The ongoing history of economic conservation laws

Heinrich, Torsten

Institute for Institutional and Innovation Economics (IINO), University of Bremen, Bremen, Germany

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Humans: correct in making leap from wealth as currency to wealth as energy. But logic failure: wealth ultimately is extension of desires, fluctuating with emotions and state of mind. Desires: when all are supported in purely adaptable system, true wealth is achieved.
1999 computer game 'Alien Crossfire' (statement given about humans by one of the game's alien characters)

Abstract

Building on Mirowski's historical analysis in the 1980s, the role of conservation laws in economic equilibrium theories is reexamined. Both static Walrasian systems and dynamic systems of intertemporal optimization are considered. While the formal derivation of conservation laws is discussed in detail, the paper also reviews the growing literature tradition in economics that is aware of the existence of conservation laws in economic equilibrium theories and tries to incorporate them into their models.

1 Introduction

It has been almost 25 years since Mirowski (1989) published his seminal study on the history of the energy-value analogy and of concepts from physics being employed in economics in general. Mirowski discusses the history of the neoclassical marginalist revolution as well as the contemporary development of both physics and economics. He arrives at the conclusion that the analogies which were explicitely drawn between the two by the proponents of the marginalist revolution were mistaken.[1] It requires unrealistic assumptions and implies conclusions that are neither necessary nor useful. The problem was perhaps aggravated when the analogy was almost forgotten by successive generations of economists while the method subsequently became the unique and almost unchallenged textbook-standard in economics - an arrangement that continues today.

*University of Bremen; torsten.heinrich@uni-bremen.de

[1]Mirowski's (1989) analysis operates both on an mathematical analytical level and on the history of though level. As shown by Hands (1993) some issues are actually more complex than discussed in Mirowski (particularly that a conservative vector field like the ones in classical mechanics is not formed by prices in general equilibrium models but rather by price-equivalents resulting from compensated demand functions) but the general analysis holds.

The one who made the analogy most explicit was probably Fisher (1892): he compiled a table of what he believed to be analogous between physics and economics. As Mirowski (1989) pointed out, many of these analogies were highly questionable. Mirowski also noted the omission of other analogies that would have logically followed from the stated ones, most notably the conservation law and its presumable Walrasian economic equivalent, conjectured by Mirowski to be the constancy of utility and expenditure. It is clear that this has no useful economic interpretation while other alternative economic conservation laws do at least come as additional constraints that the model may not violate but that do not bestow additional benefit to the usability of the model.

Since, many works have been published about Mirowski's study as well as the relation between neoclassical economics and physics in general (De Marchi, 1993, Smith and Foley, 2008).

However, at the same time and without taking note of this line of research a tradition of othodox economic literature has developed that is well aware of the fact that the use of Lagrangian and Hamiltonian mechanics in economics implies economic conservation laws of some sort (Samuelson, 1970, 1990a,b, Sato, 1981, 2006, Kataoka and Hashimoto, 1995). In fact, they derive and interpret the conservation laws for a number of advanced economic models and try to establish empirical evidence for their presence in real economic systems (Sato, 2006). The models used in this tradition belong to the dynamic infinite horizon optimization models as they are commonly used in the endogenous growth theory.

The purpose of the current paper is to reconsider the case of conservation laws in the Lagrangian and Hamiltonian systems used in both classical Walrasian equilibrium theory 2 and in modern dynamic optimization models 3. The derivation, implications, and literature will be discussed in these two sections before the paper proceeds to offer a conclusion with a more general assessment of the role of economic power laws 4.

