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ABSTRACT

Capital theory controversies and 'paradoxes' showed that, due to price-feedback effects, the wage-production price-profit rate curves may display shapes inconsistent with the requirements of the neoclassical theory of value and distribution. Subsequent findings on a number of quite diverse actual single-product economies suggested that the impact of those effects is of limited empirical significance. This paper argues that, by focusing on the distributions of the eigenvalues and singular values of the system matrices, we can further study these issues and derive some meaningful theoretical results consistent with the available empirical evidence. Consequently, the real paradox, in the sense of knowledge vacuum and, thus, requiring further research, is the distributions of the characteristic values and not really the 'paradoxes in capital theory'.

Keywords: Capital Theory, Characteristic value distributions, Hyper-basic industry, Spectral decompositions, Wage-price-profit rate curves

JEL Classification: B21, B51, C67, D57

1. Introduction

Capital theory controversies and the associated with these 'paradoxes' culminated in the decades of 1960s and 1970s. These debates started as a critique of the logical foundations of the neoclassical theory of value and distribution, and they showed that the wage-

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production price-profit rate curves can display shapes that are inconsistent with the theoretical requirements of this theory. We do know that neoclassical prices are indexes of relative scarcities; thus, it is expected that, as the profit rate rises, corresponding to a fall in the real wage rate, the prices of 'capital-intensive' ('labour-intensive') commodities rise (fall).

The movement of relative production prices of 'actual' (linear, closed and single-product) economies has been examined in a relatively large number of studies, and, subsequently, the findings were extended both theoretically and empirically (see Shaikh 1998; Tsoulfidis and Mariolis 2007; Mariolis and Tsoulfidis 2009; Mariolis et al., 2013). Thus, it has been stated that, if Sraffa's (1960, Chaps. 4-5) Standard commodity (SSC) is chosen as the standard of value or numeraire, then the price-movement is, more often than not, governed by the 'capital-intensity effect', i.e. by the difference between the industry's vertically integrated capital-intensity and the capital-intensity of the Sraffian Standard system (SSS), where the latter equals the reciprocal of the maximum profit rate. This 'traditional flavour' condition can be modified by the 'price effect', i.e. the revaluation of the industry's vertically integrated capital, which depends on the entire economic system and, therefore, is not predictable at the level of any single industry (also see Sraffa 1960, pp. 14-15; Pasinetti 1977, pp. 82-84). In effect, empirical evidence associated with quite diverse economies, and spanning different time periods, showed that the capital-intensity effect overshadows the price effect, although there are cases where the latter effect is so strong that it can supersede the former giving rise to extrema and 'price-labour value reversal reversals' (i.e. reversals in the direction of deviation between production prices and labour values). In these realistic but relatively rare instances, i.e. not significantly more than 20% of the cases tested, the price-profit rate curves are non-monotonic and have no more than one extreme point, while cases of price-labour value reversal are rarer. It then follows that the idea of representing the actual price-profit rate curves through linear or, *a fortiori*, quadratic approximations is absolutely justifiable (Bienenfeld 1988; Shaikh 2012; Iliadi et al. 2014). Moreover, from Sekerka *et al.* (1970) and Krelle (1977) onwards, a *typical* finding in many relevant studies is that, although the actual economies deviate considerably from the Ricardo-Marx-Samuelson 'equal value compositions of capital' case, the wage-profit curves (WPCs) are near-linear, i.e. the correlation coefficients between the distributive variables tend to be above 99%, and their second derivatives change sign no more than once or, very rarely, twice, irrespective of the numeraire chosen (also see Ochoa 1989; Petrović 1991; Angeloussis 2006; Han and

Schefold, 2006; Mariolis and Tsoulfidis 2015, Chaps. 3 and 5, and the references therein). All these findings imply that, although the actual economies *cannot* be analyzed on the basis of 'neoclassical parables', the role of price-feedback effects is actually of limited empirical significance.

The claim that this paper raises is that, by focusing on both the eigenvalue and singular value distributions of the system matrices, we can (i) further study these theoretical issues; and (ii) derive some meaningful theoretical results *consistent* with the available empirical evidence. More specifically, we start with a spectral representation of the price-wage-profit system of a closed economy involving only single products and 'basic' commodities (in the sense of Sraffa 1960, pp. 7-8). Then the approach is applied to the Symmetric Input-Output Table (SIOT) of the UK economy for the year 1990, which represents an ideal testing ground for our theoretical findings. It should be stressed from the outset that the results obtained are typical for a number of countries that have been hitherto tested (Mariolis and Tsoulfidis 2015, Chaps. 3 and 5-6) and are presented here for the first time. We used input-output data of the UK economy mainly for two reasons: first, the UK data have not been used in similar experiments and, second, the available capital flow matrix for the year 1990 enables to carry out our experiments in both circulating and fixed capital stock matrices. Thus, we can directly compare the results derived from both type of matrices and pinpoint their probable differences. Finally, this investigation shows that the characteristic value distributions of the matrices of vertically integrated technical coefficients make possible the mimicking of the behaviour of the entire price system through the use of a single or just a few hypothetical basic industries. Those industries bear meaningful similarities to what can be described by the Samuelson-Hicks-Spaventa or 'corn-tractor' model (see Spaventa 1970).

The remainder of the paper is structured as follows. Section 2 deals with the theoretical issues lurking behind the shapes of the wage-price-profit rate curves. Section 3 exposes the empirical results. Section 4 argues that the results could be connected to the characteristic value distributions of the system matrices. Section 5 estimates a corn-tractor approximation that tends to contain the essential properties of the original system. Finally, Section 6 concludes.

2. The Mechanics of Production Prices

2.1. Preliminaries

Consider a closed, linear system involving only single products, basic commodities and circulating capital. Assume that (i) the input-output coefficients are fixed; (ii) the system is strictly 'profitable' or, equivalently, 'viable', i.e. the Perron-Frobenius (P-F) eigenvalue of the irreducible $n \times n$ matrix of direct technical coefficients, $\mathbf{A}$, is less than 1, and 'diagonalizable', i.e. $\mathbf{A}$ has a complete set of n linearly independent eigenvectors;[1] (iii) the net product is distributed to profits and wages that are paid at the end of the common production period; (iv) labour may be treated as homogeneous because relative wage rates are invariant; and (v) the profit rate, r, is uniform.

On the basis of these assumptions we can write

$$\mathbf{p} = w\mathbf{l} + (1+r)\mathbf{pA} \tag{1}$$

where $\mathbf{p}$ denotes a vector of production prices, w the money wage rate, and $\mathbf{l}$ ($>\mathbf{0}$) the vector of direct labour coefficients. After rearrangement, equation (1) becomes

$$\mathbf{p} = w\mathbf{v} + r\mathbf{pH}$$

or

$$\mathbf{p} = w\mathbf{v} + \rho\mathbf{pJ} \tag{2}$$

or, if $\rho < 1$,

$$\mathbf{p} = w\mathbf{v}[\mathbf{I} - \rho\mathbf{J}]^{-1} \tag{2a}$$

where $\mathbf{v} \equiv \mathbf{l}[\mathbf{I} - \mathbf{A}]^{-1}$ denotes the vector of vertically integrated labour coefficients or labour values, and $\mathbf{H} \equiv \mathbf{A}[\mathbf{I} - \mathbf{A}]^{-1}$ ($>\mathbf{0}$) the vertically integrated technical coefficients matrix. Moreover, $\rho \equiv rR^{-1}$, $0 \le \rho \le 1$, denotes the 'relative (or normalized) profit rate', which equals the share of profits in the SSS, and $R \equiv \lambda_{\mathbf{A}1}^{-1} - 1 = \lambda_{\mathbf{H}1}^{-1}$ the maximum profit rate, i.e. the profit rate corresponding to [$w = 0$, $\mathbf{p} > \mathbf{0}$], which equals the ratio of the net

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[1] The transpose of a $1 \times n$ vector $\mathbf{y} \equiv [y_j]$ is denoted by $\mathbf{y}^{\mathrm{T}}$, and the diagonal matrix formed from the elements of $\mathbf{y}$ is denoted by $\hat{\mathbf{y}}$. Furthermore, $\mathbf{A}_j$ denotes the jth column of a semi-positive $n \times n$ matrix $\mathbf{A} \equiv [a_{ij}]$, $\lambda_{\mathbf{A}1}$ the P-F eigenvalue of $\mathbf{A}$ and $(\mathbf{x}_{\mathbf{A}1}^{\mathrm{T}}, \mathbf{y}_{\mathbf{A}1})$ the corresponding eigenvectors, while $\lambda_{\mathbf{A}k}$, $k = 2,...,n$ and $|\lambda_{\mathbf{A}2}| \ge |\lambda_{\mathbf{A}3}| \ge ... \ge |\lambda_{\mathbf{A}n}|$, denote the non-dominant eigenvalues, and $(\mathbf{x}_{\mathbf{A}k}^{\mathrm{T}}, \mathbf{y}_{\mathbf{A}k})$ the corresponding eigenvectors. Finally, $\mathbf{I}$ denotes the $n \times n$ identity matrix, $\mathbf{e}$ the summation vector, i.e. $\mathbf{e} \equiv [1,1,...,1]$, and $\mathbf{e}_j$ the jth unit vector. It is also noted that, given any $\mathbf{A}$ and an arbitrary $\varepsilon \ne 0$, it is possible to perturb the entries of $\mathbf{A}$ by an amount less than $|\varepsilon|$ so that the resulting matrix is diagonalizable (see, e.g. Aruka 1991, pp. 74-76).

product to the means of production in the SSS (see Sraffa 1960, pp. 21-23).[2] Finally, $\mathbf{J} \equiv R\mathbf{H}$ denotes the normalized vertically integrated technical coefficients matrix, $\lambda_{\mathbf{J}1} = R\lambda_{\mathbf{H}1} = 1$, and the moduli of the normalized eigenvalues of system (2) are less than those of system (1), i.e. $|\lambda_{\mathbf{J}k}| < |\lambda_{\mathbf{A}k}|\lambda_{\mathbf{A}1}^{-1}$ holds for all k (see, e.g. Mariolis and Tsoulfidis 2014, pp. 213-214). In the case of fixed capital *à la* Leontief (1953)-Bródy (1970), $\mathbf{H}$ is replaced by $\mathbf{H}^{\mathrm{K}} \equiv \mathbf{K}[\mathbf{I} - \mathbf{A}]^{-1}$, where $\mathbf{K}$ denotes the matrix of capital stock coefficients.

