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Technical Appendix to "Macroeconomic effects of public sector unions

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1 Technical Appendix

1.1 Optimality conditions

1.1.1 Firm's problem

The profit function is maximized when the derivatives of that function are set to zero. Therefore, the optimal amount of capital - holding the level of technology A_t and labor input N_t^p constant - is determined by setting the derivative of the profit function with respect to K_t^p equal to zero. This derivative is

$$(1-\theta)A_t(K_t^p)^{-\theta}(N_t^p)^{\theta}(K_t^g)^{\nu} - r_t \quad = \quad 0 \tag{1}$$

where $(1-\theta)A_t(K_t^p)^{-\theta}(N_t^p)^{\theta}(K_t^g)^{\nu}$ is the marginal product of capital because it expresses how much output will increase if capital increases by one unit. The economic interpretation of this First-Order Condition (FOC) is that in equilibrium, firms will rent capital up to the point where the benefit of renting an additional unit of capital, which is the marginal product of capital, equals the rental cost, i.e the interest rate.

$$r_t \quad = \quad (1-\theta)A_t(K_t^p)^{-\theta}(N_t^p)^{\theta}(K_t^g)^{\nu} \tag{2}$$

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Now, multiply by K_t^p and rearrange terms. This gives the following relationship:

$$K_t^p(1-\theta)A_t(K_t^p)^{-\theta}(N_t^p)^{\theta}(K_t^g)^{\nu} = r_tK_t^p \quad \text{or} \quad (1-\theta)Y_t = r_tK_t^p \tag{3}$$

because

$$K_t^p(1-\theta)A_t(K_t^p)^{-\theta}(N_t^p)^{\theta}(K_t^g)^{\nu} = A_t(K_t^p)^{1-\theta}(N_t^p)^{\theta}(K_t^g)^{\nu} \quad = \quad (1-\theta)Y_t$$

To derive firms' optimal labor demand, set the derivative of the profit function with respect to the labor input equal to zero, holding technology and capital constant:

$$\theta A_t(K_t^p)^{1-\theta}(N_t^p)^{\theta-1}(K_t^g)^{\nu} - w_t^p = 0 \text{ or } w_t^p = \theta A_t(K_t^p)^{1-\theta}(N_t^p)^{\theta-1}(K_t^g)^{\nu} \tag{4}$$

In equilibrium, firms will hire labor up to the point where the benefit of hiring an additional hour of labor services, which is the marginal product of labor, equals the cost, i.e the hourly wage rate.

Now multiply both sides of the equation by N_t^p and rearrange terms to yield

$$N_t^p\theta A_t(K_t^p)^{1-\theta}(N_t^p)^{\theta-1}(K_t^g)^{\nu} = w_t^pN_t^p \quad \text{or} \quad \theta Y_t = w_t^pN_t^p \tag{5}$$

Next, it will be shown that in equilibrium, economic profits are zero. Using the results above one can obtain

$$\Pi_t \quad = \quad Y_t - r_tK_t^p - w_t^pN_t^p = Y_t - (1-\theta)Y_t - \theta Y_t = 0 \tag{6}$$

Indeed, in equilibrium, economic profits are zero.

1.1.2 Consumer problem

Set up the Lagrangian

$$\begin{aligned}\mathcal{L}(C_t, K_{t+1}^p, N_t^p; \Lambda_t) = E_0\sum_{t=0}^{\infty}\Bigg\{ & \frac{\Big[(C_t+\omega G_t^c)^{\psi}(1-N_t)^{(1-\psi)}\Big]^{1-\alpha} - 1}{1-\alpha} + \\ & +\Lambda_t\Big[(1-\tau^l)(w_t^pN_t^p + w_t^gN_t^g) + (1-\tau^k)r_tK_t^p + \\ & +\tau^k\delta^pK_t^p - (1+\tau^c)C_t - K_{t+1}^p + (1-\delta)K_t^p\Big]\Bigg\}\end{aligned} \tag{7}$$

This is a concave programming problem, so the FOCs, together with the additional, boundary ("transversality") conditions for private physical capital and government bonds are both necessary and sufficient for an optimum.

To derive the FOCs, first take the derivative of the Lagrangian w.r.t C_t (holding all other variables unchanged) and set it to 0, *i.e.* $\mathcal{L}_{C_t} = 0$. That will result in the following expression

$$\beta^t \Bigg\{ \frac{1-\alpha}{1-\alpha} \Big[(C_t + \omega G_t^c)^{\psi} (1 - N_t^h)^{(1-\psi)} \Big]^{-\alpha} \times \\ \psi (C_t + \omega G_t^c)^{\psi - 1} (1 - N_t^h)^{(1-\psi)} - \Lambda_t (1 + \tau^c) \Bigg\} = 0 \tag{8}$$

Cancel the β^t and the $1 - \alpha$ terms to obtain

$$\Big[(C_t + \omega G_t^c)^{\psi} (1 - N_t)^{(1-\psi)} \Big]^{-\alpha} \psi (C_t + \omega G_t^c)^{\psi - 1} (1 - N_t)^{(1-\psi)} - \Lambda_t (1 + \tau^c) = 0 \tag{9}$$

Move Λ_t to the right so that

$$\Big[(C_t + \omega G_t^c)^{\psi} (1 - N_t)^{(1-\psi)} \Big]^{-\alpha} \psi (C_t + \omega G_t^c)^{\psi - 1} (1 - N_t)^{(1-\psi)} = \Lambda_t (1 + \tau^c) \tag{10}$$

This optimality condition equates marginal utility of consumption to the marginal utility of wealth.

Now take the derivative of the Lagrangian w.r.t K_{t+1}^p (holding all other variables unchanged) and set it to 0, *i.e.* $L_{K_{t+1}^p} = 0$. That will result in the following expression

$$\beta^t \Bigg\{ -\Lambda_t + E_t \Lambda_{t+1} \Big[(1 - \tau^k) r_{t+1} + \tau^k \delta^p + (1 - \delta^p) \Big] \Bigg\} \quad = \quad 0 \tag{11}$$

Cancel the β^t term to obtain

$$-\Lambda_t + \beta E_t \Lambda_{t+1} \Big[(1 - \tau^k) r_{t+1} + \tau^k \delta^p + (1 - \delta^p) \Big] = 0 \tag{12}$$

Move Λ_t to the right so that

$$\beta E_t \Lambda_{t+1} \Big[(1 - \tau^k) r_{t+1} + \tau^k \delta^p + (1 - \delta^p) \Big] = \Lambda_t \tag{13}$$

Using the expression for the real interest rate shifted one period forward one can obtain

$$r_{t+1} = (1-\theta)\frac{Y_{t+1}}{K^p_{t+1}}$$

$$\beta E_t \Lambda_{t+1}\left[(1-\tau^k)(1-\theta)\frac{Y_{t+1}}{K^p_{t+1}} + \tau^k\delta^p + (1-\delta^p)\right] = \Lambda_t \tag{14}$$

This is the Euler equation, which determines how consumption is allocated across periods.

Take now the derivative of the Lagrangian w.r.t N^p_t (holding all other variables unchanged) and set it to 0, *i.e.* $\mathcal{L}_{N^p_t} = 0$. That will result in the following expression

$$\begin{aligned} \beta^t \Bigg\{ \frac{1-\alpha}{1-\alpha}\left[(C_t+\omega G^c_t)^\psi (1-N_t)^{(1-\psi)}\right]^{-\alpha} \times \\ (1-\psi)(C_t+\omega G^c_t)^\psi (1-N_t)^{-\psi}(-1) + \Lambda_t(1-\tau^l)w^p_t \Bigg\} &= 0 \end{aligned} \tag{15}$$

Cancel the β^t and the $1-\alpha$ terms to obtain

$$\left[(C_t+\omega G^c_t)^\psi (1-N_t)^{(1-\psi)}\right]^{-\alpha}(1-\psi)\left[\frac{C_t+\omega G^c_t}{1-N_t}\right]^\psi(-1) + \Lambda_t(1-\tau^l)w^p_t = 0 \tag{16}$$

Rearranging, one can obtain

$$\left[(C_t+\omega G^c_t)^\psi (1-N_t)^{(1-\psi)}\right]^{-\alpha}(1-\psi)(C_t+\omega G^c_t)^\psi(1-N_t)^{-\psi} = \Lambda_t(1-\tau^l)w^p_t \tag{17}$$

Plug in the expression for w^h_t, that is,

$$w^p_t = \theta\frac{Y_t}{N^p_t} \tag{18}$$

into the equation above. Rearranging, one can obtain

$$\left[(C_t+\omega G^c_t)^\psi (1-N_t)^{(1-\psi)}\right]^{-\alpha}(1-\psi)(C_t+\omega G^c_t)^\psi(1-N_t)^{-\psi} = \Lambda_t(1-\tau^l)\theta\frac{Y_t}{N^p_t} \tag{19}$$

Transversality conditions need to be imposed to prevent Ponzi schemes, i.e borrowing bigger and bigger amounts every subsequent period and never paying it off.

$$\lim_{t\to\infty}\beta^t\Lambda_t K^p_{t+1} = 0 \tag{20}$$

1.1.3 The Objective Function of a Public Sector Union: Derivation

This subsection shows that the objective function in the government sector is a generalized version of Stone-Geary monopoly union utility function used in Dertouzos and Pencavel (1981) and Brown and Ashenfelter (1986). The utility function is

$$V(w^g, N^g) = (w^g - \bar{w}^g)^{\phi}(N^g - \bar{N}^g)^{(1-\phi)}, \tag{21}$$

where ϕ and $1-\phi$ are the weights attached to public wage and hours, respectively, and $\bar{w}^g$ and $\bar{N}^g$ denote subsistence wage rate and hours. Since there is no minimum wage in the model, $\bar{w}^g = 0$. Additionally, as public hours are assumed to be unproductive, it follows that $\bar{N}^g = 0$ as well. Therefore, the utility function simplifies to

$$V(w^g, N^g) = (w^g)^{\phi}(N^g)^{(1-\phi)}. \tag{22}$$

Doiron (1992) uses a generalized representation, which encompasses (2) as a special case when $\rho \to 0$.

$$\left[\phi(N^g)^{-\rho} + (1-\phi)(w^g - \bar{w})^{-\rho}\right]^{-1/\rho}, \tag{23}$$

when $\bar{w} = 0$, the function simplifies to

$$\left[\phi(N^g)^{-\rho} + (1-\phi)(w^g)^{-\rho}\right]^{-1/\rho}, \tag{24}$$

Union objective function used in the paper is very similar to Doiron's (1992) simplified version:

$$\left[(N^g)^{\rho} + \eta(w^g)^{\rho}\right]^{1/\rho}, \tag{25}$$

can be transformed to

$$\left[(N^g)^{\rho} + \frac{\phi}{(1-\phi)}(w^g)^{\rho}\right]^{1/\rho}, \tag{26}$$

Collecting terms under common denominator

$$\left[\frac{(1-\phi)}{(1-\phi)}(N^g)^{\rho} + \frac{\phi}{(1-\phi)}(w^g)^{\rho}\right]^{1/\rho}, \tag{27}$$

Factoring out the common term

$$\left[\frac{1}{1-\phi}\right]^{1/\rho}\left[(1-\phi)(N^g)^\rho+\phi(w^g)^\rho\right]^{1/\rho}, \tag{28}$$

Note that the constant term $\left[\frac{1}{1-\phi}\right]^{1/\rho} > 0$ can be ignored, as utility functions are invariant to positive affine transformations. After rearranging terms, the equivalent function

$$\tilde{V}=\left[\phi(w^g)^\rho+(1-\phi)(N^g)^\rho\right]^{1/\rho}. \tag{29}$$

Take natural logarithms from both sides to obtain

$$\ln\tilde{V}=\frac{1}{\rho}\ln\left[\phi(w^g)^\rho+(1-\phi)(N^g)^\rho\right]. \tag{30}$$

Take the limit $\rho\to 0$

$$\lim_{\rho\to 0}\ln\tilde{V}=\lim_{\rho\to 0}\frac{\ln\left[\phi(w^g)^\rho+(1-\phi)(N^g)^\rho\right]}{\rho} \tag{31}$$

Apply L'Hopital's Rule on the R.H.S. to obtain

$$\lim_{\rho\to 0}\ln\tilde{V}=\lim_{\rho\to 0}\frac{\frac{\partial}{\partial\rho}\ln\left[\phi(w^g)^\rho+(1-\phi)(N^g)^\rho\right]}{\frac{\partial\rho}{\partial\rho}} \tag{32}$$

