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1 December 2015

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December 9, 2015

Abstract

This paper studies a decision maker who chooses monetary bets/investment portfolios under pure uncertainty. Necessary and sufficient conditions on his preferences over these objects are provided for his choice behavior to be guided by the *maxmin expected value* rule, and therefore to exhibit both "risk neutrality" and ambiguity aversion. This result is obtained as an extension of a simple re-characterization of de Finetti's theorem on maximization of subjective expected value.

Keywords: Maxmin expected value; ambiguity aversion; risk neutrality; multiple priors; de Finetti.

JEL Classification: D01, D03, D11

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1 Introduction

In this paper we model axiomatically an individual with imprecise probabilistic beliefs over the possible states of the world, in the tradition of the multiple-priors approach to decision making under uncertainty that was pioneered by Gilboa and Schmeidler (1989). Our decision maker chooses from a linear space of monetary bets or investment portfolios and is modelled as ambiguity-averse as well as "risk-neutral" in the sense that he is assumed to maximize the minimum expected *value* according to his set of beliefs. Therefore, his utility function over wealth is assumed to be linear. Such behavior is relevant and has implications, for instance, in the theoretical analysis of financial markets with risk-neutral and ambiguity-averse sellers of securities (Araujo, Chateaneuf, and Faro, 2012, 2015).

Our result is the multi-prior extension of de Finetti's (1937) classic theorem that characterizes maximization of subjective expected value. Before stating and proving this result, however, we also provide a simple re-axiomatization of de Finetti's model in which his Additivity axiom is replaced by Independence and Homotheticity. This reformulation allows for the Gilboa-Schmeidler axiomatic approach to be suitably adapted to the present setting, and also allows for a more direct comparison between the models of single-prior subjective expected value maximization and multi-prior minimum expected value maximization.

Our decision maker's choice domain is $X := \mathbb{R}^n$ and $\succsim$ is his weak preference relation on X, with $\succ$ and $\sim$ denoting its asymmetric and symmetric parts that capture strict preference and indifference, respectively. Generic elements of X are denoted by x, y, z. We write $x \geq y$ if $x_i \geq y_i$ for all $i \leq n$. Moreover, for $c \in \mathbb{R}$ we write $\mathbf{c} := (c, \ldots, c) \in X$, and we let

$$C := \{x \in X : x = \alpha \cdot \mathbf{1} \text{ for some } \alpha \in \mathbb{R}\}.$$

The decision maker perceives uncertainty that is captured by the state space $S := \{1, \ldots, n\}$, with $\Sigma := 2^S$ being the associated algebra of events. To ease notation the pair (S, Σ) will be denoted simply by S. The vectors in X can be thought of as Savage (1954) acts $f : S \to \mathbb{R}$ and may be interpreted as monetary bets or investment portfolios that specify a payoff in each state of the world. Similarly, elements of C may be thought of as sure bets/investments that assign the same payoff in all states of the world. This setup of purely subjective uncertainty differs from the corresponding one in Casadesus-Masanell et al (2000) in that the payoffs there are assumed to be bounded above and below. Yet, it is embedded as a special case in the setup of Alon and Schmeidler (2014), as in that paper the authors allow for the outcome set to be any connected topological space.

2 A Simple Reformulation of de Finetti's Theorem

Before proceeding to our main task it will be useful first to provide a suitable reformulation of de Finetti's (1937) theorem. We begin by stating the axioms that are involved in both the original (as established in Gilboa, 2009) and the newly proposed formulation of this fundamental result.

Weak Order
$\succsim$ is complete and transitive.

Monotonicity
For all $x, y \in X$, $x \geq y$ implies $x \succsim y$.

Non-Triviality
There exist $x, y \in X$ such that $x \succ y$.

Hemicontinuity
For all $x \in X$, the sets $U_{\succsim}(x) := \{y \in X : y \succsim x\}$ and $L_{\succsim}(x) := \{y \in X : x \succsim y\}$ are closed.

