# The St. Petersburg paradox: an experimental solution

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## Abstract

The St. Petersburg paradox refers to a gamble of infinite expected value, where people are likely to spend only a small entrance fee for it. There is a huge volume of literature that mostly concentrates on the psychophysics of the game; experiments are scant. Here, rather than focusing on the psychophysics, we offer an experimental, "physical" solution as if robots played the game. After examining the time series formed by one billion plays, we: confirm that there is no characteristic scale for this game; explicitly formulate the implied power law; and identify the type of $\alpha$-stable distribution associated with the game. We find an $\alpha = 1$ and, thus, the underlying distribution of the game is a Cauchy flight, as hinted by Paul Samuelson.

*Keywords*: St. Petersburg paradox, $\alpha$-stable distributions, Cauchy flight, power laws

## 1. Introduction

The St. Petersburg game is a simple, coin-tossing game that first appeared in 1738 in a memoir by Daniel Bernoulli, who attributed it to his cousin Nicholas Bernoulli. Daniel Bernoulli first aired it as a commentary of the St. Petersburg Academy [1], thus its name. The house offers to flip a coin, and one receives the coin if it shows tails. The prize doubles with every successive toss that shows tails. The game ends when the coin shows heads for the first time. Formally [2], a single trial in the St. Petersburg game consists of tossing a true coin until it falls as heads. If it falls as heads at the *n*th throw, the gambler receives $2^{n-1}$ coins. These independent random variables assume the values $2^0, 2^1, 2^2, \ldots$ with corresponding probabilities $2^{-1}, 2^{-2}, 2^{-3}, \ldots$. Their expectation is defined by $\sum c_n f(c_n)$ with $c_n = 2^{n-1}$ and $f(c_n) = 2^{-n}$, so that each term of the series equals 1.

How much should one pay as an entrance fee for the game? Because the prize keeps growing forever, the expected value is infinite. The paradox? Most people wish to pay only a small entrance fee. The paradox is perennial in literature because authors mix the psychophysics of the game (how people mentally react to it) with the pure physics of the game. However, there is no paradox if one considers its physics properly.

Physically speaking, the first aspect to consider is that the gain has no finite expectations, and thus both the law of large numbers and the central limit theorem are inapplicable [2]. The St. Petersburg game is not "fair," so there is no "fair" entrance fee. One cannot consider the expectation $\mu = E(X) = \infty$ as a fair entrance fee. A game like this would be fair if the expectation was finite $\mu = E(X) > 0$ and, for a large number of single trials $n$, the ratio of the accumulated gain $S_n$ to the accumulated entrance fees $e_n$ approached 1. However, because $\mu$ does not exist, the entrance fees cannot be constant and should depend on the number of single trials [2]. As a result, the following limit theorem applies [2]: A game with accumulated entrance fees $e_n$ is fair if for every $\varepsilon > 0$, $P\left\{\left|\frac{S_n}{e_n} - 1\right| > \varepsilon\right\} \to 0$. This is the analogue of the law of large numbers [2], where $e_n = n\mu$, and which means (in a physical sense) that the average of $n$ independent measurements approaches $\mu$. In particular, for this limit theorem, the average of $n$ measurements approaches $e_n/n$. Interestingly, however, the St. Petersburg game turns fair if $e_n = n \log_2 n$, where $\log_2 n$ is the logarithm to the base 2 [2].

The St. Petersburg game is not frivolous in that it offers an example of the simple random walk usually considered as the prototype for many stochastic processes in physics and economics [2]. For a physics example [3], the calculation of the resistivity at the critical filling for finite lattices is shown to be simply related to the Petersburg game. In economics, the game provides insight for growth stock valuation. For example [4], consider the question: "What should one be willing to pay for a very small probability that a company can grow its cash flows by a very significant amount forever?" It is not surprising then that the game has attracted the attention of famous writers and economists from past and present-day academics, such as Cramer, De Morgan, Condorcet, Euler, Poisson, Gibbon, Cournot, Marschack, Von Mises, Ramsey, Keynes, Samuelson, Arrow, Stigler, and Aumann.

The St. Petersburg paradox is relevant for practical investors in two important ways: "The first is that the distribution of stock market returns does not follow the pattern that standard finance theory assumes [5]." The distribution is Paretian rather than Gaussian. "This deviation from theory is important for risk management, market efficiency, and individual stock selection [5]." The second idea relates to the aforementioned valuing growth stocks: "One of the major challenges in investing is how to capture (or avoid) low-probability, high-impact events. What do you pay today for a business with a low probability of an extraordinarily high payoff? This question is more pressing than ever in a world with violent value migrations and increasing returns. Consider, for example, that of the nearly 2,000 technology initial public offerings since 1980, only 5 percent account for over 100 percent of the $2-trillion-plus in wealth creation. And even within this small wealth generating group, only a handful delivered the bulk of the huge payoffs. Given the winner-take-most characteristics of many growth markets, there is little reason to anticipate a more normal wealth-creation distribution in the future. Like the St. Petersburg game, the majority of the payoffs from future deals are likely to be modest, but some will be huge [5]."