Thus

$$\ln\tilde{V}=\lim_{\rho\to 0}\frac{\left[\phi(w^g_t)^\rho\ln w^g+(1-\phi)(N^g)^\rho\ln N^g\right]/\left[\phi(w^g)^\rho+(1-\phi)(N^g)^\rho\right]}{1} \tag{33}$$

Simplify to obtain

$$\ln\tilde{V}=\frac{\lim_{\rho\to 0}\left[\phi(w^g_t)^\rho\ln w^g+(1-\phi)(N^g)^\rho\ln N^g\right]}{\lim_{\rho\to 0}\left[\phi(w^g)^\rho+(1-\phi)(N^g)^\rho\right]}=\frac{\phi\ln w^g+(1-\phi)\ln N^g}{\phi+(1-\phi)} \tag{34}$$

Therefore,

$$\ln\tilde{V}=\phi\ln w^g+(1-\phi)\ln N^g. \tag{35}$$

Exponentiate both sides of the equation to obtain

$$e^{\ln \tilde{V}} = e^{\phi \ln w^g + (1-\phi) \ln N^g}. \tag{36}$$

Thus

$$\tilde{V} = e^{\ln (w^g)^{\phi} + \ln (N^g)^{(1-\phi)}}. \tag{37}$$

or

$$\tilde{V} = e^{\ln (w^g)^{\phi} (N^g)^{(1-\phi)}}. \tag{38}$$

Finally,

$$\tilde{V} = (w^g)^{\phi} (N^g)^{(1-\phi)} \tag{39}$$

Furthermore, government period budget constraint serves the role of a labor demand function. Additionally, the public sector demand curve will be subject to shock, resulting from innovations to the fiscal shares. The balanced budget assumption is thus important in the model setup. Since wage bill is a residual, if wage rate is increased, then hours need to be decreased. Additionally, government period budget constraint can be expressed in the form $N^g = N^g(w^g)$ as

$$N^g = \frac{\tau^l w^p N^p + \tau^k (r - \delta^p) K^p + \tau^c C - G^c - G^i - G^t}{(1 - \tau^l) w^g} \tag{40}$$

Therefore, the problem in the government sector is reshaped in the standard formulation in the union literature:

$$\max_{w^g, N^g} V(w^g, N^g) \quad \text{s.t.} \quad N^g = N^g(w^g) \tag{41}$$

Since union optimizes over both the public wage and hours, the outcome is efficient. The solution pair is on the contract curve (obtained from FOCs), at the intersection point with the labor demand curve (government budget constraint).

1.1.4 Public sector union optimization problem

The union solves the following problem:

$$\max_{w_t^g, N_t^g} \left[(N_t^g)^\rho + \eta (w_t^g)^\rho \right]^{1/\rho} \tag{42}$$

s.t

$$G_t^c + G_t^t + G_t^i + w_t^g N_t^g = \tau^c C_t + \tau^k r_t K_t^p - \tau^k \delta^p K_t + \tau^l [w_t^p N_t^p + w_t^g N_t^g] \tag{43}$$

Setup the Lagrangian

$$\mathcal{V}(w_t^g, N_t^g; \nu_t) = \max_{w_t^g, N_t^g} \left\{ \left[(N_t^g)^\rho + \eta (w_t^g)^\rho \right]^{1/\rho} \right. \tag{44}$$

$$\left. -\nu_t \left[G_t^c + G_t^t + G_t^i + w_t^g N_t^g - \tau^c C_t - \tau^k r_t K_t^p + \tau^k \delta^p K_t - \tau^l [w_t^p N_t^p + w_t^g N_t^g] \right] \right\}$$

Optimal public employment is obtained, when the derivative of the government Lagrangian is et to zero, i.e $\mathcal{V}_{N_t^g} = 0$

$$(1/\rho) \left[(N_t^g)^\rho + \eta (w_t^g)^\rho \right]^{(1/\rho)-1} \rho (N_t^g)^{\rho-1} - (1-\tau^l)\nu_t w_t^g = 0 \tag{45}$$

or, when ρ is canceled out and $(1-\tau^l)\nu_t w_t^g$ put to the right

$$\left[(N_t^g)^\rho + \eta (w_t^g)^\rho \right]^{(1/\rho)-1} (N_t^g)^{\rho-1} = (1-\tau^l)\nu_t w_t^g \tag{46}$$

Optimal public wage is obtained, when the derivative of the government Lagrandean is et to zero, i.e $\mathcal{V}_{w_t^g} = 0$

$$(1/\rho) \left[(N_t^g)^\rho + \eta (w_t^g)^\rho \right]^{(1/\rho)-1} \eta\rho (w_t^g)^{\rho-1} - (1-\tau^l)\nu_t N_t^g = 0 \tag{47}$$

or, when ρ is canceled out and$(1-\tau^l)\nu_t N_t^g$ term put to the right

$$\left[(N_t^g)^\rho + \eta (w_t^g)^\rho \right]^{(1/\rho)-1} \eta (w_t^g)^{\rho-1} = (1-\tau^l)\nu_t N_t^g \tag{48}$$

Divide (11.1.46) and (11.1.48) side by side to obtain

$$\frac{\left[(N_t^g)^\rho + \eta (w_t^g)^\rho \right]^{(1/\rho)-1} (N_t^g)^{\rho-1}}{\left[(N_t^g)^\rho + \eta (w_t^g)^\rho \right]^{(1/\rho)-1} \eta (w_t^g)^{\rho-1}} = \frac{(1-\tau^l)\nu_t w_t^g}{(1-\tau^l)\nu_t N_t^g} \tag{49}$$

Cancel out the common terms

$$\frac{(N_t^g)^{\rho-1}}{\eta(w_t^g)^{\rho-1}} = \frac{w_t^g}{N_t^g} \tag{50}$$

Now cross-multiply to obtain

$$\frac{(N_t^g)^{\rho}}{\eta} = (w_t^g)^{\rho} \tag{51}$$

Hence

$$w_t^g = \left(\frac{1}{\eta}\right)^{1/\rho} N_t^g \tag{52}$$

The wage bill expression, which is obtained after simple rearrangement of the government budget constraint, is as follows

$$w_t^g N_t^g = \frac{\tau^c C_t + \tau^k r_t K_t^p - \tau^k \delta^p K_t + \tau^l w_t^p N_t^p - G_t^c - G_t^t - G_t^i}{1-\tau^l} \tag{53}$$

Use the wage bill equation and the relationship between public wage and employment in order to obtain

$$w_t^g = \eta^{-\frac{1}{2\rho}} \left[\frac{\tau^c C_t + \tau^k r_t K_t^p - \tau^k \delta^p K_t + \tau^l w_t^p N_t^p - G_t^c - G_t^t - G_t^i}{1-\tau^l}\right]^{\frac{1}{2}} \tag{54}$$

and

$$N_t^g = \eta^{\frac{1}{2\rho}} \left[\frac{\tau^c C_t + \tau^k r_t K_t^p - \tau^k \delta^p K_t + \tau^l w_t^p N_t^p - G_t^c - G_t^t - G_t^i}{1-\tau^l}\right]^{\frac{1}{2}} \tag{55}$$

1.2 Log-linearized model equations

1.2.1 Linearized market clearing

$$c_t + k^p_{t+1} + g^c_t + g^i_t - (1-\delta^p)k^p_t \;=\; y_t \tag{56}$$

Take logs from both sides to obtain

$$\ln[c_t + k^p_{t+1} + g^c_t + g^i_t - (1-\delta^p)k^p_t] \;=\; \ln(y_t) \tag{57}$$

Totally differentiate with respect to time

$$\frac{d\ln[c_t + k^p_{t+1} + g^c_t + g^i_t - (1-\delta^p)k^p_t]}{dt} \;=\; d\ln(y_t) \tag{58}$$

$$[\frac{1}{c+g^c+g^i+\delta^p k^p}][\frac{dc_t}{dt}\frac{c}{c} + \frac{dg^c_t}{dt}\frac{g}{g} + \frac{dg^i_t}{dt}\frac{g^i}{g^i} + \frac{dk^p_{t+1}}{dt}\frac{k^p}{k^p} - (1-\delta^p)\frac{dk^p_t}{dt}\frac{k^p}{k^p}] = \frac{dy_t}{dt}\frac{1}{y} \tag{59}$$

Define $\hat{z} = \frac{dz_t}{dt}\frac{1}{z}$. Thus passing to log-deviations

$$\frac{1}{y}[\hat{c}_t c + \hat{g}^c_t g^c + \hat{g}^i_t g^i + \hat{k}^p_{t+1}k^p - (1-\delta^p)\hat{k}^p_t k^p] \;=\; \hat{y}_t \tag{60}$$

$$\hat{c}_t c + \hat{g}^c_t g^c + \hat{g}^i_t g^i + \hat{k}^p_{t+1}k^p - (1-\delta^p)\hat{k}^p_t k^p \;=\; y\hat{y}_t \tag{61}$$

$$k^p\hat{k}^p_{t+1} \;=\; y\hat{y}_t - c\hat{c}_t - g^c\hat{g}^c_t - g^i\hat{g}^i_t + (1-\delta^p)k^p\hat{k}^p_t \tag{62}$$

1.2.2 Linearized production function

$$y_t \;=\; a_t(k^p_t)^{1-\theta}(n^p_t)^{\theta}(k^g_t)^{\nu} \tag{63}$$

Take natural logs from both sides to obtain

$$\ln y_t \;=\; \ln a_t + (1-\theta)\ln k^p_t + \theta\ln n^p_t + \nu\ln k^g_t \tag{64}$$

Totally differentiate with respect to time to obtain

$$\frac{d\ln y_t}{dt} \;=\; \frac{d\ln a_t}{dt} + (1-\theta)\frac{d\ln k^p_t}{dt} + \theta\frac{d\ln n^p_t}{dt} + \nu\frac{d\ln k^g_t}{dt} \tag{65}$$

$$\frac{1}{y}\frac{dy_t}{dt} \;=\; \frac{1}{a}\frac{da_t}{dt} + \frac{1-\theta}{k^p}\frac{dk^p_t}{dt} + \frac{\theta}{n^p}\frac{dn^p_t}{dt} + \frac{\nu}{k^g}\frac{dk^g_t}{dt} \tag{66}$$

Pass to log-deviations to obtain

$$0 \;=\; -\hat{y}_t + (1-\theta)\hat{k}^p_t + \hat{a}_t + \theta\hat{n}^p_t + \nu\hat{k}^g_t \tag{67}$$

1.2.3 Linearized FOC consumption

$$[(c_t+\omega g_t^c)^{\psi}(1-n_t)^{(1-\psi)}]^{-\alpha}\psi(c_t+\omega g_t^c)^{\psi-1}(1-n_t)^{(1-\psi)} \;=\; (1+\tau^c)\lambda_t \tag{68}$$

Simplify to obtain

$$\psi(c_t+\omega g_t^c)^{\psi-1-\alpha\psi}(1-n_t)^{(1-\alpha)(1-\psi)} \;=\; (1+\tau^c)\lambda_t \tag{69}$$

Take natural logs from both sides to obtain

$$\ln\psi(c_t+\omega g_t^c)^{\psi-1-\alpha\psi}(1-n_t)^{(1-\alpha)(1-\psi)} \;=\; \ln(1+\tau^c)+\ln\lambda_t \tag{70}$$

$$\ln(c_t+\omega g_t^c)^{\psi-1-\alpha\psi}(1-n_t)^{(1-\alpha)(1-\psi)} \;=\; \ln(1+\tau^c)+\ln\lambda_t \tag{71}$$

$$(\psi-1-\alpha\psi)\ln(c_t+\omega g_t^c)+(1-\alpha)(1-\psi)\ln(1-n_t) \;=\; \ln(1+\tau^c)+\ln\lambda_t \tag{72}$$