Additivity
For all $x, y, z \in X$, $x \succsim y$ implies $x + z \succsim y + z$.

Homotheticity
For all $x, y \in X$ and $\alpha > 0$, $x \succsim y$ implies $\alpha x \succsim \alpha y$.

Independence
For all $x, y, z \in X$ and all $\alpha \in (0, 1)$, $x \succsim y$ implies $\alpha x + (1 - \alpha)z \succsim \alpha y + (1 - \alpha)z$.

The first four axioms are standard in the literature. Additivity and its risk-neutrality implications are discussed in Gilboa (2009). Homotheticity is familiar from consumer theory where it is known to result in demand functions that are linear in income. Independence is a well-known normative axiom in both the domain of objective lotteries of von Neumann-Morgenstern (1944) expected utility theory as well as in the domain of acts of the Anscombe-Aumann (1963) type. An agent whose preferences violate Independence can in theory be exploited by a "Dutch book".

Theorem 1 (de Finetti). *The following are equivalent:*
1. $\succsim$ satisfies Weak Order, Monotonicity, Non-Triviality, Hemicontinuity and Additivity.
2. There exists a unique probability measure p on S such that, for all $x, y \in X$

$$x \succsim y \iff px \geq py. \tag{1}$$

3. $\succsim$ satisfies Weak Order, Monotonicity, Non-Triviality, Hemicontinuity, Homotheticity and Independence.

Proof.

$1 \Rightarrow 2$. See pp. 91-92 in Gilboa (2009).

$2 \Rightarrow 3$. This is straightforward and omitted.

$3 \Rightarrow 1$. It suffices to show that Additivity is satisfied. Suppose $x \succsim y$. From Independence, $\alpha x + (1 - \alpha)z \succsim \alpha y + (1 - \alpha)z$ for all $\alpha \in (0, 1)$ and $z \in X$. From Homotheticity, $\frac{\alpha}{1-\alpha}x + z \succsim \frac{\alpha}{1-\alpha}y + z$. When $\alpha = \frac{1}{2}$, this reduces to $x + z \succsim y + z$. ■

The addition of the third statement in this result, and the replacement of Additivity by Homotheticity and Independence, allows for clarifying the relationship between the multiple-priors model that is axiomatized in the next section with de Finetti's baseline model of subjective expected value maximization. Such a comparison is not possible with the first statement.

For reasons that are discussed below it will be useful to also consider the implications of the following strengthening of Monotonicity:

Strict Monotonicity:
For all $x, y \in X$, $x \geq y$ and $x \neq y$ implies $x \succ y$.

Corollary 2. *The following are equivalent:*
1. $\succsim$ satisfies Weak Order, Strict Monotonicity, Non-Triviality, Hemicontinuity and Additivity.
2. $\succsim$ satisfies Weak Order, Strict Monotonicity, Non-Triviality, Hemicontinuity, Homotheticity and Independence.
3. $\succsim$ can be represented as in (1) with a strictly positive probability measure p on S.

3 A Characterization of Maxmin Expected Value

We now proceed to stating suitably modified versions of the two key axioms that were introduced in Schmeidler (1989) and Gilboa and Schmeidler (1989). We refer the reader to Gilboa (2009) for intuitive discussions of these axioms.

Certainty Independence:
For all $x, y \in X$, $\mathbf{c} \in C$ and $\alpha \in (0, 1)$, $x \succsim y$ implies $\alpha x + (1 - \alpha)\mathbf{c} \succsim \alpha y + (1 - \alpha)\mathbf{c}$.

Ambiguity Aversion:
For all $x, y \in X$ and all $\alpha \in (0, 1)$, $x \sim y$ implies $\alpha x + (1 - \alpha)y \succsim x$.