We move on and present the game in more detail [6] before discussing its statistical physics. Suppose the house flips a coin $n+1$ times. A gambler wins $2^{n-1}$ coins for the $n$ tails that occur before the first heads occurs. If heads appears in the first attempt, another attempt is allowed until it shows tails, when $n=1$ (first row and column I in Table 1). The house will flip the coin in a second attempt, because it always flips it $n+1$ times. If heads appears after one tails, the gambler wins one coin:

$2^{n-1} = 2^{1-1} = 2^0 = 1$ (first row and column II in Table 1). The odds of tails is $\frac{1}{2}$ for each independent toss (first row and column III). The gambler's expected gain (first row and column IV) is the expected outcome (column II) times the probability of each outcome (column III). The gambler always wins a coin if it shows tails with probability $\frac{1}{2}$.The gambler does not win if it shows heads with probability $\frac{1}{2}$. Thus, $\left(1\times\frac{1}{2}\right)+\left(0\times\frac{1}{2}\right)=\frac{1}{2}$.

Now consider the second row in Table 1. Two tails appear before it shows heads. The gambler wins two coins: $2^{n-1} = 2^{2-1} = 2^1 = 2$. The odds are $\frac{1}{4}$: the chance of appearing the first tails ($\frac{1}{2}$) times the chance of appearing the second tails ($\frac{1}{2}$), because these are independent events. The expected gain is then $2\times\frac{1}{4}=\frac{1}{2}$. For the remaining rows, the expected gain will always be $\frac{1}{2}$. Thus, the cumulative expected gain will be infinite: $\frac{1}{2}+\frac{1}{2}+\cdots=\infty$. From the house's point of view, its cumulative expected loss will be equally infinite. (Ref. [7] provides lots of detailed examples of variants of the St. Petersburg game.)

The early difficulties in dealing with the paradox may have come from the fact that the notion of expectation in the classical theory of probability was not clearly disassociated from the definition of probability itself, and no mathematical treatment existed to surpass the difficulty posed by the paradox [2]. Once this formalism became available, it is surprising that one can still see any paradox in the St. Petersburg game [2]; physically speaking, not psychologically.

Table 1. The St. Petersburg game

| *I* <br> *Number of times the coin is tossed, n* | *II* <br> *Quantity of coins that can be won,* $c_n$ | *III* <br> *Probability of winning, w* | $IV = II \times III$ <br> *Expected gain* |
|---|---|---|---|
| 1 | 1 | $\frac{1}{2}$ | $1\times\frac{1}{2}=\frac{1}{2}$ |
| 2 | 2 | $\frac{1}{4}=\left(\frac{1}{2}\right)^2$ | $2\times\frac{1}{4}=\frac{1}{2}$ |
| 3 | 4 | $\frac{1}{8}=\left(\frac{1}{2}\right)^3$ | $4\times\frac{1}{8}=\frac{1}{2}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $n$ | $2^{n-1}$ | $\left(\frac{1}{2}\right)^n$ | $\frac{1}{2}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

From the point of view of modern statistical physics, and econophysics in particular, one can say there is no fair entrance fee for the St. Petersburg game because it is a stochastic process with no characteristic scale [6]. In making this connection, here we unveil the particular power law implicated in the St. Petersburg game. Moreover, we exactly identify the type of $\alpha$-stable distribution underlying the game. The next section justifies the methodology employed, while the results are shown in Section 3. A discussion follows (Section 4), and then we conclude the study.

## 2. Materials and methods

Let $p_n = P(N = n) = 2^{-n}$, $n = 1, 2, \ldots$ , be the distribution of the number of coin flipping $N$ until the first heads appears. If the gambler wins randomly $X = 2^{N-1}$ coins, as observed, his probability distribution is straightforward: $f(c_n) = P(X = c_n) = 2^{-n}$.

However, because $c_n = 2^{n-1}$ then $n = 1 + \log_2 c_n$. Therefore, in terms of the quantity of coins that can be won $c_n$, there is the power law distribution:

$$w = P(X = c_n) = 2^{-n} = 2^{-(1+\log_2 c_n)} = \frac{1}{2c}, \tag{1}$$

where $c_n = 1, 2, 4, 8, 16, 32, \ldots$. Equation (1) gives the law governing the St. Petersburg game. It shows $c_n$ and $w$ inversely related precisely as follows.

*St. Petersburg game power law: Double the probability of winning and the prize (quantity of coins that can be won) is reduced by half.*

To illustrate it, one can look at the game backwards in Table 1 to realize that as $w$ is doubled from, say, $\frac{1}{8}$ to $\frac{1}{4}$, then $c_n$ is cut by half, from 4 to 2.

