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Abstract

Games with incomplete preferences are normal-form games where the preferences of the players are defined as partial orders over the outcomes of the game. We define rationality in these games as follows. A rational player forms a set-valued belief of possible strategies selected by the opponent(s) and selects a strategy that is not dominated with respect to this belief. Here, we say a strategy is dominated with respect to the set-valued belief if the player has another strategy that would yield a better outcome according to the player's preference relation, no matter which strategy combination the opponent(s) play

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among those contained in the belief. We define rationalizable strategies as the logical implication of common knowledge of this rationality. We show that the sets of rationalizable strategies are the maximal mutually nondominated sets, i.e., the maximal sets that contain no dominated strategies with respect to each other. We show that no new rationalizable strategies appear when additional preference information is included. We consider multicriteria games as a special case of games with incomplete preferences and introduce a way of representing incomplete preference information in multicriteria games by sets of feasible weights of the criteria.

Key— Normal-form games, incomplete preferences, rationalizable strategies, multicriteria games

1 Introduction

Perea [2014] concludes his review on epistemic game theory by stating that an important future research topic is to present novel natural collections of epistemic assumptions and characterize the choices resulting from such assumptions. In this paper, we take this approach in considering normal-form games with incomplete preferences as defined by Bade [2005]. In such a game, two or more players simultaneously select their strategies from fixed sets of available strategies. The combination of the selected strategies determines the outcome of the game. The preferences of the players are incomplete, i.e., the players are not able to state preferences between all pairs of outcomes. Bade [2005] showed that the Nash equilibria of this game correspond to the Nash equilibria of the completions of the game.

The game with incomplete preferences is also a generalization of ordinal games, where ordinal referes to the idea that there are no numeric values associated to the outcomes but only a preference order. For a discussion of ordinal games, see Durieu et al. [2008] and references therein. However, there preferences were assumed to be complete. In the recent literature, incomplete preferences have been formulated by Li [2013] [see Xie et al., 2013, for a generalized definition] as the so-called nonmonetized game. The nonmonetized game is a similar concept to the game of incomplete preferences of Bade [2005] except formulated slightly differently. In nonmonetized games, the incomplete preference order is defined in a common outcome space for all players while in the game with incomplete preferences, different preference orders for each player are defined directly over the strategy combinations.

Bernheim [1984] noted that equilibrium behavior is not a necessary consequence of rationality. Instead, the justifications of Nash equilibrium require that the players anticipate that the equilibrium solution will be selected. In the case of no common expectations formed by, e.g., pre-play communication, common knowledge of rationality and the game does not imply Nash equilibrium but rather that the players play rationalizable strategies [Bernheim, 1984, Pearce, 1984]. That is, strategies that can be justified with a hierarchy of beliefs consistent with common knowledge of rationality.

We assume that the incompleteness of the players' preferences is not only the perception of an outsider observer but that the players themselves have incomplete preferences related to the outcomes. Then, if the preferences of a player do not specify whether he prefers his equilibrium strategy to some other response to the other players' equilibrium strategies, the player has no clear incentive not to deviate from the equilibrium. Thus, the concept of equilibrium seems even more questionable in games with incomplete preferences and there is a strong motivation to study rationalizable strategies instead of equilibria.

In the game with incomplete preferences, we argue that one cannot reasonably assume any preferences over lotteries, as even the preferences over outcomes are not completely specified. Hence, rationality cannot be interpreted as the maximization of expected utility. Furthermore, forming a probabilistic belief about the strategies selected by the other players would seem to necessitate a probabilistic belief about complete preferences of the other players. If such beliefs exist, the game setting may be analyzed as a Bayesian game [Harsanyi, 1967] instead of a game with incomplete preferences.

Due to the reasons mentioned above, we apply a more robust concept of rationality in this paper. Players are not assumed to have probability distributions over the strategies of the opponents but only set-valued beliefs of possible strategies of the opponents. The players select nondominated strategies, i.e., a player does not select a strategy if another strategy yields a better outcome with all combinations of the strategies of the other players that he considers possible. Nothing else is assumed about, e.g., the risk attitudes of the players. Then, we define rationalizability with these set-valued beliefs.