If commodity $\mathbf{z}^{\mathrm{T}}$, with $\mathbf{v}\mathbf{z}^{\mathrm{T}} = 1$, is chosen as the numeraire, i.e. $\mathbf{p}\mathbf{z}^{\mathrm{T}} = 1$, then equation (2a) implies

$$w = (\mathbf{v}[\mathbf{I} - \rho\mathbf{J}]^{-1}\mathbf{z}^{\mathrm{T}})^{-1} \tag{3}$$

and

$$\mathbf{p} = (\mathbf{v}[\mathbf{I} - \rho\mathbf{J}]^{-1}\mathbf{z}^{\mathrm{T}})^{-1}\mathbf{v}[\mathbf{I} - \rho\mathbf{J}]^{-1} \tag{4}$$

Equation (3) gives a trade-off between w and ρ, known as the WPC, which may admit up to $3n-6$ inflection points, and equation (4) gives the production prices as functions of ρ, which may admit up to $2n-4$ extremum points (though it is not certain that they will all occur in $0 \le \rho \le 1$; Garegnani 1970, p. 419). It then follows that $w(0) = 1$, $\mathbf{p}(0) = \mathbf{v}$, $w(1) = 0$ and $\mathbf{p}(1) = (\mathbf{y}_{\mathbf{J}1}\mathbf{z}^{\mathrm{T}})^{-1}\mathbf{y}_{\mathbf{J}1}$. If SSC is chosen as the numeraire, i.e. $\mathbf{z}^{\mathrm{T}} = [\mathbf{I} - \mathbf{A}]\mathbf{x}_{\mathbf{A}1}^{\mathrm{T}}$, with $\mathbf{l}\mathbf{x}_{\mathbf{A}1}^{\mathrm{T}} = 1$, then

$$w = w^{\mathrm{S}} \equiv 1 - \rho \tag{5}$$

and

$$\mathbf{p} = (1-\rho)\mathbf{v} + \rho\mathbf{p}\mathbf{J} \tag{6}$$

or, if $\rho < 1$,

$$\mathbf{p} = (1-\rho)\mathbf{p}(0)[\mathbf{I} - \rho\mathbf{J}]^{-1} = \mathbf{p}(0)\sum_{h=0}^{+\infty}(1-\rho)(\rho\mathbf{J})^{h} \tag{6a}$$

Equation (5) gives a linear WPC. Equation (6) indicates that p_j is a convex combination of v_j and $\mathbf{p}\mathbf{J}_j$, where the latter equals the ratio of means of production in the vertically integrated industry producing commodity j to means of production in the SSS. Finally, equation (6a) is the reduction of prices to 'dated quantities of embodied labour' (Kurz and Salvadori 1995, p. 175) in terms of SSC (for a thorough analysis of this equation, see Steedman 1999).

[2] If wages are paid *ex ante*, then ρ is no greater than the share of profits in the SSS.

2.2. Spectral decompositions

When (i) the system is 'regular' (in the sense of Schefold 1971, pp. 11-23), i.e. $\mathbf{l}\mathbf{x}_{\mathbf{J}k}^{\mathrm{T}} \neq 0$ for all k (which is the empirically relevant case); and (ii) $\mathbf{y}_{\mathbf{J}i}$, $[\mathbf{I}-\mathbf{A}]\mathbf{x}_{\mathbf{J}i}^{\mathrm{T}}$ are normalized by setting $\mathbf{y}_{\mathbf{J}i}[\mathbf{I}-\mathbf{A}]\mathbf{x}_{\mathbf{J}i}^{\mathrm{T}} = 1$ and $\mathbf{v}[\mathbf{I}-\mathbf{A}]\mathbf{x}_{\mathbf{J}i}^{\mathrm{T}} = 1$, then the WPC and the price-relative profit rate relationships can be expressed in the following 'spectral forms' (Schefold 2008, pp. 14-20; Mariolis and Tsoulfidis 2011, pp. 91-92; Mariolis 2015, pp. 280-282):

$$w = [(1-\rho)^{-1} d_1 + \Lambda_k^w]^{-1},\ \Lambda_k^w \equiv \sum_{k=2}^{n} (1-\rho\lambda_{\mathbf{J}k})^{-1} d_k,\ \sum_{i=1}^{n} d_i = 1,\ d_1 = \mathbf{y}_{\mathbf{J}1}\mathbf{z}^{\mathrm{T}} \tag{7}$$

$$\mathbf{p} = w[(1-\rho)^{-1}\mathbf{y}_{\mathbf{J}1} + \mathbf{\Lambda}_k^{\mathbf{p}}],\ \mathbf{\Lambda}_k^{\mathbf{p}} \equiv \sum_{k=2}^{n} (1-\rho\lambda_{\mathbf{J}k})^{-1}\mathbf{y}_{\mathbf{J}k},\ \sum_{i=1}^{n} \mathbf{y}_{\mathbf{J}i} = \mathbf{p}(0) \tag{8}$$

where the terms Λ_k^w and $\mathbf{\Lambda}_k^{\mathbf{p}}$ represent the effects of non-dominant eigenvalues on WPC and commodity prices, respectively, the d_i denote the coordinates of $\mathbf{z}^{\mathrm{T}}$ in terms of the right eigenbasis $[\mathbf{I}-\mathbf{A}]\mathbf{x}_{\mathbf{J}i}^{\mathrm{T}}$, and $\mathbf{p}(1) = d_1^{-1}\mathbf{y}_{\mathbf{J}1}$.

Matrix $\mathbf{J}$ can be decomposed as ('spectral representation'; see, e.g. Meyer 2001, 517-518)

$$\mathbf{J} = (\mathbf{y}_{\mathbf{J}1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}})^{-1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{y}_{\mathbf{J}1} + \sum_{k=2}^{n} \lambda_{\mathbf{J}k}(\mathbf{y}_{\mathbf{J}k}\mathbf{x}_{\mathbf{J}k}^{\mathrm{T}})^{-1}\mathbf{x}_{\mathbf{J}k}^{\mathrm{T}}\mathbf{y}_{\mathbf{J}k}$$

If there are strong quasi-linear dependencies amongst the technical conditions of production in all the vertically integrated industries, then $rank[\mathbf{J}] \approx 1$ or $|\lambda_{\mathbf{J}k}| \approx 0$ for all k, and, therefore, $\mathbf{J} \approx \mathbf{J}^{\mathrm{A}} \equiv (\mathbf{y}_{\mathbf{J}1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}})^{-1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{y}_{\mathbf{J}1}$.[3] Thus, from equations (5) to (8) it follows that $\Lambda_k^w \approx 1 - d_1$, $\mathbf{\Lambda}_k^{\mathbf{p}} \approx \mathbf{p}(0) - \mathbf{y}_{\mathbf{J}1}$ and *both* the WPC and the relative price-profit rate relationships tend to be rational functions of degree 1 (homographic functions):

$$w \approx w^{\mathrm{A}} \equiv [(1-\rho)^{-1} d_1 + \sum_{k=2}^{n} d_k] = w^{\mathrm{S}}[1+\rho(d_1-1)]^{-1} \tag{9}$$

$$\mathbf{p} \approx \mathbf{p}^{\mathrm{A}} \equiv w^{\mathrm{S}}[1+\rho(d_1-1)]^{-1}[(1-\rho)^{-1} d_1\mathbf{p}(1) + \sum_{k=2}^{n}\mathbf{y}_{\mathbf{J}k}]$$

or

$$\mathbf{p} \approx \mathbf{p}^{\mathrm{A}} = [1+\rho(d_1-1)]^{-1}[\mathbf{p}(0) + \rho(d_1\mathbf{p}(1) - \mathbf{p}(0))] \tag{10}$$

[3] As we will see in the next sections of the present paper, this heuristic case is more geared towards reality than one at first sight might think.

That is, the system tends to behave as a reducible two-industry economy without 'self-reproducing non-basics' (in the sense of Sraffa 1960, pp. 90-91). These eigenvalue decomposition rank-one approximations have the following properties:

(i). Their accuracy is directly related to the magnitudes of $|\lambda_{\mathbf{J}k}|^{-1}$.

(ii). They are exact at the extreme, economically significant, values of ρ.

(iii). When $rank[\mathbf{J}]=1$, i.e. $\mathbf{J}=\mathbf{J}^{\mathrm{A}}$, they become exact for all ρ. In that case, $\mathbf{J}$ can be transformed, via a semi-positive similarity matrix $\mathbf{T}$, into (Schur triangularization theorem; see, e.g. Meyer 2001, pp. 508-509)

$$\tilde{\mathbf{J}} \equiv \mathbf{T}^{-1}\mathbf{J}\mathbf{T} = \begin{bmatrix} 1 & \tilde{\mathbf{J}}_{12} \\ \mathbf{0}_{(n-1)\times 1} & \mathbf{0}_{(n-1)\times(n-1)} \end{bmatrix} \tag{11}$$

where the first column of $\mathbf{T}$ is $\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}}$ (the remaining columns are arbitrary), and $\tilde{\mathbf{J}}_{12}$ is a $1\times(n-1)$ *positive* vector: if, for instance,

$$\mathbf{T}=[\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}},\mathbf{e}_2^{\mathrm{T}},...,\mathbf{e}_n^{\mathrm{T}}] \tag{11a}$$

then

$$\tilde{\mathbf{J}}_{12}=(\mathbf{y}_{\mathbf{J}1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}})^{-1}[y_{2\mathbf{J}1},y_{3\mathbf{J}1},...,y_{n\mathbf{J}1}] \tag{11b}$$

That is, the original system is economically equivalent to an $n\times n$ corn-tractor system, even if $\mathbf{J}$ is irreducible (Mariolis 2013). Thus, the first row in the transformed matrix $\tilde{\mathbf{J}}$ represents an industry which can be characterized as '*hyper*-basic', and the price system (2) is transformed to

$$\mathbf{p}=w\mathbf{v}+\rho\mathbf{p}\mathbf{T}\tilde{\mathbf{J}}\mathbf{T}^{-1}$$

or

$$\boldsymbol{\pi}=w\boldsymbol{\omega}+\rho\boldsymbol{\pi}\tilde{\mathbf{J}} \tag{12}$$

where $\boldsymbol{\pi}\equiv\mathbf{p}\mathbf{T}$, $\boldsymbol{\omega}\equiv\mathbf{v}\mathbf{T}$, $\pi_1=\mathbf{p}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}}$ and $\omega_1=\mathbf{v}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}}$. The first equation in system (12) corresponds to the 'tractor industry', which is no more than the SSS, while the remaining equations correspond to *non*-uniquely determined 'corn-industries'.[4]

An alternative but rather different rank-one approximation can be deduced from the singular value decomposition of $\mathbf{J}$, i.e. $\mathbf{J}=\mathbf{U}\hat{\boldsymbol{\sigma}}_{\mathbf{J}}\mathbf{V}^{\mathrm{T}}$, where $\mathbf{U}$ and $\mathbf{V}^{\mathrm{T}}$ are real and orthogonal *nxn* matrices (i.e. $\mathbf{U}^{\mathrm{T}}=\mathbf{U}^{-1}$ and $\mathbf{V}^{\mathrm{T}}=\mathbf{V}^{-1}$), and $\hat{\boldsymbol{\sigma}}_{\mathbf{J}}\equiv[\sigma_{\mathbf{J}i}]$, $\sigma_{\mathbf{J}1}>\sigma_{\mathbf{J}2}\geq...\geq\sigma_{\mathbf{J}n}\geq 0$, is a diagonal matrix with non-negative numbers on its diagonal,

[4] For other heuristic cases, which are also interesting, both theoretically and empirically, and include more than one hyper-basic industry, see Mariolis and Tsoulfidis (2014, pp. 214-215; 2015, Chap. 5).