The same power law given by Eq. (1) can be found from the first 100 realizations of the game as in Table 1. First, we hypothesize the law describing the relationship between $c_n$ and $w$ as a power law of the form:

$$c_n = aw^b. \tag{2}$$

This means $c_n$ changes as if it were a power of $w$. The problem is then to verify the conjecture by determining $a$ and $b$.

Taking the logarithm to base 10 on both sides of Eq. (2):

$$\log c_n = b \log w + \log a. \tag{3}$$

(Any base works, of course, including base 2). Figure 1 shows a straight line in a log-log plot of Eq. (3), where $b$ is the slope, and $\log a$ is the *y*-intercept = *d*. Thus, $a = 10^d$.

From a fitting line of the observations in Figure 1, we find:

$$\log c_n = -\log w - 0.301, \tag{4}$$

and $a$ and $b$ can be found in turn. The slope is $b = -1$, and $d = -0.301$; thus, $a = 10^{-0.301} = 0.5$. The values for $a$ and $b$ can then be inserted back into Eq. (2) to exactly produce the power law in Eq. (1).

As observed [8], "despite the age and the importance of the problem only a few experiments on the Petersburg gamble have been documented." Here, we perform such an experiment. To dismiss any psychophysics explanations from the start, we offer a "physical" solution, as if robots played the game. We run one billion single trials of the game, collect each prize, and build a time series of the prizes. This approach is the only one that is empirically viable, and therefore is preferable to considering the realizations of the game as displayed in Table 1. This is so because the series grows explosively in the realizations in Table 1. As a result, one cannot get a time series large enough to allow for an analysis of the tails distribution of the game. This can be appreciated in Figure 2, which shows a histogram of the first 100 realizations of the game.

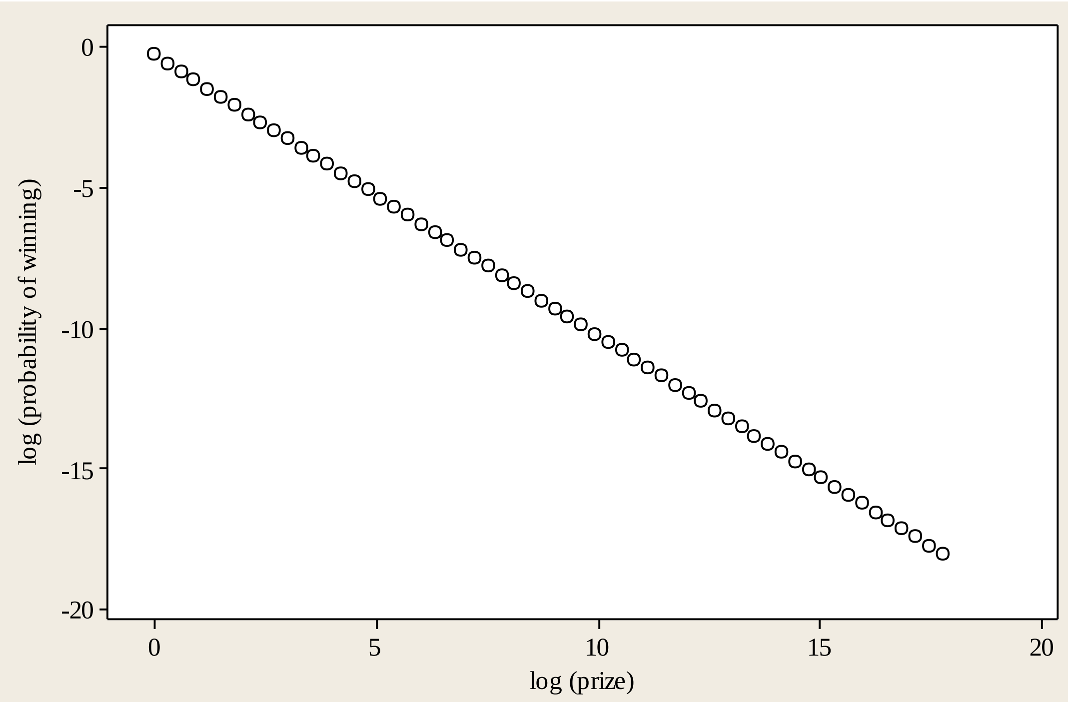

Figure 1. The St. Petersburg game power law for the first 100 realizations.

However, even for 100 realizations, Table 2 suggests some lessons can be learned. Table 2 presents the results for 30, 60, and 100 realizations, which are amenable to analysis. It suggests the realizations of the game are not Gaussian, as the excess kurtosis departs from three and keeps growing far away from it. Not only fat tails are involved, but also the probability density function is heavily skewed (Figure 2 again). A test of Gaussianity shows the distribution of the first 100 realizations fail to be Gaussian. We consider an Anderson-Darling normality test, which modifies the Kolmogorov-Smirnov test to give more weight to the tails. Its critical values are distribution-free, and a $p$-value $< 0.05$ suggests, the distribution is not Gaussian. The data for the 100 realizations present a $p$-value $< 0.004$, and thus are unlikely to be modeled by the Gaussian. This result suggests the St. Petersburg game power law reflects non-Gaussian scaling.