2 Conservation Laws in Walrasian Equilibria

In classical mechanics, Lagrangians are constructed as dynamic equations of kinetic (T) and potential energy (V) in space for the purpose of deriving the movement path of a particle. The space may or may not be constrained (by a constraint F). While kinetic energy depends on space and the dynamic change(s) of this (vector of) variable(s) $\frac{\partial x_i}{\partial t} = \dot{x_i}$, hence $T(x_i, \dot{x_i})$, potential energy depends only on the position in space, hence $V(x_i)$.[2] The Lagrangian is constructed as

$$\mathcal{L}(x_i, \dot{x_i}, \lambda) = T(x_i, \dot{x_i}) - V(x_i) - \lambda F(x_i) \tag{1}$$

where λ is the Lagrangian multiplier.
Since the Lagrangian is not directly dependent on time t, a Hamiltonian can be constructed as the Legendre transformation of the Lagrangian with respect to the dynamic change variable(s) $\dot{x_i}$. The value of that Hamiltonian must be constant over time and over all optimal paths which constitutes the law of the conservation of energy. The Legendre transformation is an involutive transfromation of a monotonic function $f(x)$ of a variable x such that the transformation

[2]For details on the mathematical methodology, see Zia et al. (2009) or Arnol'd (1991).

is a function $f_L(x_L, x) = x_L x - f(x)$ of $x_L = \min(px - f(x))$. Since it is involutive the Legendre transformation of the function f returns the function f itself. The Legendre transformation may be stated as $f_L(x) = f_L(x_L(x))$, yielding

$$f_L(x) = x\frac{\partial f(x)}{\partial x} - f(x).$$

Consequently, the Hamiltonian corresponding to the above Lagrangian is

$$\mathcal{H}(x_i, \dot{x}_i, \lambda) = \sum_i \dot{x}_i \frac{\partial \mathcal{L}(x_i, \dot{x}_i, \lambda)}{\partial \dot{x}_i} - \mathcal{L}(x_i, \dot{x}_i, \lambda) = \text{const.} \tag{2}$$

$$\mathcal{H}(x_i, \dot{x}_i, \lambda) = \sum_i \dot{x}_i \frac{\partial \mathcal{T}(x_i, \dot{x}_i, \lambda)}{\partial \dot{x}_i} - T(x_i, \dot{x}_i) + V(x_i) + \lambda F(x_i) = \text{const.} \tag{3}$$

where $\sum_i \dot{x}_i \frac{\partial \mathcal{T}(x_i, \dot{x}_i, \lambda)}{\partial \dot{x}_i} = 2T(x_i, \dot{x}_i$, thus

$$\mathcal{H} = 2T - T + V + \lambda F = T + V + \lambda F = \text{const.}$$

which in unconstrained problems (i.e. without constraint F) reduces to $\mathcal{H} = T + V = \text{const.}$

In 1892, Fisher (1892) had compiled a table of which concepts in economics he thought te be equivalent to which concepts in classical mechanics. There, he reiterated the alleged analogy of utility to energy or work. This appears to be based on the assessment, that in classical mechanics, work is the product of force and space, i.e. with the x being space and $\mathcal{L} = T(x) - V(x)$ being the Lagrangian composed of kinetic (T) and potential energy (V) function, hence force $\frac{\partial \mathcal{L}}{\partial x}$, $x\frac{\partial \mathcal{L}}{\partial x}$. For economics, his table gives utility analogously to work or force in physics as the product of commodity space and marginal utility, i.e. with commodity space denoted by x and the utility function $U(x)$, $x\frac{\partial U}{\partial x}$. This does, however, only hold if the marginal utility were constant in x (which is usually not the case since utility functions are usually assumed to be concave) and would otherwise conflict with the utility function $U(x)$.

There has been some confusion about how to interpret the equivalent to kinetic energy in Walrasian economic models. Mirowski (1989) conjectured it to be expenditure so that the economic conservation law would require the sum of utility and expenditure to be constant. However, the Walrasian economic optimization problem is not dynamic. Its Lagrangian is constructed from the said utility function $U(x_i)$ (or equivalently shaped production function in production theory) and a budget constraint $F(x_i) = \sum_i p_i x_i - \overline{W} \leq 0$ (where the p_i are the prices of goods or input factors i and $\overline{W}$ is the budget). Since the constraint is linear, in order for the problem at hand to yield one unique optimum the utility or production function has to be monotonically increasing, concave and twice differentiable - which is a pretty strong assumption. Hence, we have a Lagrangian

$$\mathcal{L}(x_i, \lambda) = U(x_i) - \lambda F(x_i) \tag{4}$$

and (as the Lagrangian does not directly depend on time t) the corresponding Hamiltonian

$$\mathcal{H}(x_i, \dot{x}_i, \lambda) = \sum_i \dot{x}_i \frac{\partial \mathcal{L}(x_i, \lambda)}{\partial \dot{x}_i} - \mathcal{L}(x_i, \lambda) = \text{const.} \tag{5}$$

