



Munich Personal RePEc Archive

Charity Auctions for the Happy Few

Bos, Olivier

Panthéon-Assas University

1 December 2015

Online at <https://mpra.ub.uni-muenchen.de/68212/>
MPRA Paper No. 68212, posted 05 Dec 2015 14:15 UTC

Charity Auctions for the Happy Few*

Olivier Bos[†]

This version: December, 2015

Abstract

Recent literature has shown that all-pay auctions raise more money for charity than either winner-pay auctions or lotteries. We demonstrate that first-price and second-price winner-pay auctions have a better revenue performance than first-price and second-price all-pay auctions when bidders are sufficiently asymmetric. Lotteries can also provide higher revenue than all-pay auctions. To prove this, we consider a framework with complete information. Complete information is helpful and may reflect events that occur, for instance, in a local service club (such as a voluntary organization) or at a show-business dinner.

KEYWORDS: All-pay auctions, charity, complete information, externalities

JEL CLASSIFICATION: D44, D62, D64

*Financial support from the ANR DBCPG and the CESifo is gratefully acknowledged. This research has been conducted as part of the Labex MME-DII (ANR11-LBX-0023-01) project. I am grateful for very helpful comments to two anonymous referees. I would also like to thank Lawrence Ausubel, Gabrielle Demange, Damian Damianov, Fernanda Estevan, David Ettinger, Arye Hillman, Philippe Jehiel, Laurent Lamy, Lucie Ménager, Konrad Mierendorff, Sander Onderstal, Christina Pawlowitsch, Martin Ranger, and Nora Szech as well as the participants at the Paris School of Economics, IZA and WZB seminars and the participants at the SAET Conference (Kos), the Third World Congress of the Game Theory Society, ESEM (Barcelona), and the CESifo Workshop on Advances in the Theory of Contests and its Applications for helpful conversations and comments. All errors are mine.

[†]Panthéon-Assas University. Address: Panthéon-Assas University – LEMMA, 4 rue Blaise Desgoffe, 75006 Paris France. E-mail: olivier.bos@u-paris2.fr. Tel: +33170232005.

1 Introduction

The recent literature of fundraising mechanisms is not conclusive about the relative performance of all-pay auctions, winner-pay auctions and lotteries. Theoretical results (Goeree, Maasland, Onderstal and Turner (2005), Engers and McManus (2007)) and experimental results (Schram and Onderstal (2009)) show, in a symmetric independent private values model, that all-pay auctions raise more money for charity than winner-pay auctions and lotteries. A field experiment by Carpenter et al. (2008) concludes in favor of winner-pay auctions instead of all-pay auctions.

We show that the asymmetry among bidders' values in a complete information framework can lead winner-pay auctions to raise more money for charity than all-pay auctions and lotteries. This invalidates the theoretical and experimental results of Goeree et al. (2005), Engers and McManus (2007) and Schram and Onderstal (2009), and might support the field experiment results of Carpenter et al. (2008). More precisely, our purpose is twofold. First, we would like to determine if the results found theoretically and confirmed in a lab experiment that all-pay auctions raise more money for charity than winner-pay auctions and lotteries are robust. Agents do not usually have the same preferences. Thus a way to test the robustness of results found in theoretical and experimental literature is to consider asymmetry either in the evaluation for the item sold or in altruism. Second, we would like to investigate if bidders' asymmetry could explain the results from the field experiment (Carpenter, Homes and Matthews (2008)), which are that the winner-pay auctions can raise more money than the first-price all-pay auction. Indeed, Carpenter, Homes and Matthews's (2010) theoretical investigation of endogenous participation shows that participations cost do not provide a convincing explanation. Therefore, we compare five mechanisms: the first and second-price all-pay auctions, the first and second-price winner-pay auctions and the lotteries in a complete information framework.

Why are charity auctions interesting to analyze? Charity auctions have been held in the United States and in Europe for many years now.¹ At such auctions, an item (for example a key case of zero value or an item donated by a luxury brand) is sold and the proceeds go to charity. Although many charity auctions are held on the Internet some are conducted among wealthy guests at charity dinners. These events may occur at local service clubs (such as the Rotary Club², the Lions Club³ and other (types of) voluntary organizations) or at show-business dinners. Potential bidders tend to be acquainted with each other in varying degrees. Beyond the item's value, the valuations of potential bidders vary with their interest in the voluntary organization (their altruism or philanthropy). Thus, potential bidders make a trade-off between giving money for fundraising and keeping it for some other personal use. Unlike non-charity auctions, though, here the amount paid is "never lost". Since the money raised is used to finance

¹Although historically more common in the United States, charity auctions have long been held in Europe, e.g. the *Hospices de Beaune wine auctions*, <http://www.france.fr/en/celebrations-and-festivals/hospices-de-beaune-wine-auctions-ancestral-event.html>.

²The *Rotary Club* is a worldwide organization of business and professional leaders that provides humanitarian services, encourages high ethical standards in all vocations, and helps build goodwill and peace in the world. There are about 32,000 clubs in 200 countries and geographical areas and 1,000 clubs in France including in Paris, of course, but also small towns like Niort. <http://www.rotary.org/>

³<http://www.lionsclubs.org/>

a charitable purpose, every participant in the charity auction may benefit from it, independently of the winner’s identity. More specifically, the money raised by each potential bidder impacts the utility of all participants as they take advantage of an externality of the money raised for charity.

Under complete information, such auctions can be compared to those described in [Ettinger \(2010\)](#) who analyzes a general winner-pay auction framework with financial externalities.⁴ These externalities are independent of the winner’s identity and can be applied to charity auctions in which only the winner pays.⁵ Moreover, he shows that there is no “revenue equivalence” with these externalities.⁶ [Maasland and Onderstal \(2007\)](#) investigate winner-pay auctions with this kind of linear externalities in an independent private signals model. Their paper can also be applied to charities. They find similar qualitative predictions to [Ettinger \(2010\)](#): the second-price winner-pay auction outperforms⁷ the first-price winner-pay auction. [Goeree et al. \(2005\)](#) analyze charity auctions in the symmetric independent private values model. They show that, given the externality, all-pay auctions raise more money for charity than both winner-pay auctions (second-price outperforms first-price) and lotteries. In particular, they determine that the optimal fundraising mechanism is the lowest-price all-pay auction with an entry fee and a reserve price. The lab experiment conducted by [Schram and Onderstal \(2009\)](#) show results similar to [Goeree et al. \(2005\)](#), namely that the first-price all-pay auction leads to a higher revenue than winner-pay auctions and lotteries. [Bos \(2011\)](#) compares in a complete information framework the first-price all-pay auction and lotteries with asymmetric agents. In this paper, it is shown that the result of [Goeree et al. \(2005\)](#) can be reversed under complete information, which means that lotteries outperform all-pay auctions, if agents are asymmetric enough. [Engers and McManus \(2007\)](#) report findings similar to [Goeree et al. \(2005\)](#). Contrary to [Goeree et al. \(2005\)](#), a psychological effect comes into play: the winner benefits from a higher externality with her own bid, the others’ bids having a lower effect on her. In their setting, as in [Goeree et al. \(2005\)](#), first-price all-pay auctions and second-price winner-pay auctions raise more money than first-price winner-pay auctions. Moreover, first-price all-pay auctions outperform each winner-pay auction only for a sufficiently high number of bidders.

[Carpenter et al. \(2008\)](#) report testing the predictions of [Engers and McManus \(2007\)](#) and [Goeree et al. \(2005\)](#) in a field experiment. Similar objects were sold in four American pre-schools through three different mechanisms which were the first-price all-pay auction and the first-price and second-price winner-pay auctions. They studied the determinants of the bidders’ behavior and the revenue raised. Contrary to the theoretical predictions, first-price all-pay auctions did not produce higher revenues than winner-pay auctions. One main explanation for the gap be-

⁴To the best of our knowledge, [Ettinger \(2010\)](#) is the only one to consider general externalities which could be non-linear.

⁵Actually, [Ettinger \(2010\)](#) investigates a framework with two kinds of externalities. One is independent of the winner’s identity and the other depends on the winner’s identity.

⁶The *revenue equivalence principle* (see [Myerson \(1981\)](#)) is one of the most famous and important results in auction theory. It determines that every auction mechanism, under some assumptions such as available information on the bidders’ type and neutral-risk aversion, leads to the same expected revenue independently of the payment rule. For more details the textbook of [Krishna \(2009\)](#) makes for useful preliminary reading.

⁷In the following, *outperform* means *generate higher revenue than*.

tween theory and the field experiment could be a non-participation effect (see [Carpenter, Homes and Matthews \(2010\)](#)), due to unfamiliarity with these mechanisms and their complexity: the participants did not know the all-pay design and few took part in second-price auctions on the Internet.