Experimentally, one can detect a Gaussian after a closer look at its tails [9]. Take the variable height. The standard deviation away from the average ($\sigma$) is around 8 to 10 centimeters for height, and the probability of exceeding:

$0\sigma$ = 1 in 2 times
$1\sigma$ = 1 in 6.3 times
$2\sigma$ = 1 in 44 times
$3\sigma$ = 1 in 740 times
$4\sigma$ = 1 in 32,000 times
$5\sigma$ = 1 in 3,500,000 times
$6\sigma$ = 1 in 1,000,000,000 times
$7\sigma$ = 1 in 780,000,000,000 times
$8\sigma$ = 1 in 1,600,000,000,000,000 times
$9\sigma$ = 1 in 8,900,000,000,000,000,000 times
$10\sigma$ = 1 in 130,000,000,000,000,000,000, 000 times
…
$20\sigma$ = 1 in
36,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,00
0,000,000,000,000, 000 times.

The tails decay grows exponentially. By contrast, the distribution of wealth, for instance, is not like the distribution of height, where the tails decay is exponential. It is Paretian rather than Gaussian. Take the odds of encountering a millionaire in Europe [9]:

Richer than 1 million: 1 in 62.5
Richer than 2 million: 1 in 250
Richer than 4 million: 1 in 1,000
Richer than 8 million: 1 in 4,000
Richer than 16 million: 1 in 16,000
Richer than 32 million: 1 in 64,000
Richer than 320 million: 1 in 6,400,000.

Thus, the analysis of tails decay shows that when the number is doubled, the incidence goes down by four. The tails decay is constant. This is a power law distribution.

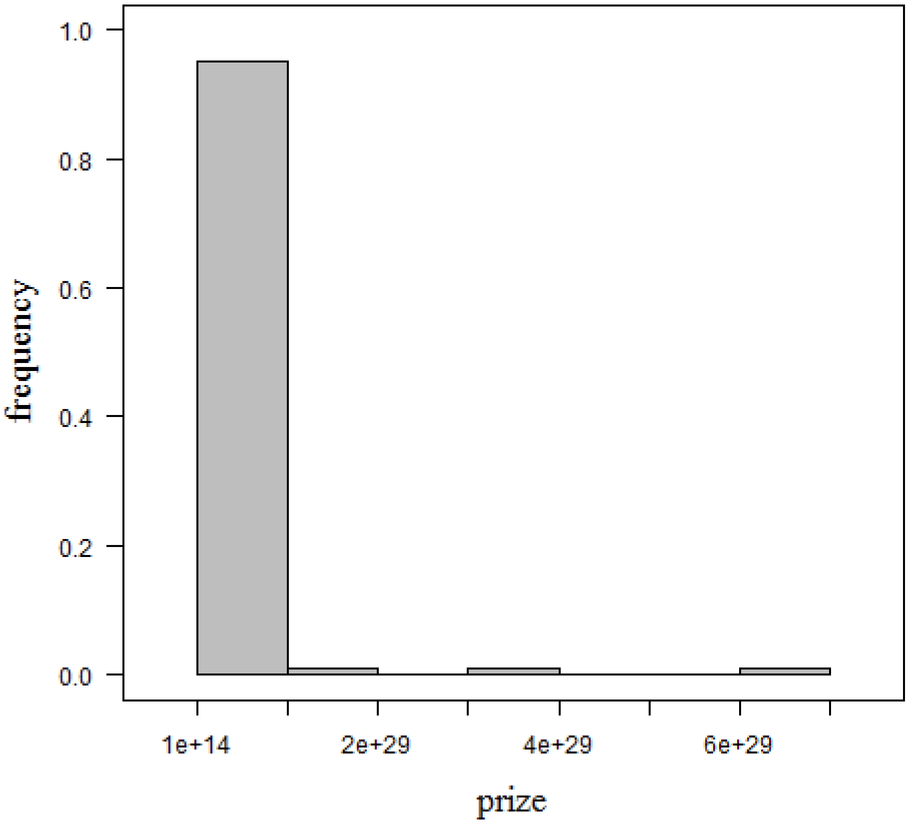


Figure 2. Histogram of the first 100 realizations of the St. Petersburg game.

Table 2. Descriptive statistics for the first 100 realizations of the St. Petersburg game.

| *Statistic* | *Realizations* | | |
|---|---|---|---|
| Sample size | 30 | 60 | 100 |
| Range | 5.37E+8 | 5.76E+17 | 6.34E+29 |
| Mean | 3.58E+7 | 1.92E+16 | 1.27E+28 |
| Variance | 1.19E+16 | 7.12E+33 | 5.25E+57 |
| Standard deviation | 1.09E+8 | 8.44E+16 | 7.24E+28 |
| Coefficient of variation | 3.05 | 4.40 | 5.71 |
| Standard error | 1.99E+7 | 1.09E+16 | 7.24E+27 |
| Skewness | 3.9249 | 5.652 | 7.348 |
| Coefficient of skewness | 0.985 | 0.682 | 0.525 |
| Excess kurtosis | 16.375 | 34.413 | 58.432 |

Note: As the sample size increases, the four moments of the underlying distribution also increase. The coefficient of skewness is the only quantity that declines.