Theorem 3. *The following are equivalent:*
1. $\succsim$ satisfies Weak Order, Strict Monotonicity, Hemicontinuity, Homotheticity, Certainty Independence and Ambiguity Aversion.
2. There exists a compact convex set P of strictly positive probability measures on S such that, for all $x, y \in X$,

$$x \succsim y \iff \min_{p \in P} px \geq \min_{p \in P} py. \tag{2}$$

Proof.

Necessity of the axioms is straightforward to establish. The sufficiency proof combines a utility representation theorem of Debreu (1954) with suitable adaptations of the arguments in the proof of Lemma 3.3 in Gilboa and Schmeidler (1989) and application of Theorem 2 in Marinacci (1998). We re-state the latter result as a Lemma:

Lemma (Marinacci). *Let $(X, \geq)$ be a partially ordered real topological vector space and consider a functional $u : X \to \mathbb{R}$ that satisfies the following:*
(i) $u(\mathbf{0}) = 0$ and $u(x) \geq u(y)$ if $x \geq y$ and $x, y \in X$.
(ii) $u(x + y) \geq u(x) + u(y)$ for all $x, y \in X$.
(iii) For some $\overline{x} \in X$ it holds that $u(\alpha x + \beta \overline{x}) = \alpha u(x) + \beta u(\overline{x})$ for all $\alpha \geq 0$, $\beta \in \mathbb{R}$ and $x \in X$.
Then, there exists a unique convex and weak*-compact set $\mathcal{L}$ of positive linear functionals L on X such that

$$L(\overline{x}) = u(\overline{x}) \quad \text{for all } L \in \mathcal{L} \tag{3}$$

$$u(x) = \min_{L \in \mathcal{L}} L(x) \quad \text{for all } x \in X \tag{4}$$

We start by noticing that Weak Order, Hemicontinuity and the fact that X is a connected and separable metric space ensure that there exists an ordinally unique continuous utility function $u : X \to \mathbb{R}$ such that $x \succsim y$ iff $u(x) \geq u(y)$ (Debreu, 1954; Theorem I). In the sequel we normalize such a u so that $u(\mathbf{0}) = 0$ and $u(\mathbf{1}) = 1$.

Clearly, Strict Monotonicity ensures that $u(x) \geq u(y)$ if $x \geq y$, with strict inequality whenever $x \neq y$. Thus, condition (i) of the Lemma is satisfied. Moreover, it is a trivial consequence of Homotheticity that $u(\alpha x) = \alpha u(x)$ for all $x \in X$ and $\alpha > 0$. Thus, u is homogeneous of degree one.

We will show next that u is C-independent, i.e. that $u(x + \mathbf{c}) = u(x) + u(\mathbf{c})$ for all $x \in X$ and $\mathbf{c} \in C$. Let $u(2x) := k$ and define $\mathbf{e} := 2\mathbf{c} \in C$. Clearly, $u(\mathbf{e}) = 2c$ and $2x \sim \mathbf{k}$. Therefore, from Certainty Independence, $\frac{1}{2}(2x) + \frac{1}{2}\mathbf{e} \sim \frac{1}{2}\mathbf{k} + \frac{1}{2}\mathbf{e}$. Repeated application of the homogeneity property of u now yields

$$u(x + \mathbf{c}) \;=\; u(\tfrac{1}{2}\mathbf{k} + \mathbf{c}) \;=\; \tfrac{1}{2}u(\mathbf{k} + 2\mathbf{c}) \;=\; \tfrac{1}{2}u(\mathbf{k}) + u(\mathbf{c})$$

$$=\quad \tfrac{1}{2}k + c \quad = \quad u(x) + c$$

as required. In view of homogeneity and C-independence of u, condition (iii) of the Lemma is also satisfied.