We consider a complete information framework to analyze the revenue performance of all-pay auctions, winner pay-auctions and lotteries. Complete information can help to provide analyses which are not available from the usual incomplete information.

First, complete information makes it easier to analyze asymmetries among bidders in a charity setting. In the current setting we are able to distinguish how altruism and asymmetry can independently affect bidding strategies and expected revenue.

Second, as recently pointed out by [Damianov and Peeters \(2012\)](#), complete information leads to a better understanding of the payment rule effects (through altruism) on bidding strategies and revenue raised. [Damianov and Peeters \(2012\)](#) distinguish three externalities attributable to the payment rule which can be analyzed separately in a complete information setting. In the second-price all-pay and winner-pay auctions, the highest bidder benefits from a positive externality: an increase in her bid raises her probability of winning without affecting her payment. A second positive externality is attributable to the bid of the second highest bidder. Any increase in her bid will raise the winner's payment by the same amount and so improve the second highest bidder's payoff. Finally, following [Damianov and Peeters \(2012\)](#) and [Morgan \(2000\)](#), the second-price all-pay auction leads to a negative externality due to the expected increase of revenue relative to the first-price all-pay auction.

Third, as [Damianov and Peeters \(2012\)](#) write, "the complete information model helps us further clarify the reasons for the superiority of auctions". While their setting features symmetric participants, their intuition to explain why all-pay auctions can outperform lotteries can still be applied here. Following [Morgan \(2000\)](#), they suggest that externalities are greater from auctions than lotteries.

We analyze first-price and second-price all-pay auctions for charity and compare this analysis to known results of winner-pay auctions and lotteries. In this framework, the externalities are such that every bidder derives as much advantage (obtains as much utility) from her own bid as from her rival's bid. Additionally, bidder i 's adjusted-value is the ratio of her valuation of the item sold and the fraction of her payment which she perceives as a cost given her altruism for the charitable cause. Bidders are then arranged in such a way that the adjusted-values and valuations are ranked in the same order. This ranking and its consequences are discussed.

The first-price all-pay auction equilibrium is characterized and the expected revenue computed. As in a case without externalities, there is no pure strategy Nash equilibrium and only the two bidders with the highest adjusted-values are active.

The equilibrium in the second-price all-pay auction is also characterized and the expected revenue computed. The results are then compared to those of [Ettinger \(2010\)](#) and [Bos \(2011\)](#) who analyze winner-pay auctions and lotteries with externalities that do not depend on the winner's identity and which could be applied to charity auctions.

The expected revenue from all-pay auctions can be dominated by the revenue from winner-

pay auctions and lotteries contrary to the results of [Goeree et al. \(2005\)](#). Actually, above a certain threshold of asymmetry in the bidders' valuations, winner-pay auctions and lotteries can raise more money for charity than all-pay auctions. These results might also be related to the work of [Carpenter et al. \(2008\)](#). Their findings could be due to strong asymmetry between bidders, even if unlike the present results they have more than two participants, which makes a big difference. The present results are examined by analyzing the bidders' altruism. Additionally, revenues of winner-pay auctions and lotteries are compared.

2 The model

At a charity dinner, an indivisible object (or prize) is sold through an auction. This prize is allocated to one of the potential bidders $N = \{1, \dots, n\}$ contingent upon their bids $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}_+^n$. As the bidders usually meet each other at these kinds of events, the willingness to pay and the valuation ranking of each bidder, $v_1 > v_2 > \dots > v_n$, are common knowledge.⁸ In the following the function t_i represents for each bidder i her payment $t_i(\mathbf{x})$ to the charity organization for the vector of bids \mathbf{x} .

Bidders look to raise the maximum amount of money for charity and are risk neutral. Each bidder benefits from her own participation in the charity auction and from others' as well. In other words, the money raised by each potential bidder impacts the utility of all the participants including herself. Thus, the bidder's utility function includes an externality which depends on the amount of money raised for the charity. Accordingly we consider the externality as a function with a single argument $\sum_{j=1}^n t_j(\mathbf{x})$. The externality is independent of the winner's identity and only takes into account the amount raised. The externality that participants benefit is given by $\alpha_i \sum_{j=1}^n t_j(\mathbf{x})$ where $\alpha_i \geq 0$ is the coefficient of bidder i 's altruism for the charity.⁹

It follows that bidder i 's utility is given by¹⁰

$$U_i(\mathbf{x}) = \begin{cases} v_i - t_i(\mathbf{x}) + \alpha_i \sum_{j=1}^n t_j(\mathbf{x}) & \text{if } i \text{ is the only winner} \\ \frac{v_i}{k} - t_i(\mathbf{x}) + \alpha_i \sum_{j=1}^n t_j(\mathbf{x}) & \text{if } i \text{ is one of } k \text{ winners} \\ -t_i(\mathbf{x}) + \alpha_i \sum_{j=1}^n t_j(\mathbf{x}) & \text{otherwise} \end{cases} \quad (1)$$

Assumption 1 (A1). *Bidder i 's utility decreases with her payment $t_i(\mathbf{x})$ in the charity auction.*

This assumption means that each bidder has a strict preference to keep one euro for her own use rather than to give it to the charity auction. This is the limit to the bidders' altruism in

⁸Although valuations are common knowledge among the potential bidders, the seller has no information about them.

⁹We make a linearity assumption regarding the form of the externality. To the best of our knowledge all theoretical papers on charity auctions make a similar assumption, apart from [Ettinger \(2010\)](#) on winner-pay auctions with complete information. Moreover, an equilibrium existence result for the first-price all-pay auction with non-linear externalities is available in a previous version of this paper ([Bos, 2010](#)).

¹⁰There may well be not one but k winners. In this case, the k bidders submitted the same highest bid such that $k = \#\{j | j = \arg \max\{x_i, i \in N\}\}$.

giving money for charity. This limit is affected by the payment rule. In the second-price all-pay auction, bidder i could bid more than the other bidders but pay her opponents' highest bid.

Thus, a change in the payment rule leads to a new limit on the bidders' altruism: in first-price auctions it is $\alpha_i < 1$ while in second-price auctions $\alpha_i < 1/2$.

As established below, for some auctions, pure strategy Nash equilibria either do not exist or are degenerate. In such cases, we look for Nash equilibria in mixed strategies. In the following we denote $F_i(x) \equiv \mathbb{P}(X_i \leq x)$ the cumulative distribution functions such that bidder i decides to submit a bid lower than x . F_1, \dots, F_n can be interpreted as the bidding strategies where the support is \mathbb{R}_+ .

Let us denote by $\frac{v_i}{1-\alpha_i}$ bidder i 's adjusted-value. Bidders i 's adjusted-value is defined as the ratio of her valuation of the item sold and the fraction of her payment which she perceives as a cost given her altruism for the charitable cause. We can observe this adjusted-value in the expected utility with a normalization by dividing it by $1-\alpha_i$. As bidders are *ex ante* asymmetric, we arrange them such that $\frac{v_i}{1-\alpha_i}$ decreases with the suffix i and without equality. This is common knowledge. Thus,

$$\frac{v_1}{1-\alpha_1} > \frac{v_2}{1-\alpha_2} > \dots > \frac{v_n}{1-\alpha_n}$$

3 The First-Price All-Pay Auction

In this section, we study the most popular all-pay auction design, *i.e.* the first-price all-pay auction. Each bidder pays her own bid $t_i(\mathbf{x}) = x_i \ \forall i \in N$, but only the highest bidder wins.

As we noticed in the previous section, assumption *A1* implies that $\alpha_i < 1$. So there is no pure strategy Nash equilibrium. This is a well known result when there is no externality (see [Hillman and Riley \(1989\)](#) and [Baye et al. \(1996\)](#)). We only provide a sketch of the proof of this result with two bidders for the first-price all-pay auction with externalities.

Let us assume that $x_i \geq x_j$, then either of two cases may arise. First, if bidder j can overbid, then her best reply is $x_i + \varepsilon$, for $\varepsilon > 0$ such that $v_j - (x_i + \varepsilon) + \alpha_j(2x_i + \varepsilon) \geq -x_j + \alpha_j(x_i + x_j)$. Hence, it is impossible for $x_i \geq x_j$. Second, if j cannot overbid, then her best reply is to offer zero since, given assumption *A1*, $\alpha_j x_i > -x_j + \alpha_j(x_i + x_j)$. Consequently, i 's best reply is to offer $\varepsilon > 0$. As a result, the equilibrium is unstable and there is no pure strategy Nash equilibrium.