We characterize the sets of rationalizable strategies in terms of mutual nondominance as follows. We define mutual nondominance so that sets of strategies are mutually nondominated if they contain no dominated strategies with respect to each other. We show that the sets of rationalizable strategies are the maximal mutually nondominated sets.

We consider the effect of additional preference information in the following sense. All original preferences are maintained and new preferences are added over some pairs of outcomes for which a player was previously indecisive. These new preferences are assumed to be consistent with the original preferences, i.e., maintain transitivity and irreflexivity. We show that all rationalizable strategies of the modified game are rationalizable strategies of the original game. Therefore, the incompleteness of preference information does not cause excluding any strategies that would be contained in the solutions with more complete preference information. On the other hand, additional preference information may lead to a more accurate solution in the sense that some strategies contained in the solution of the original game are excluded.

In addition to the aforementioned contributions, we consider multicriteria games [Shapley, 1959, Blackwell, 1956, Corley, 1985, Borm et al., 1988, de Marco and Morgan, 2007] as a special case of games with incomplete preferences. In the multicriteria games, the preferences are modeled by representing the outcomes as pascyoff vectors, where each component of a payoff vector describes the goodness of the corresponding outcome with respect to a particular criterion. The main solution concept in the existing literature on multicriteria games is the multicriteria extension of Nash equilibrium [e.g. Shapley, 1959, Corley, 1985, Borm et al., 1988, de Marco and Morgan, 2007]. A combination of strategies is an equilibrium if the strategy of each player is nondominated when other players play the equilibrium strategies. Nondominance here means that no other strategy yields a payoff that is at least as good with respect to all criteria and better with respect to at least one criterion. This equilibrium concept does not take into account information about relative importance of criteria. In preference programming [Salo and Hämäläinen, 1995], incomplete preference information in the context of multicriteria decision problems is represented by a set of feasible weights of an additive value function. The weights represent the relative importance of the criteria. A decision alternative is preferred to another if it is at least as good with all the weights in the set of the feasible weights and better with some weights in the set of the feasible weights. We apply this technique of representing incomplete information about preferences to multicriteria games. While Monroy et al. [2009] considered sets of feasible weights in a bargaining setting, to our knowledge, sets of feasible weights have not been applied to noncooperative multicriteria games. This novel idea enables analyzing the impact of additional preference information about the relative importance of the criteria to the solutions of the multicriteria games.

The paper is structured as follows. Games with incomplete preferences as well as the rationality concept are defined in Section 2, and rationalizable strategies are defined in Section 3. Section 4 provides properties of the sets of rationalizable strategies. First, the characterization in terms of mutual dominance is given in Section 4.1. Then, the relation between rationalizable strategies and the iterative elimination of dominated strategies is discussed in Section 4.2. The existence of the rationalizable strategies in the case of finite strategy sets and possible nonexistence in the infinite case are shown in Section 4.3. The result that adding preference information does not lead to additional rationalizable strategies is shown in Section 4.4. Multicriteria games with incomplete preference information are discussed in Section 5. Examples of a game with incomplete preferences with a finite number of strategies as well as of a multicriteria game with an infinite number of strategies are given in Section 6. Finally, concluding remarks are presented in Section 7.

2 Game with Incomplete Preferences

2.1 Elements of the game

Definition 1. *A game with incomplete preferences [Bade, 2005] consists of the following components:*

- *The set of players* $I = \{1, \dots, n\}$.
- *For each player* $i \in I$*, the set of strategies* S_i.
- *For each player* $i \in I$*, the preference relation: a transitive and irreflexive binary relation* $\succ_i$ *defined on* $S = S_1 \times \dots \times S_n$.

The players are assumed to select their strategies simultaneously. The combination of the selected strategies then implies the outcome of the game, and thus the outcome is formally defined as the combination of the selected strategies. Player $i \in I$ prefers the outcome implied by strategies $(s_1, \dots, s_n) \in S$ to the outcome implied by strategies $(s'_1, \dots, s'_n) \in S$, if $(s_1, \dots, s_n) \succ_i (s'_1, \dots, s'_n)$. The preference relations are assumed to be transitive and irreflexive, but may be incomplete. That is, the players may be indecisive over some outcomes or it may not be commonly known which outcomes the players prefer.