To confirm the St. Petersburg power law stated in the previous section for 100 realizations in Table 1, we consider one billion single trials and find a distribution similar to the second example above, as follows.

Prize of 32: 1 in 64
Prize of 64: 1 in 128
Prize of 128: 1 in 256
Prize of 256: 1 in 512
Prize of 512: 1 in 1,024
Prize of 1,024: 1 in 2,048.

Thus, as the prize (the quantity of coins that can be won) doubles, the probability of winning is reduced by half.

## 3. Results

Figures 3 and 4 first show one million single trials. (The dataset is publicly available at http://dx.doi.org/10.6084/m9.figshare.1468405.) Figure 3 shows the number of rounds actually played, and Figure 4 shows the quantity of coins actually won ($\times 10^5$).

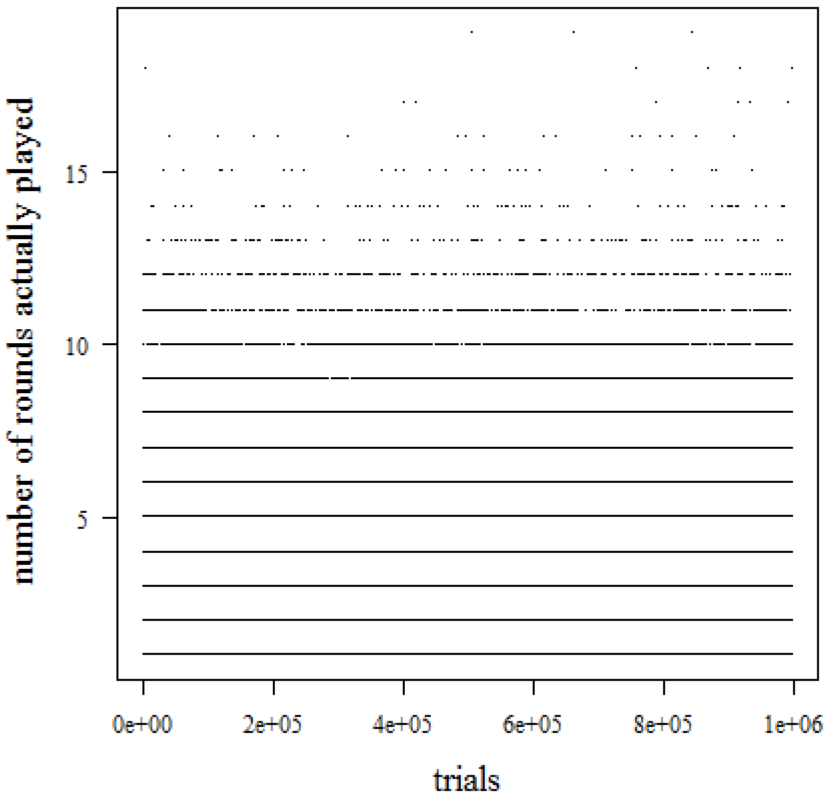


Figure 3. Number of rounds actually played from one million single trials of the St. Petersburg game.

Figure 5 replicates the power law of the game by considering one billion single trials. The straight line is given by:

$$\log c_n = -\log w - 1. \tag{5}$$

Thus, $\alpha = 1$, which means a stable Cauchy.