If bidder i offers x_i , then j will offer less with probability $F_j(x_i)$ and will offer more with probability $1 - F_j(x_i)$. Whatever the outcome, bidder i benefits from the sum of all bids, including her own. When computing her expected utility, she takes the amount paid by each opponent into account. Bidder i 's expected utility with n potential competitors is given by

$$\mathbb{E}U_i(x_i, \mathbf{X}_{-i}) = \prod_{j \neq i} F_j(x_i) v_i - (1 - \alpha_i) x_i + \alpha_i \sum_{j \neq i} \mathbb{E}X_j$$

with $\mathbf{X}_{-i} = (X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n)$. A potential bidder takes part in the auction if for some bids her expected utility is equal to or greater than the externalities from which she benefits when her bid is zero. Formally, a bidder takes part in the auction if

$$\exists x \text{ such that } \mathbb{E}U_i(x, \mathbf{X}_{-i}) \geq \alpha_i \sum_{j \neq i} \mathbb{E}X_j$$

with $\alpha_i \sum_{j \neq i} \mathbb{E}X_j$ bidder i 's expected reservation utility when she takes part in the auction. We call the highest price at which a given bidder is ready to take part in the auction her *indifference price*. i 's *indifference price* is denoted by \tilde{x}_i and satisfies $\mathbb{E}U_i(\tilde{x}_i) = \alpha_i \sum_{j \neq i} \mathbb{E}X_j$.

Proposition 1. *In the first-price all-pay auction, there is a unique Nash equilibrium and the mixed strategies are given by*

$$F_1(x) = \frac{1 - \alpha_2}{v_2} x \quad \forall x \in \left[0, \frac{v_2}{1 - \alpha_2}\right] \quad \text{and} \quad F_2(x) = 1 - \frac{1 - \alpha_1}{1 - \alpha_2} \frac{v_2}{v_1} + \frac{1 - \alpha_1}{v_1} x \quad \forall x \in \left(0, \frac{v_2}{1 - \alpha_2}\right]$$

All other bidders use the pure strategy of bidding zero and do not take part in the auction: $F_j(0) = 1$ for $j \in \{3, \dots, n\}$. The expected revenue is given by $\mathbb{E}R = \frac{1}{2} \frac{v_2}{1 - \alpha_2} \left(\frac{1 - \alpha_1}{1 - \alpha_2} \frac{v_2}{v_1} + 1 \right)$.

Dividing bidder i 's expected utility by $1 - \alpha_i$ we obtain an affine transformation of the expected utility without externality as described by Hillman and Riley (1989) and Baye et al. (1996). In our case the adjusted-value $\frac{v_i}{1 - \alpha_i}$ plays the same role as the value v_i . Moreover, we have an additional term $\frac{\alpha_i}{1 - \alpha_i} \sum_{j \neq i} \mathbb{E}X_j$ which is constant in the equilibrium. As the result of Baye et al. (1996) is invariant with respect to positive affine transformations of expected utility, the mixed strategies are invariant with respect to dividing by $1 - \alpha_i$ and adding a constant to the expected utility. Then our result follows.

Corollary 1. *In the first-price all-pay auction, all bidders obtain a positive payoff. Indeed, the bidders with the two highest adjusted-values obtain a positive payoff $U_1^* = v_1 - \frac{1 - \alpha_1}{1 - \alpha_2} v_2 + \frac{\alpha_1}{2} \frac{1 - \alpha_1}{v_1} \left(\frac{v_2}{1 - \alpha_2} \right)^2$ and $U_2^* = \frac{v_2}{2} \frac{\alpha_2}{1 - \alpha_2}$ and their competitors get $U_i^* = \frac{\alpha_i}{2} \frac{v_2}{1 - \alpha_2} \left(\frac{1 - \alpha_1}{v_1} \frac{v_2}{1 - \alpha_2} + 1 \right)$ for $i \in \{3, \dots, n\}$.*

The proof of Corollary 1 is by direct calculation. Contrary to the case with no externality (see Hillman and Riley (1989) and Baye et al. (1996)), the highest bidder's opponents get a positive payoff. This is a consequence of the externalities: bidders benefit from their competitors' behavior and so have outside options when not participating in the auction.

Remark 1. *Let us assume that the difference between α_1 and α_2 is large enough for bidder 1's adjusted-value to be ranked second such that the two highest adjusted-values are permuted. Then bidder 1 can obtain a lower payoff than in the case with no externality if and only if her altruism level is lower than $\tilde{\alpha} \equiv 2 \frac{v_1 - v_2}{3v_1 - 2v_2}$. We notice that this threshold does not depend on her rival's level of altruism, while the changes in the ranking of the adjusted-values are due only to the difference in how altruistic the players are.*

Notice here that there are two opposing effects. Because of the externalities, the value of one euro that is invested in the auction is less than one euro. Thus, the bidders may choose more aggressive offers. However, each bidder knows that her competitor is more aggressive and this will affect her probability of winning. Given an increase in her competitor's aggressiveness, the bidder's best reply could be to raise or lower her bid.

4 The Second-Price All-Pay Auction

In the second-price all-pay auction, the losers still pay their own bids, $t_i(\mathbf{x}) = x_i$, but the winner pays the second highest bid, $t_i(\mathbf{x}) = x^{(2)}$ where $x^{(2)}$ is the second order statistic of the sample \mathbf{x} . Our purpose is now to determine the mixed strategy Nash equilibrium and the expected revenue.¹¹

It is not necessary to find each agent's support of the probability distribution in order to determine the mixed strategy Nash equilibrium. We need only to assume that each bidder i 's offer, x_i , belongs to a strategy space $[0, +\infty)$. For the same reasons as in the first-price auction, the bidders' minimum valuations are zero. As noticed before, assumption A1 allows us to write that $\alpha_i < 1/2$.

As the support of the strategies is \mathbb{R}_+ , the strategies are completely mixed. They are therefore continuous, atomless, gapless (see Moulin (1986) for details). In the two-bidders case the payment rule leads to the winner paying her rival's bid and both pay the same amount. In addition, each bidder benefits from two externalities, one associated with her own bid, and the other associated with her rival's bid. Therefore the expected utility is given by

$$\mathbb{E}U_i(x_i, \mathbf{X}_{-i}) = \int_0^{x_i} (v_i - (1 - 2\alpha_i)x) dF_j(x) - (1 - 2\alpha_i)x_i(1 - F_j(x_i)).$$

Dividing by $1 - 2\alpha_i$, this expected utility is qualitatively equivalent to the expected utility without any externalities (see for example Vartiainen (2007)).¹² However, this is no longer the case for n bidders. Externalities have a different effect depending on whether bidder i is the winner, the second highest bidder, or a loser with a bid lower than the second highest bid. Accordingly the expected utility and the equilibrium, then, are non-intuitive and difficult to compute with n bidders. Let us denote $G_i(x) = \prod_{j \neq i} F_j(x)$. It follows from equation (1) that the expected utility can be written

$$\begin{aligned} \mathbb{E}U_i(x_i, \mathbf{X}_{-i}) &= \int_0^{x_i} (v_i - (1 - \alpha_i)x) dG_i(x) - (1 - \alpha_i)x_i(1 - G_i(x_i)) \\ &+ \alpha_i \sum_{l \neq i} \int_{\mathbb{R}_+} x_l \left(1 - \mathbb{1}_{x_i \leq x_l} \prod_{k \neq l, i} F_k(x_l) \right) dF_l(x_l) \\ &+ \alpha_i \sum_{l \neq i} \left(\int_{\mathbb{R}_+} \int_{x_i}^{x_l} \sum_{k \neq l, i} x_k \prod_{\substack{m \neq i, k, l \\ k \neq l}} F_m(x_k) dF_k(x_k) dF_l(x_l) + x_i \prod_{m \neq i, l} F_m(x_i)(1 - F_l(x_i)) \right) \end{aligned} \quad (2)$$

The transition from equation (1) to equation (2) is explained in the proof of Proposition 2 given in the appendix. The two terms in the first line represent bidder i 's payoff depending on whether she wins or loses the auction, given the externality that arises from her own action. The other lines represent the externalities that derive from her competitors' actions (whether they lose or win).

¹¹Pure strategy Nash equilibria in the second-price all-pay auction and their consequences in terms of expected revenues are discussed at the end of the next section.

¹²Vartiainen (2007) investigates first-price and second-price all-pay auctions with general (non-linear) cost functions. Interestingly, bidders may have asymmetric costs, which is qualitatively equivalent to the asymmetric valuation case when the cost functions are linear.