2.2 Rationality Concept

We do not assume that the players possess probabilistic assessments about what strategies the opponents may select. Instead, we assume only that a player holds a set-valued belief of possible selections of strategies taken by the opponents. With no probabilities, the only assumption that can be made about preferences over strategies is that the rational players do not select strategies that lead to worse outcomes no matter which strategies the opponents select among those that are possible according to the belief. Thus, we take rationality to mean that the players select nondominated strategies with respect to their beliefs. The set of nondominated strategies for player i with belief $B_i \subseteq S_{-i}$ is[1] hereafter denoted by $\mathsf{ND}(i, B_i)$ and defined as

$$\mathsf{ND}(i, B_i) = \{s_i \in S_i \mid \nexists s'_i \in S_i : \forall s_{-i} \in B_i : (s'_i, s_{-i}) \succ_i (s_i, s_{-i})\}. \tag{1}$$

A useful property of ND is that all strategies nondominated with respect to a belief remain nondominated if the belief is replaced with another belief containing additional possible strategies of the opponents. This is formalized in the following remark.

Remark 1. *Let A and B be two beliefs of player i, and $A \subseteq B$. Then, $\mathsf{ND}(i, A) \subseteq \mathsf{ND}(i, B)$.*

Our rationality concept formulated in this section can be seen as a relaxation of a similar rationality concept based on set-valued beliefs, viz. rationality* defined by Chen et al. [2007]. Here, the difference is that we allow the preference relation to be incomplete.

3 Definition of Rationalizable Strategies

Bernheim [1984] motivates the concept of rationalizable strategies as the logical conclusion of assuming that the players view the selections of the

[1] We use the notational convention that S_{-i} denotes the $(n-1)$-tuple of sets of strategies S_k for $k \neq i$. Expressions such as (s_i, s_{-i}) are implicitly assumed to refer to the tuple $(s_1, \ldots, s_n)$ in the correct order.

opponents' strategies as uncertain events, that the players follow Savage's axioms of individual rationality, and that the latter as well as the game (i.e., the strategies and the payoffs) are common knowledge. In this paper, we assume selection of nondominated strategies in the sense of Section 2.2 instead of Savage rationality. Furthermore, in our case, the game being common knowledge means that the strategies and the preference relations are common knowledge. The definition of rationalizable strategies used here is obtained by modifying the definition in Bernheim [1984] accordingly.

The following notation is taken from Bernheim [1984]. Let Δ_I^i be the set of sequences $(i_1, \ldots, i_m)$, $i_j \in I$ for $1 \leq j \leq m$, where $1 \leq m < \infty$, $i_1 \neq i$, and $i_j \neq i_{j+1}$ for all $1 \leq j \leq m$. Denote the last element of $\delta \in \Delta_I^i$ by $l(\delta)$, and a sequence formed by adding j to the end of δ by $\delta + (j)$. Similarly, a sequence formed by concatenating δ_1 and δ_2 is denoted by $\delta_1 + \delta_2$.

Definition 2. *A mapping $\Theta : \Delta_I^i \to P(S_1) \cup \ldots \cup P(S_n)$, where $P(S_k)$ denotes the power set of S_k, is called a system of beliefs for player i iff $\forall \delta \in \Delta_I^i : \ \Theta(\delta) \subseteq S_{l(\delta)}$.*

The interpretation is that for $\delta = (i_1, \ldots, i_m)$, $\Theta(\delta)$ is the set of strategies s for which player i believes that i_1 may believe that i_2 may believe that ... that i_{m-1} may believe that i_m may select s. Naturally, such strategies must belong to the set of strategies of player i_m, i.e., $\Theta(\delta)$ is a subset of $S_{l(\delta)}$. Bernheim [1984] requires $\Theta(\delta)$ to be a Borel subset. In our nonprobabilistic framework, no such assumption is necessary and thus the systems of beliefs are allowed to contain any subsets of the strategy sets S_k.