which are known as the singular values of $\mathbf{J}$. The columns of $\mathbf{U}$ (of $\mathbf{V}$) are particular choices of the right eigenvectors of $\mathbf{J}\mathbf{J}^{\mathrm{T}}$ (of $\mathbf{J}^{\mathrm{T}}\mathbf{J}$), which is a positive symmetric matrix. The nonzero singular values of $\mathbf{J}$ are the square roots of the nonzero eigenvalues of either $\mathbf{J}\mathbf{J}^{\mathrm{T}}$ or $\mathbf{J}^{\mathrm{T}}\mathbf{J}$, and, therefore,

$$\sqrt{n} \geq \sigma_{\mathbf{J}1} \geq \lambda_{\mathbf{J}1} = 1 > |\lambda_{\mathbf{J}k}| \geq \sigma_{\mathbf{J}n}$$

$$|\det \mathbf{J}| = \prod_{i=1}^{n} \sigma_{\mathbf{J}i} = \left| \prod_{i=1}^{n} \lambda_{\mathbf{J}i} \right|$$

$$rank[\mathbf{J}] = rank[\hat{\boldsymbol{\sigma}}_{\mathbf{J}}] \geq rank[\hat{\boldsymbol{\lambda}}_{\mathbf{J}}]$$

where $\hat{\boldsymbol{\lambda}}_{\mathbf{J}}$ denotes the diagonal matrix formed from the eigenvalues of $\mathbf{J}$.[5] Now, let $\hat{\boldsymbol{\sigma}}_{\mathbf{J}}^{[1]}$ be the matrix derived from $\hat{\boldsymbol{\sigma}}_{\mathbf{J}}$ by replacing the last $n-1$ singular values by zeroes. Then the positive matrix

$$\overline{\mathbf{J}}^{\mathrm{A}} \equiv \mathbf{U}\hat{\boldsymbol{\sigma}}_{\mathbf{J}}^{[1]}\mathbf{W}^{\mathrm{T}} \tag{13}$$

is the closest rank-one matrix to $\mathbf{J}$ in both the spectral (SP) and the Frobenius (F) norms (Schmidt-Eckart-Young Theorem):

$$\left\| \mathbf{J} - \overline{\mathbf{J}}^{\mathrm{A}} \right\|_{\mathrm{SP}} = \sigma_{\mathbf{J}2}$$

$$\left\| \mathbf{J} - \overline{\mathbf{J}}^{\mathrm{A}} \right\|_{\mathrm{F}} = \sqrt{\sigma_{\mathbf{J}2}^2 + \ldots + \sigma_{\mathbf{J}n}^2}$$

Finally, the index of 'inseparability',

$$\varepsilon_{\mathbf{J}1} \equiv 1 - \sigma_{\mathbf{J}1}^2 \left(\sum_{i=1}^{n} \sigma_{\mathbf{J}i}^2\right)^{-1}, \; 0 \leq \varepsilon_{\mathbf{J}1} < 1$$

is a convenient scalar-valued measure of the truncation error. Low values of $\varepsilon_{\mathbf{J}1}$, say less than 0.010, indicate that $\overline{\mathbf{J}}^{\mathrm{A}}$ represents $\mathbf{J}$ adequately (see Treitel and Shanks 1971, pp. 12-15).

It goes without saying that matrices $\hat{\boldsymbol{\lambda}}_{\mathbf{J}}$ and $\hat{\boldsymbol{\sigma}}_{\mathbf{J}}$ provide the basis for constructing respective higher-rank approximations of prices (for those approximations and their relationships with Bienenfeld's 1988 and Steedman's 1999 polynomial approximations, see Mariolis and Tsoulfidis 2015, Chap. 5).

[5] See, e.g. Horn and Johnson (1991, Chap. 3) and take into account that $\mathbf{J}$ is similar to the column stochastic matrix $\hat{\mathbf{y}}_{\mathbf{J}1}\mathbf{J}\hat{\mathbf{y}}_{\mathbf{J}1}^{-1}$.

2.3. Price-movements

If SSC is chosen as the numeraire, then $d_1 = 1$ and, therefore, approximation (10) becomes

$$\mathbf{p} \approx \mathbf{p}^{\mathrm{A}} = \mathbf{p}(0) + \rho(\mathbf{p}(1) - \mathbf{p}(0)) \tag{14}$$

which coincides with Bienenfeld's (1988) linear approximation formula for the price vector. By contrast, from equations (8), $d_1 = 1$ and $d_k = 0$, it follows that

$$\dot{\mathbf{p}} \equiv d\mathbf{p} / d\rho = -\mathbf{\Lambda}_k^{\mathbf{p}} + (1-\rho)\dot{\mathbf{\Lambda}}_k^{\mathbf{p}}$$

which implies that the individual components of $\mathbf{p}$ can change in a complicated way as ρ changes.[6] For the purposes of this paper, however, it suffices to focus only on equation (6), from which we get

$$\mathbf{p}\hat{\mathbf{v}}^{-1} = (1-\rho)\mathbf{e} + \rho\mathbf{pJ}\hat{\mathbf{v}}^{-1}$$

or

$$\mathbf{p}\hat{\mathbf{v}}^{-1} - \mathbf{e} = \rho R(\mathbf{pH}\hat{\mathbf{v}}^{-1} - R^{-1}\mathbf{e})$$

or, in terms of an industry j,

$$p_j v_j^{-1} - 1 = \rho R(k_j - R^{-1}) \tag{15}$$

where $k_j \equiv \mathbf{pH}_j v_j^{-1}$ denotes the capital-intensity of the vertically integrated industry producing commodity j, and R^{-1} the capital-intensity of the SSS. From equation (15) it directly follows that (i) when $k_j(\rho^*) = R^{-1}$, where $\rho^* > 0$, there is a price-labour value reversal at $\rho = \rho^*$; and (ii) when k_j is a constant function of ρ, the $p_j - \rho$ curve is linear.

Differentiation of equation (15) with respect to ρ gives

$$\dot{p}_j v_j^{-1} = Rk_j(e_{kj} - D_j) \tag{16}$$

where $e_{kj} \equiv \dot{k}_j \rho k_j^{-1}$ denotes the elasticity of k_j with respect to ρ, and $D_j \equiv (Rk_j)^{-1} - 1$ the percentage deviation of the capital-intensity of the SSS from k_j (for an exploration of these relationships, see Mariolis and Tsoulfidis 2009, pp. 5-7). The term e_{kj} represents the price effect, while D_j represents the capital-intensity effect. From equation (16) it follows

[6] Nevertheless, C. Bidard, H. G. Ehrbar, U. Krause and I. Steedman have detected some 'monotonicity (theoretical) laws' for the relative prices (see Bidard and Ehrbar 2007, and the references therein).

that the existence of a value of ρ such that e_{kj} and D_j have the same sign is the necessary condition for the violation of the traditional implication

$$D_j < (>)\ 0 \Leftrightarrow \dot{p}_j > (<)\ 0 \tag{17}$$

while the relevant sufficient condition is $e_{kj} < (>)\ D_j < (>)\ 0$ or, alternatively,

$$e_{kj} D_j^{-1} = \dot{k}_j \rho (R^{-1} - k_j)^{-1} > 1 \tag{18}$$

Condition (18) signifies that the violation of the traditional condition is 'more unlikely': (i) the smaller is the value of the relative profit rate; and/or (ii) the greater is the difference between the capital-intensity of the SSS and the capital-intensity of the vertically integrated industry under consideration.

3. Empirical Results

In this section we apply our analysis of the price-wage-relative profit rate curves to the SIOT of the UK for the year 1990 ($n = 33$), by distinguishing between the cases of circulating and fixed capital.[7] It is noted at the outset that, in the former case, the 'Hilbert distance' (consider Bidard and Ehrbar 2007, pp. 183-188) between $\mathbf{p}(1)$ and $\mathbf{l}$ is 1.049, and the 'normalized $d-$ distance' (Mariolis and Soklis 2010, p. 94) is 0.446. In the latter case, the figures are 1.441 and 0.557, respectively. Thus, it could be said that these systems deviate considerably from the equal value compositions of capital case. Furthermore, in the former case, the '*actual*' relative profit rate (see the Appendix) is approximately equal to 0.310, while in the latter case, is approximately equal to 0.281. To the best of our knowledge, there is no relevant empirical study where the actual ρ has been estimated as less than 0.170 and considerably greater than 0.500. Taking into account that ρ is no greater than the share of profits in the SSS, this seems to be in accordance with many well-known estimations of the share of profits (approximated by the net operating surplus) in actual economies (for further discussion and estimates for various actual economies, see Mariolis and Tsoulfidis 2015, Chap. 3).

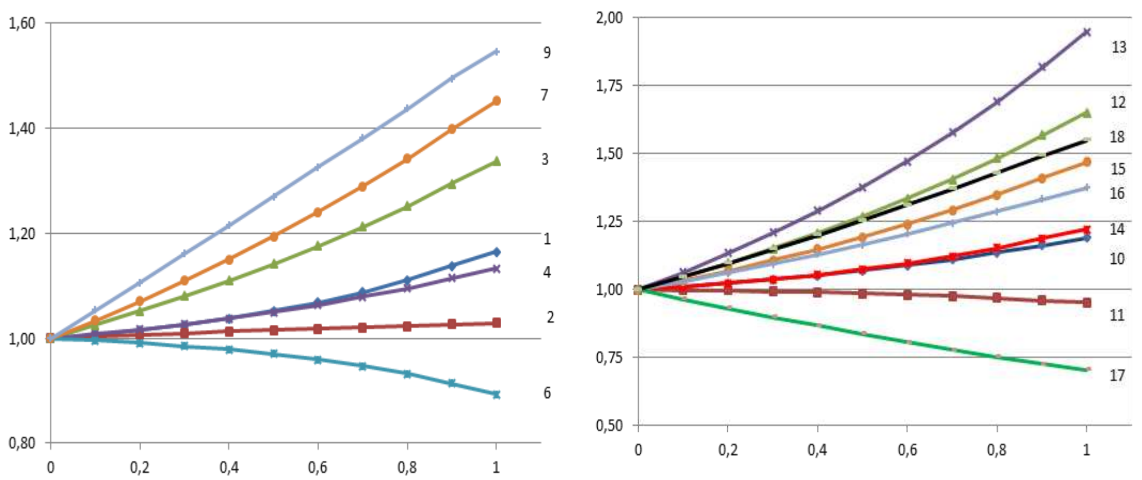
3.1. Price curves in the circulating capital case

The changes in the price-labour value ratios, in terms of SSC, induced by hypothetical changes in the relative profit rate ($0 \le \rho \le 1$) are in most cases monotonic, as this is displayed in Figure 1 (consider equation (16) and the traditional condition (17)). There are five 'exceptions' (or $n = 5/33 \cong 15\%$), which are displayed in Figure 2 (consider

[7] Information on the sources of data and the construction of variables is available in the Appendix.

equations (15) and (18)), while three from these curves cross the line of equality between prices and labour values (price-labour value reversal).[8] The 'mirror images' of the price-movements are given in Figure 3 (associated with the monotonic price curves) and Figure 4 (associated with the non-monotonic price curves), which display the changes in the vertically integrated capital-intensities, k_j, induced by changes in ρ, and the capital-intensity of the SSS, R^{-1}, which in our case is approximately equal to 1.149.

It is thus observed that (i) for instance, in industries 2, 9 and 23, the term k_j is almost constant and, therefore, the price curves are near-linear; (ii) in industries 5, 20, and 28 (in industry 31), where the price-movement is non-monotonic, the term k_j is a strictly increasing (decreasing) function of ρ; (iii) in industry 8, both p_j and k_j are non-monotonic functions; and (iv) as ρ increases, industries 5 and 20 are transformed (industry 31 is transformed) to capital (to labour) intensive relative to the SSS, whereas industry 8 (industry 28) remains capital (labour) intensive.



[8] The price curve corresponding to industry 8 displays a maximum at $\rho \cong 0.955$, which is not visible in Figure 2. It is deduced from more detailed data.