Totally differentiate with respect to time to obtain

$$\begin{aligned}(\psi-1-\alpha\psi)\frac{d\ln(c_t+\omega g_t^c)}{dt}+(1-\alpha)(1-\psi)\frac{d\ln(1-n_t)}{dt} &= \\ &= \frac{d\ln(1+\tau^c)}{dt}+\frac{d\ln\lambda_t}{dt}\end{aligned} \tag{73}$$

$$(\psi-1-\alpha\psi)\frac{1}{c+\omega g^c}\left(\frac{dc_t}{dt}+\omega\frac{dg_t^c}{dt}\right)+(1-\alpha)(1-\psi)\frac{-1}{1-n}\frac{dn_t}{dt} \;=\; \frac{d\lambda_t}{dt}\frac{1}{\lambda} \tag{74}$$

$$\begin{aligned}\frac{(\psi-1-\alpha\psi)}{c+\omega g^c}\frac{dc_t}{dt}\frac{c}{c}+\frac{\omega(\psi-1-\alpha\psi)}{c+\omega g^c}\frac{dg_t^c}{dt}\frac{g^c}{g^c}+& \\ -(1-\alpha)(1-\psi)\frac{1}{1-n}\frac{dn_t}{dt}\frac{n}{n} &= \frac{d\lambda_t}{dt}\frac{1}{\lambda}\end{aligned} \tag{75}$$

$$\frac{c(\psi-1-\alpha\psi)}{c+\omega g^c}\hat{c}_t+\frac{\omega g^c(\psi-1-\alpha\psi)}{c+\omega g^c}\hat{g}_t^c-(1-\alpha)(1-\psi)\frac{n}{1-n}\hat{n} \;=\; \hat{\lambda}_t \tag{76}$$

Since

$$\hat{n} \;=\; \frac{n^p}{n^p+n^g}\hat{n}^p+\frac{n^g}{n^p+n^g}\hat{n}^g=\frac{n^p}{n}\hat{n}^p+\frac{n^g}{n}\hat{n}^g, \tag{77}$$

and consumers choose n^p only, pass to log-deviations to obtain

$$\frac{c(\psi-1-\alpha\psi)}{c+\omega g^c}\hat{c}_t+\frac{\omega g^c(\psi-1-\alpha\psi)}{c^c+\omega g}\hat{g}_t^c-(1-\alpha)(1-\psi)\frac{n}{1-n}\frac{n^p}{n^p+n^g}\hat{n}^p \;=\; \hat{\lambda}_t \tag{78}$$

Since $n=n^p+n^g$, it follows that

$$\frac{c(\psi-1-\alpha\psi)}{c+\omega g^c}\hat{c}_t+\frac{\omega g^c(\psi-1-\alpha\psi)}{c+\omega g^c}\hat{g}_t^c-(1-\alpha)(1-\psi)\frac{n^p}{1-n}\hat{n}^p \;=\; 0 \tag{79}$$

1.2.4 Linearized no-arbitrage condition for capital

$$\lambda_t \quad = \quad \beta E_t \lambda_{t+1}[(1-\tau^k) r_{t+1} + \tau^k \delta^p + (1-\delta^p)] \tag{80}$$

Substitute out r_{t+1} on the right hand side of the equation to obtain

$$\lambda_t \quad = \quad \beta E_t[\lambda_{t+1}((1-\tau^k)(1-\theta)\frac{y_{t+1}}{k^p_{t+1}} + \tau^k \delta^p + 1 - \delta^p)] \tag{81}$$

Take natural logs from both sides of the equation to obtain

$$\ln \lambda_t \quad = \quad \ln E_t[\lambda_{t+1}((1-\tau^k)(1-\theta)\frac{y_{t+1}}{k^p_{t+1}} + \tau^k \delta^p + 1 - \delta^p)] \tag{82}$$

Totally differentiate with respect to time to obtain

$$\frac{d \ln \lambda_t}{dt} = \frac{d \ln E_t[\lambda_{t+1}((1-\tau^k)(1-\theta)\frac{y_{t+1}}{k^p_{t+1}} + \tau^k \delta^p + 1 - \delta^p)]}{dt} \tag{83}$$

$$\begin{aligned}\frac{1}{\lambda}\frac{d\lambda_t}{dt} = E_t \Bigg\{ & \frac{1}{\lambda((1-\tau^k)(1-\theta)\frac{y}{k^p} + 1 - \delta^p + \tau^k \delta^p} \times \\ & \Bigg[((1-\tau^k)(1-\theta)\frac{y}{k^p} + \tau^k \delta^p + 1 - \delta^p)\frac{d\lambda_{t+1}}{dt}\frac{\lambda}{\lambda} \\ & + \frac{\lambda(1-\tau^k)(1-\theta)}{k^p}\frac{dy_{t+1}}{dt}\frac{y}{y} - \left[\frac{\lambda(1-\tau^k)(1-\theta)y}{(k^p)^2}\right]\frac{dk^p_{t+1}}{dt}\frac{k^p}{k^p}\Bigg]\Bigg\}\end{aligned} \tag{84}$$

Pass to log-deviations to obtain

$$\begin{aligned}\hat{\lambda}_t = E_t \Bigg\{ \hat{\lambda}_{t+1} + \Bigg[& \frac{(1-\tau^k)(1-\theta)y}{((1-\tau^k)(1-\theta)\frac{y_{t+1}}{k^p_{t+1}} + \tau^k \delta^p + 1 - \delta^p)k^p}\hat{y}_{t+1} \\ & - \frac{(1-\tau^k)(1-\theta)y}{((1-\theta)\frac{y_{t+1}}{k^p_{t+1}} + \tau^k \delta^p + 1 - \delta^p)k^p}\hat{k}^p_{t+1}\Bigg]\Bigg\}\end{aligned} \tag{85}$$

Observe that

$$(1-\tau^k)(1-\theta)\frac{y_{t+1}}{k^p_{t+1}} + \tau^k \delta^p + 1 - \delta^p \quad = \quad 1/\beta \tag{86}$$

Plug it into the equation to obtain

$$\hat{\lambda}_t \quad = \quad E_t\left[\hat{\lambda}_{t+1} + \frac{\beta(1-\tau^k)(1-\theta)y}{k^p}\hat{y}_{t+1} - \frac{\beta(1-\tau^k)(1-\theta)y}{k^p}\hat{k}^p_{t+1}\right] \tag{87}$$

$$\hat{\lambda}_t \quad = \quad E_t\hat{\lambda}_{t+1} + \frac{\beta(1-\tau^k)(1-\theta)y}{k^p}E_t\hat{y}_{t+1} - \frac{\beta(1-\tau^k)(1-\theta)y}{k^p}E_t\hat{k}^p_{t+1} \tag{88}$$

1.2.5 Linearized MRS

$$(1-\psi)(c_t+\omega g_t^c) \quad = \quad \psi(1-n_t)\frac{(1-\tau^l)}{(1+\tau^c)}\theta\frac{y_t}{n_t^p} \tag{89}$$

Take natural logs from both sides of the equation to obtain

$$\ln(1-\psi)(c_t+\omega g_t^c) \quad = \quad \ln\psi(1-n_t)\frac{(1-\tau^l)}{(1+\tau^c)}\theta\frac{y_t}{n_t^p} \tag{90}$$

$$\ln(c_t+\omega g_t^c) \quad = \quad \ln(1-n_t)+\ln y_t-\ln n_t^p \tag{91}$$

Totally differentiate with respect to time to obtain

$$\frac{d\ln(c_t+\omega g_t^c)}{dt} \quad = \quad \frac{d\ln(1-n_t)}{dt}+\frac{d\ln y_t}{dt}-\frac{d\ln n_t^p}{dt} \tag{92}$$

$$\frac{1}{c+\omega g^c}(\frac{dc_t}{dt}+\omega\frac{dg_t^c}{dt}) \quad = \quad -\frac{1}{1-n}\frac{dn_t}{dt}+\frac{1}{y}\frac{dy_t}{dt}-\frac{1}{n^p}\frac{dn_t^p}{dt} \tag{93}$$

$$\frac{1}{c+\omega g^c}\frac{dc_t}{dt}\frac{c}{c}+\frac{\omega}{c+\omega g^c}\frac{dg_t^c}{dt}\frac{g^c}{g^c} \quad = \quad -\frac{1}{1-n}\frac{dn_t}{dt}\frac{n}{n}+\frac{1}{y}\frac{dy_t}{dt}-\frac{1}{n^p}\frac{dn_t^p}{dt} \tag{94}$$

$$\frac{c}{c+\omega g^c}\frac{dc_t}{dt}\frac{1}{c}+\frac{\omega g^c}{c+\omega g^c}\frac{dg_t^c}{dt}\frac{1}{g^c} \quad = \quad -\frac{n}{1-n}\frac{dn_t}{dt}\frac{1}{n}+\frac{1}{y}\frac{dy_t}{dt}-\frac{1}{n^p}\frac{dn_t^p}{dt} \tag{95}$$

Pass to log-deviations to obtain

$$\frac{c}{c+\omega g^c}\hat{c}_t+\frac{\omega g^c}{c+\omega g}\hat{g}_t^c \quad = \quad -\frac{n}{1-n}\hat{n}+\hat{y}_t-\hat{n}_t^p \tag{96}$$

Since

$$\hat{n}=\frac{n^p}{n^p+n^g}\hat{n}^p+\frac{n^g}{n^p+n^g}\hat{n}^g, \tag{97}$$

and noting that consumers are only choosing n^p, then

$$\frac{c}{c+\omega g^c}\hat{c}_t+\frac{\omega g^c}{c+\omega g^c}\hat{g}_t^c \quad = \quad -\frac{n}{1-n}\frac{n^p}{n^p+n^g}\hat{n}^p+\hat{y}_t-\hat{n}_t^p \tag{98}$$

$$\frac{c}{c+\omega g^c}\hat{c}_t+\frac{\omega g^c}{c+\omega g^c}\hat{g}_t^c \quad = \quad -\frac{n}{1-n}\frac{n^p}{n^p+n^g}\hat{n}^p+\hat{y}_t-\hat{n}_t^p \tag{99}$$

$$\frac{c}{c+\omega g^c}\hat{c}_t+\frac{\omega g^c}{c+\omega g^c}\hat{g}_t^c \quad = \quad -\left(1+\frac{n}{1-n}\frac{n^p}{n^p+n^g}\right)\hat{n}^p+\hat{y}_t \tag{100}$$

Since $n=n^p+n^g$, it follows that

$$\frac{c}{c+\omega g^c}\hat{c}_t+\frac{\omega g^c}{c+\omega g^c}\hat{g}_t^c \quad = \quad -\left(1+\frac{n^p}{1-n}\right)\hat{n}^p+\hat{y}_t \tag{101}$$

$$\frac{c}{c+\omega g^c}\hat{c}_t+\frac{\omega g^c}{c+\omega g^c}\hat{g}_t^c+\left(1+\frac{n^p}{1-n}\right)\hat{n}^p-\hat{y}_t \quad = \quad 0 \tag{102}$$

1.2.6 Linearized private capital accumulation

$$k^p_{t+1} \;=\; i_t + (1-\delta^p)k^p_t \tag{103}$$

Take natural logs from both sides of the equation to obtain

$$\ln k^p_{t+1} \;=\; \ln(i_t + (1-\delta^p)k^p_t) \tag{104}$$

Totally differentiate with respect to time to obtain

$$\frac{d\ln k^p_{t+1}}{dt} \;=\; \frac{1}{i+(1-\delta^p)k^p}\frac{d(i_t+(1-\delta^p)k^p_t)}{dt} \tag{105}$$

Observe that since

$$i = \delta^p k^p, \text{ it follows that } \quad i+(1-\delta^p)k^p = \delta^p k^p + (1-\delta^p)k^p = k^p. \text{ Then} \tag{106}$$

$$\frac{dk^p_{t+1}}{dt}\frac{1}{k^p} \;=\; \frac{1}{k^p}\frac{di_t}{dt}\frac{i}{i} + \frac{k^p}{i+(1-\delta^p)k^p_t}\frac{dk^p_t}{dt}\frac{k^p}{k^p} \tag{107}$$

Pass to log-deviations to obtain

$$\hat{k}^p_{t+1} \;=\; \frac{\delta^p k^p}{k^p}\hat{i}_t + \frac{(1-\delta^p)k^p}{k^p}\hat{k}^p_t \tag{108}$$

$$\hat{k}^p_{t+1} \;=\; \delta^p\hat{i}_t + (1-\delta^p)\hat{k}^p_t \tag{109}$$

1.2.7 Linearized government capital accumulation

$$k^g_{t+1} \;=\; g^i_t + (1-\delta^g)k^g_t \tag{110}$$

Take natural logs from both sides to obtain

$$\ln k^g_{t+1} \;=\; \ln(g^i_t + (1-\delta^g)k^g_t) \tag{111}$$

Totally differentiate with respect to time to obtain

$$\frac{d\ln k^g_{t+1}}{dt} \;=\; \frac{1}{g^i+(1-\delta^g)k^g}\frac{d(g^i_t+(1-\delta^g)k^g_t)}{dt} \tag{112}$$