The first of these two lines describes the situation in which bidder l ($l \neq i$) loses the auction. In the last line bidder l wins the auction; we distinguish situations where bidder i 's bid is the second highest offer from situations where it is not. Each bidder's offer could be the second highest bid and we take account of this (sum operator under the integral). The bidder who makes an offer between bidder i and bidder l 's offers makes the second highest bid. The other part of the equation gives the amount of money that bidder l has to pay when i offers the second highest bid. Indeed, $\prod_{m \neq i, l} F_m(x_i)(1 - F_l(x_i))$ is the probability that every bidder except l makes a lower bid than i . This probability is multiplied by bidder i 's bid.

Note that this expression for the expected utility is valid for at least four bidders. In order to study the three bidders case, it is necessary to change the third line (slightly). To do this, we have to stop at the second line of the computation of the term B_I in the appendix. Thus, this term is written as $\alpha_i \sum_{l \neq i} \left(\int_{\mathbb{R}_+} \int_{x_i}^{x_l} x_k dF_k(x_k) dF_l(x_l) + x_i F_k(x_i)(1 - F_l(x_i)) \right)$, where k is neither i nor l .

Proposition 2. *In the second-price all-pay auction only two bidders, named i and j , among n participate actively. Bidder i 's mixed strategy is given by the cumulative distribution $F_i(x) = 1 - \exp\left(-\frac{1 - 2\alpha_j}{v_j}x\right) \forall x \in [0, +\infty)$ and the expected revenue by $\mathbb{E}R = \frac{2v_i v_j}{(1 - 2\alpha_i)v_j + (1 - 2\alpha_j)v_i}$.*

Proof. See Appendix ■

The weakness of this result is that we do not know which bidders are going to participate. Thus, it might be that the two bidders with the highest values participate or the ones with the lowest values. This has consequences on the expected revenue.

5 Revenue Comparisons

In this section, we investigate the revenue performance of first-price and second-price all-pay auctions, first-price and second-price winner-pay auctions and lotteries.

We determined the equilibria in the first-price and second-price all-pay auctions in the previous sections. To have a complete overview of all mechanisms discussed below, we describe in this paragraph the rules and equilibria of winner-pay auctions and lotteries.

In the winner-pay auctions the winner, as in all auctions, is the bidder with the highest bid submitted. Yet, contrary to the all-pay auctions, only the winner pays. Then, in the first-price winner-pay auction, the winner pays his own bid and in the second-price winner-pay auction, she pays the second-highest bid. [Ettinger \(2010\)](#) determines that in the first-price winner-pay auction, bidding strategies are not affected by the charity purpose. Thus, the winner, the bidder with the highest value, submits a bid equal to the second highest value, v_2 , and all the losers bid their own value.¹³ He also shows that in the second-price winner-pay auction the only equilibrium is such that at least two bidders submit the highest value, *ie* v_1 .

In a lottery, each participant i buys a certain number of tickets l_i , which are not refundable, and the winning ticket is drawn among all tickets. Therefore, each participant has a probability

¹³More details are provided in the working paper version of [Ettinger \(2010\)](#), in [Ettinger \(2002\)](#).

of winning $\frac{l_i}{\sum_{j=1}^n l_j}$. [Bos \(2011\)](#) determines the unique Nash equilibrium in a charitable lottery, given by

$$l_i = (n^p - 1) \left(\sum_{j=1}^{n^p} \frac{1}{\frac{v_j}{1-\alpha_j}} \right)^{-1} \left(1 - (n^p - 1) \left(\frac{v_j}{1-\alpha_i} \sum_{j=1}^{n^p} \frac{1}{\frac{v_j}{1-\alpha_j}} \right)^{-1} \right)$$

for all $i \leq n^p$ with n^p the number of active participants.¹⁴

In the following, we assume that bidders are equally altruistic i.e. $\alpha_1 = \alpha_2 = \dots = \alpha_n = \alpha$. Hence, the bidder with the highest value is also the one with the highest adjusted-value. The expected revenues become

$$\mathbb{E}R^{AP1} = \frac{1}{2} \frac{v_2}{1-\alpha} \left(\frac{v_2}{v_1} + 1 \right) \text{ and } \mathbb{E}R^{AP2} = \frac{2}{1-2\alpha} \frac{v_i v_j}{v_i + v_j} \quad i, j \in N$$

Indices AP_i and WP_i correspond to i^{th} -price all-pay and winner-pay auctions, and LOT corresponds to lotteries. If bidders are thoroughly altruistic, i.e. $\alpha^{AP1} \rightarrow 1$ and $\alpha^{AP2} \rightarrow 1/2$, the expected revenues diverge as [Goeree et al. \(2005\)](#) predicted. Thus, the altruism level is an essential element in determining the expected revenue. When bidders' altruism levels are the same, the revenue from the auction is at least equal to the revenue that would be obtained with non-altruistic bidders.

In the following, we compare our results on all-pay auctions with externalities to [Ettinger \(2010\)](#) and [Bos' \(2011\)](#) results on winner-pay auctions and lotteries with externalities. These results are summed up in Table 1:

| $v_1 > v_2 > v_3 > v_i \forall i > 3$ | R^{WP1} | R^{WP2} | R^{LOT} | $\mathbb{E}R^{AP1}$ | $\mathbb{E}R^{AP2}$ |
|---------------------------------------|-----------|-----------|---|---|---|
| $\alpha > 0$ | v_2 | v_1 | $(n^p - 1) \left(\sum_{i=1}^{n^p} \frac{1}{\frac{v_i}{1-\alpha}} \right)^{-1}$ | $\frac{1}{2} \frac{v_2}{1-\alpha} \left(\frac{v_2}{v_1} + 1 \right)$ | $\frac{2}{1-2\alpha} \frac{v_1 v_i}{v_1 + v_i}, i \neq 1$ |
| $\alpha = 0$ | v_2 | v_2 | $(n^p - 1) \left(\sum_{i=1}^{n^p} \frac{1}{v_i} \right)^{-1}$ | $\frac{v_2}{2} \left(\frac{v_2}{v_1} + 1 \right)$ | $2 \frac{v_1 v_i}{v_1 + v_i}, i \neq 1$ |

Table 1: Revenues and expected revenues

[Bos \(2011\)](#) investigates the comparison between the first-price all-pay auction and lotteries with heterogeneous values and externalities. He shows and discusses that homogeneous values and charity components (externalities) lead to the same qualitative result as [Goeree et al. \(2005\)](#), i.e. the first-price all-pay auction dominates lotteries. Externalities improve the revenue performance of both mechanisms in the same way and so do not change their ranking. Yet in a framework with no externality and heterogeneous participants, [Fang \(2002\)](#) established that the first-price all-pay auction can lead to a higher expected revenue than lotteries if and only if the participants are asymmetric enough. [Bos \(2011\)](#) confirmed this result with externalities, which contradict [Goeree et al.'s \(2005\)](#) qualitative results. This result is recalled in Table 2. However, the case of the second-price all-pay auction has not been compared to lotteries. For

¹⁴In the lottery, independently of the charity component the number of participants is denoted n^p , which is the highest integer of $m \in \{2, \dots, n\}$ such that $m \leq 2 + v_m \sum_{i=1}^{m-1} \frac{1}{v_i}$.

homogeneous values with externalities, the same qualitative results as [Goeree et al. \(2005\)](#) are determined here; charity components improve the revenue performance of the second-price all-pay auctions relative to lotteries. Yet in a framework with no externality and heterogeneous values, the second-price all-pay auction also outperforms lotteries as long as the bidders with the two highest adjusted-values take part in the auction. Therefore the result with externalities and heterogeneous participants can be intuitively determined as asymmetry and charity components have the same effects on the revenue comparison.

From the *revenue equivalence principle*, we know that all auction designs with homogeneous values and without externalities generate the same revenue. Moreover, if we consider homogeneous values with charity components, we find the same qualitative results as [Goeree et al. \(2005\)](#). So, externalities improve the revenue performance of all-pay auctions relative to winner-pay auctions. On the contrary, in a framework with no externality and heterogeneous bidders, winner-pay auctions outperform first-price all-pay auctions and could outperform second-price all-pay auctions (it depends which bidders are going to participate). Therefore, the asymmetry component improves the revenue performance of winner-pay auctions relative to first-price all-pay auctions and could improve it relative to second-price all-pay auctions. Hence, asymmetry and charity have opposing effects on the revenue comparison among all-pay and winner-pay auctions. Thus, the revenue comparison result with charitable and asymmetric bidders is not obvious.

Moreover, although our framework might be suitable for charity dinners with complete information (for example dinners held by a local Rotary Club), first-price and second-price all-pay auctions contradict [Goeree et al.'s \(2005\)](#) qualitative results. In order to analyze the impact of asymmetry on revenues, we use the following definition.

