack{(S',Q')\in EC \\ |S'|=1, \lambda_{Q'}=[1,1,\ldots,1]}} \sum_{\theta \in S_n} \{\theta\}_{(S,Q)} \{\theta\}_{(S',Q')}
\end{aligned}$$

For $x \in S$:

$$\sum_{\theta \in S_n} \{\theta\}_{(S,Q)} \{\theta\}_{(S',Q')} = \{\theta \in S_n \mid \theta(Q) = Q,\ \theta(S) = S,\ \theta(x) = x\}$$

Without loss of generality, suppose $|S| = k = \lambda_1$ and take the case $m_k^\lambda = m_{\lambda_1}^\lambda = 1$. Here, $M = H_S = \{\theta \in S_n \mid \theta(Q) = Q,\ \theta(S) = S\}$ acts on $S \in Q \in Q_\lambda$ and take $x \in S$. The orbit of x under M is

$$Mx = \{\theta(x) \mid \theta \in M\} = S$$

and the isotropy subgroup of x is

$$M_x = \{\theta \in M \mid \ \theta(x) = x\} = \{\theta \in S_n \mid \theta(Q) = Q,\ \theta(S) = S,\ \theta(x) = x\}$$

By Lagrange theorem,

$$|Mx| = \frac{|M|}{|M_x|} = \frac{|H_S|}{|M_x|} = k \quad \Rightarrow \quad |M_x| = \frac{|H_S|}{k} = \frac{n!}{k\,|Q_\lambda|\,m_k^\lambda}$$

Thus,

$$\begin{aligned}
\left\langle \chi_{G_\lambda^k}, \chi_{\mathbb{R}^n} \right\rangle &= \frac{1}{n!} \sum_{\substack{(S,Q)\in EC \\ |S|=k, \lambda_Q=\lambda}} \left[k\frac{n!}{k\,|Q_\lambda|\,m_k^\lambda} + m_{\lambda_2}^\lambda \lambda_2 \frac{n!}{m_{\lambda_2}^\lambda \lambda_2\,|Q_\lambda|\,m_k^\lambda} + \cdots + m_{\lambda_l}^\lambda \lambda_l \frac{n!}{m_{\lambda_l}^\lambda \lambda_l\,|Q_\lambda|\,m_k^\lambda} \right] \\
&= \frac{1}{n!} \sum_{\substack{(S,Q)\in EC \\ |S|=k, \lambda_Q=\lambda}} |\lambda^\circ|\,\frac{n!}{|Q_\lambda|\,m_k^\lambda} = \frac{1}{n!}\,|Q_\lambda|\,m_k^\lambda \left[|\lambda^\circ|\,\frac{n!}{|Q_\lambda|\,m_k^\lambda} \right] \\
&= |\lambda^\circ|
\end{aligned}$$

Finally, without loss of generality, suppose $|S| = k = \lambda_1$ and take the case $m_k^\lambda > 1$. Following the same line of reasoning as above, we obtain

$$\begin{aligned}
\left\langle \chi_{G_\lambda^k}, \chi_{\mathbb{R}^n} \right\rangle &= \frac{1}{n!} \sum_{\substack{(S,Q)\in EC \\ |S|=k, \lambda_Q=\lambda}} \left[\begin{array}{c} k\frac{n!}{k|Q_\lambda|m_k^\lambda} + k(m_k^\lambda - 1)\frac{n!}{k(m_k^\lambda-1)|Q_\lambda|m_k^\lambda} + \\ m_{\lambda_2}^\lambda \lambda_2 \frac{n!}{m_{\lambda_2}^\lambda \lambda_2 |Q_\lambda| m_k^\lambda} + \cdots + m_{\lambda_l}^\lambda \lambda_l \frac{n!}{m_{\lambda_l}^\lambda \lambda_l |Q_\lambda| m_k^\lambda} \end{array} \right] \\
&= \frac{1}{n!} \sum_{\substack{(S,Q)\in EC \\ |S|=k, \lambda_Q=\lambda}} (|\lambda^\circ| + 1)\frac{n!}{|Q_\lambda|\,m_k^\lambda} = \frac{1}{n!}\,|Q_\lambda|\,m_k^\lambda \left[(|\lambda^\circ| + 1)\frac{n!}{|Q_\lambda|\,m_k^\lambda} \right] \\
&= |\lambda^\circ| + 1
\end{aligned}$$

In summary,

$$\left\langle \chi_{G_\lambda^k}, \chi_U \right\rangle = 1 \quad \text{and} \quad \left\langle \chi_{G_\lambda^k}, \chi_V \right\rangle = |I_{\lambda,k}|$$

The next task is to identify such copies of U and V inside G_λ^k. For that end, define the functions $L_{\lambda,k} : \mathbb{R}^n \to G_\lambda^k$ by $L_{\lambda,k}(x) = x^{(\lambda,k)}$. These maps are isomorphisms between U_λ^k and U (similarly between $\left\{ x_j^{(\lambda,k)} \mid x_j \in V \right\}$ and V, for each $j \in I_{\lambda,k}$) since they are linear, S_n-equivariant and $1-1$. Thus, inside of G_λ^k, we have the images of these two subspaces: $U_\lambda^k = L_{\lambda,k}(U)$ and $V_\lambda^k = L_{\lambda,k}(V)$.