Observe that since

$$g^i = \delta^g k^g, \tag{113}$$

it follows that

$$g^i + (1-\delta^g)k^g = \delta^g k^g + (1-\delta^g)k^g = k^g. \tag{114}$$

Hence,

$$\frac{dk^g_{t+1}}{dt}\frac{1}{k^g} = \frac{1}{k^g}\frac{dg^i_t}{dt}\frac{g^i}{g^i} + \frac{k^g}{x+(1-\delta^g)}\frac{dk^g_t}{dt}\frac{k^g}{k^g} \tag{115}$$

Pass to log-deviations to obtain

$$\hat{k}^g_{t+1} \quad = \quad \frac{\delta^g k^g}{k^g}\hat{g}^i_t + \frac{(1-\delta^g)k^g}{k^g}\hat{k}^g_t \tag{116}$$

Cancel out the k^g terms to obtain

$$\hat{k}^g_{t+1} \quad = \quad \delta^g \hat{g}^i_t + (1-\delta^g)\hat{k}^g_t \tag{117}$$

1.2.8 Public wage rate rule

$$w^g_t = \eta^{-\frac{1}{2\rho}}\left[\frac{\tau^c c_t + \tau^k r_t k^p_t - \tau^k \delta^p k^p_t + \tau^l w^p_t n^p_t - g^c_t - g^t_t - g^i_t}{1-\tau^l}\right]^{\frac{1}{2}} \tag{118}$$

Take logs from both sides to obtain

$$\begin{aligned} \ln w^g_t = -\frac{1}{2\rho}\ln\eta - \frac{1}{2}\ln(1-\tau^l) + \\ \frac{1}{2}\ln\left\{\tau^c c_t + \tau^k r_t k^p_t - \tau^k \delta^p k^p_t + \tau^l w^p_t n^p_t - g^c_t - g^t_t - g^i_t\right\} \end{aligned} \tag{119}$$

Totally differentiate with respect to time to obtain

$$\frac{d\ln w^g_t}{dt} = \frac{1}{2}\frac{d}{dt}\ln\left\{\tau^c c_t + \tau^k r_t k^p_t - \tau^k \delta^p k^p_t + \tau^l w^p_t n^p_t - g^c_t - g^t_t - g^i_t\right\} \tag{120}$$

Observe that

$$\begin{aligned} \tau^k r_t k^p_t - \tau^k \delta^p k_t + \tau^l w^p_t n^p_t = \tau^k(1-\theta)y_t + \tau^l\theta y_t - \tau^k\delta^p k^p_t = \\ = \left[\tau^k(1-\theta) + \tau^l\theta\right]y_t - \tau^k\delta^p k^p_t \end{aligned} \tag{121}$$

Also

$$(1-\tau^l)w^g n^g = \tau^c c + [\tau^k(1-\theta) + \tau^l\theta]y - \tau^k\delta^p k^p - g^c - g^i - g^t_t \tag{122}$$

Thus

$$\frac{dw_t^g}{dt}\frac{1}{w^g} = \frac{1}{2}\frac{1}{(1-\tau^l)w^g n^g}\left\{\tau^c\frac{dc_t}{dt} + [\tau^k(1-\theta)+\tau^l\theta]\frac{dy_t}{dt} - \tau^k\delta^p\frac{dk_t^p}{dt} - \frac{dg_t^c}{dt} - \frac{dg_t^i}{dt} - \frac{dg_t^t}{dt}\right\} \tag{123}$$

$$\frac{dw_t^g}{dt}\frac{1}{w^g} = \frac{1}{2}\frac{1}{(1-\tau^l)w^g n^g}\times$$
$$\left\{\tau^c\frac{dc_t}{dt}\frac{c}{c} + \left[\tau^k(1-\theta)+\tau^l\theta\right]\frac{dy_t}{dt}\frac{y}{y} - \tau^k\delta^p\frac{dk_t^p}{dt}\frac{k^p}{k^p} - \frac{dg_t^c}{dt}\frac{g^c}{g^c} - \frac{dg_t^i}{dt}\frac{g^i}{g^i} - \frac{dg_t^t}{dt}\frac{g^t}{g^t}\right\} \tag{124}$$

$$\begin{aligned}\frac{dw_t^g}{dt}\frac{1}{w^g} = {} & \frac{(1/2)\tau^c c}{(1-\tau^l)w^g n^g}\frac{dc_t}{dt}\frac{1}{c} + \frac{(1/2)\left[\tau^k(1-\theta)+\tau^l\theta\right]y}{(1-\tau^l)w^g n^g}\frac{dy_t}{dt}\frac{1}{y} \\ & - \frac{(1/2)\tau^k\delta^p k^p}{(1-\tau^l)w^g n^g}\frac{dk_t^p}{dt}\frac{1}{k^p} - \frac{(1/2)g^c}{(1-\tau^l)w^g n^g}\frac{dg_t^c}{dt}\frac{1}{g^c} \\ & - \frac{(1/2)g^i}{(1-\tau^l)w^g n^g}\frac{dg_t^i}{dt}\frac{1}{g^i} - \frac{(1/2)g^t}{(1-\tau^l)w^g n^g}\frac{dg_t^t}{dt}\frac{1}{g^t}\end{aligned} \tag{125}$$

Pass to log-deviations to obtain

$$\hat{w}_t^g = \frac{(1/2)\tau^c c}{(1-\tau^l)w^g n^g}\hat{c}_t + \frac{(1/2)\left[\tau^k(1-\theta)+\tau^l\theta\right]y}{(1-\tau^l)w^g n^g}\hat{y}_t$$
$$-\frac{(1/2)\tau^k\delta^p k^p}{(1-\tau^l)w^g n^g}\hat{k}_t - \frac{(1/2)g^c}{(1-\tau^l)w^g n^g}\hat{g}_t^c - \frac{(1/2)g^i}{(1-\tau^l)w^g n^g}\hat{g}_t^i - \frac{(1/2)g^t}{(1-\tau^l)w^g n^g}\hat{g}_t^t \tag{126}$$

1.2.9 Public hours/employment rule

$$n_t^g = \eta^{\frac{1}{\rho}} w_t^g \tag{127}$$

Take logs from both sides to obtain

$$\ln n_t^g = \frac{1}{\rho}\ln\eta + \ln w_t^g \tag{128}$$

Totally differentiate both sides to obtain

$$\frac{d\ln n_t^g}{dt} = \frac{d\ln w_t^g}{dt} \tag{129}$$

$$\frac{dn_t^g}{dt}\frac{1}{n^g} = \frac{dw_t^g}{dt}\frac{1}{w^g} \tag{130}$$

Pass to log-deviations to obtain

$$\hat{n}_t^g = \hat{w}_t^g \tag{131}$$

1.2.10 Total hours/employment

$$n_t = n_t^g + n_t^p \tag{132}$$

Take logs from both sides to obtain

$$\ln n_t = \ln(n_t^g + n_t^p) \tag{133}$$

Totally differentiate to obtain

$$\frac{d\ln n_t}{dt} = \frac{d\ln(n_t^g + n_t^p)}{dt} \tag{134}$$

$$\frac{dn_t}{dt}\frac{1}{n} = \left(\frac{dn_t^g}{dt} + \frac{dn_t^p}{dt}\right)\frac{1}{n} \tag{135}$$

$$\frac{dn_t}{dt}\frac{1}{n} = \left(\frac{dn_t^g}{dt}\frac{n^g}{n^g} + \frac{dn_t^p}{dt}\frac{n^p}{n^p}\right)\frac{1}{n} \tag{136}$$

$$\frac{dn_t}{dt}\frac{1}{n} = \frac{dn_t^g}{dt}\frac{1}{n^g}\frac{n^g}{n} + \frac{dn_t^p}{dt}\frac{1}{n^p}\frac{n^p}{n} \tag{137}$$

Pass to log-deviations to obtain

$$\hat{n}_t = \frac{n^g}{n}\hat{n}_t^g + \frac{n^p}{n}\hat{n}_t^p \tag{138}$$

1.2.11 Linearized private wage rate

$$w_t^p = \theta\frac{y_t}{n_t^p} \tag{139}$$

Take natural logarithms from both sides to obtain

$$\ln w_t^p = \ln\theta + \ln y_t - \ln n_t^p \tag{140}$$

Totally differentiate with respect to time to obtain

$$\frac{d\ln w_t^p}{dt} = \frac{d\ln\theta}{dt} + \frac{d\ln y_t}{dt} - \frac{d\ln n_t^p}{dt} \tag{141}$$

Simplify to obtain

$$\frac{dw_t^p}{dt}\frac{1}{w^p} = \frac{dy_t}{dt}\frac{1}{y} - \frac{dn_t^p}{dt}\frac{1}{n^p} \tag{142}$$

Pass to log-deviations to obtain

$$\hat{w}_t^p = \hat{y}_t - \hat{n}_t^p \tag{143}$$

1.2.12 Linearized real interest rate

$$r_t = \theta \frac{y_t}{k_t^p} \tag{144}$$

Take natural logarithms from both sides to obtain

$$\ln r_t = \ln \theta + \ln y_t - \ln k_t^p \tag{145}$$

Totally differentiate with respect to time to obtain

$$\frac{d \ln r_t}{dt} = \frac{d \ln \theta}{dt} + \frac{d \ln y_t}{dt} - \frac{d \ln k_t^p}{dt} \tag{146}$$

Simplify to obtain

$$\frac{dr}{dt}\frac{1}{r} = \frac{dy_t}{dt}\frac{1}{y} - \frac{dk_t^p}{dt}\frac{1}{k^p} \tag{147}$$

Pass to log-deviations to obtain

$$\hat{r}_t = \hat{y}_t - \hat{k}_t^p \tag{148}$$

1.2.13 Public/private wage ratio

$$rw_t = w_t^g / w_t^p \tag{149}$$

Take logs from both sides of the equation

$$\ln rw_t = \ln w_t^g - \ln w_t^p \tag{150}$$

Totally differentiate to obtain

$$\frac{d \ln rw_t}{dt} = \frac{d \ln w_t^g}{dt} - \frac{d \ln w_t^p}{dt} \tag{151}$$

$$\frac{drw_t}{dt}\frac{1}{rw} = \frac{dw_t^g}{dt}\frac{1}{w^g} - \frac{dw_t^p}{dt}\frac{1}{w^p} \tag{152}$$

Pass to log-deviations to obtain

$$r\hat{w}_t = \hat{w}_t^g - \hat{w}_t^p \tag{153}$$

1.2.14 Public/private hours/employment ratio

$$rl_t = n_t^g / n_t^p \tag{154}$$

Take logs from both sides of the equation

$$\ln rl_t = \ln n_t^g - \ln n_t^p \tag{155}$$

Totally differentiate to obtain

$$\frac{d \ln rl_t}{dt} = \frac{d \ln n_t^g}{dt} - \frac{d \ln n_t^p}{dt} \tag{156}$$

$$\frac{drl_t}{dt}\frac{1}{rl} = \frac{dn_t^g}{dt}\frac{1}{n^g} - \frac{dn_t^p}{dt}\frac{1}{n^p} \tag{157}$$

Pass to log-deviations to obtain

$$\hat{rl}_t = \hat{n}_t^g - \hat{n}_t^p \tag{158}$$

1.2.15 Linearized technology shock process

$$\ln a_{t+1} = \rho_a \ln a_t + \epsilon_{t+1}^a \tag{159}$$

Totally differentiate with respect to time to obtain

$$\frac{d \ln a_{t+1}}{dt} = \rho_a \frac{d \ln a_t}{dt} + \frac{d\epsilon_{t+1}^a}{dt} \tag{160}$$

$$\frac{da_{t+1}}{dt} = \rho_a \frac{da_t}{dt} + \epsilon_{t+1}^a \tag{161}$$

where for $t = 1$ $\frac{d\epsilon_{t+1}^a}{dt} \approx \ln(e^{\epsilon_{t+1}^a}/e^{\epsilon^a}) = \epsilon_{t+1}^a - \epsilon^a = \epsilon_{t+1}^a$ since $\epsilon^a = 0$. Pass to log-deviations to obtain

$$\hat{a}_{t+1} = \rho_a \hat{a}_t + \epsilon_{t+1}^a \tag{162}$$

1.2.16 Linearized stochastic process for government consumption/output share

$$\ln g_{t+1}^{cy} = (1-\rho^g)\ln g^{cy} + \rho^g \ln g_t^{cy} + \epsilon_{t+1}^c \tag{163}$$