However, since the Lagrangian does also not depend on $\dot{x}$, the sum in the first term of the Hamiltonian is not the expenditure; instead it is always zero for this kind of problems. The Kuhn-Tucker conditions require further $\lambda F(x_i)$ to be zero in equilibrium, thus we effectively obtain a conservation law of

$$\mathcal{H}(x_i) = -\mathcal{L}(x_i) = -U(x_i) = \text{const.}, \tag{6}$$

i.e. for all optimum points the total utility is constant - a tautologically true statement.

However, Walrasian optimization problems could also be constructed as dynamic problems, for instance, if the constraint were to depend on the change of commodity quantities in time. Consider a situation where the utility does not arise from the consumption of quantities x_i of goods i but rather from posession of those goods (say, real estate or luxury goods the agent has no practical use for, i.e. which are only for display of wealth). In this case the agent pays only her additional purchases of goods x_i which may be negative if she sells some of a particular good again, i.e. $\dot{x_i}$. She still has a constant stream of income of $\overline{W}$. The budget constraint is then $F(\dot{x_i}) = \sum_i \dot{x}_i p_i - \overline{W} \leq 0$; the Lagrangian is

$$\mathcal{L}(x_i, \lambda) = U(x_i) - \lambda(\sum_i \dot{x}_i p_i - \overline{W}) \tag{7}$$

with the Hamiltonian

$$\mathcal{H}(x_i) = \sum_i \dot{x}_i \frac{\partial \mathcal{L}(x_i, \lambda)}{\partial \dot{x}_i} - \mathcal{L}(x_i) = \sum_i \dot{x}_i \lambda \frac{\partial F(\dot{x}_i}{\partial \dot{x}_i} - U(x_i) = \sum_i \lambda \dot{x}_i p_i - U(x_i) = \text{const.} \tag{8}$$

For this, in turn, Mirowski's proposition is (almost) true, since $\lambda \sum_i \dot{x}_i p_i$ is the net expenditure, weighted with the constraint's shadow price λ which allows this monetary quantity to be transformed into a utility-like measure. The difference of this (summed up over the goods) and the utility must be constant (over all points in time along the system's development trajectory). Within the assumptions of the theory and the particular budget function considered here, this conservation law makes perfect sense; it serves as a condition for the intertemporal optimum.

Given that some of the assumptions[3] of the theory are, however, rather heroic, the theory as a whole seems oddly fragile. The so derived conservation law makes this all the more obvious.

3 Dynamic Conservation Laws in Economics

There is another class of neoclassical economic conservation laws. They are a consequence of the theory of endogenous growth, so to speak the queen of neoclassics that does not only define static situations but allows the derivation of a dynamic optimal path and thus conclusions with respect to economic growth and development. Endogenous growth models typically maximize a discounted

[3]The monotonically increasing, concave, twice differentiable utility function which supposedly guides perfectly rational and (for free) well-informed agents who are willing and able to constantly perform computations of infinite-dimensional optimization problems in order to make decisions.