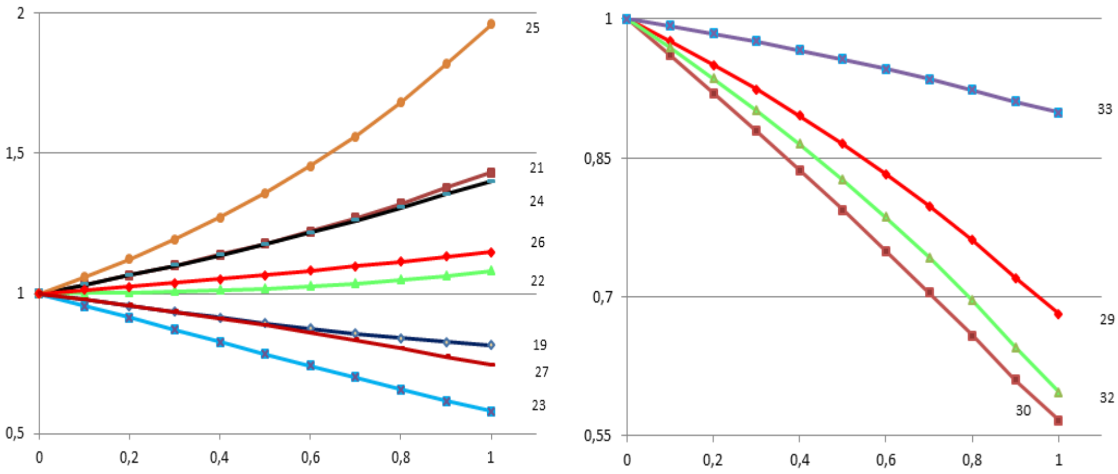


Figure 1. Monotonic price curves; circulating capital model

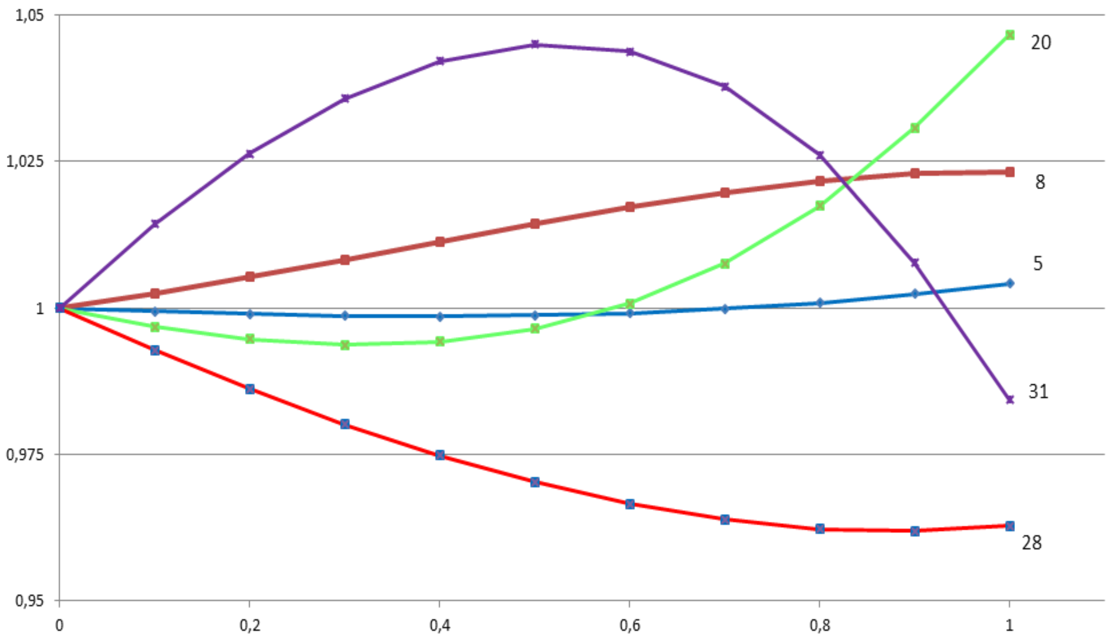


Figure 2. Non-monotonic price curves; circulating capital model

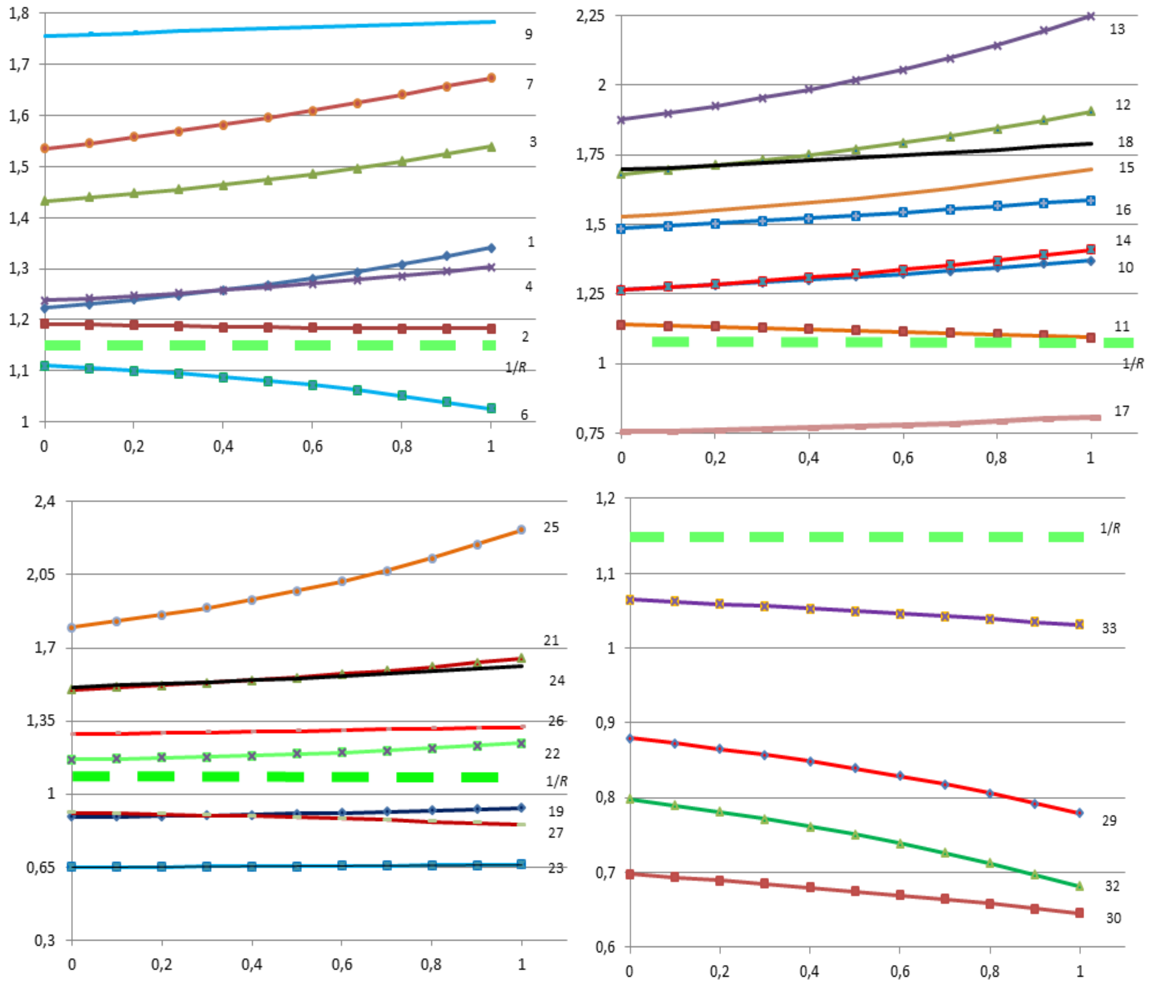


Figure 3. Vertically integrated capital-intensities associated with the monotonic price curves, and the capital-intensity of the SSS; circulating capital model

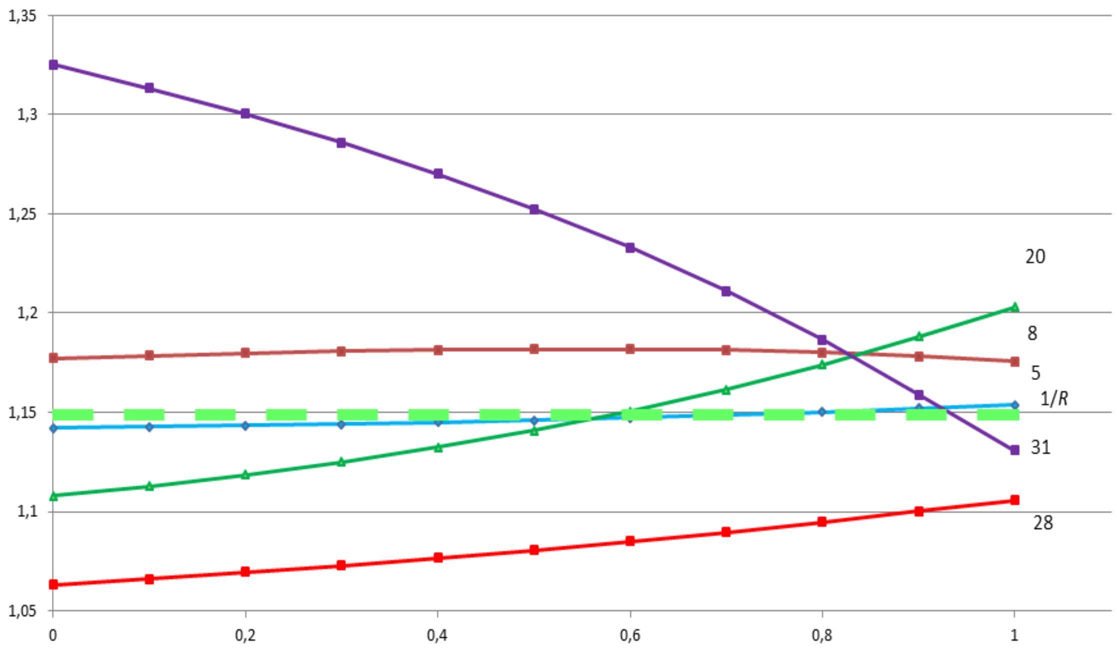


Figure 4. Vertically integrated capital-intensities associated with the non-monotonic price curves, and the capital-intensity of the SSS; circulating capital model

3.2. Price curves in the fixed capital case

Figures 5 and 6 display, respectively, the changes in the price-labour value ratios and in the vertically integrated capital-intensities in terms of SSC (now, R^{-1} is approximately equal to 3.065). It is observed that: (i) there are only two non-monotonic price curves, i.e. those corresponding to industries 15 and 30, which display a maximum (at $\rho \cong 0.923$ and $\rho \cong 0.533$, respectively); (ii) there is no price-labour value reversal; (iii) the differences $\left|k_j - R^{-1}\right|$ tend to be larger than those of the circulating capital case;[9] and (iv) the number of near-linear price curves is remarkably greater than that of the circulating capital case.