Definition. *The level of asymmetry between bidders' valuations will be considered very high if $v_1 - v_2 > 2\alpha(v_1 + v_2)$, high if $v_1 - v_2 > 2\alpha v_1$, medium if $2\alpha v_1 > v_1 - v_2 > 2\alpha v_1 - v_1 + v_2 \frac{v_2}{v_1}$, and low if $v_1 - v_2 < 2\alpha v_1 - v_1 + v_2 \frac{v_2}{v_1}$.*

Proposition 3. *We assume that $\alpha_i = \alpha$ for all bidders and that the bidder with the highest value takes part in the second-price all-pay auction. $\mathbb{E}R^{AP2} > R^{WP2}$ if and only if the level of asymmetry between valuations is not very high, $\mathbb{E}R^{AP2} > R^{WP1}$ and $\mathbb{E}R^{AP2} > \mathbb{E}R^{AP1}$ independently of the level of asymmetry.*

Assuming that the bidders with the two highest values take part in the second-price all-pay auction then $\mathbb{E}R^{AP2} > R^{LOT}$ independently of the level of asymmetry.

$\mathbb{E}R^{AP1} > R^{WP2}$ if and only if the level of asymmetry between valuations is low, $R^{WP2} > \mathbb{E}R^{AP1} > R^{WP1}$ if and only if this level is medium, and $R^{WP1} > \mathbb{E}R^{AP1}$ if and only if it is high.

Proposition 3 can be proved by directly comparing the revenues and expected revenues listed in Table 1. Results of Proposition 3 and Bos (2011) are summarized in Table 2:

| Level of asymmetry | low | medium | high | very high |
|-------------------------------|---|---|-------------------------------|-----------|
| $AP1$ versus $\{WP1, WP2\}$ | $\mathbb{E}R^{AP1} > R^{WP2}$ | $R^{WP2} > \mathbb{E}R^{AP1} > R^{WP1}$ | $R^{WP1} > \mathbb{E}R^{AP1}$ | |
| $AP1$ versus LOT | $R^{LOT} > \mathbb{E}R^{AP1}$ if and only if asymmetry between the two highest values is strong enough.** | | | |
| $AP2$ versus $WP2$ * | $\mathbb{E}R^{AP2} > R^{WP2}$ | | $WP2 > \mathbb{E}R^{AP2}$ | |
| $AP2$ versus $\{AP1, WP1\}$ * | $\mathbb{E}R^{AP2} > \{\mathbb{E}R^{AP1}, R^{WP1}\}$ | | | |
| $AP2$ versus LOT *** | $\mathbb{E}R^{AP2} > R^{LOT}$ | | | |

*The bidder with the highest value takes part in the auctions.

** This level of asymmetry determined by Bos (2011) is discussed below.

***The bidder with the two highest values takes part in the auction.

Table 2: Impact of asymmetry on the revenue comparisons

Bos (2011) determines the level of asymmetry between the two highest values such that lotteries outperform first-price all-pay auctions in terms of revenue. If there are more than two participants in the lottery, the level of asymmetry involves the values of other participants and cannot be compared to the different levels proposed in Definition 1. Yet, if there are only two participants in the lottery then this level of asymmetry could be higher than the “very high” level of asymmetry defined here.¹⁵ Unlike in winner-pay auctions, the revenue comparisons with lotteries are independent of the level of altruism.

The second-price all-pay auction leads to a higher revenue than the lottery independently of asymmetry and altruism level as long as the bidders with the highest two adjusted-values take part in the auction. If one of them does not then the revenue comparison is affected by the identity of the participants in the second-price all-pay auctions but also the number of participants in the lottery.

The second-price all-pay auction generates a higher revenue than the second-price winner-pay auction if and only if the level of asymmetry is not very high and it generates a higher revenue than all the other auction designs as long as the bidder with the highest adjusted-value takes part in the auction. Moreover, the revenue performance of the second-price all-pay auctions can be interpreted in another way when the bidder with the highest adjusted-value participates in the auction. Given v_1 and v_2 , the second-price all-pay auction outperforms the second-price winner-pay auction when the bidders’ altruism level is superior to $\frac{1}{2} \frac{v_1 - v_2}{v_1 + v_2}$.

On the contrary, when this bidder does not take part in the auction, the ranking of the expected revenue raised in the second-price all-pay auction depends on the asymmetry between bidders’ valuations.

As for second-price all-pay auctions, we can interpret the revenue performance of the first-price all-pay auction relative to the winner-pay auctions in two independent ways.

¹⁵Interestingly, lotteries with two participants outperform first-price all-pay auctions if and only if $v_1 - v_2 > \frac{v_2}{v_1} (v_1 + v_2)$. For all $\alpha \in [\frac{1}{2}, 1)$ this threshold of asymmetry is lower than the “very high” threshold $2\alpha(v_1 + v_2)$. If $\alpha \in [0; \frac{1}{2})$ this threshold may be higher or lower than the very high threshold $2\alpha(v_1 + v_2)$.

- First, given the altruism level α , the (first-price) all-pay auction is dominated by the first-price winner-pay auction when asymmetry is high. Furthermore, this all-pay auction raises more money than the second-price winner-pay auction when asymmetry is low. Thus, in order to determine which design is better at raising money for charity, we need to know the level of asymmetry between bidders.
- Given v_1 and v_2 , the (first-price) all-pay auction is dominated by first and second-price winner-pay auctions when the bidders' altruism level is less than $\frac{1}{2}(1 - \frac{v_2}{v_1})$. Yet, the all-pay auction outperforms the first-price auction and is dominated by the second-price auction when the bidders' altruism level is inferior to $1 - \frac{1}{2} \frac{v_2}{v_1} (\frac{v_2}{v_1} + 1)$ and superior to $\frac{1}{2}(1 - \frac{v_2}{v_1})$. In particular, the threshold above which this all-pay auction raises more money than the first-price winner-pay auction is less than $\frac{1}{2}$. Lastly, the first-price all-pay auction outperforms the winner-pay auctions when $\alpha > 1 - \frac{1}{2} \frac{v_2}{v_1} (\frac{v_2}{v_1} + 1)$.

The greater the asymmetry, the higher the level of altruism needs to be for the first-price all-pay auction to yield a higher revenue than the winner-pay auctions and for the second-price all-pay auction to yield a higher revenue than the second-price winner-pay auction. The difference between the expected revenue of all-pay auctions and the revenue of winner-pay auctions are depicted in Figures 1, 2, and 3. These figures show the limits (in terms of revenue domination) for the first-price and second-price all-pay auctions. We use two parameters: the altruism level on the x-axis and the asymmetry among bidders' values on each curve (from left to right, $\frac{v_2}{v_1}$ varies from 0.9 to its limit in zero with a 0.1 step). The difference $ER^{APi} - R^{WPj}$ is on the vertical axis.