intertemporal utility or income function (income often being written as the growth of $\dot{K}$)

$$\int \dot{K} dt \tag{9}$$

subject to some intertemporal production function

$$F(K, \dot{K}) = 0 \tag{10}$$

that defines the limit of available capital. The function may be extended to two or more capital variables of which all, some, or only one may be included into the objective function: Limiting the discussion here to a model with the growth of only one sort of capital in the target function[4], say $\dot{\kappa}$,

$$\int \dot{\kappa} dt \tag{11}$$

subject to

$$F(\kappa, \dot{\kappa}, K_i, \dot{K}_i) = 0 \tag{12}$$

the Lagrangian is obtained as

$$\mathcal{L}(\kappa, \dot{\kappa}, K_i, \dot{K}_i, \lambda) = \int \dot{\kappa} - \lambda_t F(\kappa, \dot{\kappa}, K_i, \dot{K}_i) dt. \tag{13}$$

Applying the Legendre transformation, the Hamiltonian follows as[5]

$$\mathcal{H} = -\mathcal{L} + \dot{\kappa} \frac{\partial \mathcal{L}}{\partial \kappa} + \sum_i \dot{K}_i \frac{\partial \mathcal{L}}{\partial \dot{K}_i} = \text{const.} \tag{14}$$

$$\mathcal{H} = -\dot{\kappa} + \lambda_t F + \dot{\kappa} 1 - \lambda_t \dot{\kappa} \frac{\partial F}{\partial \kappa} - \lambda_t \sum_i \dot{K}_i \frac{\partial F}{\partial \dot{K}_i} = \text{const.} \tag{15}$$

$$\mathcal{H} = -\lambda_t \left(\dot{\kappa} \frac{\partial F}{\partial \kappa} + \sum_i \dot{K}_i \frac{\partial F}{\partial \dot{K}_i} \right) = \text{const.} \tag{16}$$

$$\frac{\mathcal{H}}{\lambda_t} = \dot{\kappa} - \frac{\partial F}{\partial \kappa} + \sum_i \dot{K}_i - \frac{\partial F}{\partial \dot{K}_i}. \tag{17}$$

The right hand side of this last equation is usually interpreted as the total income at time t. $\dot{\kappa}$ and $\dot{K}_i$ are the changes in the stocks of all capital goods; for the $\frac{\partial F}{\partial \kappa}$ and $\frac{\partial F}{\partial \dot{K}_i}$ the heroic interpretation is made that they are, in fact, the prices of the respective capital goods (Samuelson, 1970, Kataoka and Hashimoto, 1995). Note that this says not much with respect to an intertemporal conservation laws since λ_t can change over time. However, since λ_t depends on the rest of the system of equations, its possible variation is limited and linked to the optimization problem. For instance Samuelson (1970) - making the bold assumption that $\lambda(t)$ is continuous - used the Euler-Lagrange equation in the system's optimum

[4]This is a relatively simple setup. The derivation of the conservation law roughly follows Samuelson (1970, 1990a,b).

[5]Since $\mathcal{L}$ does not directly depend on t.

path[6] (i.e. with $\mathcal{L} = 0$) to obtain an differential equation $\frac{\dot{\lambda}}{\lambda}$ which he then transforms into an exponential function by defining the right-hand side (with additional assumptions) to be the growth rate of the price-weighted capital stock $\sum_i -\frac{\partial F}{\partial \dot{K}_i} K_i - \frac{\partial F}{\partial \dot{\kappa}} \kappa$ ('national wealth'). Having total income and the capital stock so similarly defined (just one with K the other one with $\dot{K}$) he proceeds to show that the development of income is then governed by the same function and that the ratio between the two must therefore be constant at all times (given that the economy follows its equilibrium path).

More elaborate versions of such models by Sato (2006) and by Kataoka and Hashimoto (1995) (those drawing on Sato (1981)) use Lie groups to derive the invariance conditions that must be satisfied in order to obtain the conservation law(s). Kataoka and Hashimoto (1995) arrive at the same conclusion as Samuelson (1970, 1990a,b) in a very different way. Here, the exponential depreciation in the original present value target function is preserved in the conservation law which results in both capital and income following the inverse of the exponential depreciation (e^{rt} following the present value target function e^{-rt} with r being the factor of depreciation). That is, the main result of Samuelsons 1970 paper, the constant relation of income and capital, remains unchanged in those theories, though this conjecture is derived in a different way and the underlying model is completely different.[7] It is obvious that the exponential function is merely a convenient consequence of the way the time-preferences are constructed - it resembles the exponential growth paths predicted by the endogeous growth models of the time which were used to approximate the empirical GDP development in the decades following World War II. With different time prefecence terms, the shape of the function would change but the constant ratio of income and capital would be preserved at least as long as the function remains monotonic.