Finally, the invariant inner product $\langle , \rangle$ gives an equivariant isomorphism, in particular must preserve the decomposition. This implies orthogonality of the decomposition. ■

Proposition 8 *Let $w \in G$. Then*

$$w = \sum_{(\lambda,k)\in C_n} a_{\lambda,k} u_\lambda^k + \sum_{(\lambda,k,j)\in D_n} z_{\lambda,k,j}^{(\lambda,k)} + r$$

where,

1. *$a_{\lambda,k}$ is the average of the values $w(S,Q)$ with $|S| = k$ and $\lambda_Q = \lambda$:*

$$a_{\lambda,k} = \frac{\sum\limits_{\substack{(S,Q)\in EC \\ |S|=k, \lambda_Q=\lambda}} w(S,Q)}{|(S,Q) \in EC \mid |S| = k, \lambda_Q = \lambda|}$$

2. For every $(\lambda, k, j) \in D_n$, $z_{\lambda,k,j} \in V$ *is given by:*

$$(z_{\lambda,k,j})_i = \sum_{\substack{(S,Q)\in EC\\ S\ni i, |S|=k\\ \lambda_Q=\lambda}} \sum_{\substack{T\in Q\setminus\{S\}\\ |T|=j}} \frac{j}{n} w(S,Q) - \sum_{\substack{(S,Q)\in EC\\ S\not\ni i, |S|=k\\ \lambda_Q=\lambda, |Q^i|=j}} \frac{k}{n} w(S,Q)$$

3. r *may be computed as "the rest", i.e.,*

$$r = w - \sum_{(\lambda,k)\in C_n} a_{\lambda,k} u_\lambda^k - \sum_{(\lambda,k,j)\in D_n} z_{\lambda,k,j}^{(\lambda,k)}$$

Proof. We start by computing the orthogonal projection of w onto U_G. Notice that $\{u_\lambda^k\}$ is an orthogonal basis for U_G, and that $\left\|u_\lambda^k\right\|^2 = m_k^\lambda |Q_\lambda|$.

Thus, the projection of w onto U_G is

$$\sum_{(\lambda,k)\in C_n} \frac{\langle w, u_\lambda^k\rangle}{\langle u_\lambda^k, u_\lambda^k\rangle} u_\lambda^k$$

and so,

$$a_{\lambda,k} = \frac{\langle w, u_\lambda^k\rangle}{\langle u_\lambda^k, u_\lambda^k\rangle} = \frac{\sum\limits_{|S|=k, \lambda_Q=\lambda} w(S,Q)}{|(S,Q)\in EC \mid |S|=k, \lambda_Q=\lambda|}$$

Now, for each $(\lambda, k) \in C_n$, we define $f^{\lambda,k} : G \to \mathbb{R}^n$ as

$$f_i^{\lambda,k}(w) = \sum_{\substack{(S,Q)\in EC\\ S\ni i, |S|=k\\ \lambda_Q=\lambda}} w(S,Q)$$

where each $f^{\lambda,k}$ is S_n-equivariant and observe that $f^{[n],n}(w) = w(N, \{N\})(1, ..., 1)$. Let $z \in V$, then $f^{\lambda,k}(z_{\gamma,i,j}^{(\gamma,i)}) = 0$ if $\lambda \neq \gamma$ or $i \neq k$, whereas (by Schur's Lemma) $f^{\lambda,k}(z_{\lambda,k,j}^{(\lambda,k)}) = z_{\lambda,k,j} \in V$.

Let $p : \mathbb{R}^n \to V$ be the projection of $\mathbb{R}^n$ onto V given by

$$p_i(x) = x_i - \frac{1}{n}\sum_{j=1}^n x_j$$

This projection is equivariant, sends U to zero and it is the identity on V.

Next, define $L^{\lambda,k} : G \to V$ as $L^{\lambda,k} = p \circ f^{\lambda,k}$. Observe that

$$L^{\lambda,k}(w) = \sum_{j\in I_{\lambda,k}} z_{\lambda,k,j}$$

since by equivariance, $f^{\lambda,k}(U_G) \subset U$ and $f^{\lambda,k}(W_G) = 0$. Moreover, $f^{\lambda,k}(z_{\gamma,i,j}^{(\gamma,i)}) = 0$ if $\lambda \neq \gamma$ or $i \neq k$. Then,

$$\begin{aligned} L_i^{\lambda,k}(w) &= p_i(f^{\lambda,k}(w)) = \sum_{\substack{(S,Q)\in EC\\ S\ni i, |S|=k\\ \lambda_Q=\lambda}} w(S,Q) - \frac{1}{n}\sum_{l\in N} \sum_{\substack{(S,Q)\in EC\\ S\ni l, |S|=k\\ \lambda_Q=\lambda}} w(S,Q) \\ &= \frac{n-k}{n} \sum_{\substack{(S,Q)\in EC\\ S\ni i, |S|=k\\ \lambda_Q=\lambda}} w(S,Q) - \frac{k}{n} \sum_{\substack{(S,Q)\in EC\\ S\not\ni i, |S|=k\\ \lambda_Q=\lambda}} w(S,Q) \end{aligned}$$

The value of the component $(z_{\lambda,k,j})_i$ follows from the fact that the last expression can be written as

$$\sum_{j\in I_{\lambda,k}}\left[\sum_{\substack{(S,Q)\in EC\\ S\ni i,|S|=k\\ \lambda_Q=\lambda}}\sum_{\substack{T\in Q\setminus\{S\}\\ |T|=j}}\frac{j}{n}w(S,Q)-\sum_{\substack{(S,Q)\in EC\\ S\not\ni i,|S|=k\\ \lambda_Q=\lambda,|Q^i|=j}}\frac{k}{n}w(S,Q)\right]$$

■

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