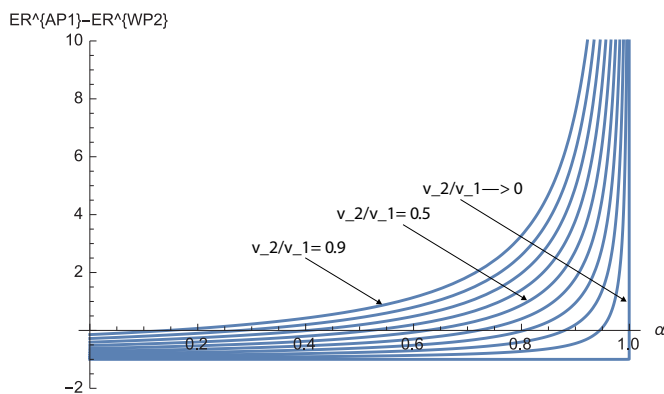


Figure 1: $\mathbb{E}R^{AP1} > R^{WP2}$

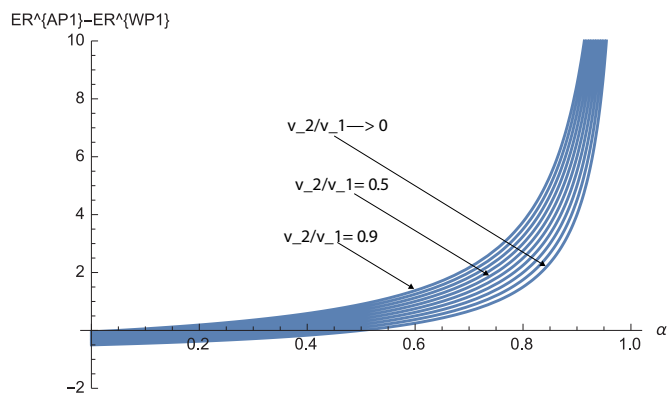


Figure 2: $\mathbb{E}R^{AP1} > R^{WP1}$

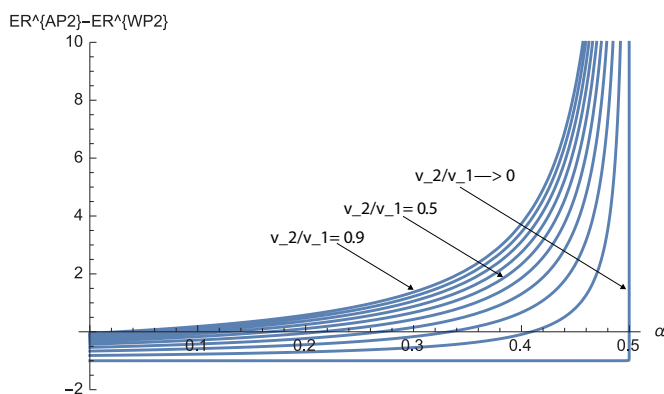


Figure 3: $\mathbb{E}R^{AP2} > R^{WP2}$

Figure comments: In each figure, each curve represents the asymmetry among bidders' values. Bidders are almost symmetric on the left-most curve such that $\frac{v_2}{v_1} = 0.9$. On each curve from left to right, the asymmetry increases by 10%. Therefore, the right-most curve represents the greatest asymmetry $\frac{v_2}{v_1} \rightarrow 0$.

As a consequence, in order to determine which design is better at raising money for charity we need to know both levels of asymmetry and altruism. Contrary to the results of [Goeree et al. \(2005\)](#), here the all-pay auctions do not always outperform the winner-pay auctions.

As in [Goeree et al. \(2005\)](#), in our framework winner-pay auctions eliminate positive effects that all-pay auctions engender. In winner-pay auctions when a bidder tops the highest bid of a competitor to win the object, she eliminates the positive externality she would have derived from this bid. However, this effect is counterbalanced here by the impact of the asymmetry. The greater the asymmetry among the bidders, the lower the expected revenues of the all-pay auctions and the revenues of the winner-pay auctions. Furthermore, the revenues of the winner-pay auctions decrease more slowly than the expected revenues of the all-pay auctions and slowly enough to counterbalance the externality effect above. Thus, if the asymmetry is sufficiently high winner-pay auctions outperform all-pay auctions in terms of revenue.

Other equilibria could appear in the second-price all-pay auction even if they are degenerate. There is a continuum of pure strategy equilibria as in the situations without externalities. [Hendricks et al. \(1988\)](#) show such equilibria are never subgame perfect in the dynamic version

of the auction which is strategically equivalent to the static version.¹⁶ We provide an intuitive argument for the two-bidders case only. Bidder i 's expected utility is given by

$$U_i(x) = \begin{cases} v_i + (2\alpha_i - 1)x_j & \text{if } x_i > x_j \\ \frac{v_i}{2} + (2\alpha_i - 1)x_i & \text{if } x_i = x_j \\ (2\alpha_i - 1)x_i & \text{if } x_i < x_j \end{cases}$$

As before, we denote \tilde{x}_i bidder i 's *indifference price*, such that $\tilde{x}_1 > \tilde{x}_2$. Let x_i be bidder i 's offer. Thus, pure strategy Nash equilibria are

$$\begin{aligned} (0, \beta_1) & \text{ with } \beta_1 \in (\tilde{x}_1, +\infty) \\ (\beta_2, 0) & \text{ with } \beta_2 \in (\tilde{x}_2, +\infty). \end{aligned}$$

These equilibria lead to a revenue of zero and then the second-price all-pay auction is always dominated by other mechanisms in terms of revenue.

As our objective is to determine the revenue performance of the all-pay auctions relative to other designs we did not compare revenues between lotteries and winner-pay auctions. We can see from Table 1 that the charity component improves the revenue performance of lotteries relative to winner-pay auctions for homogenous participants. Indeed for a high enough level of altruism $\alpha > \frac{1}{n^p} \equiv \underline{\alpha}$, the result with no externality is reversed and lotteries outperform winner-pay auctions for homogenous values. The minimum level of altruism to obtain this result decreases with the number of participants in lotteries.

Furthermore, in a framework with heterogeneous participants and no externality, winner-pay auctions outperform lotteries. Yet it is unclear whether the asymmetry improves the relative revenue performance of the winner-pay auctions in a framework with no externality and then has an opposite effect to the charity component. In a setting with heterogenous participants and externalities, lotteries can again outperform winner-pay auctions when the level of altruism is high enough. The next proposition summarizes these results.

Proposition 4. *We assume that $\alpha_i = \alpha$ for all potential participants. In a framework with asymmetry among the participants' values, the lottery raises more money for charity than the second-price winner-pay auction if and only if the level of altruism is high enough such that*

$$\alpha > 1 - \frac{(n-1)\prod_{i>1}^{n^p} v_i}{\sum_{k=1}^{n^p} \prod_{i \neq k} v_i} \equiv \bar{\alpha}.$$

Also the lottery raises more money for charity than the first-price winner-pay auction if and only if the level of altruism is high enough such that

$$\alpha > 1 - \frac{(n-1)v_1 \prod_{i>2}^{n^p} v_i}{\sum_{k=1}^{n^p} \prod_{i \neq k} v_i}.$$

¹⁶Hendricks et al. (1988) propose a very general analysis of a two-player war of attrition which is the dynamic version of the second-price all-pay auction. Players have to choose at what point in time they will concede to the other. Moving first leads to a lower payoff.

Proposition 4 can be proved by directly comparing the revenues listed in Table 1.

Using Proposition 4, it can be established that asymmetry can improve the relative revenue performance of lotteries. If the threshold of altruism $\bar{\alpha}$ is lower than $\underline{\alpha}$, then asymmetry improves the revenue performance of the lottery relative to the second-price winner-pay auction. Unfortunately, it cannot be established neither that asymmetry improves the revenue performance of the lottery relative to the first-price winner-pay auction nor the opposing result.

6 Conclusion

This paper shows that all-pay auctions do not always raise higher revenue for charity than winner-pay auctions and lotteries. This result depends on the asymmetry between bidders. In particular, winner-pay auctions and lotteries outperform the first-price all-pay auction when the asymmetry between bidders is strong. This contradicts [Goeree et al. \(2005\)](#)'s results. Externalities work in favor of all-pay auctions whereas asymmetry works in favor of winner-pay auctions. When the asymmetry is strong enough, bidders are aware that they have few chances to win against the bidder with the highest value. In the all-pay auctions, the bidder with the second highest value, who is the closest competitor to the highest value bidder, believes she will probably lose even though she will have to pay her bid. Hence she has an incentive to reduce her bid. The best reply of the bidder with the highest value is then to submit a low bid too. Contrary to that, in a winner-pay auction, a bidder with a low value has no reason to submit a very low bid given she has to pay only if she wins. As externalities affect positively the bids of the bidders with low values, the highest value bidder must bid more aggressively. Therefore, an all-pay auction provides a lower average revenue than a winner-pay auction.

Our work may be likened to [Carpenter et al. \(2008\)](#). In the field experiment that they report, they find results similar to ours, namely that the first-price winner-pay auction outperforms the first-price all-pay auction.¹⁷ [Carpenter et al. \(2008\)](#) suggest that this could be due to an endogenous participation. In a lab experiment [Carpenter, Holmes and Matthews \(2010\)](#) investigate 15 charity mechanisms to understand if results in the field, especially the low participation in the first-price all-pay auction, could be explained by an endogenous participation.¹⁸ Bidders' asymmetry could also explain the results of [Carpenter et al. \(2008\)](#) in the field. Yet, [Carpenter et al. \(2008\)](#) did not control for asymmetry between bidders and then it is unclear whether or not asymmetry is one of the driving forces. A lab experiment, controlling for bidders' asymmetry, would be a relevant investigation. It would be interesting to determine in the lab if there is any empirical foundation for our findings, that is only two bidders submit a positive bid at the equilibrium. Beyond [Carpenter, Holmes and Matthews \(2010\)](#), two lab experiments ([Orzen \(2008\)](#) and [Schram and Onderstal \(2009\)](#)) have been conducted on charity auctions, with opposing results compared to the [Carpenter et al.'s \(2008\)](#) field experiment.¹⁹ [Orzen \(2008\)](#) investigates a

¹⁷However, contrary to the theoretical predictions (under complete information) of [Ettinger \(2010\)](#), [Carpenter et al. \(2008\)](#) find that the first-price winner-pay auction generates more revenue than the second-price winner-pay auction.

¹⁸This lab experiment is partially based on the theoretical paper of [Carpenter, Homes and Matthews \(2010\)](#).