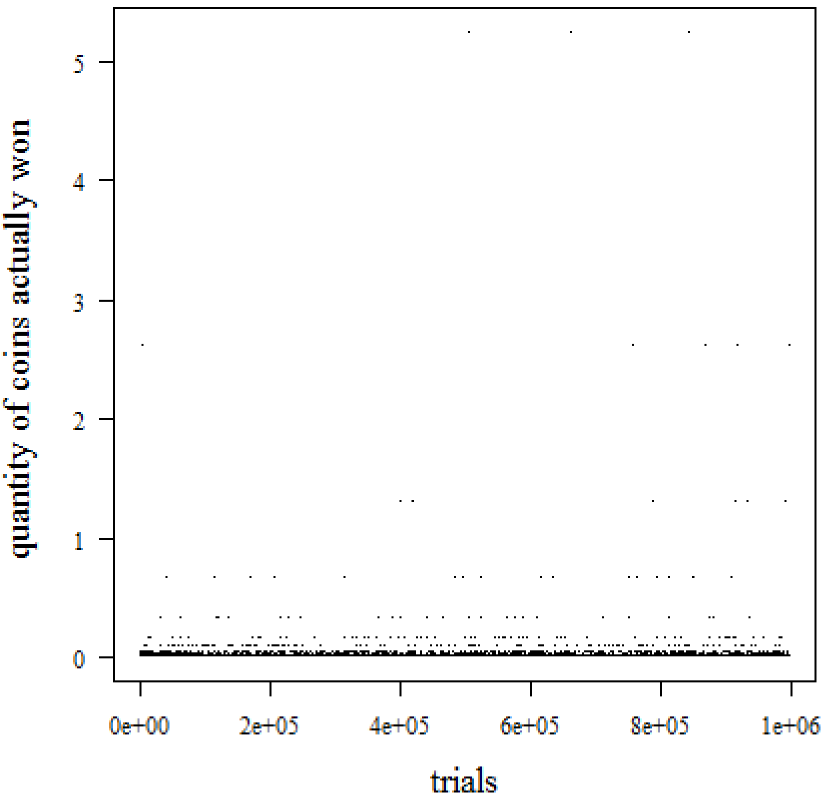

Figure 4. Quantity of coins actually won ($\times 10^5$) from one million single trials of the St. Petersburg game.

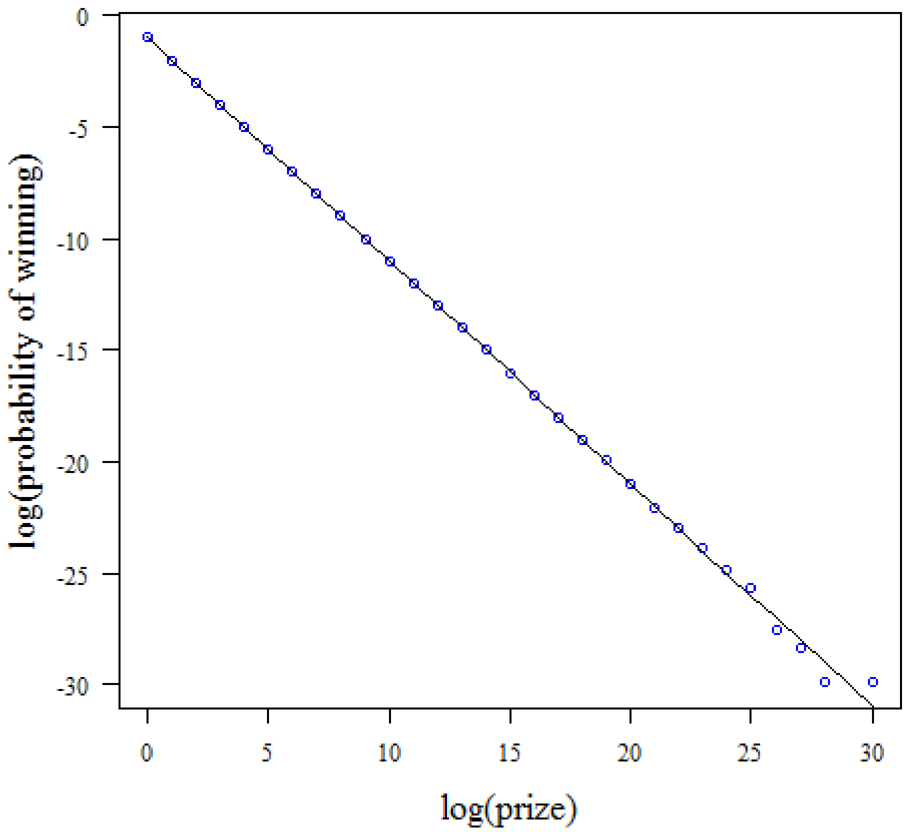

Figure 5. The St. Petersburg game power law for one billion single trials.

Figure 6 shows a standardized histogram of the one billion single trials, and Table 3 shows a statistical summary. Of note, the quartiles stabilize as the number of single trials $n$ grows in Table 3. Such a result is analytically expected. By definition [10], for discrete random variables geometrically distributed, the $q$th quartile of the number of single trials $n$ is a number $\xi_q$, such that $P(n \geq \xi_q) \geq \frac{4-q}{4}$ and $P(n \leq \xi_q) \geq \frac{q}{4}$, with $q = 1, 2, 3$. In particular, $\xi_1 = 1$, $\xi_2 = 1$, and $\xi_3 = 2$. Thus, because $X = 2^{n-1}$ is a one-to-one transformation, the first, the median, and the third quartile of $X$ are, respectively, 1, 1, and 2.