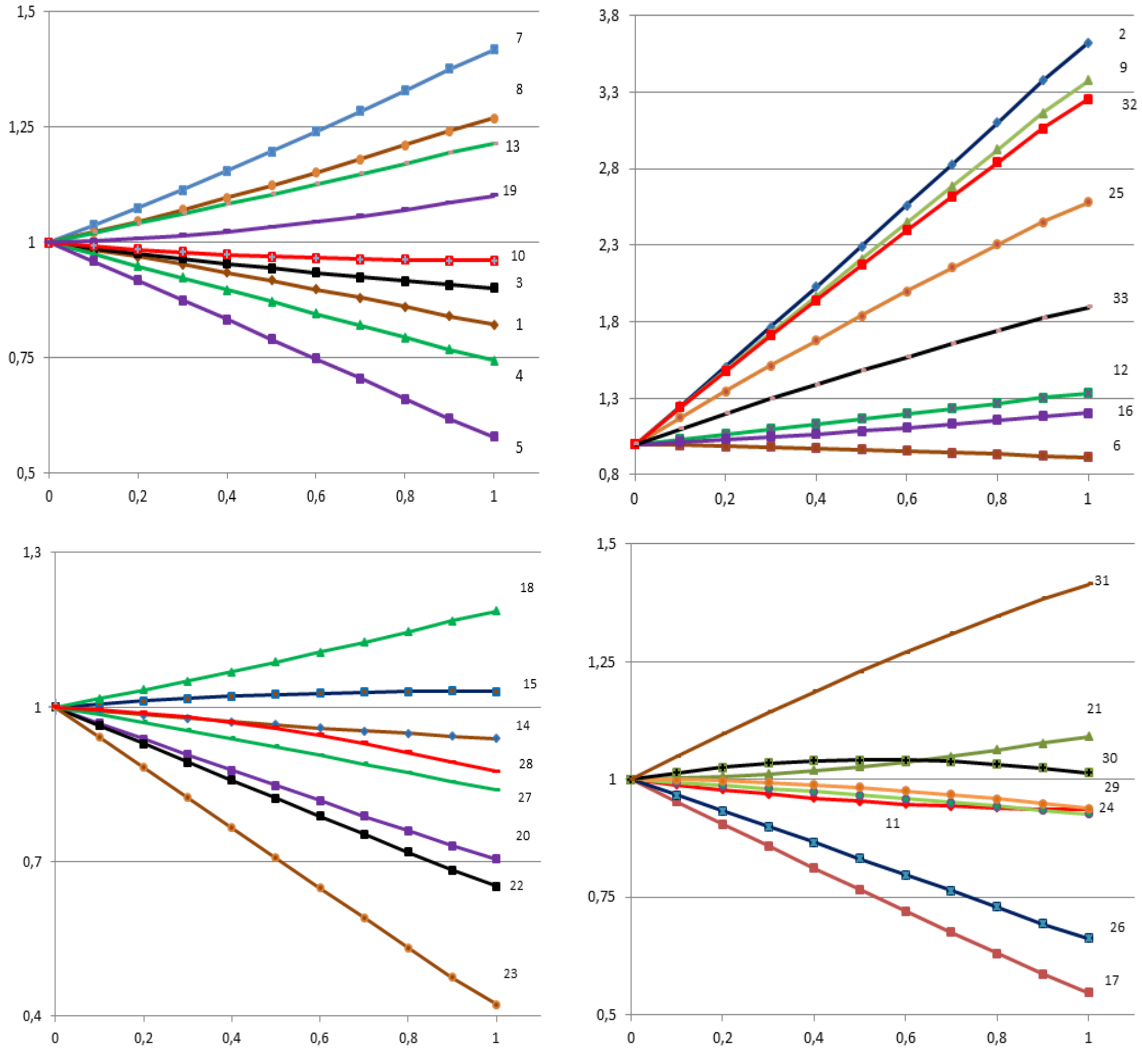


Figure 5. Price curves; fixed capital model

[9] In effect, we found that, at $\rho = 0$, the vertically integrated capital-intensities in the circulating (fixed) capital case gave an arithmetic mean equal to 1.25 (to 3.87) and a standard deviation of 0.32 (of 2.52), with a coefficient of variation of 0.26 (of 0.65).

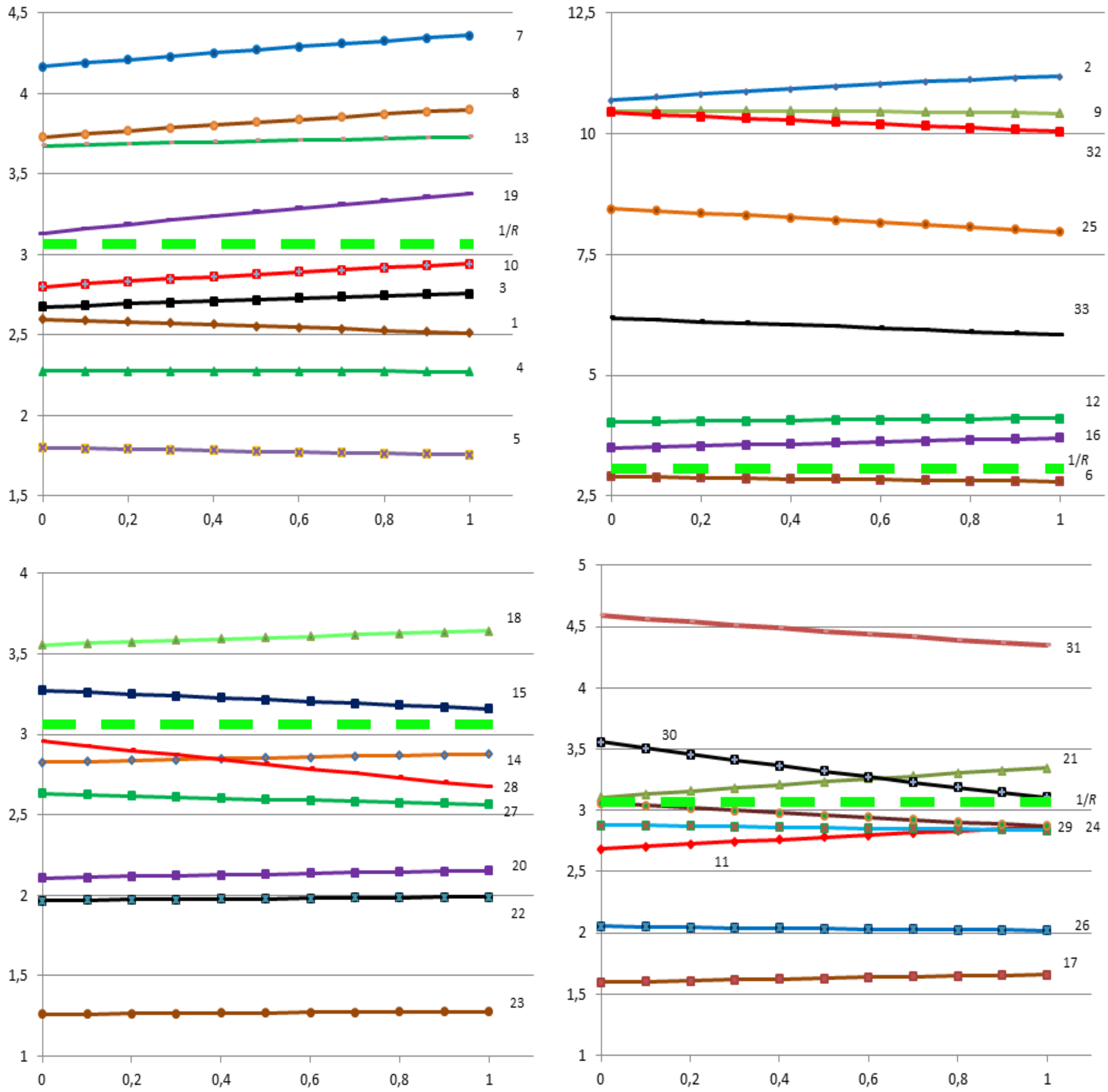


Figure 6. Vertically integrated capital-intensities and the capital-intensity of the SSS; fixed capital model

3.3. Linear approximation of prices

The already identified 'smooth' patterns of the price-movements indicate that the linear approximation (14) would not be without some empirical validity. Figure 7 displays the errors in this approximation, i.e. $p_j^{\mathrm{A}} - p_j$, $j = 1, 2, ..., 33$, as functions of the relative profit rate (hereafter, the symbol 'CC' ('FC') will indicate the circulating (fixed) capital model).[10] A visual inspection of this figure reveals that, in both models, the majority of the errors is concentrated in narrow bands and, therefore, the 'crude' approximation (14)

[10] Only a single curve, corresponding to industry 2 in the circulating capital case, crosses the horizontal axis (at $\rho \cong 0.639$).

works well (especially in the more realistic case of fixed capital, where the errors are less than ± 0.004).

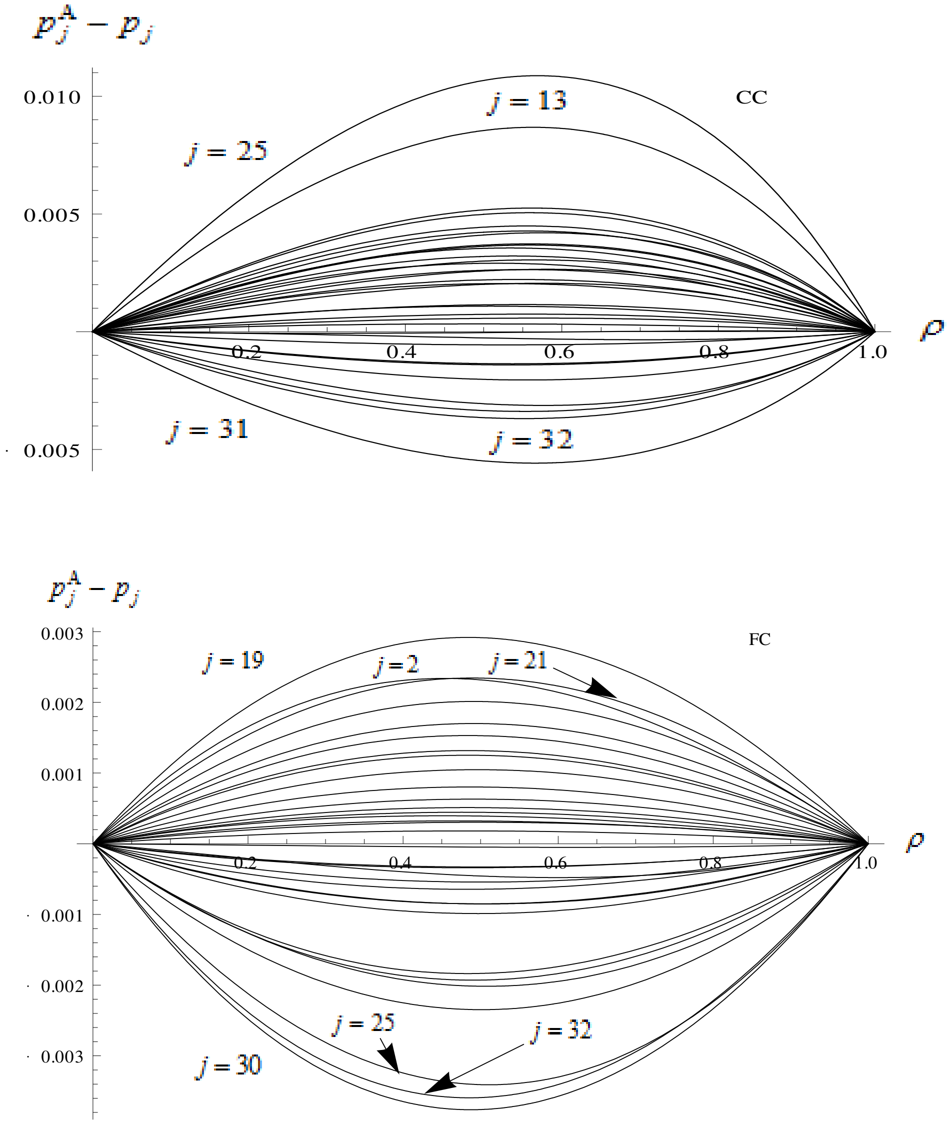
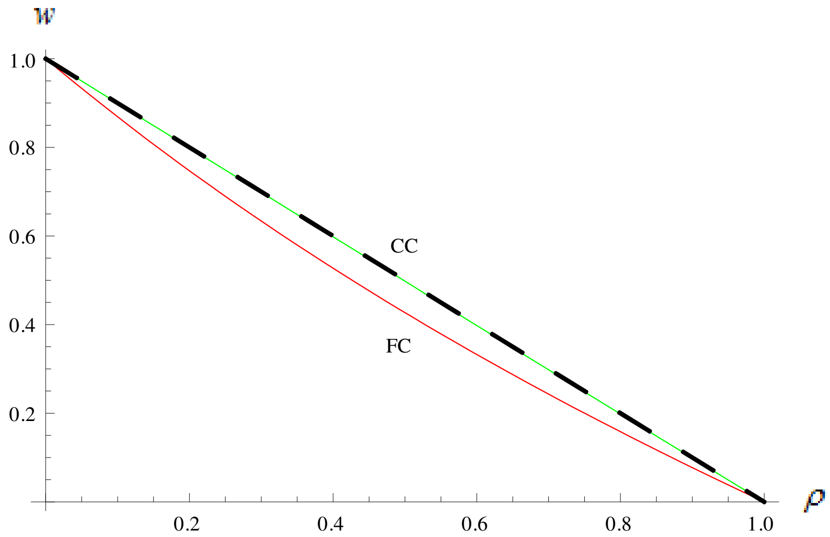