Common knowledge of rationality implies that i believes that i_1 believes that ... i_{m-1} believes that i_m is rational. Hence, the strategies that i believes that i_1 believes that ... i_{m-1} believes that i_m may select must be nondominated with respect to the strategies that i believes that i_1 believes that ... i_{m-1} believes that i_m believes that his opponents may select. This implies the following consistency condition for the systems of beliefs.

Definition 3. *A system of beliefs of player i, denoted by Θ, is consistent iff $\forall \delta \in \Delta_I^i$*

$$\Theta(\delta) \subseteq \mathsf{ND}(l(\delta), \times_{j \neq l(\delta)} \Theta(\delta + (j))). \tag{2}$$

Bernheim [1984] requires that each strategy in $\Theta(\delta)$ is a best response to some probability distribution over the strategies of $l(\delta)$'s opponents, where only strategies that i believes that i_1 believes that i_2 believes that ... that $l(\delta)$

considers possible may have nonzero probability. Our definition captures the idea without probability distributions and with nondominance instead of best responses. Finally, the rationalizable strategies of player i are the strategies that are nondominated with respect to some consistent system of beliefs for player i, which leads to the following definition.

Definition 4. *Strategy $s_i \in S_i$ is rationalizable iff a consistent system of beliefs Θ of player i exists such that*

$$s_i \in \mathsf{ND}(i, \times_{j \neq i} \Theta((j))). \tag{3}$$

The set of rationalizable strategies for player i is hereafter denoted by S_i^{R}. The following remark shows that the strategies contained in a consistent system of beliefs are rationalizable.

Remark 2. *Let Θ be a consistent system of beliefs for player i and $\delta \in \Delta_I^i$. Then, $\Theta(\delta) \subseteq S_{l(\delta)}^{\mathsf{R}}$.*

Proof. If $\delta' \in \Delta_I^{l(\delta)}$, then $\delta + (l(\delta)) + \delta' \in \Delta_I^i$. Therefore, we may define a system of beliefs Θ' for player $l(\delta)$ as follows:

$$\Theta'(\delta') = \Theta(\delta + (l(\delta)) + \delta'). \tag{4}$$

Consistency of Θ then implies consistency of Θ' as well as that the condition of Eq. 3 is fulfilled for any $s_j \in \Theta(\delta)$. Thus, any such s_j is rationalizable. $\square$

4 Properties of Rationalizable Strategies

In this section, we first characterize the sets of rationalizable strategies as the largest mutually nondominated sets of players' strategies. Then, we show that the iterative elimination of dominated strategies never removes rationalizable strategies. Furthermore, in the case of finite strategy sets, the iterative elimination converges exactly to the rationalizable strategies, which in turn implies the existence of rationalizable strategies in the finite case.

These results are close in spirit to those of Chen et al. [2007] who showed that their iterative elimination concept, IESDS*, converges to the largest stable set with respect to dominance, which in turn is the implication of common knowledge of rationality* as defined in their paper. We give independently developed proofs of our results since our rationality concept differs from rationality* in that we allow the preferences to be incomplete.

Note that while IESDS* uses uncountably infinite number of iterations, we focus only on countable iteration, since our main motivation for studying the iterative elimination is to show the existence of rationalizable strategies in the finite case, and to use it as a practical algorithm for actually finding the rationalizable strategies in games.

In Section 4.4, we formulate and prove the result that adding preference information to the preference relations does not lead to new rationalizable strategies. This result is novel and has no correspondence in the rationality* framework of Chen et al. [2007] since they do not consider incomplete preferences.

4.1 Characterization

The sets of rationalizable strategies are characterized here in terms of mutually nondominated subsets. We define mutually nondominated subsets as subsets consisting of strategies that are nondominated with respect to belief that the opponents select from the same mutually nondominated sets. This concept is an adaptation of the best response property used by Bernheim [1984] to the rationality concept used in this paper. In the following, we show 1) that the sets of rationalizable strategies must be mutually nondominated, 2) that any mutually nondominated sets are subsets of the rationalizable strategies, and 3) that the union of all mutually nondominated sets is mutually nondominated. This implies that the set of rationalizable strategies is the union of all mutually nondominated sets. In other words, the rationalizable strategies comprise the maximal mutually nondominated set.