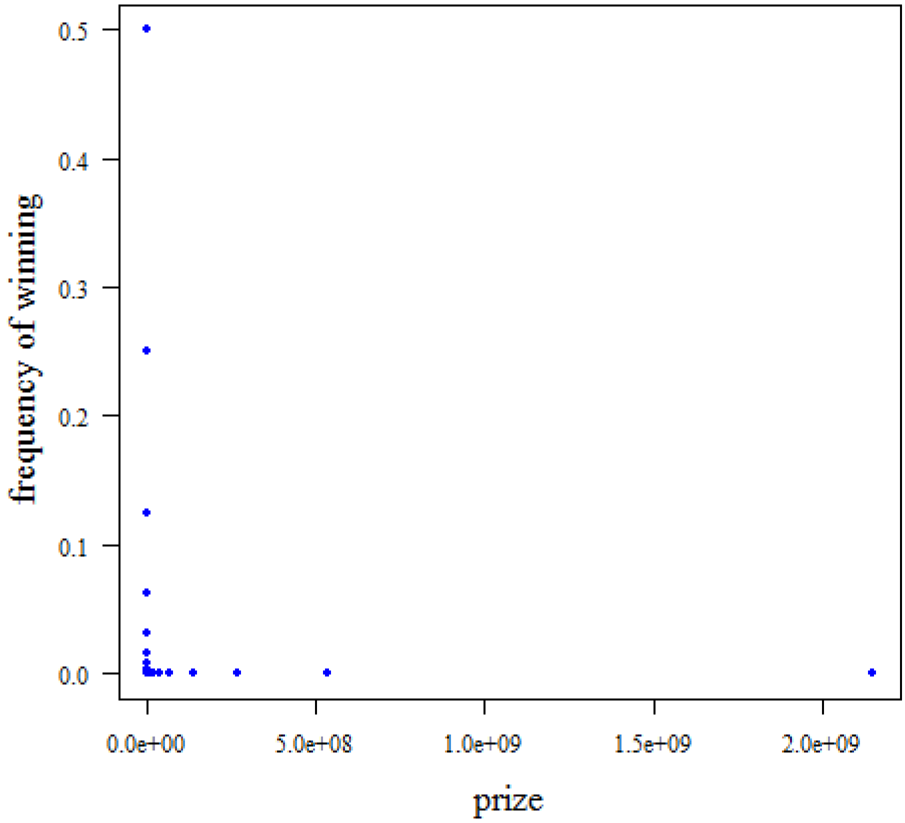

Figure 6. Standardized histogram of one billion single trials of the St. Petersburg game.

## 4. Discussion

In his classic statistics textbook, Feller [2] warned us that there is no paradox in the St. Petersburg game simply because it is not a fair game and thus there is no such thing as a fair entrance fee. It makes no sense to consider the mean as the fee because the law of large numbers and the central limit theorem are applicable only for finite expectations, and, in the St. Petersburg game, expectations are infinite. Here, we explored further the statistical physics of the game, having in mind Samuelson's observation [11] that "when you have defanged a paradox with the texture of the St. Petersburg puzzle, the problem does not disappear or fade away into banality." By considering one billion single trials, we could generate a time series large enough to be amenable to the analysis of the tails distribution. We showed the game is Paretian and its underlying distribution is scale-free. We then unveiled the St. Petersburg power law: When the probability of winning doubles, the prize is reduced by half.

Talking about the game, Samuelson [11] had already speculated that it is governed by a Cauchy: "Mother Nature might present Paul with a Cauchy distribution for extreme positive gains in the right-hand tail: Peter, the casino owner, would never be willing to do so or even ever be able to do so." The Cauchy (or Lorentzian for the physicist) is a member of the Lévy family [12] of stable distributions, which presents non-Gaussian scaling and thus has no characteristic scale [6]. The Gaussian is also a member of the Lévy family, though it presents characteristic scale. The Pareto stable distribution [13] encompass the Lévy family. Strictly speaking, the Cauchy distribution is defined for continuous random variables. Benoit Mandelbrolt [14] used the term "Cauchy flight" for the case where the distribution of discrete step sizes is a Cauchy distribution. Thus, for the Petersburg game we should use the term Cauchy flight.

Kenneth Arrow [15] observed that "not all stochastic processes can be ordered by the expected value of their utility outcomes." This is most certainly true regarding the St. Petersburg game despite that, since its very beginning, discussions of the paradox raised by the game have received rationales based on its psychophysics.

Daniel Bernoulli himself observed that the utility of the prizes grows much more slowly than the prize itself, which explains why the gamble is not attractive and most people do not think the gamble is worth more than a few coins. Bernoulli suggested a gamble should be assessed not by its expected value (a weighted average of the possible outcomes, where each outcome is weighted by its probability), but by the psychological values of the outcomes: their utilities. Because people dislike risk, if they are offered a choice between a gamble and an amount equal to its expected value, they will go for the sure thing [16]. Bernoulli then launched the expected utility theory still taught in today's financial textbooks, almost 300 years later.