Figure 7. The errors in the linear approximation of prices as functions of the relative profit rate

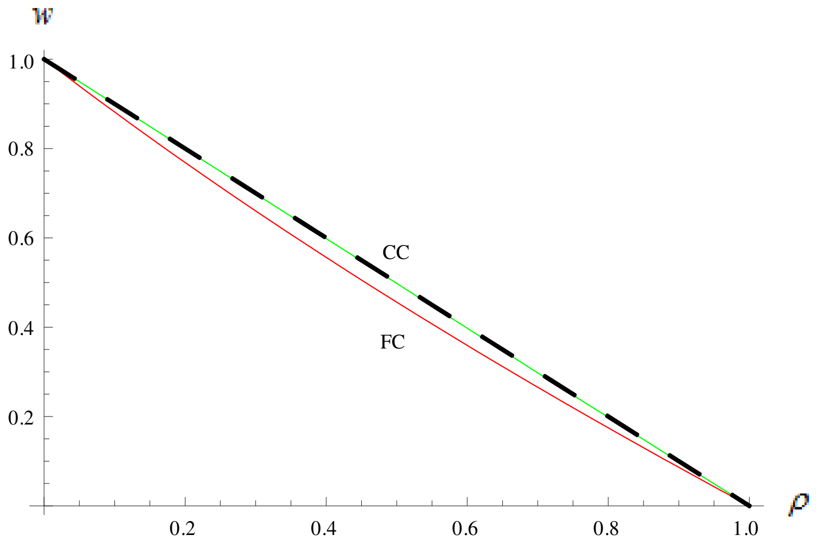
3.4. Wage-profit curves

In order to complete the picture, Figure 8 displays the WPCs (equation (3)), measured in terms of three alternative numeraires that are of particular significance, i.e. the actual gross output, net output and real wage rate vectors, and the w^S curve, which is depicted by a dashed line. This figure also reports the sign of the second derivative, $\ddot{w}$, the maximum absolute deviation from the straight line, i.e. $\max\left|w - w^S\right|$, and the mean value of this deviation, i.e. $MAD \equiv \int_0^1 |(w - w^{\mathrm{S}})| d\rho$. It is thus observed that there is only one inflection point, which occurs at a rather low value of ρ and, in fact, less than the actual one. Nevertheless, as experiments showed, the relevant price effects, in the six cases already examined, are relatively weak and, therefore, for low values of ρ, we can safely write $w = 1 - \rho \mathbf{pJz}^{\mathrm{T}} \approx 1 - \rho \mathbf{p}(0)\mathbf{Jz}^{\mathrm{T}}$ (see equation (2)).



Numeraire: actual gross output

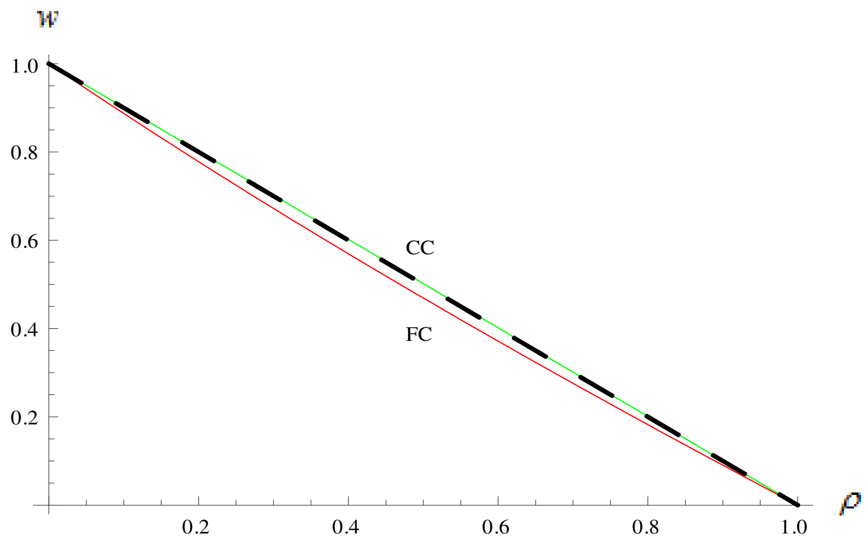
$\ddot{w} > 0$, $\max|w - w^{\mathrm{S}}| \cong 0.0021$ (at $\rho \cong 0.59$), $MAD \cong 0.0014$; CC $\ddot{w} > 0$, $\max|w - w^{\mathrm{S}}| \cong 0.0736$ (at $\rho \cong 0.45$), $MAD \cong 0.0489$; FC



Numeraire: actual net output

$\ddot{w} < (>)\ 0$ for $\rho < (>)\ \tilde{\rho} \cong 0.164$, $\max|w - w^{\mathrm{S}}| \cong 0.0022$ (at $\rho \cong 0.63$), $MAD \cong 0.0014$; CC

$\ddot{w} > 0$, $\max|w - w^{\mathrm{S}}| \cong 0.0443$ (at $\rho \cong 0.46$), $MAD \cong 0.0294$; FC



Numeraire: actual real wage rate

$\ddot{w} < 0$, $\max\left|w - w^{S}\right| \cong 0.0020$ (at $\rho \cong 0.38$), $MAD \cong 0.0012$; CC

$\ddot{w} > 0$, $\max\left|w - w^{S}\right| \cong 0.0312$ (at $\rho \cong 0.45$), $MAD \cong 0.0207$; FC

Figure 8. The WPCs in terms of composite commodities

Further experiments, with all the individual commodities as numeraires, gave somewhat similar results: consider, for instance, Figure 9 which displays the deviations of the WPCs, measured in terms of individual commodity j, from the w^{S} curve as functions of ρ.[11] It is added that, in the circulating capital case (i) the WPCs in terms of commodities $j=$ 5, 20 and 31 cross the w^{S} curve at the points where occur price-labour value reversals (see Figure 2), and an inflection point appears on each WPC for $0.57 < \rho < 0.70$, i.e. at values of ρ which are greater than the actual one; (ii) an inflection point also appears on the WPCs in terms of commodities $j=$ 6, 11 and 22, for $0.10 < \rho < 0.30$, i.e. at values of ρ which are less than the actual one; and (iii) the maximum value of $\max\left|w^{j} - w^{S}\right|$ is 0.140 (for $j = 13$ and 23; the relevant *MADs* are 0.092). In the fixed capital case, (i) an inflection point appears on the WPCs in terms of

[11] If p_j^{S} denotes the price of commodity j in terms of SSC, and w^{j} denotes the money wage rate corresponding to the normalization equation $p_j = v_j$, then $(w^{j})^{-1} p_j = (w^{S})^{-1} p_j^{S}$ or $w^{j} - w^{S} = w^{j}(1 - p_j^{S} v_j^{-1})$ or, recalling equation (15), $w^{j} - w^{S} = w^{j} \rho R (R^{-1} - k_j)$. It then follows that (i) $p_j^{S} v_j^{-1} = 1$ implies $w^{j} = w^{S}$; and (ii) $w^{j} - w^{S}$ is directly related to $R^{-1} - k_j$ (compare the outer curves in Figure 9 with the relevant differences $R^{-1} - k_j$ in Figures 3 and 6).

commodities $j =$ 19, 21, 28 and 29; (ii) for $j = 19$, 21 and 28, the inflection point occurs at $\rho \cong$ 0.25, 0.25 and 0.22, respectively, i.e. at values of ρ which are less than the actual one, while for $j = 29$, it occurs at $\rho \cong 0.32$, i.e. at a value of ρ which is greater than the actual one; and (iii) the maximum value of $\max\left|w^j - w^S\right|$ is 0.310 (for $j = 2$; the relevant *MAD* is 0.201).

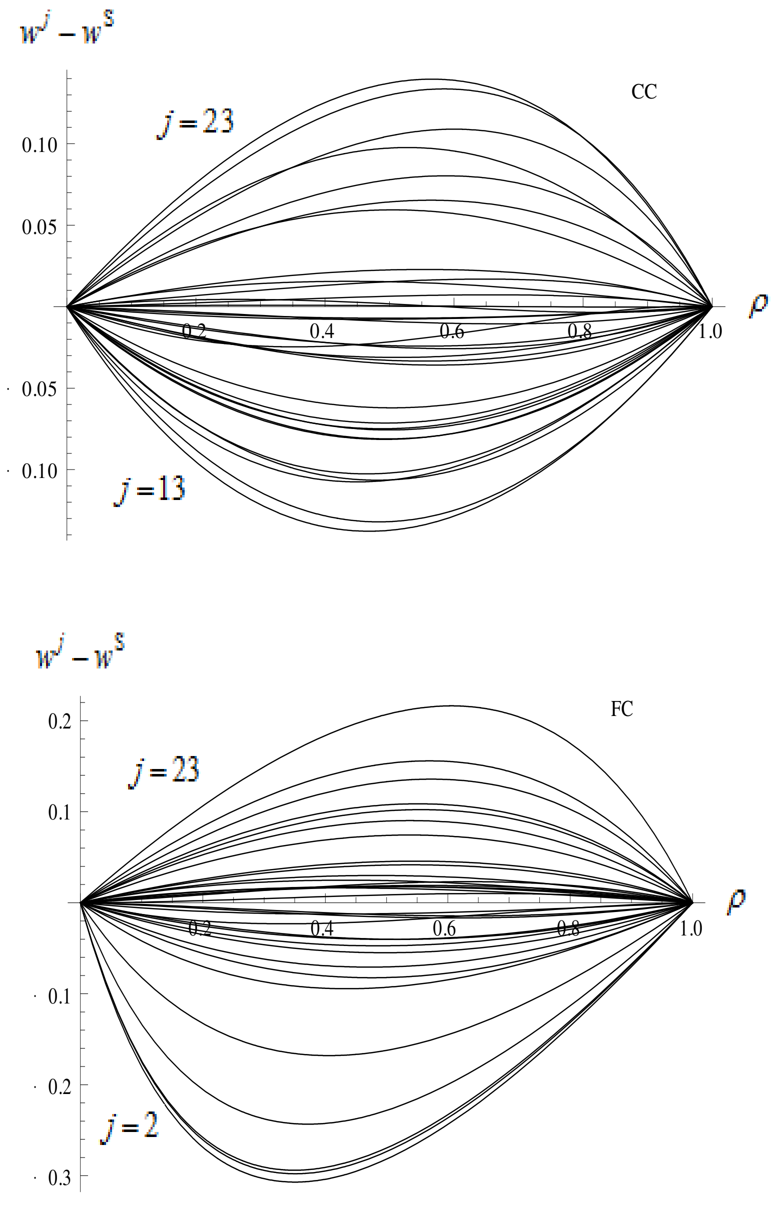


Figure 9. The deviations of the WPCs, in terms of individual commodities, from the straight line as functions of the relative profit rate

4. Characteristic Value Distributions

Summing up, our empirical results show that, *within* the economically relevant interval of ρ, there are 7 cases (or $7/66 \cong 11\%$ of the cases tested) of non-monotonic price-movement, with no more than one extreme point, and 3 cases (or 5%) of price-labour value reversals. Both phenomena occur at values of ρ which are greater than the actual one. Moreover, the linear approximation of the prices works well. Finally, the WPCs alternate in curvature in 11 cases (or $11/72 \cong 15\%$ of the cases tested), in 7 cases (or 10%) the inflection point occurs at values of ρ which are less than the actual one, while it has not been found any case where the curvature switches more than once. In general, the price effects along the WPCs could be considered relatively weak.