We will now show that $u(x + y) \geq u(x) + u(y)$ for all $x, y \in X$. Given that u is homogeneous, it suffices to show that $u(\frac{1}{2}x + \frac{1}{2}y) \geq \frac{1}{2}u(x) + \frac{1}{2}u(y)$. If $u(x) = u(y)$, then $x \sim y$ holds and the claim readily follows from Ambiguity Aversion. Suppose now, with no loss of generality, that $u(x) > u(y)$ and let $c := u(x) - u(y)$. Define $z := y + \mathbf{c}$ and observe that, from C-independence of u, $u(z) = u(y + \mathbf{c}) = u(y) + c = u(x)$. Moreover,

$$u\left(\tfrac{1}{2}x + \tfrac{1}{2}y\right) + \tfrac{1}{2}c \;=\; u\left(\tfrac{1}{2}x + \tfrac{1}{2}z\right) \;\geq\; \tfrac{1}{2}u(x) + \tfrac{1}{2}u(z) \;=\; \tfrac{1}{2}u(x) + \tfrac{1}{2}u(y) + \tfrac{1}{2}c$$

where the inequality again follows from Ambiguity Aversion after noting that $x \sim z$, while the equalities follow from homogeneity of u. This shows that condition (ii) of the Lemma is satisfied too.

Since $X = \mathbb{R}^n$ is a real topological vector space that has been endowed with the canonical partial order $\geq$, and u satisfies conditions (i)–(iii) of the Lemma, there exists a unique convex and compact (in the Euclidean topology) set $\mathcal{L}$ of positive linear functionals L on X such that (3) and (4) hold. Moreover, the element(s) $\overline{x} \in X$ in the Lemma coincide with the elements $\mathbf{c} \in C$. Therefore, for each $L \in \mathcal{L}$ we have $L(x) = \beta_L \cdot x$ for some $\beta_L \in \mathbb{R}^n_+$. Moreover, a straightforward modification in the proof of Theorem 2 in Marinacci (1998) shows that Strict Monotonicity actually implies that $\beta_L \in \mathbb{R}^n_{++}$, and therefore that $\mathcal{L}$ consists of strictly positive linear functionals.

Now, for each β_L, define $p_L \in \mathbb{R}^n_{++}$ by

$$p_L^i = \frac{\beta_L^i}{\sum_{j=1}^n \beta_L^j}.$$

Clearly, p_L is a strictly positive probability measure on S. For each $L \in \mathcal{L}$ define $\widehat{L}$ by $\widehat{L}(x) := p_L \cdot x$ and let $\widehat{\mathcal{L}}$ be the set of all such $\widehat{L} \equiv p_L$. Finally, let P be the convex hull of $\widehat{\mathcal{L}}$. Clearly, P is convex and compact (see, for instance, Corollary 5.33 in Aliprantis and Border, 2006). It remains to be shown that

$$\min_{L \in \mathcal{L}} L(x) \geq \min_{L \in \mathcal{L}} L(y) \iff \min_{p \in P} px \geq \min_{p \in P} py.$$

But this equivalence must be true because for each $p \in P$, either p can be derived from some $L \in \mathcal{L}$ as above, or, from Carathéodory's Theorem, p is a convex combination of finitely many elements of P that can be derived in this way (see, for instance, p. 184 in Aliprantis and Border, 2006). ∎

The polar opposite of the Ambiguity Aversion axiom that drives the pessimistic behavior captured by the maxmin choice rule is the following:

Ambiguity Inclination:
For all $x, y \in X$ and all $\alpha \in (0,1)$, $x \sim y$ implies $x \succsim \alpha x + (1-\alpha)y$.

It has been known in the literature that followed Gilboa and Schmeidler (1989) that, jointly with the other axioms of the maxmin model (all of which are stated in the Anscombe-Aumann (1963) setting of choice under uncertainty), Ambiguity Inclination characterizes the optimistic decision rule that is associated with *maxmax expected utility*. We note without proof that a similar conclusion is derivable in the present setting of pure uncertainty with the maxmax expected *value* rule.