Table 3. Statistical summary of one billion single trials of the St. Petersburg game.

| $n$ | $c_n$ | *Counts* | *Relative frequency* | *Cumulative frequency* |
|---|---|---|---|---|
| 1 | 1 | 499989691 | 5.00E-01 | 5.00E-01 |
| 2 | 2 | 250016460 | 2.50E-01 | 7.50E-01 |
| 3 | 4 | 124993624 | 1.25E-01 | 8.75E-01 |
| 4 | 8 | 62492742 | 6.25E-02 | 9.37E-01 |
| 5 | 16 | 31255491 | 3.13E-02 | 9.69E-01 |
| 6 | 32 | 15629774 | 1.56E-02 | 9.84E-01 |
| 7 | 64 | 7812976 | 7.81E-03 | 9.92E-01 |
| 8 | 128 | 3904127 | 3.90E-03 | 9.96E-01 |
| 9 | 256 | 1953042 | 1.95E-03 | 9.98E-01 |
| 10 | 512 | 975896 | 9.76E-04 | 9.99E-01 |
| 11 | 1024 | 487560 | 4.88E-04 | 1.00E+00 |
| 12 | 2048 | 244569 | 2.45E-04 | 1.00E+00 |
| 13 | 4096 | 122154 | 1.22E-04 | 1.00E+00 |
| 14 | 8192 | 61122 | 6.11E-05 | 1.00E+00 |
| 15 | 16384 | 30439 | 3.04E-05 | 1.00E+00 |
| 16 | 32768 | 15185 | 1.52E-05 | 1.00E+00 |
| 17 | 65536 | 7590 | 7.59E-06 | 1.00E+00 |
| 18 | 131072 | 3719 | 3.72E-06 | 1.00E+00 |
| 19 | 262144 | 1893 | 1.89E-06 | 1.00E+00 |
| 20 | 524288 | 995 | 9.95E-07 | 1.00E+00 |
| 21 | 1048576 | 473 | 4.73E-07 | 1.00E+00 |
| 22 | 2097152 | 227 | 2.27E-07 | 1.00E+00 |
| 23 | 4194304 | 122 | 1.22E-07 | 1.00E+00 |
| 24 | 8388608 | 67 | 6.70E-08 | 1.00E+00 |
| 25 | 1.7E+07 | 33 | 3.30E-08 | 1.00E+00 |
| 26 | 3.4E+07 | 19 | 1.90E-08 | 1.00E+00 |
| 27 | 6.7E+07 | 5 | 5.00E-09 | 1.00E+00 |
| 28 | 1.3E+08 | 3 | 3.00E-09 | 1.00E+00 |
| 29 | 2.7E+08 | 1 | 1.00E-09 | 1.00E+00 |
| 31 | 1.1E+09 | 1 | 1.00E-09 | 1.00E+00 |

It is then no surprise that, as Samuelson [11] observed, "even for Lévy distributions with no finite integral moments, the expectation of the utility of wealth is finite, being bounded by the bounds of the utility of wealth." In particular, Bernoulli used what is called today Weber's law, according to which most psychophysical functions relating the subjective quantity in the observer's mind and the objective quantity in the material world are logarithm. Coins in the material world and the utility of coins in people's mind are related by a logarithm function.

By considering logs, one can turn the St. Petersburg game fair, as observed by Feller [2]. So, by proposing expected utility theory, Bernoulli turned the St. Petersburg game fair. However, this is psychophysics. In line with Bernoulli, a huge literature followed through the centuries by considering only the psychophysics of the game. The psychophysics of the St. Petersburg game is valuable in itself, and is now maturing through a neuroscience perspective [7]. However, the original game is not a fair one, and the psychophysics approach is talking about a different game.

In this connection, take this comment by Samuelson [11]: "A different line of reasoning, which is less to my liking, runs as follows: Because of the need to avoid the St. Petersburg paradox, it is necessary in axiomatizing stochastic choice theory to assume the axiom that people do not have linear utility." In terms of our findings, if the St. Petersburg has any implication for stochastic choice theory, that is, the game offers a counter-example that no theory of risky choice can afford to neglect. One theory cannot be extended to random variables with infinite expectations, in which case there is no characteristic scale, and the mean and higher moments are not a meaningful way to characterize data.

Moreover, an infinite mean would be feasible only in the presence of infinite single trials, which is unfeasible in practice. Thus, because the quartiles of the distribution do not grow (as we showed), a theoretical mean only signifies the distribution of a quantity of coins does not reach an equilibrium. For practical purposes, then, taking $n = \log_2 c_n$ makes the moments finite. Such a procedure is analogous to the common practice in finance of taking log returns of financial time series.

Finally, we have to mention that the authors in Refs. [5] and [17] already came across the power law in Figure 1 considering one million single trials. However, they did not formulate the law explicitly, as we did. Nor did they perform the statistical physics analysis to uncover the underlying stable distribution of the game.

## 5. Conclusion

The St. Petersburg paradox presents a counter-example for any devised theory of risky choice in which random variables with infinite expectations are present, and thus there is no characteristic scale. No choice theory can exist in such a polar situation. Despite the huge volume of literature of the paradox, there are few examples of empirical studies. Here, we offer an empirical approach where one billion single trials of the game generate a time series that can be analyzed through statistical physics methods. We formulate the power law governing the St. Petersburg game ("when the probability of winning doubles, the prize is reduced by half") and show that its underlying stable distribution is a Cauchy flight, as hinted in the past by Paul Samuelson.

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