It seems, therefore, that the actual economy under consideration behaves as a low-dimensional system, that is, with no more than three industries. The spectral forms (7) and (8) suggest that this experimental finding, which is in line with the overall available evidence, could be connected to the characteristic value distributions of matrices $\mathbf{J}$. Figures 10 and 11 (the horizontal axes are plotted in *logarithmic* scale) display the moduli of the eigenvalues and the normalized singular values, $\sigma_{\mathbf{J}i}\sigma_{\mathbf{J}1}^{-1}$, respectively, as well as the relevant arithmetic (*AM*) and geometric (*GM*) means of the non-zero non-dominant values, and the indexes of inseparability.[12] It is interesting to note that the moduli of the second eigenvalue (of the second normalized singular value) of the fixed capital model is much lower than that of the circulating capital model, i.e. 0.099 (0.207) versus 0.427 (0.417). This finding is closely related to the fact that the matrix of capital stock coefficients, $\mathbf{K}$, which contains many zero and near-zero elements, imposes its reducibility form *and* spectral attributes onto the matrix $\mathbf{H}^{\mathrm{K}} \equiv \mathbf{K}[\mathbf{I}-\mathbf{A}]^{-1}$. This claim may be ascertained by merely comparing the graphs and the levels of the indexes in Figures 10 and 11 with those in Figure 12, which displays the moduli of the normalized eigenvalues and the normalized singular values of $\mathbf{K}$ (the horizontal axis is plotted in logarithmic scale). It may be recalled that

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[12] As is well-known, the geometric mean is more appropriate for detecting the central tendency of an exponential set of numbers. In our case, it can be written as

$$GM = \left|\det \mathbf{J}\right|^{(n-1)^{-1}}$$

We define the index of inseparability associated with the eigenvalues as

$$\varepsilon_{\mathbf{J}1}^{\lambda} \equiv 1 - \lambda_{\mathbf{J}1}\left(\sum_{i=1}^{n}\left|\lambda_{\mathbf{J}i}\right|\right)^{-1} = 1 - \left(1 + \sum_{k=2}^{n}\left|\lambda_{\mathbf{J}k}\right|\right)^{-1}$$

$$rank[\mathbf{K}] + rank[\mathbf{I}-\mathbf{A}] - n \le rank[\mathbf{H}^{\mathrm{K}}] \le \min\{rank[\mathbf{K}],\ rank[\mathbf{I}-\mathbf{A}]\}$$

(see, e.g. Meyer 2001, p. 211), and in our case,

$$rank[\mathbf{K}] = 30,\ |\lambda_{\mathbf{K}2}|\lambda_{\mathbf{K}1}^{-1} \cong 0.285,\ |\lambda_{\mathbf{K}3,4}|\lambda_{\mathbf{K}1}^{-1} \cong 0.055,$$

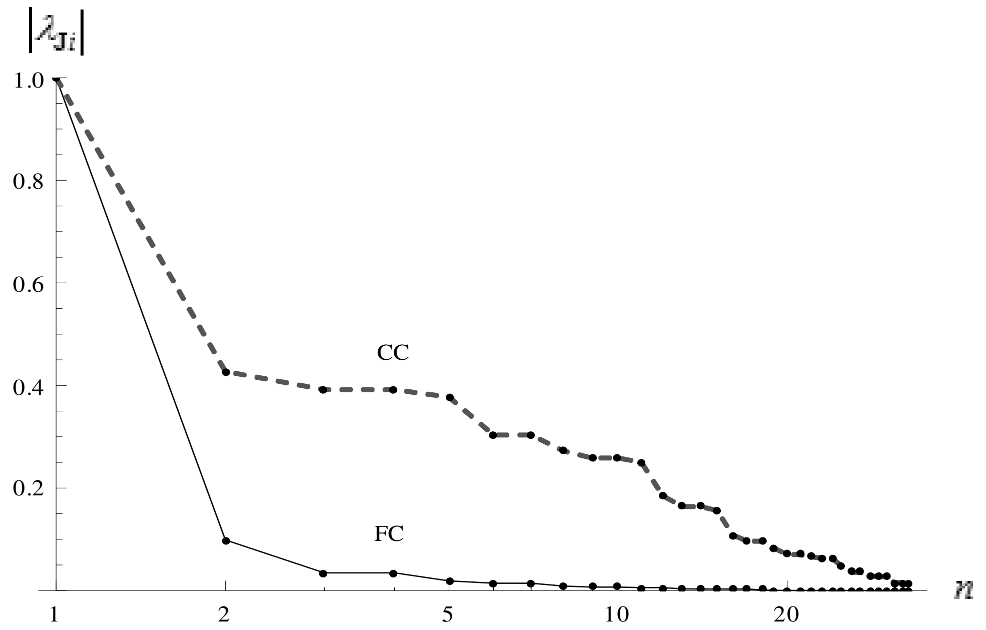
$$\sigma_{\mathbf{K}2}\sigma_{\mathbf{K}1}^{-1} \cong 0.302,\ \sigma_{\mathbf{K}3}\sigma_{\mathbf{K}1}^{-1} \cong 0.236,$$

$$rank[\mathbf{H}^{\mathrm{K}}] = 30,\ rank[\mathbf{I}-\mathbf{A}] = 33.$$

Thus, matrix $\mathbf{H}^{\mathrm{K}}$ is reducible without self-reproducing non-basics, and the decay of its characteristic values is remarkably faster than that of the characteristic values of $\mathbf{H} \equiv \mathbf{A}[\mathbf{I}-\mathbf{A}]^{-1}$.

These characteristic value configurations, which are in absolute accordance with those detected in other studies of quite diverse actual single-product economies (see Mariolis and Tsoulfidis 2015, Chaps. 5-6, and the references therein), indicate that the *effective* ranks of the system matrices are relatively low and, therefore, our statements about the shapes of the price-wage-profit rate curves of the UK economy would not be so sensitive to the numeraire choice.[13]

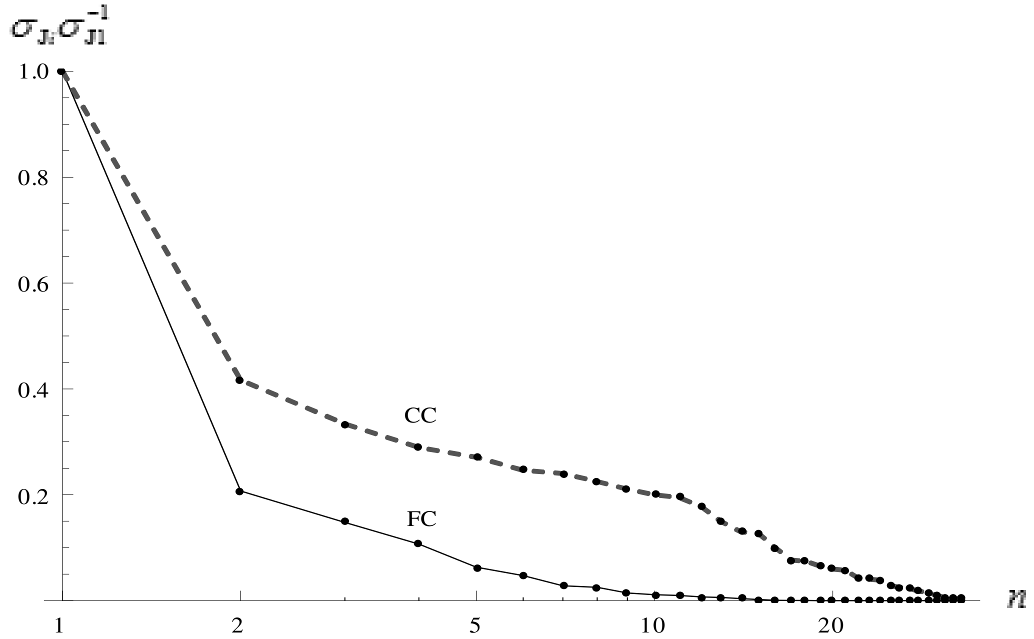
[13] There is, however, a caveat for the SIOTs as adequate representations of actual economies. It is well known that these tables can be derived from the 'System of National Accounts' framework of Supply and Use Tables (SUTs), where the latter tables could be considered as the counterpart of joint production systems *à la* v. Neumann (1937) and Sraffa (1960). The hitherto evidence from the SUTs indicates that actual economies do not necessarily have the usual properties of single-product systems (Mariolis and Soklis 2010; Soklis 2011, 2015); for instance, there are cases in which elements in the (positive) vector of 'labour-commanded prices', $w^{-1}\mathbf{p}$, decrease with the profit rate and, therefore, the monotonicity of the WPC depends on the numeraire choice.



$AM(|\lambda_{\mathbf{J}k}|) \cong 0.152$, $GM(|\lambda_{\mathbf{J}k}|) \cong 0.097$, $\varepsilon_{\mathbf{J}1}^{\lambda} \cong 0.830$; CC

$AM(|\lambda_{\mathbf{J}k}|) \cong 0.009$, $GM(|\lambda_{\mathbf{J}k}|) \cong 0.001$, $\varepsilon_{\mathbf{J}1}^{\lambda} \cong 0.211$; FC

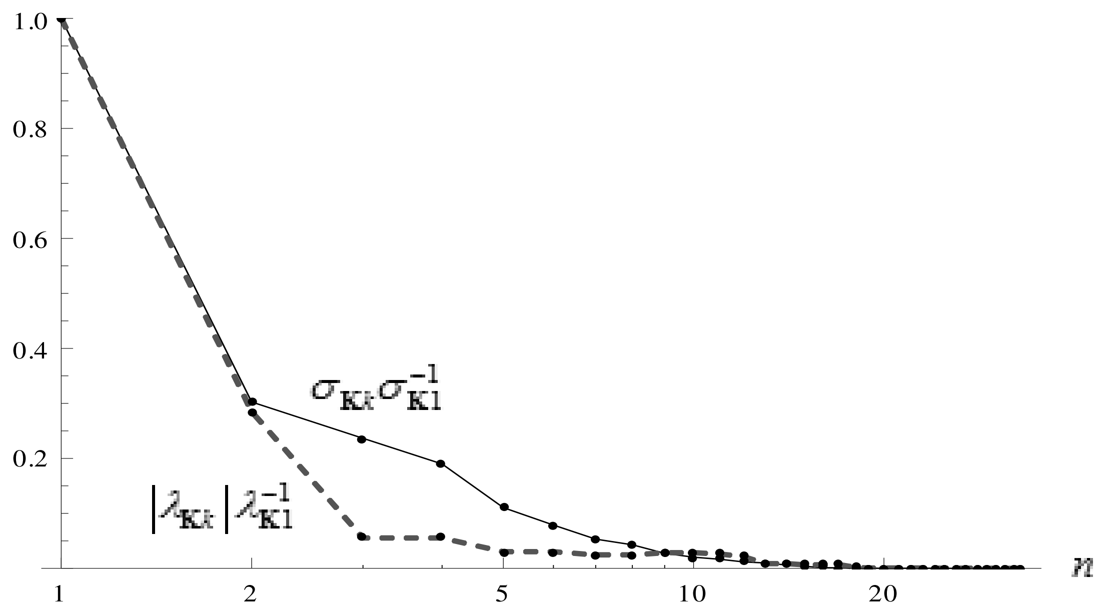
Figure 10. The moduli of the eigenvalues of the normalized vertically integrated technical coefficients matrices