Definition 5. *An n-tuple of sets of strategies for each player $(S'_1, \dots, S'_n)$ is mutually nondominated, denoted by $(S'_1, \dots, S'_n) \in \mathsf{MND}$, if*

$$\forall i \in I: \; S'_i \subseteq \mathsf{ND}(i, S'_{-i}). \tag{5}$$

Lemma 1. *The sets of rationalizable strategies are mutually nondominated, i.e., $S^{\mathsf{R}} = (S^{\mathsf{R}}_1, \dots, S^{\mathsf{R}}_n) \in \mathsf{MND}$.*

Proof. All strategies contained in consistent systems of beliefs must be rationalizable (Remark 2). On the other hand, a rationalizable strategy must be nondominated with respect to strategies contained in a consistent system of beliefs. Therefore, a rationalizable strategy is nondominated with respect to a subset of the rationalizable strategies. Based on Remark 1, this implies that a rationalizable strategy is nondominated with respect to the rationalizable strategies. Hence, the sets of rationalizable strategies are mutually nondominated. $\square$

Lemma 2. *All strategies contained in an n-tuple of mutually nondominated sets of strategies are rationalizable, i.e.,*
$\forall (S'_1, \ldots, S'_n) \in \mathsf{MND}: \ S'_1 \subseteq S^{\mathsf{R}}_1, \ldots, S'_n \subseteq S^{\mathsf{R}}_n.$

Proof. Define the following systems of beliefs for all players: $\forall \delta \in \Delta^i_I: \ \Theta(\delta) = S'_{l(\delta)}$. Mutual nondominance implies consistency of Θ and that Θ rationalizes all strategies in $(S'_1, \ldots, S'_n)$. $\square$

Definition 6. *We denote by $\bigcup \mathsf{MND}$ the ordered n-tuple*
$(\bigcup \mathsf{MND}_1, \bigcup \mathsf{MND}_2, \ldots, \bigcup \mathsf{MND}_n)$ *such that for all $i \in I$,*
$\bigcup \mathsf{MND}_i = \bigcup \{S'_i \mid \exists S' = (S'_1, \ldots, S'_n) \in \mathsf{MND}\}$*. Note that this is an abuse of notation since $\bigcup \mathsf{MND}$ is not technically the union of MND but the tuple of the unions of the components of the members of MND.*

Lemma 3. $\bigcup \mathsf{MND} \in \mathsf{MND}$.

Proof. For any strategy s_i in $\bigcup \mathsf{MND}$, there exists a $S' \in \mathsf{MND}$ such that $s_i \in S'_i$. By the definition of MND, this implies

$$s_i \in \mathsf{ND}(i, S'_{-i}). \tag{6}$$

Furthermore, ND has the property that all nondominated strategies remain nondominated if the belief set is replaced with its superset. Therefore,

$$s_i \in \mathsf{ND}(i, \bigcup \mathsf{MND}_{-i}), \tag{7}$$

and thus $\bigcup \mathsf{MND}$ satisfies Definition 5. $\square$

Theorem 1. $S^{\mathsf{R}} = \bigcup \mathsf{MND}$.

Proof. Lemma 3 states that $\bigcup \mathsf{MND} \in \mathsf{MND}$. Hence, Lemma 2 implies that $\bigcup \mathsf{MND} \subseteq S^{\mathsf{R}}$. On the other hand, Lemma 1 states that $S^{\mathsf{R}} \in \mathsf{MND}$. Hence, $S^{\mathsf{R}} \subseteq \bigcup \mathsf{MND}$. Therefore, $S^{\mathsf{R}} = \bigcup \mathsf{MND}$. $\square$

4.2 Iterative elimination of dominated strategies

Next, we show that the iterative elimination of dominated strategies never removes rationalizable strategies. If the strategy sets are finite, the iterative elimination of dominated strategies converges to the rationalizable strategies. However, in the case of infinite strategy sets, nonrationalizable strategies may survive the iterative elimination.