Totally differentiate with respect to time to obtain

$$\frac{d\ln g_{t+1}^{cy}}{dt} = (1-\rho^g)\frac{d\ln g^{cy}}{dt} + \rho^g\frac{d\ln g_t^{cy}}{dt} + \frac{d\epsilon_{t+1}^c}{dt} \tag{164}$$

$$\frac{dg_{t+1}^{cy}}{dt} = \rho_g\frac{dg_t^{cy}}{dt} + \epsilon_{t+1}^c \tag{165}$$

where for $t=1$ $\frac{d\epsilon_{t+1}^c}{dt} \approx \ln(e^{\epsilon_{t+1}^c}/e^{\epsilon^c}) = \epsilon_{t+1}^c - \epsilon^c = \epsilon_{t+1}^c$ since $\epsilon^c = 0$. Pass to log-deviations to obtain

$$\hat{g}_{t+1}^{cy} = \rho_g\hat{g}_t^{cy} + \epsilon_{t+1}^c \tag{166}$$

1.2.17 Linearized level of government consumption

$$g_t^c = g_t^{cy}y_t \tag{167}$$

Take natural logarithms from both sides to obtain

$$\ln g_t^c = \ln g_t^{cy} + \ln y_t \tag{168}$$

Totally differentiate with respect to time to obtain

$$\frac{d\ln g_t^c}{dt} = \frac{d\ln g_t^{cy}}{dt} + \frac{d\ln y_t}{dt} \tag{169}$$

$$\frac{dg_t^c}{dt}\frac{1}{g^c} = \frac{dg_t^{cy}}{dt}\frac{1}{g^c} + \frac{dy_t}{dt}\frac{1}{y} \tag{170}$$

Pass to log-deviations to obtain

$$\hat{g}_t^c = \hat{g}_t^{cy} + \hat{y}_t \tag{171}$$

1.2.18 Linearized stochastic process for the government investment/output ratio

$$\ln g^{iy}_{t+1} = (1-\rho^i)\ln g^{iy} + \rho^i \ln g^{iy}_t + \epsilon^i_{t+1} \tag{172}$$

Totally differentiate with respect to time to obtain

$$\frac{d\ln g^{iy}_{t+1}}{dt} = (1-\rho^i)\frac{d\ln g^{iy}}{dt} + \rho^i \frac{d\ln g^{iy}_t}{dt} + \frac{d\epsilon^i_{t+1}}{dt} \tag{173}$$

$$\frac{dg^{iy}_{t+1}}{dt} = \rho_g \frac{dg^{iy}_t}{dt} + \epsilon^i_{t+1} \tag{174}$$

where for $t=1$ $\frac{d\epsilon^i_{t+1}}{dt} \approx \ln(e^{\epsilon^i_{t+1}}/e^{\epsilon^i}) = \epsilon^i_{t+1} - \epsilon^i = \epsilon^i_{t+1}$ since $\epsilon^i = 0$. Pass to log-deviations to obtain

$$\hat{g}^{iy}_{t+1} = \rho_i \hat{g}^{iy}_t + \epsilon^i_{t+1} \tag{175}$$

1.2.19 Linearized level of government investment

$$g^i_t = g^{iy}_t y_t \tag{176}$$

Take natural logarithms from both sides to obtain

$$\ln g^i_t = \ln g^{iy}_t + \ln y_t \tag{177}$$

Totally differentiate with respect to time to obtain

$$\frac{d\ln g^i_t}{dt} = \frac{d\ln g^{iy}_t}{dt} + \frac{d\ln y_t}{dt} \tag{178}$$

$$\frac{dg^i_t}{dt}\frac{1}{g^i} = \frac{dg^{iy}_t}{dt}\frac{1}{g^i} + \frac{dy_t}{dt}\frac{1}{y} \tag{179}$$

Pass to log-deviations to obtain

$$\hat{g}^i_t = \hat{g}^{iy}_t + \hat{y}_t \tag{180}$$

1.2.20 Linearized level of government transfers

$$g_t^t = g^{ty} y_t \tag{181}$$

Take natural logarithms from both sides to obtain

$$\ln g_t^t = \ln g^{ty} + \ln y_t \tag{182}$$

Totally differentiate with respect to time to obtain

$$\frac{d \ln g_t^t}{dt} = \frac{d \ln g^{ty}}{dt} + \frac{d \ln y_t}{dt} \tag{183}$$

$$\frac{dg_t^t}{dt} \frac{1}{g^t} = \frac{dy_t}{dt} \frac{1}{y} \tag{184}$$

Pass to log-deviations to obtain

$$\hat{g}_t^t = \hat{y}_t \tag{185}$$

1.3 Auto- and cross-correlation functions

As an additional test of model fit, this appendix compares auto- and cross-correlation functions generated from the model with collective bargaining and Finn (1998) calibrated for Germany, with their empirical counterparts. The main emphasis in this subsection is on the ACFs and CCFs of labor market variables. In particular, close attention is paid to cyclical properties of public and private wage rates and hours. To establish 95% confidence intervals for the theoretical ACFs and CCFs, as in Gregory and Smith (1991), the simulated time series are used to obtain 1000 ACFs and CCFs. The mean ACFs and CCFs are computed by averaging across simulations, as well as the corresponding standard error across simulations. Those moments allow for the lower and upper bounds for the ACFs confidence intervals to be estimated. The empirical ACFs and CCFs are then plotted, together with the theoretical ones. If empirical ACFs lie within the confidence region, this means that the calibrated model fits data well.

Empirical ACFs and CCFs were generated from a Vector Auto-Regressive (VAR) process of order 1. Since ACFs and CCFs are robust to identifying restrictions (Canova (2007), Ch.7), the VAR(1) was left unrestricted. The figures on the following pages display empirical ACFs (solid line), together with the simulated average ACFs (dashed line) and the corresponding stochastic error bounds (dotted lines). This is done for the union model first , and then for the calibration using Finn's (1998) framework.

The model with the public sector union calibrated for Germany outperforms Finn (1998), especially in the prediction of the dynamic behavior of labor market variables. In terms of capturing the autocorrelation structure of the variables, the union model fits data quite well. One exception is the public sector wage: in data, it is highly autocorrelated, while the model generates low persistence. A possible explanation could be that the public union puts weight also on last year's public sector wage level, *i.e.* the union bargains over the public wage increase rate, and not just the wage level. Public and total hours are also borderline cases, as employment rates in data were used instead. In addition, the union model predicts