Consequently, the proponents of this theory tradition attempted to show that this constancy of the income to capital ratio also holds empirically. Sato (2006), for instance, finds roughly constant but different ratios for the pre-World War II (0.23) and the post-World War II US economy (0.33) as well as mixed results for other OECD countries. This is explained (particularly for the Japanese case in which a continuing decline of the supposedly constant rate is found) by suggesting that the environment of the optimization problem (prices, time-preference factors, etc.) may not have been constant thus resulting in a shift of the optimization problem. It is not unlikely that the ratios have recently - with the ongoing financial crisis and recession - become more erratic. Further, for the cases in which the ratio has indeed been constant for a prolonged period of time, it might be prudent to also consider alternative explanations such as firm's reinvestment behavior or expectations (perhaps even self-fulfilling expectations) in economically calm times. This would not necessarily imply that intertemporal neoclassical models and the conservation laws that follow from them are well-founded.

[6] The Euler-Lagrange equations are defined as $0 = \frac{\partial \mathcal{L}}{\partial K_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{K}_i}$ for all i (for all input factors) and equivalently for (output) κ. Differentiating the term $\lambda_t F(\kappa, \dot{\kappa}, K_i, \dot{K}_i)$ results according to the product rule in a term which contains both λ and $\dot{\lambda}$ and can be rewritten in the form $\frac{\dot{\lambda}}{\lambda} = ...$

[7] In fact, Kataoka and Hashimoto (1995) derive two different conservation laws which do however always include or may be transformed into this constant ratio.

4 Conclusion: Can Economic Conservation Laws be Preserved?

The two traditions of theory - Mirowski and critics of mechanical models and conservation laws in economics on the one hand and those who try to derive and interpret the conservation laws on the other hand - have so far mostly ignored each other. The reevaluation of both traditions in the current paper suggests that conservation laws in economic equilibrium models do generally have a useful interpretation and fit well into the theory. In fact, they might offer additional insight and make the general equilibrium theory easier to understand both as such and in relation to similar models in classical mechanics.

The problems that remain are, of course, the assumptions on which the general equilibrium theory rests and that have been sharply criticised by entire traditions of economists starting with Sraffa (1926, 1960) and Young (1928) to Kaldor (1972), institutionalists ((Ayres, 1935, Elsner, 2008)), and Keynesians (Robinson, 1978) to evolutionary economists, (Nelson and Winter, 1974), ecological economists (Georgescu-Roegen, 1975), and, more recently, complexity economists (Kirman, 1997, Foley, 1998, Keen, 2001, Chen, 2010, Foley, 2010) as well as many others. As these critics have continued to point out, there is no evidence that either the aggregated (demand, supply, utility) functions exist, mor that they are well-behaved, nor that any perfectly rational optimization takes place on the part of the individuals.[8]

Thus the question to be asked may be: Can economic conservation laws of either the Walrasian or the Samuelsonian form be, well, conserved? It should be noted that general equilibrium theory continues to be the only integrated approach that allows to conveniently grasp the economy as a whole and assess the interdependence and interaction of different sectors and levels of aggregation. The price paid for this is the simplification necessary for the assumptions of the theory - conservation laws are an integral part of it. The problems of their interpretation are evidenced by the worrying omission of the subject throughout much of the history of general equilibrium theory in economics as discussed by Mirowski (1989) and others (De Marchi, 1993). The careless application of conclusions drawn from economic conservation laws to economic reality should probably be avoided.

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[8]Psychological economic approaches do, in fact, suggest otherwise. See Tversky and Kahneman (1972), Kahneman and Tversky (1973).

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