¹⁹The interested reader by lab experiments on charity auctions might also read the related experimental papers on dynamic charity auctions from [Davis et al. \(2006\)](#) and [Isaac and Schneur \(2005\)](#). The former compare lotteries

symmetric complete information framework to compare four fundraising mechanisms, voluntary contributions, lotteries, first-price and lowest-price all-pay auctions. [Orzen \(2008\)](#) and [Schram and Onderstal \(2009\)](#) find similar results to [Goeree et al. \(2005\)](#). However, our results are quite different from [Goeree et al.'s \(2005\)](#) because of the introduction of asymmetric valuations and the information setting. That is another reason why it would be interesting to test our prediction by introducing asymmetry between bidders' valuations: all-pay auctions can be dominated by winner-pay auctions and lotteries. Finally, theoretical and experimental work should be done on the form of the externalities which are mainly assumed in the literature as linear.

Appendix

Proof of Proposition 2.

Lemma 1. *Let us consider i and j the two potential participants. Bidder i 's mixed strategy is given by $F_i(x) = 1 - \exp\left(-\frac{1-2\alpha_j}{v_j}x\right) \forall x \in [0, +\infty)$ and the expected revenue by $\mathbb{E}R = \frac{2v_i v_j}{(1-2\alpha_i)v_j + (1-2\alpha_j)v_i}$.*

Proof. The expected utility of bidder i is given by

$$\mathbb{E}U_i(x_i, X_j) = \int_0^{x_i} (v_i - (1-2\alpha_i)x) dF_j(x) - (1-2\alpha_i)x_i(1-F_j(x_i))$$

Then, dividing the expected utility by $1-2\alpha_i$ and considering the adjusted values $\frac{v_i}{1-2\alpha_i}$ instead of the values v_i we get a positive transformation of the expected utility without any externalities. The mixed strategies in the equilibrium would not be altered by this transformation. \parallel

Lemma 2. *Let n be the number of potential participants. Then, only two out of n bidders participate actively in the auction.*

Proof. By equation (1) the expected utility follows:

$$\mathbb{E}U_i(x_i, \mathbf{X}_{-i}) = v_i \prod_{j \neq i} dF_j(x_j) - (1-\alpha_i) \underbrace{\int_{\mathbb{R}_+^{n-1}} t_i(\mathbf{x}) \prod_{j \neq i} dF_j(x_j)}_A + \alpha_i \underbrace{\int_{\mathbb{R}_+^{n-1}} \sum_{j \neq i} t_j(\mathbf{x}) \prod_{j \neq i} dF_j(x_j)}_B$$

with $\mathbf{X}_{-i} = (X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n)$. We notice that the events $\{i \text{ is the only winner}\}$ and $\{i \text{ is one of } k \text{ winners}\}$ are disjoint. Thus, for the latter event the value of the integral is zero. Indeed, a tie is a zero measure event. A represents bidder i 's expected payment when we take into account her own external effect. The term B is the expected payment of bidder i 's rivals. $\alpha_i B$ is the sum of the externalities of bidder i 's rivals from which i benefits.

We can write A again as follows

$$\underbrace{\int_{\mathbb{R}_+^{n-1}} x^{(2)} \mathbb{1}_{x_i \geq x_j \forall j \neq i} \prod_{j \neq i} dF_j(x_j)}_{A_I} + \underbrace{\int_{\mathbb{R}_+^{n-1}} x_i \mathbb{1}_{\exists k / x_k > x_i, k \neq i} \prod_{j \neq i} dF_j(x_j)}_{A_{II}}$$

and ascending auctions and latter lead lab and field experiment on silent auctions (variant of English auction involving the simultaneous sale of multiple items). Interestingly, [Isaac and Schneur \(2005\)](#) analyze jump-bidding effects.

The term A_I is i 's expected payment when she wins *i.e.* she pays the second highest bid. A_{II} is i 's expected payment when she loses. She could then either be the second highest bidder or a lower bidder.

$$\begin{aligned}
A_I &= \int_{\mathbb{R}_+^{n-1}} \sum_{j \neq i} x_j \mathbb{1}_{\substack{x_k \leq x_j \leq x_i \\ \forall k \neq \{j, i\}, j \neq i}} \prod_{j \neq i} dF_j(x_j) \\
&= \int_{\mathbb{R}_+} \sum_{j \neq i} x_j \mathbb{1}_{x_j \leq x_i} \left\{ \int_{\mathbb{R}_+^{n-2}} \prod_{k \neq i, j} \mathbb{1}_{x_k \leq x_j \leq x_i} \prod_{k \neq i, j} dF_k(x_k) \right\} dF_j(x_j) \\
&= \int_{\mathbb{R}_+} \sum_{j \neq i} x_j \mathbb{1}_{x_j \leq x_i} \left\{ \prod_{k \neq i, j} \int_{\mathbb{R}} \mathbb{1}_{x_k \leq x_j \leq x_i} dF_k(x_k) \right\} dF_j(x_j) \\
&= \int_{\mathbb{R}_+} \sum_{j \neq i} x_j \mathbb{1}_{x_j \leq x_i} \prod_{k \neq i, j} F_k(x_j) dF_j(x_j) \\
&= \int_0^{x_i} x dG_i(x)
\end{aligned}$$

We get the first line from the fact that $x^{(2)} \mathbb{1}_{x_i \geq x_j} = \sum_{j \neq i} x_j \mathbb{1}_{\substack{x_k \leq x_j \leq x_i \\ \forall k \neq \{j, i\}, j \neq i}}$. The independence of the distribution functions explains how we get from the second to the third line. By denoting $dG_i(x) = \sum_{j \neq i} \prod_{k \neq i, j} F_k(x) dF_j(x)$, we obtain the final result.

$$\begin{aligned}
A_{II} &= \int_{\mathbb{R}_+^{n-1}} x_i (1 - \mathbb{1}_{i \in Q(x)}) \prod_{j \neq i} dF_j(x_j) \\
&= x_i - x_i \prod_{j \neq i} F_j(x_i) \\
&= x_i (1 - G_i(x_i))
\end{aligned}$$

The independence of the distribution functions, explains how we get from the first line to the second.

B can also be written as

$$\begin{aligned}
B &= \sum_{l \neq i} \int_{\mathbb{R}_+^{n-1}} t_l(\mathbf{x}) \prod_{j \neq i} dF_j(x_j) \\
&= \sum_{l \neq i} \left\{ \underbrace{\int_{\mathbb{R}_+^{n-1}} x^{(2)} \mathbb{1}_{\substack{x_l \geq x_k \\ \forall k \neq l}} \prod_{j \neq i} dF_j(x_j)}_{B_I} + \underbrace{\int_{\mathbb{R}_+^{n-1}} x_l \mathbb{1}_{\exists k/x_l < x_k} \prod_{j \neq i} dF_j(x_j)}_{B_{II}} \right\}
\end{aligned}$$

We add all the expected external effects. The case where player $l \neq i$ takes the second highest bid is distinguished from the others.

$$\begin{aligned}
B_I &= \int_{\mathbb{R}_+^{n-1}} \sum_{k \neq l} x_k \mathbb{1}_{x_m \leq x_k \leq x_l} \prod_{\substack{\forall m \neq \{k,l\} \\ j \neq i}} dF_j(x_j) \\
&= \int_{\mathbb{R}_+^{n-1}} \sum_{k \neq l} x_k \prod_{m \neq \{k,l\}, k \neq l} \mathbb{1}_{x_m \leq x_k \leq x_l} \prod_{j \neq i} dF_j(x_j) \\
&= \int_{\mathbb{R}_+^{n-1}} \sum_{k \neq i, l} x_k \prod_{\substack{m \neq i, k, l \\ k \neq l}} \mathbb{1}_{x_m \leq x_k \leq x_l} dF_m(x_m) \mathbb{1}_{x_i \leq x_k \leq x_l} dF_k(x_k) dF_l(x_l) \\
&\quad + \int_{\mathbb{R}_+^{n-1}} x_i \prod_{m \neq i, l} \mathbb{1}_{x_m \leq x_i \leq x_l} \prod_{j \neq i} dF_j(x_j) \\
&= \int_{\mathbb{R}_+^2} \sum_{k \neq i, l} x_k \int_{\mathbb{R}_+^{n-3}} \prod_{\substack{m \neq i, k, l \\ k \neq l}} \mathbb{1}_{x_m \leq x_k} dF_m(x_m) \mathbb{1}_{x_i \leq x_k \leq x_l} dF_k(x_k) dF_l(x_l) \\
&\quad + x_i \int_{\mathbb{R}_+} \prod_{m \neq i, l} \left\{ \int_0^{x_i} dF_m(x_m) \right\} \mathbb{1}_{x_i \leq x_l} dF_l(x_l) \\
&= \int_{\mathbb{R}_+^2} \sum_{k \neq i, l} x_k \prod_{\substack{m \neq i, k, l \\ k \neq l}} F_m(x_k) \mathbb{1}_{x_i \leq x_k \leq x_l} dF_k(x_k) dF_l(x_l) + x_i \prod_{m \neq i, l} F_m(x_i) (1 - F_l(x_i)) \\
&= \int_{\mathbb{R}_+} \int_{x_i}^{x_l} \sum_{k \neq i, l} x_k \prod_{\substack{m \neq i, k, l \\ k \neq l}} F_m(x_k) dF_k(x_k) dF_l(x_l) + x_i \left(\prod_{m \neq i, l} F_m(x_i) - G_i(x_i) \right).
\end{aligned}$$