Figure 1: Theoretical and empirical ACFs for Germany: Union

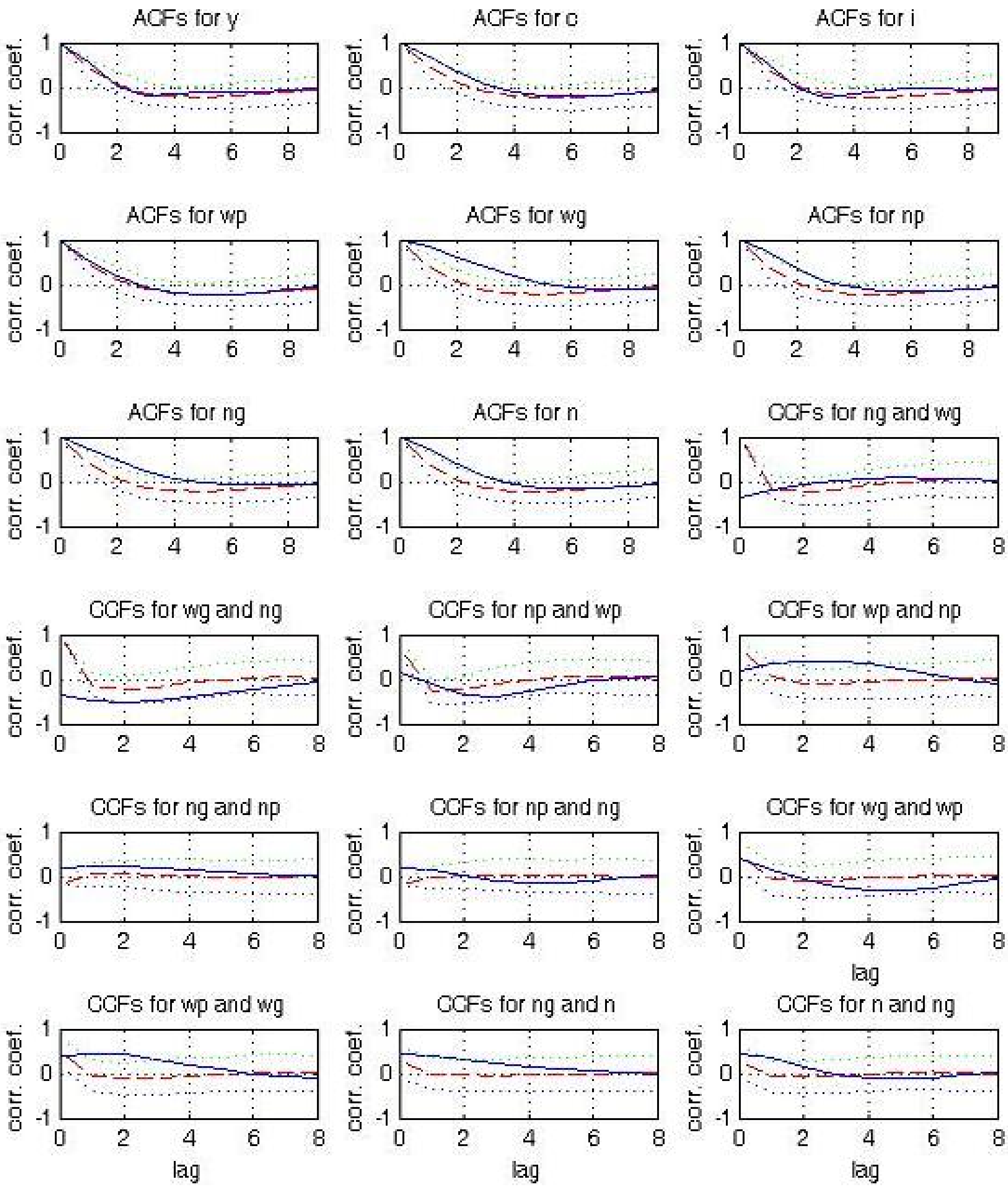


Figure 2: Theoretical and empirical ACFs for Germany: Union

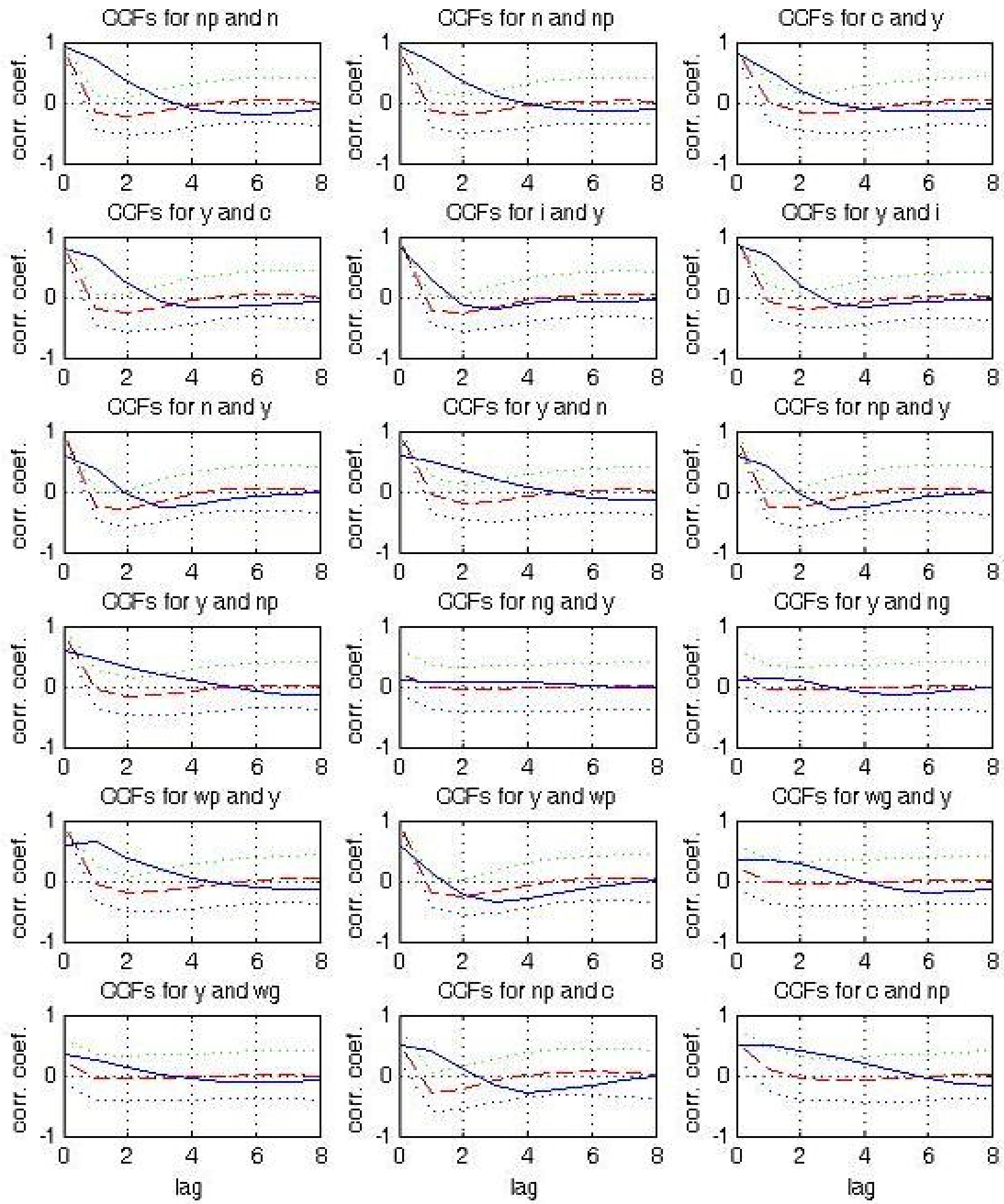


Figure 3: Theoretical and empirical ACFs for Germany: Union

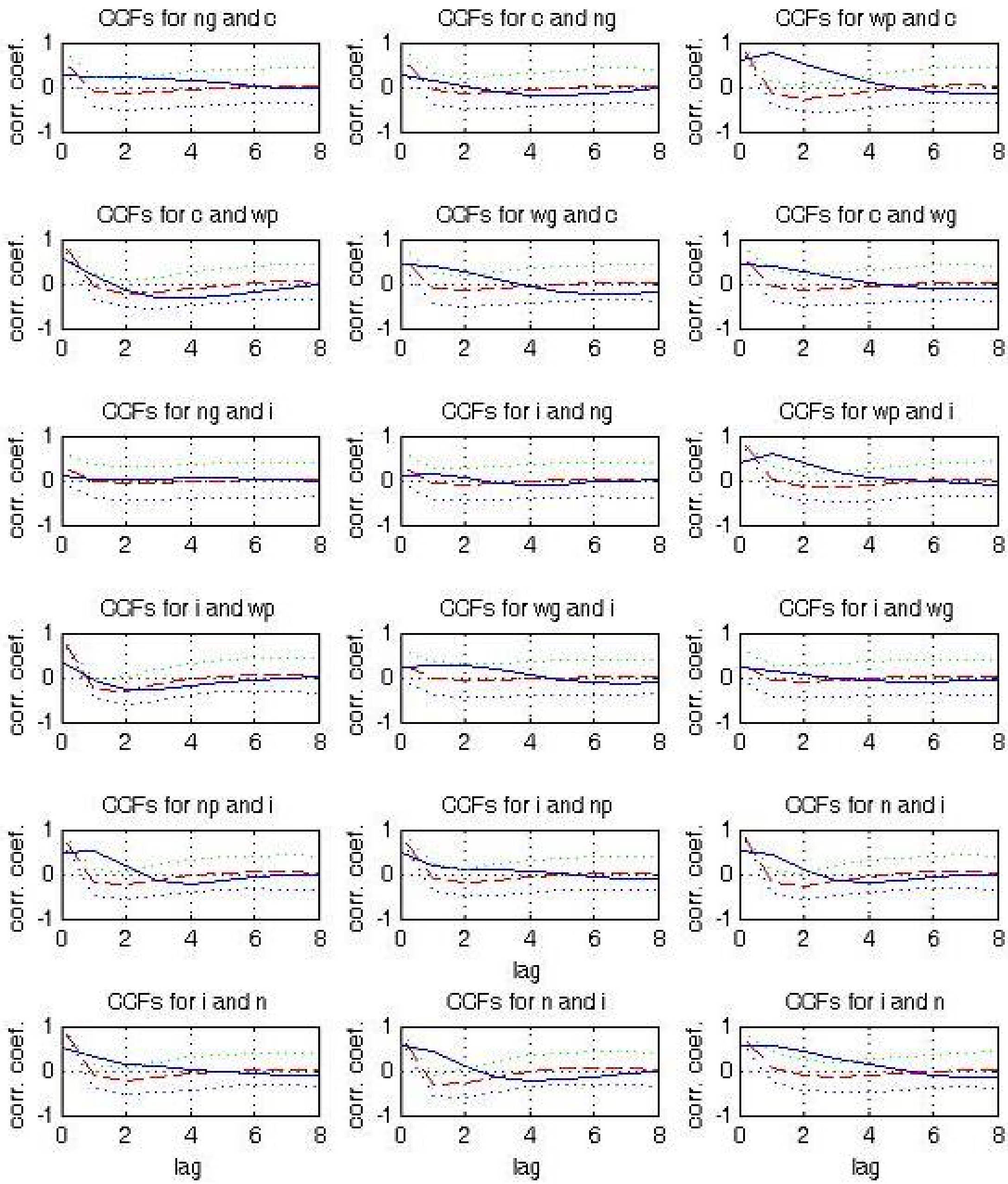


Figure 4: Theoretical and empirical ACFs for Germany: Finn

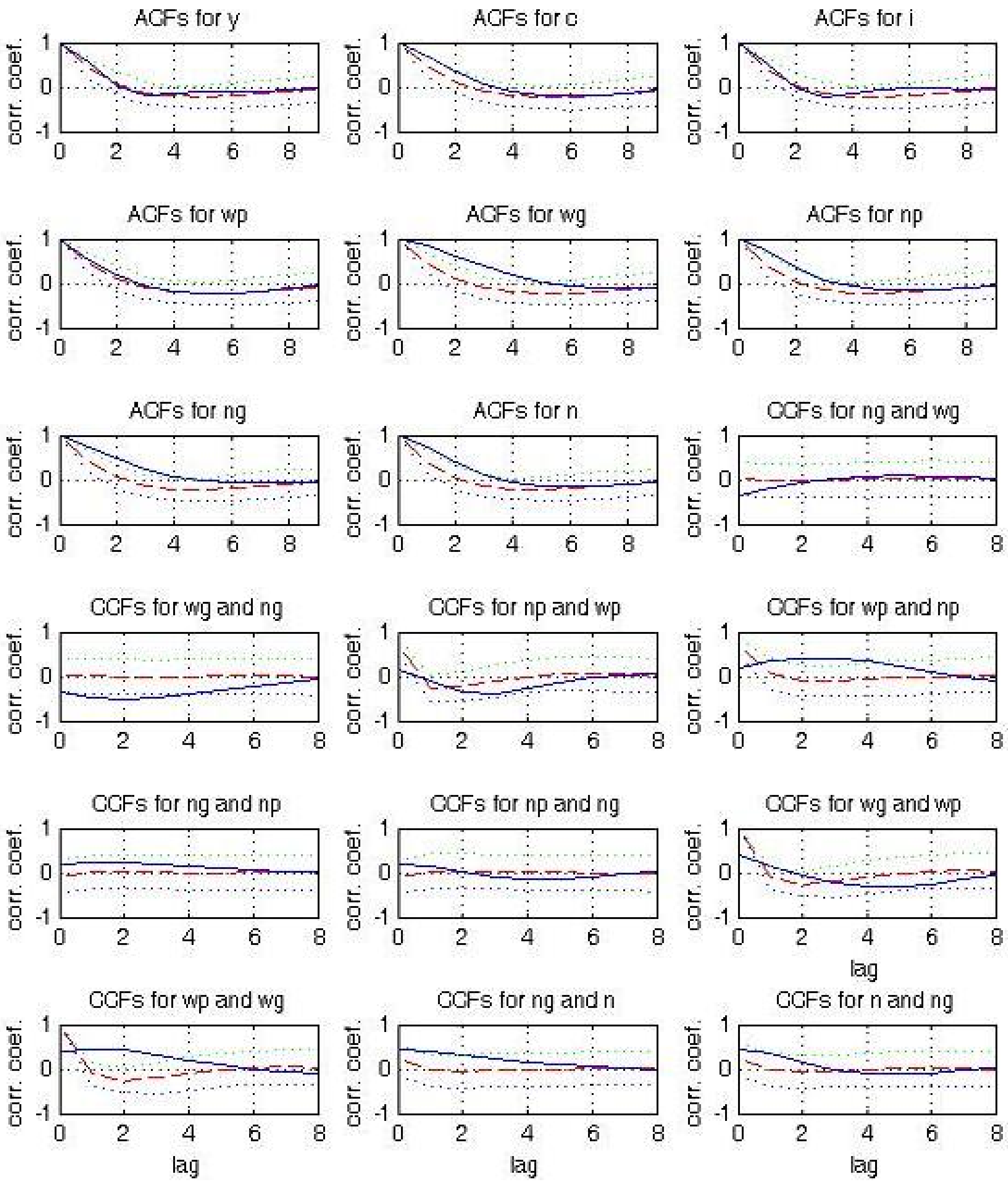


Figure 5: Theoretical and empirical ACFs for Germany: Finn

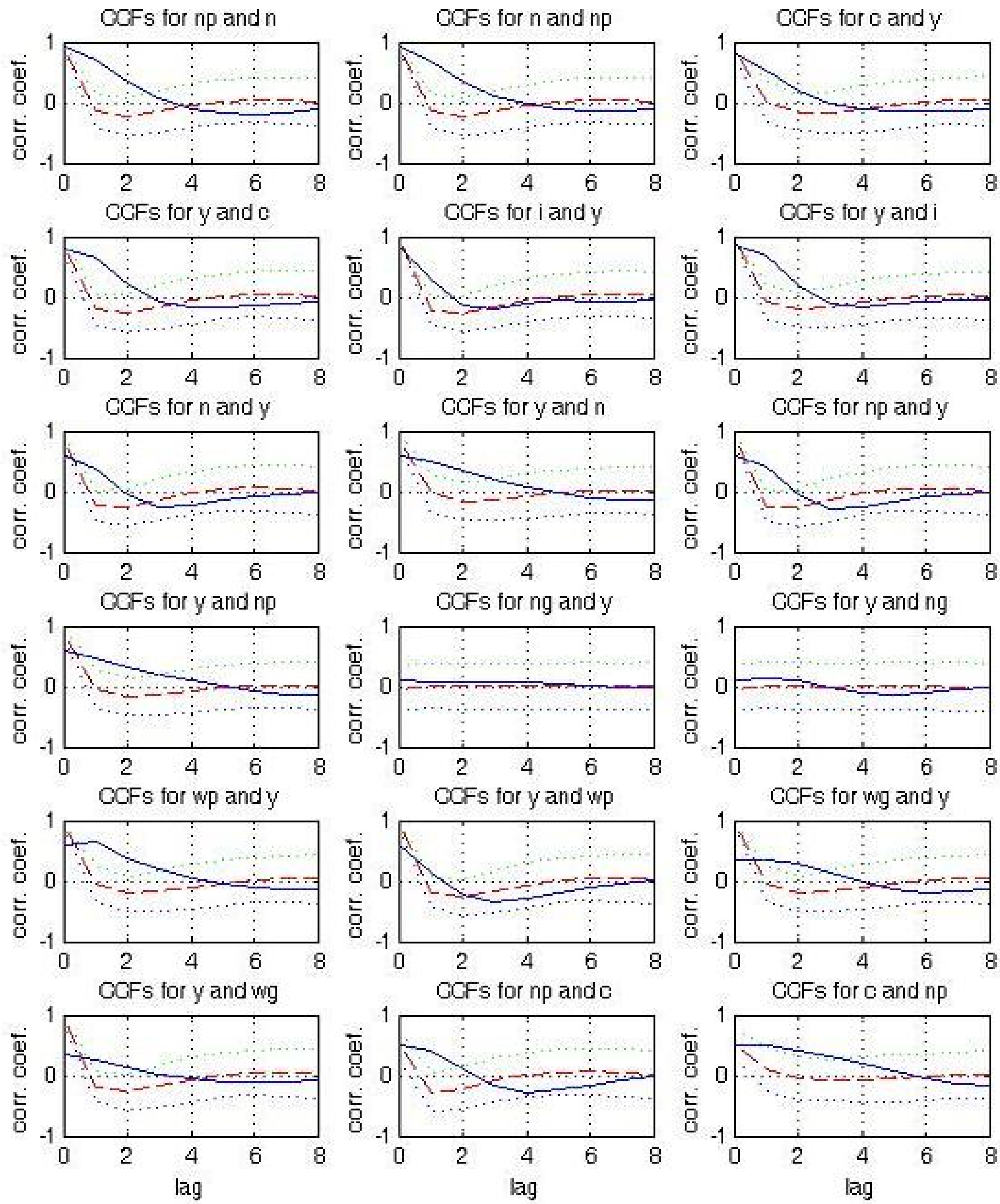
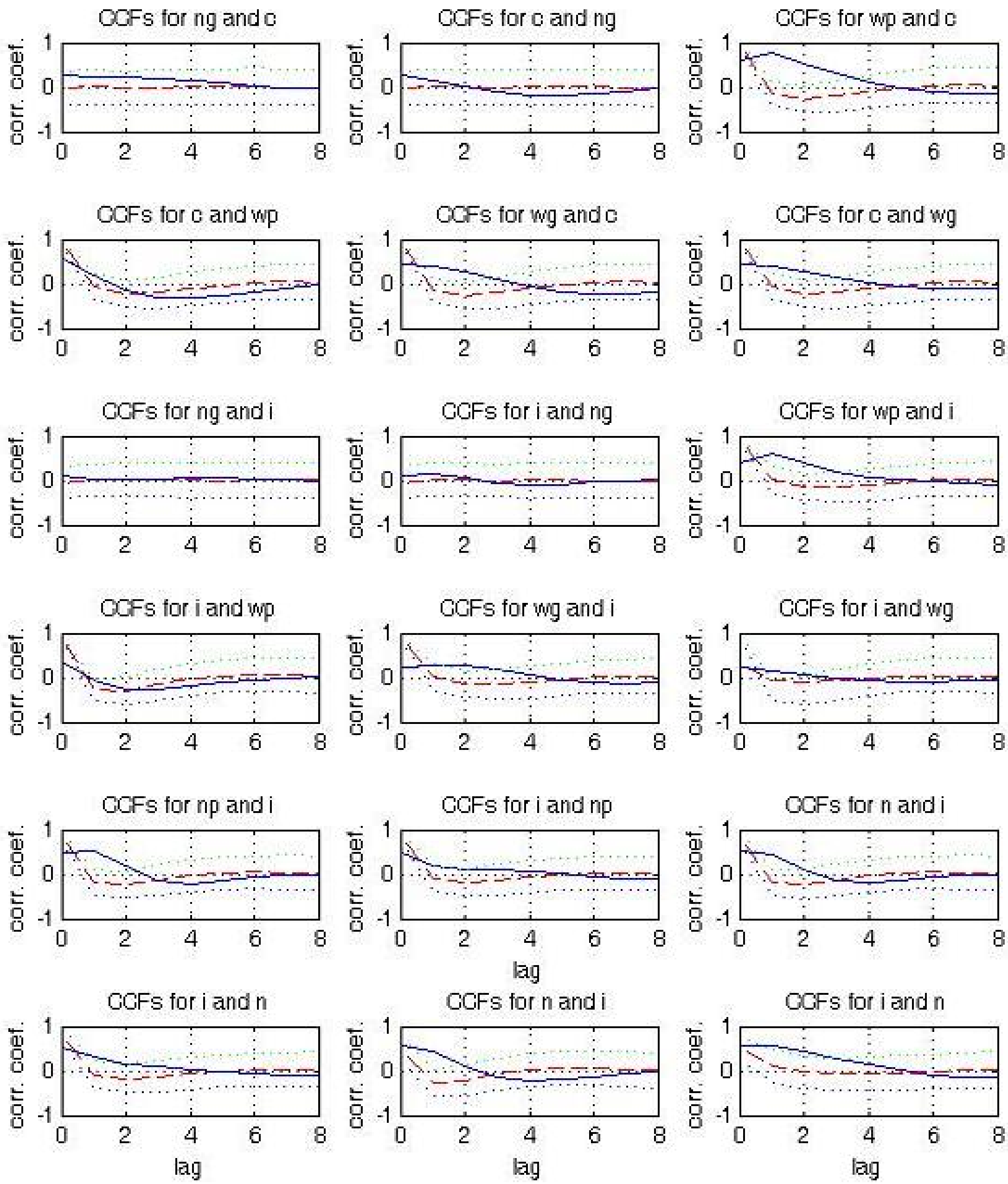


Figure 6: Theoretical and empirical ACFs for Germany: Finn



perfect positive contemporaneous correlation between public wages and hours, while in data, it is negative. Overall, the model with public sector union calibrated for Germany captures the dynamic co-movement of hours and wages with output, consumption and investment. In addition, union model is able to address and match some new dimensions such as the dynamic correlation of the two wage rates and the pair of hours worked.

1.4 Sensitivity analysis

To evaluate the effect of structural parameters on the shape of the Laffer curves, this section performs sensitivity analysis for different values of model parameters and how those affect tax revenues. The two parameters of interest are the curvature parameter of household's Cobb-Douglas utility function α, as well as the weight on composite consumption, ψ. Interestingly, as α is allowed to vary, steady-state revenues are essentially unchanged. Even an implausibly high value, $\alpha = 50$, does not produce any difference in steady state tax revenues. In both models considered in this paper, the preference parameter is not important for steady-state fiscal policy effect. This result is not surprising in the literature, as Trabandt and Uhlig (2010) obtain a very similar finding in their paper.[1]

In contrast, changes in the second parameter, ψ, yield significant differences. Both the capital and labor tax Laffer curves, and the responses of the other tax bases to capital and labor income tax rate are affected when ψ is allowed to vary.[2] Higher values of ψ shift up the Laffer curve and make it steeper, without significant change in its peak. The difference between Finn and the model with endogenous public employment becomes significant for implausibly high values of ψ, i.e. $\psi > 0.5$. (As explained in the calibration section, parameter $\psi = 0.296$, describing household's preference was calculated as the ratio of hours of work out of total potential hours in the model.) Intuitively, a higher ψ corresponds to a lower weight to leisure, $(1 - \psi)$, in the household's utility function. In other words, a higher ψ

[1] Parameter α is important for model dynamics, though.

[2] Consumption tax Laffer curve proves to be very sensitive to ψ parameter. In the majority of the cases it breaks down for values outside the benchmark value. This is also a typical result in the literature, e.g. Trabandt and Uhlig (2010).