Corollary 4. *The following are equivalent:*
1. $\succsim$ satisfies Weak Order, Strict Monotonicity, Hemicontinuity, Homotheticity, Certainty Independence and Ambiguity Inclination.
2. There exists a compact convex set P of strictly positive probability measures on S such that for all $x, y \in X$

$$x \succsim y \iff \max_{p \in P} px \geq \max_{p \in P} py. \tag{5}$$

4 Degenerate Beliefs, Monotonicity and Indifference

It is worth noting that if Strict Monotonicity is replaced by Monotonicity in the statements of Theorem 3 and Corollary 4, this would open up the possibility for the set P in (2) and (5) to contain degenerate priors that assign probability one to some state. In the extreme case where all n degenerate priors are in P (and hence $P = \Delta(S)$, the set of all probability measures on S), some behaviorally counterintuitive predictions obtain that concern the indifference relation.

Observation. *Suppose $P = \Delta(S)$. Then, for all $x, y \in X$,*

$$x \sim y \iff \begin{cases} \min_{i \leq n} x_i = \min_{i \leq n} y_i & \text{if (2) holds} \\[2mm] \max_{i \leq n} x_i = \max_{i \leq n} y_i & \text{if (5) holds} \end{cases}$$

Indeed, suppose (2) holds and let $u(x) = \min_{p \in P} px$. Since $P = \Delta(S)$, $u(x) = e_j x$, where e_j is the probability measure that assigns probability 1 to state j, and $x_j = \arg\min_{i \leq n} x_i := \underline{x}$. Therefore, $x \sim y \Leftrightarrow u(x) = u(y) \Leftrightarrow \underline{x} = \underline{y}$. The argument for (5) is symmetric.

Strict Monotonicity is a sufficient (although clearly not necessary) condition to rule out the above counter-intuitive predictions. To the extent that more wealth is preferred to less regardless of which state of the world obtains, this axiom is a behaviorally relevant one too.

References

ALIPRANTIS, C. D., AND K. C. BORDER (2006): *Infinite Dimensional Analysis.* Berlin Heidelberg: Springer, 3rd edn.

ALON, S., AND D. SCHMEIDLER (2014): "Purely Subjective Maxmin Expected Utility," *Journal of Economic Theory*, 152, 382–412.

ANSCOMBE, F. J., AND R. J. AUMANN (1963): "A Definition of Subjective Probability," *Annals of Mathematical Statistics*, 14, 477–482.

ARAUJO, A., A. CHATEANEUF, AND J. H. FARO (2012): "Pricing Rules and Arrow-Debreu Ambiguous Valuation," *Economic Theory*, 49, 1–35.

——— (2015): "Efficient Complete Markets are the Rule Rather than the Exception," *mimeo.*

CASADESUS-MASANELL, R., P. KLIBANOFF, AND E. OZDENOREN (2000): "Maxmin Expected Utility Over Savage Acts with a Set of Priors," *Journal of Economic Theory*, 92, 35–65.

DE FINETTI, B. (1937): "La Prévision: Ses Lois Logiques, Ses Sources Subjectives," *Annales de l'Institut Henri Poincaré*, 7, 1–68.

DEBREU, G. (1954): "Representation of a Preference Ordering by a Numerical Function," in *Decision Processes*, ed. by R. M. Thrall, C. H. Coombs, AND R. L. Lavis, pp. 159–165. Wiley.

GILBOA, I. (2009): *Theory of Decision under Uncertainty*, Econometric Society Monographs 45. New York: Cambridge University Press.

GILBOA, I., AND D. SCHMEIDLER (1989): "Maxmin Expected Utility with Non-Unique Prior," *Journal of Mathematical Economics*, 18, 141–153.

MARINACCI, M. (1998): "A Simple Proof of a Basic Result for Multiple Priors," *mimeo.*

SAVAGE, L. J. (1954): *The Foundations of Statistics.* New York: Wiley.

SCHMEIDLER, D. (1989): "Subjective Probability and Expected Utility without Additivity," *Econometrica*, 57, 571–587.

VON NEUMANN, J., AND O. MORGENSTERN (1944): *Theory of Games and Economic Behavior.* Princeton: Princeton University Press.