$AM(\sigma_{\mathbf{J}k}\sigma_{\mathbf{J}1}^{-1}) \cong 0.122$, $GM(\sigma_{\mathbf{J}k}\sigma_{\mathbf{J}1}^{-1}) \cong 0.065$, $\varepsilon_{\mathbf{J}1} \cong 0.462$; CC

$AM(\sigma_{\mathbf{J}k}\sigma_{\mathbf{J}1}^{-1}) \cong 0.023$, $GM(\sigma_{\mathbf{J}k}\sigma_{\mathbf{J}1}^{-1}) \cong 0.0007$, $\varepsilon_{\mathbf{J}1} \cong 0.078$; FC

Figure 11. The normalized singular values of the normalized vertically integrated technical coefficients matrices



$AM(|\lambda_{\mathbf{K}k}|\lambda_{\mathbf{K}1}^{-1}) \cong 0.022$, $GM(|\lambda_{\mathbf{K}k}|\lambda_{\mathbf{K}1}^{-1}) \cong 0.003$, $\varepsilon_{\mathbf{K}1}^{\lambda} \cong 0.394$

$AM(\sigma_{\mathbf{K}k}\sigma_{\mathbf{K}1}^{-1}) \cong 0.038$, $GM(\sigma_{\mathbf{K}k}\sigma_{\mathbf{K}1}^{-1}) \cong 0.001$, $\varepsilon_{\mathbf{K}1} \cong 0.172$

Figure 12. The moduli of the normalized eigenvalue and the normalized singular values of the capital stock coefficients matrix

5. The Hyper- Basic Industry

Having established that $\mathbf{J}^{\mathrm{A}} \equiv (\mathbf{y}_{\mathbf{J}1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}})^{-1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{y}_{\mathbf{J}1}$ is a good approximation of $\mathbf{J}$, in the sense that both matrices give rise to price trajectories close to each other, we shall focus on the fixed capital case and apply the transformations (11)-(11a) to these matrices. It then follows that:

(i). The first row of the semi-positive and rank-1 matrix $\tilde{\mathbf{J}}^{\mathrm{A}} \equiv \mathbf{T}^{-1}\mathbf{J}^{\mathrm{A}}\mathbf{T}$ is

1.000	0.792	0.355	0.302	0.297	0.362	0.463	0.384	0.654	0.370	0.356
0.350	0.321	0.345	0.305	0.463	0.346	0.348	0.535	0.331	0.432	0.227
0.292	0.284	0.673	0.279	0.345	0.283	0.374	0.346	0.455	1.234	0.420

(ii). The first row of the *non*-semipositive and rank-30 matrix $\tilde{\mathbf{J}} \equiv \mathbf{T}^{-1}\mathbf{J}\mathbf{T}$ is

1.000	0.791	0.336	0.276	0.299	0.375	0.366	0.302	0.555	0.309	0.286
0.302	0.283	0.296	0.295	0.358	0.293	0.318	0.346	0.280	0.337	0.208
0.281	0.265	0.715	0.290	0.382	0.391	0.425	0.205	0.550	1.575	0.694

(iii). The Euclidean norm of the difference between these rows is 0.585. The spectral norm of matrix $\tilde{\mathbf{J}} - \tilde{\mathbf{J}}^{\mathrm{A}}$ is 0.760, while that of matrix $\mathbf{J} - \mathbf{J}^{\mathrm{A}}$ is 0.631.

Thus, it is concluded that the transformed approximate matrix $\tilde{\mathbf{J}}^{\mathrm{A}}$ extracts the essential information embedded in the original system and that, in the fixed capital case, even the original system tends to be economically equivalent to an $n \times n$ corn-tractor system. For the circulating capital case, this statement is ascertained only in a weaker sense.

Finally, the same holds true for the singular value decomposition rank-one approximation matrix of $\mathbf{J}$ (see equation (13)). The first row of the semi-positive and rank-1 matrix $\tilde{\bar{\mathbf{J}}}^{\mathrm{A}} \equiv \bar{\mathbf{T}}^{-1}(\lambda_{\bar{\mathbf{J}}^{\mathrm{A}}1}^{-1}\bar{\mathbf{J}}^{\mathrm{A}})\bar{\mathbf{T}}$, where $\bar{\mathbf{T}} = [\mathbf{x}_{\bar{\mathbf{J}}^{\mathrm{A}}1}^{\mathrm{T}}, \mathbf{e}_2^{\mathrm{T}}, ..., \mathbf{e}_n^{\mathrm{T}}]$ and $\lambda_{\bar{\mathbf{J}}^{\mathrm{A}}1} \cong 0.973$, is

1.000	0.799	0.371	0.313	0.320	0.402	0.405	0.360	0.605	0.356	0.321
0.329	0.304	0.324	0.314	0.414	0.322	0.340	0.488	0.302	0.378	0.214
0.295	0.293	0.720	0.295	0.376	0.360	0.423	0.248	0.532	1.414	0.602

The Euclidean norm of the difference between this row and the first row of $\tilde{\mathbf{J}}$ is 0.286. The spectral norm of matrix $\tilde{\mathbf{J}} - \tilde{\bar{\mathbf{J}}}^{\mathrm{A}}$ is 0.627, while that of matrix $\mathbf{J} - \lambda_{\bar{\mathbf{J}}^{\mathrm{A}}1}^{-1}\bar{\mathbf{J}}^{\mathrm{A}}$ is 0.548.

6. Concluding Remarks

Main aspects of the wage-price-profit rate relationships in linear single-product systems are regulated to a great extent by the distributions of the characteristic values contained in their input-output structure. It has been repeatedly shown that, across countries and over time, the moduli of the eigenvalues and the singular values of actual economies follow nearly 'L-shaped' patterns, while, when the capital stock matrices are taken into account, the decay of the characteristic values is remarkably faster. Thus, only a few eigenvalues really matter for the particular shapes of the wage-price-profit rate curves, which is another way to say that these curves tend to be similar to those of low-dimensional systems. In fact, it seems that similarity transformations that result in one (or a few) hyper-basic industry (industries) extract the essential features contained in the original system and provide the basis for constructing reliable approximations of the actual price curves. Consequently, the real paradox we are confronted with is the actual skew distributions of the characteristic values and not really the 'paradoxes in capital theory'. Hence, future research efforts should focus on the proximate determinants of these characteristic value distributions.

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Appendix: Data Sources and Construction of Variables

The input-output table of UK for the year 1990 is available from the OECD STAN database (http://www.oecd.org), and the degree of disaggregation is such that 33 product/industry groups are identified (see Table A.1).

The market prices of all products are taken to be equal to one; that is to say, the physical unit of measurement of each product is that unit which is worth of a monetary unit (in the SIOT of the UK economy, the unit is set to one million pounds). The various variables used in our estimations were constructed as follows:

(i). The matrix of direct technical coefficients, $\mathbf{A}$, is obtained from the SIOT by dividing each industry's inputs by their respective gross output.

(ii). The matrix of capital stocks, $\mathbf{S}$, is constructed by using the capital flows table, which allocates the gross investment flows, I_{ij}, of each industry to itself and to the other industries. With the aid of the capital flows table we form a matrix of weights by dividing the elements of each column by the respective column sum, i.e. $\mathbf{I}_{\mathrm{w}} \equiv [I_{ij}(\sum_{i=1}^{n} I_{ij})^{-1}]$. By assuming that the capital stocks are allocated amongst industries in a way similar to that of investment flows, we can write $\mathbf{S} \equiv \mathbf{I}_{\mathrm{w}}\hat{\boldsymbol{\kappa}}$, where $\hat{\boldsymbol{\kappa}}$ denotes the diagonal matrix formed from the vector of net capital stocks, $\boldsymbol{\kappa}$ (both the capital flows table for the year 1990 and $\boldsymbol{\kappa}$ are available in the OECD STAN data base). The so derived matrix of capital stocks is subsequently divided by the actual gross output vector, $\bar{\mathbf{x}}$, to obtain the matrix of capital stock coefficients, i.e. $\mathbf{K} \equiv \mathbf{S}\hat{\bar{\mathbf{x}}}^{-1}$. In similar fashion we could construct the matrix of depreciation coefficients, but the lack of depreciation data at the required industry detail did not allow the construction of such matrix.

(iii). The vector of direct labor coefficients, $\mathbf{l}$, is estimated using the wage bill (the product of annual wage times the number of employees) of each of our 33 industries. The problem with this estimation is that the self-employed population is not accounted for. Fortunately, the OECD data base provides information on both the total employment and the number of employees for each of our 33 industries. From the available data, we estimate the average industry wage and we divide it by the economy-wide minimum wage, the so estimated relative industry wages are subsequently multiplied by the total employment (employed plus self-employed) and so we derive the homogenized industry employment. This reduction, of course, is only meaningful when the relative wages express with sufficient precision the differences in skills and intensity of labour. The adjusted for skills total employment is divided by the industry total output to obtain the vector of direct labor coefficients.

(iv). By assuming that all wages are consumed and that consumption out of wages has the same composition as the vector of the final consumption expenditures of the household sector, $\mathbf{c}^{\mathrm{T}}$, directly available in the SIOT, the commodity vector defining the 'actual' real wage rate is estimated as $\mathbf{b}^{\mathrm{T}} = [\min_j \{w_{\mathrm{m}j}\}(\mathbf{p}_{\mathrm{m}}\mathbf{c}^{\mathrm{T}})^{-1}]\mathbf{c}^{\mathrm{T}}$, where $\min_j \{w_{\mathrm{m}j}\}$ denotes the economy-wide minimum money wage rate in terms of market prices, and $\mathbf{p}_{\mathrm{m}}$ the vector of market prices, which is identified with $\mathbf{e}$.

(v). Substituting $w = \mathbf{p}\mathbf{b}^{\mathrm{T}}$ in $\mathbf{p} = w\mathbf{v} + \rho\mathbf{p}\mathbf{J}$ yields $\mathbf{p} = \rho\mathbf{p}\mathbf{J}[\mathbf{I} - \mathbf{b}^{\mathrm{T}}\mathbf{v}]^{-1}$. Thus the 'actual' relative profit rates for both models are estimated as the reciprocals of the P-F eigenvalues of matrices $\mathbf{J}[\mathbf{I} - \mathbf{b}^{\mathrm{T}}\mathbf{v}]^{-1}$.

Table A.1. Industry Classification

1	Agriculture, forestry & fishing
2	Mining & quarrying
3	Food, beverages & tobacco
4	Textiles, apparel & leather
5	Wood products & furniture
6	Paper, paper products & printing
7	Industrial chemicals
8	Drugs & medicines
9	Petroleum & coal products
10	Rubber & plastic products
11	Non-metallic mineral products
12	Iron & steel
13	Non-ferrous metals
14	Metal products
15	Non-electrical machinery
16	Office & computing machinery
17	Electrical apparatus, nec
18	Radio, TV & communication equipment
19	Shipbuilding & repairing
20	Other transport
21	Motor vehicles
22	Aircraft
23	Professional goods
24	Other manufacturing
25	Electricity, gas & water
26	Construction
27	Wholesale & retail trade
28	Restaurants & hotels
29	Transport & storage
30	Communication
31	Finance & insurance
32	Real estate & business services
33	Community, social & personal services