Figure 7: Sensitivity analysis: Union

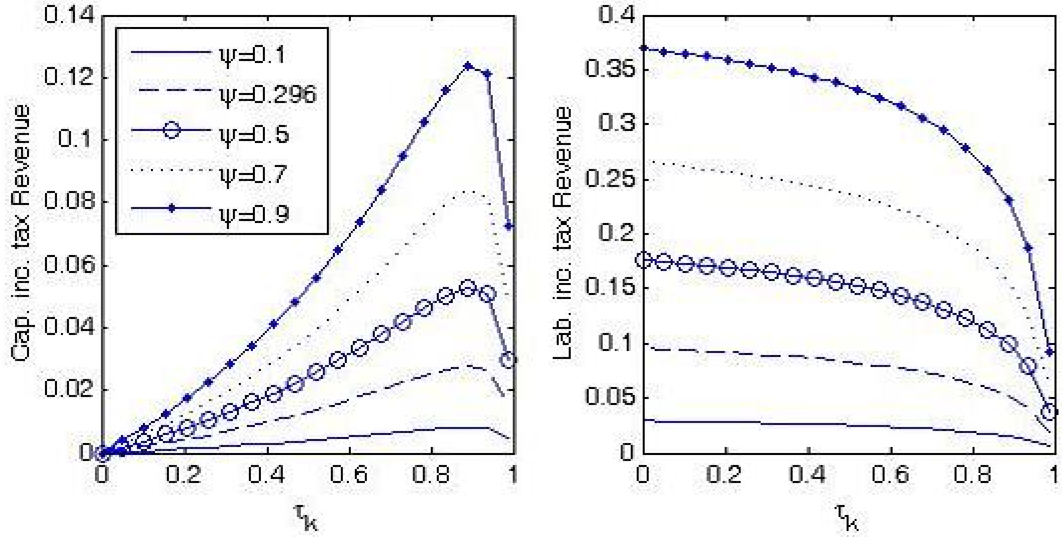


Figure 8: Sensitivity analysis: Union

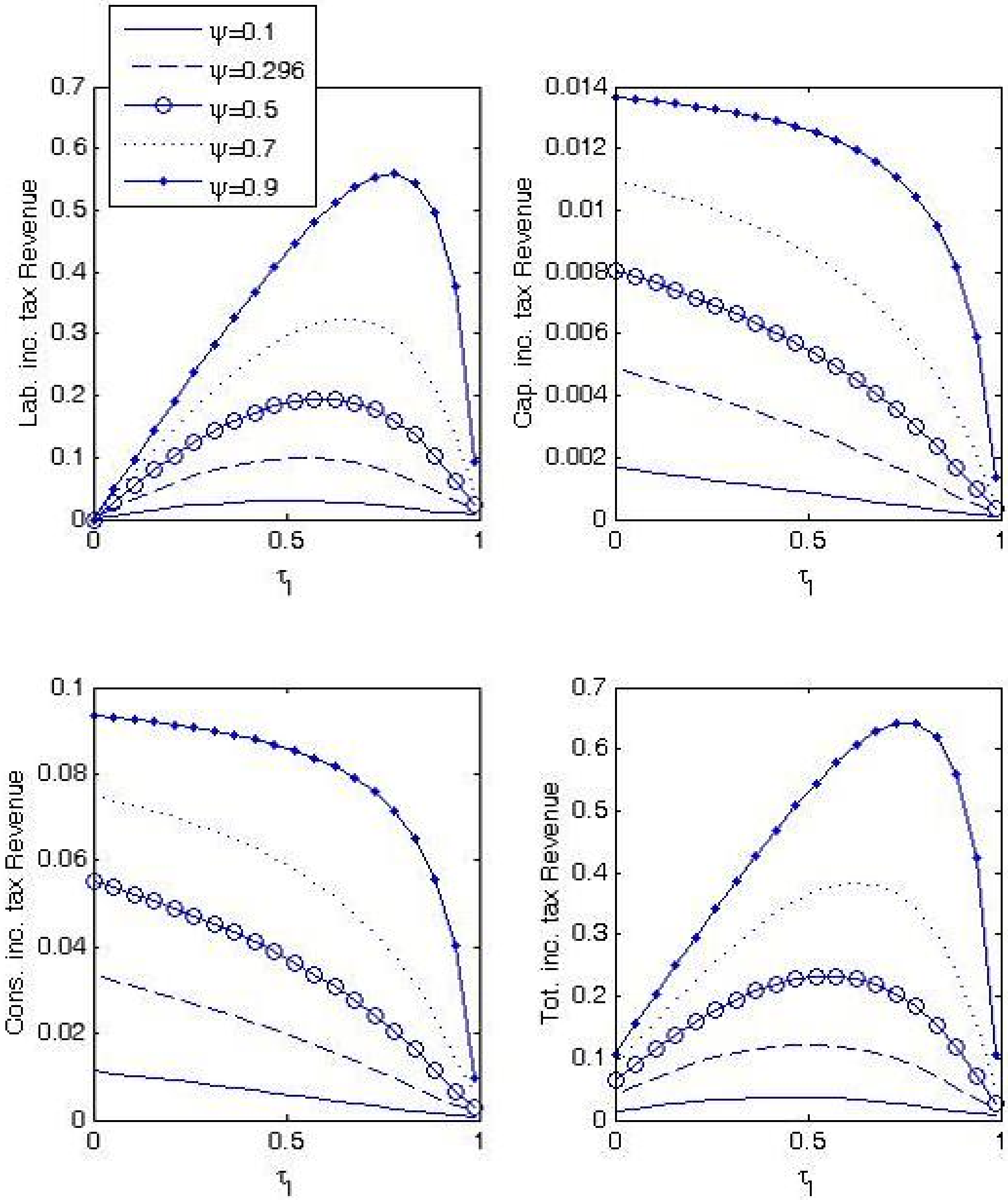


Figure 9: Sensitivity analysis: Finn

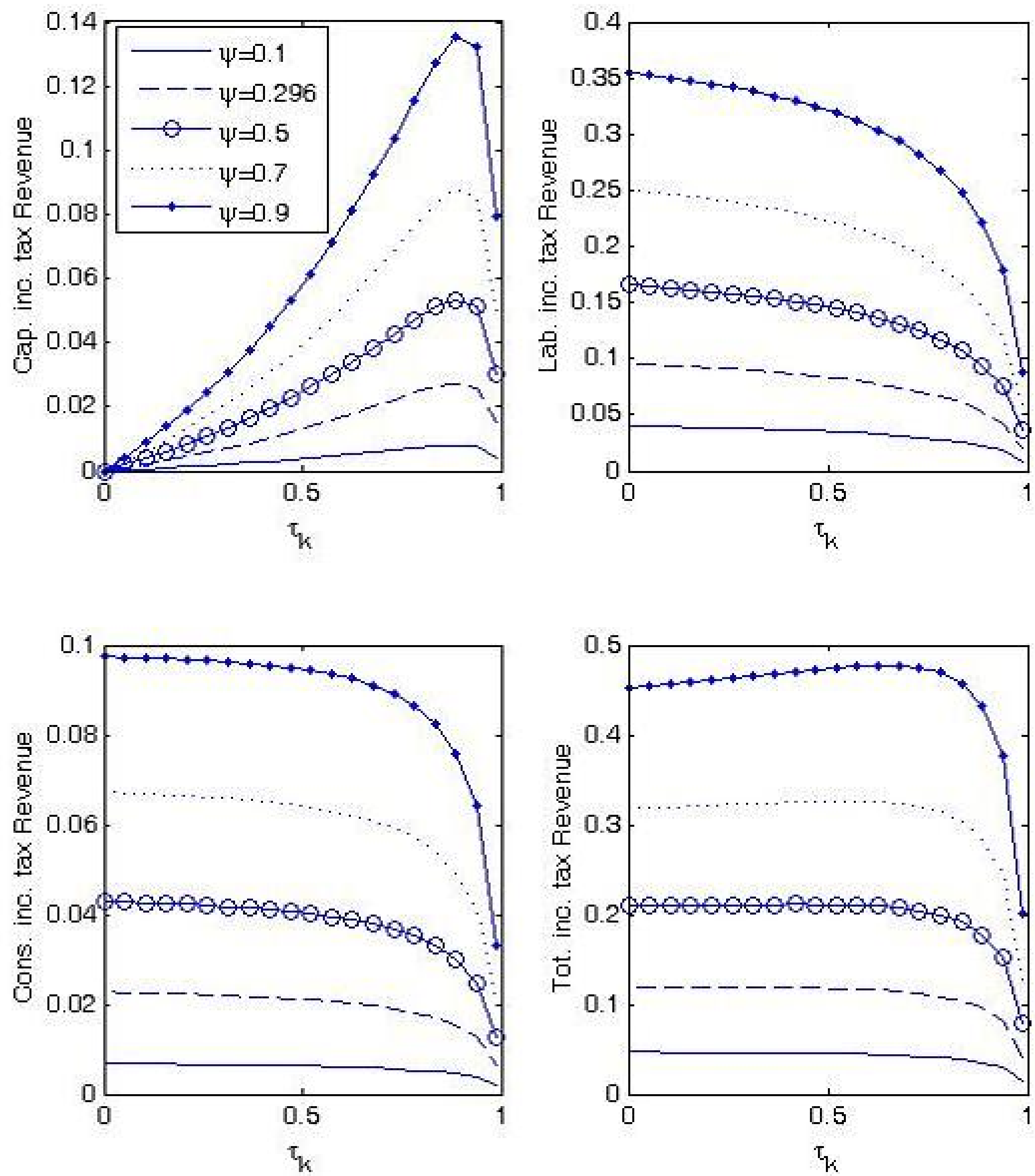
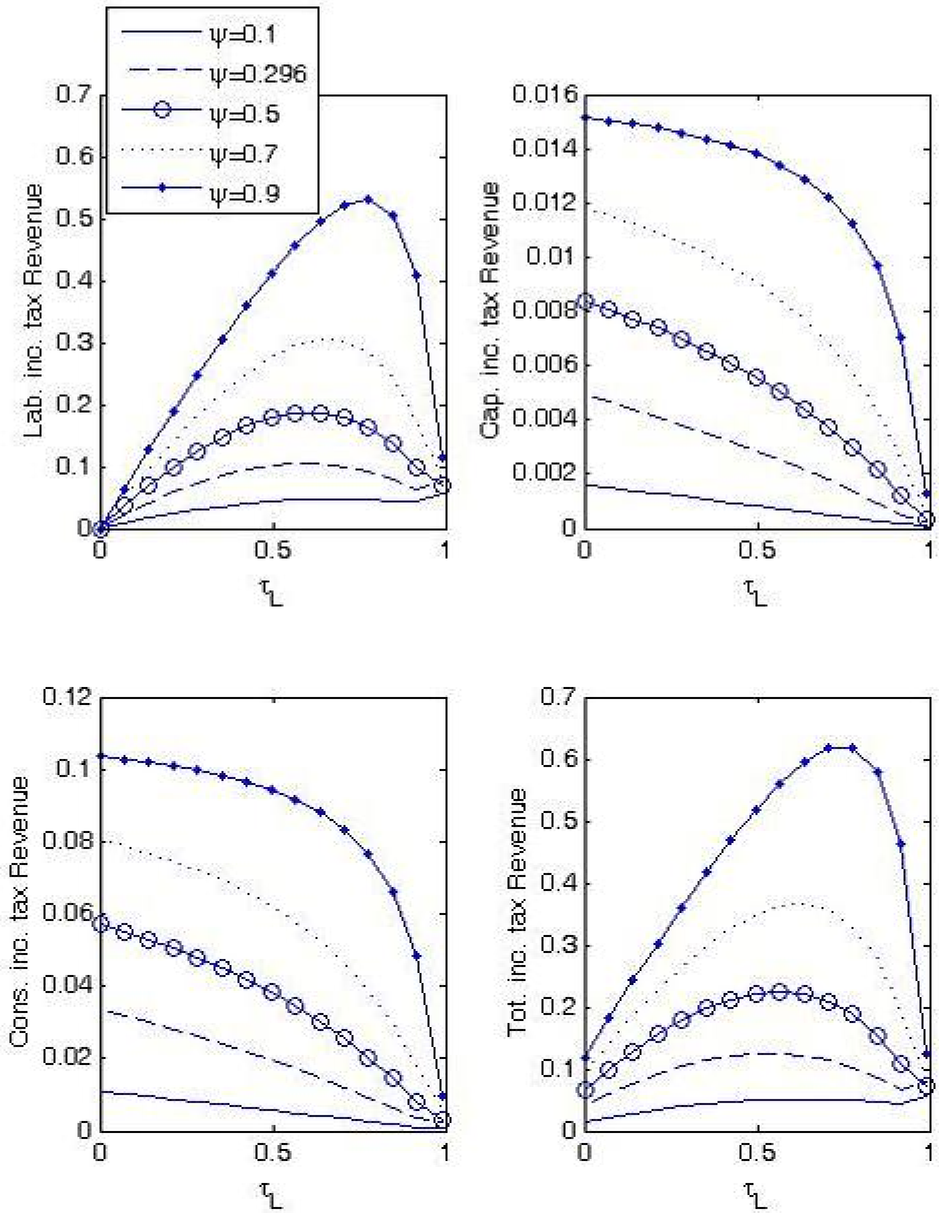


Figure 10: Sensitivity analysis ψ: Finn



decreases the elasticity of private labor supply. Intuitively, when labor tax rate increases, or equivalently, after tax private wage falls, private hours respond less, thus increasing labor income tax revenue, as well as total tax revenue.

The effect of higher ψ on capital tax Laffer curve is similar to ψ's effect on the labor tax Laffer curve above. When τ^k is allowed to vary, a higher weight attached to consumption in household's utility function, together with the optimality condition for the marginal rate of substitution between consumption and hours require private higher capital stock to finance private consumption. Therefore, a higher ψ shifts the capital tax Laffer curve upward as well.

1.5 Measuring conditional welfare

In steady state

$$u(c, g^c, 1-n) = \frac{[(c+\omega g^c)^{\psi}(1-n)^{(1-\psi)}]^{(1-\alpha)} - 1}{1-\alpha} \tag{186}$$

Let A and B denote two different regimes. The welfare gain, ζ, is the fraction of consumption that is needed to complement household's steady-state consumption in regime B so that the household is indifferent between the two regimes. Thus

$$\frac{[(c^A+\omega g^{c,A})^{\psi}(1-n^A)^{(1-\psi)}]^{(1-\alpha)} - 1}{1-\alpha} = \frac{[((1+\zeta)c^B+\omega g^{c,B})^{\psi}(1-n^B)^{(1-\psi)}]^{(1-\alpha)} - 1}{1-\alpha} \tag{187}$$

Multiply both sides by $(1-\alpha)$ to obtain

$$[(c^A+\omega g^{c,A})^{\psi}(1-n^A)^{(1-\psi)}]^{(1-\alpha)} - 1 = [((1+\zeta)c^B+\omega g^{c,B})^{\psi}(1-n^B)^{(1-\psi)}]^{(1-\alpha)} - 1 \tag{188}$$

Cancel -1 terms at both sides to obtain

$$[(c^A+\omega g^{c,A})^{\psi}(1-n^A)^{(1-\psi)}]^{(1-\alpha)} = [((1+\zeta)c^B+\omega g^{c,B})^{\psi}(1-n^B)^{(1-\psi)}]^{(1-\alpha)} \tag{189}$$

Raise both sides to the power $\frac{1}{1-\alpha}$ to obtain

$$(c^A+\omega g^{c,A})^{\psi}(1-n^A)^{(1-\psi)} = ((1+\zeta)c^B+\omega g^{c,B})^{\psi}(1-n^B)^{(1-\psi)} \tag{190}$$

Divide throughout by $(1-n^B)^{(1-\psi)}$ to obtain

$$((1+\zeta)c^B+\omega g^{c,B})^{\psi} = (c^A+\omega g^{c,A})^{\psi}\left(\frac{1-n^A}{1-n^B}\right)^{(1-\psi)}$$

Raise both sides to the power $1/\psi$ to obtain

$$(1+\zeta)c^B + \omega g^{c,B} = (c^A + \omega g^{c,A})\left(\frac{1-n^A}{1-n^B}\right)^{\frac{(1-\psi)}{\psi}} \tag{191}$$

Move $\omega g^{c,B}$ term to the right to obtain

$$(1+\zeta)c^B = (c^A + \omega g^{c,A})\left(\frac{1-n^A}{1-n^B}\right)^{\frac{(1-\psi)}{\psi}} - \omega g^{c,B} \tag{192}$$

Divide both sides by c^B to obtain

$$1+\zeta = \frac{1}{c^B}\left\{(c^A + \omega g^{c,A})\left(\frac{1-n^A}{1-n^B}\right)^{\frac{(1-\psi)}{\psi}} - \omega g^{c,B}\right\} \tag{193}$$

Thus

$$\zeta = \frac{1}{c^B}\left\{(c^A + \omega g^{c,A})\left(\frac{1-n^A}{1-n^B}\right)^{\frac{(1-\psi)}{\psi}} - \omega g^{c,B}\right\} - 1 \tag{194}$$

Note that if $\zeta > 0 (< 0)$, there is a welfare gain (loss) of moving from B to A. In this paper B is the initial scenario, while A will be the fiscal regime change.