$$\begin{aligned}
B_{II} &= \int_{\mathbb{R}_+^{n-1}} x_l (1 - \mathbb{1}_{l \in Q(x)}) \prod_{j \neq i} dF_j(x_j) \\
&= \int_{\mathbb{R}_+^{n-1}} x_l \prod_{j \neq i} dF_j(x_j) - \int_{\mathbb{R}_+^{n-1}} x_l \prod_{k \neq i, l} \left(\mathbb{1}_{x_k \leq x_l} dF_k(x_k) \right) \mathbb{1}_{x_i \leq x_l} dF_l(x_l) \\
&= \int_{\mathbb{R}_+^{n-1}} x_l \prod_{j \neq i} dF_j(x_j) - \int_{\mathbb{R}_+} x_l \mathbb{1}_{x_i \leq x_l} \left\{ \int_{\mathbb{R}_+^{n-2}} \prod_{k \neq i, l} \mathbb{1}_{x_k \leq x_l} dF_k(x_k) \right\} dF_l(x_l) \\
&= \int_{\mathbb{R}_+} x_l dF_l(x_l) - \int_{\mathbb{R}_+} x_l \mathbb{1}_{x_i \leq x_l} \prod_{k \neq i, l} F_k(x_l) dF_l(x_l) \\
&= \int_{\mathbb{R}_+} x_l (1 - \mathbb{1}_{x_i \leq x_l} \prod_{k \neq i, l} F_k(x_l)) dF_l(x_l).
\end{aligned}$$

Hence:

$$\begin{aligned}
\mathbb{E}U_i(x_i, \mathbf{X}_{-i}) &= \int_0^{x_i} (v_i - (1 - \alpha_i)x) dG_i(x) - (1 - \alpha_i)x_i(1 - G_i(x_i)) \\
&\quad + \alpha_i \sum_{l \neq i} \int_{\mathbb{R}_+} x_l (1 - \mathbb{1}_{x_i \leq x_l} \prod_{k \neq i, l} F_k(x_l)) dF_l(x_l) \\
&\quad + \alpha_i \sum_{l \neq i} \left(\int_{\mathbb{R}_+} \int_{x_i}^{x_l} \sum_{k \neq i, l} x_k \prod_{\substack{m \neq i, k, l \\ k \neq l}} F_m(x_k) dF_k(x_k) dF_l(x_l) + x_i \prod_{m \neq i, l} F_m(x_i) (1 - F_l(x_i)) \right).
\end{aligned}$$

Next, we note:

$$G_{il}(x) = \prod_{k \neq i, l} F_k(x) \text{ et } G'_{il}(x) = \sum_{j \neq i, l} \prod_{k \neq i, l, j} F_k(x) dF_j(x).$$

As the expected utility is constant in the equilibrium, the FOC leads to

$$v_i G'_i(x) - (1 - \alpha_i)(1 - G_i(x)) + \alpha_i \sum_{l \neq i} G_{il}(x) - \alpha_i \sum_{l \neq i} G_{il}(x) F_l(x) - \alpha_i x \sum_{l \neq i} G'_{il}(x) F_l(x) = 0.$$

Notice that $(n - 1)G_i(x) = \sum_{l \neq i} G_{il}(x) F_l(x)$ and $(n - 2)G'_i(x) = \sum_{l \neq i} G'_{il}(x) F_l(x)$ henceforth:

$$(v_i - \alpha_i x(n - 2))G'_i(x) + (1 - \alpha_i n)G_i(x) = (1 - \alpha_i) - \alpha_i \sum_{l \neq i} G_{il}(x) \quad \forall i \in \{1, \dots, n\}. \quad (\text{A1})$$

This result is true for all $n > 3$. The closed characterization of the solution is very difficult. Yet, we can deduce the solution in an alternative way. Let F_i and F_j be the mixed strategies of the two bidders i and j . We notice that the derivative of the expected utility of a third bidder k $H_k(x) = \frac{\partial EU_k}{\partial x}(x_i, X_1, X_2)$ is a monotone increasing function. Furthermore, $H_k(0) = -(1 - \alpha_k)$ and $\lim_{x \rightarrow +\infty} H_k(x) = 0$. Thus, given the mixed strategies of i and j , k does not participate. This result can easily be extended to a number n of bidders. For that, we should use recurrence.
||

The result follows from Lemmas 1 and 2. ■

References

- Baye, M., Kovenock, D. and de Vries, C. (1996), ‘The all-pay auction with complete information’, *Economic Theory* **8**(2), 291–305.
- Bos, O. (2010), Charity auctions for the happy few. Working Paper Series in Economics 45, University of Cologne.
- Bos, O. (2011), ‘How lotteries outperform auctions’, *Economic Letters* **110**, 262–264.
- Carpenter, J., Holmes, J. and Matthews, P. (2010), Charity auctions in the experimental lab, in ‘Research in Experimental Economics’, Vol. 13, Bingley: Emerald Group Publishing, pp. 201–249.
- Carpenter, J., Homes, J. and Matthews, P. H. (2008), ‘Charity auctions: A field experiment’, *Economic Journal* **118**, 92–113.
- Carpenter, J., Homes, J. and Matthews, P. H. (2010), ‘Endogenous participation in charity auctions’, *Journal of Public Economics* **94**, 921–935.
- Damianov, D. and Peeters, R. (2012), The lowest-bid all-pay-auction as a fund-raising mechanism: Theoretically optimal but behaviorally fragile. Research Memorandum 051, Maastricht University.
- Davis, D., Razzolini, L., Reilly, R. and Wilson, B. (2006), Raising revenues for charity: Auctions versus lotteries, in D. Davis and R. M. Isaac, eds, ‘Research in experimental economics’, New York: JAI Press., pp. 49–95.
- Engers, M. and McManus, B. (2007), ‘Charity auctions’, *International Economic Review* **48**(3), 953–994.

- Ettinger, D. (2002), Bidding among friends and enemies. FEEM Working Paper No. 23.2003.
- Ettinger, D. (2010), ‘Bidding among friends and enemies with symmetric information’, *Journal of Institutional and Theoretical Economics* **166**(2), 365–385.
- Fang, H. (2002), ‘Lottery versus all-pay auction models of lobbying’, *Public Choice* **112**, 351–371.
- Goeree, J. K., Maasland, E., Onderstal, S. and Turner, J. L. (2005), ‘How (not) to raise money’, *Journal of Political Economy* **113**(4), 897–918.
- Hendricks, K., Weiss, A. and Wilson, C. (1988), ‘The war of attrition in continuous time with complete information’, *International Economic Review* **29**, 663–680.
- Hillman, A. and Riley, J. (1989), ‘Politically contestable rents and transfers’, *Economics and Policy* **1**, 17–39.
- Isaac, R. M. and Schneir, K. (2005), ‘Silent auctions in the field and in the laboratory’, *Economic Inquiry* **43**(4), 715–733.
- Krishna, V. (2009), *Auction Theory*, second edn, Academic Press.
- Maasland, E. and Onderstal, S. (2007), ‘Auctions with financial externalities’, *Economic Theory* **32**, 551–574.
- Morgan, J. (2000), ‘Financing public goods by means of lotteries’, *Review of Economic Studies* **67**, 761–784.
- Moulin, H. (1986), *Game Theory for the Social Sciences*, New York University Press.
- Myerson, R. (1981), ‘Optimal auction design’, *Mathematics of Operations Research* **6**, 58–73.
- Orzen, H. (2008), Fundraising through competition: Evidence from the lab. CeDEx Discussion paper no. 2008–11.
- Schram, A. and Onderstal, S. (2009), ‘Bidding to give : an experimental comparaisn of auctions for charity’, *International Economic Review* **50**, 431–457.
- Vartiainen, H. (2007), Comparing of all-pay auctions under complete information. Working Paper.