Definition 7. *We define the sets of strategies surviving k steps of the iterative elimination recursively as follows. $\forall i \in I : S_i^0 = S_i$ and*

$$\forall i \in I, k \in \mathbb{N} : \; S_i^{k+1} = \mathsf{ND}(i, S_{-i}^k). \tag{8}$$

Note that clearly $\forall i, k : S_i^{k+1} \subseteq S_i^k$. Then, the strategies surviving the iterative elimination of dominated strategies are

$$\forall i : \; S_i^\infty = \bigcap_{k \in \mathbb{N}} S_i^k. \tag{9}$$

Theorem 2. *All rationalizable strategies survive the iterative elimination of dominated strategies, i.e., $S_i^{\mathsf{R}} \subseteq S_i^\infty$.*

Proof. We show by induction that the set of rationalizable strategies of player i is a subset of the strategies surviving k steps of the iterative elimination for all k. Initially, $S_i^0 = S_i$ and thus $S_i^0 \subseteq S_i^{\mathsf{R}}$. Assume that $\forall i \in I : S_i^{\mathsf{R}} \subseteq S_i^k$. The sets of rationalizable strategies are mutually nondominated and thus also nondominated with respect to S_{-i}^k. Hence $\forall i \in I : \; S_i^{\mathsf{R}} \subseteq S_i^{k+1}$. □

Theorem 3. *If the strategy sets S_i are finite, $\exists K : \; k > K \Rightarrow \forall i \; S_i^k = S_i^{\mathsf{R}}$.*

Proof. The strategy sets S_i are finite and thus strategies can be removed from them for only a finite number of times. Therefore, $\exists K : \; \forall k > K : \forall i \; S_i^k = S_i^{k+1}$. Assume $k > K$. Then, because $\forall i \; S_i^k = S_i^{k+1}$, the sets S_i^k are mutually nondominated, and therefore by Lemma 2, $\forall i \; S_i^k \subseteq S_i^{\mathsf{R}}$. On the other hand, according to Theorem 2, $\forall i \; S_i^{\mathsf{R}} \subseteq S_i^\infty$. Therefore, $\forall i, \forall k > K : \; S_i^k = S_i^{\mathsf{R}}$. □

Remark 3. *There exists a game with incomplete preferences for which $S_i^R \neq S_i^\infty$.*

Proof. Consider the following game with two players where the players pick numbers from the set of natural numbers extended with a "small infinity"

(∞') and a "big infinity" (∞). The player who selects the largest number wins. A player always prefers a win to a tie and a tie to a loss. Furthermore, between two losing outcomes, the players prefer ones where they select larger numbers. However, the players have no preference between two winning outcomes. Formally, the sets of strategies are $S_1 = S_2 = \{0, 1, 2, \ldots, \infty', \infty\}$ and the preference relation of player 1, $\succ_1$, is defined so that $(n_1, n_2) \succ_1 (m_1, m_2)$ if and only if

- $m_1 < m_2$ and $n_1 \geq n_2$, or
- $m_1 = m_2$ and $n_1 > n_2$, or
- $m_1 < m_2$ and $n_1 > m_1$,

where the relations $>$ and $<$ are extended so that $\forall k \in \mathbb{N} : \ \infty, \infty' > k$ and $\infty > \infty'$. The iterative elimination removes at each step the smallest number from the remaining strategy sets of both players, and thus

$$\forall k \in \mathbb{N} : \ S_1^k = S_2^k = \{k, k+1, \ldots, \infty', \infty\}, \tag{10}$$

and therefore $S_1^\infty = S_2^\infty = \{\infty', \infty\}$. However, these sets are not mutually nondominated as only ∞ is nondominated with respect to the belief that the opponent selects from $\{\infty', \infty\}$. □

4.3 Existence

Here, we show in Theorem 4 that when the strategy sets are finite, the sets of rationalizable strategies are nonempty. However, in the case of infinite strategy sets, the existence of rationalizable strategies is not guaranteed, which is illustrated in Remark 4.

Theorem 4. *If the strategy sets are finite, the sets of rationalizable strategies are nonempty.*

Proof. Theorem 3 implies that the iterative elimination of dominated strategies converges to the set of rationalizable strategies. When the strategy sets are finite, a step of the iterative elimination of dominated strategies will never remove all strategies. Hence, the sets of rationalizable strategies are nonempty. □

Remark 4. *When the strategy sets are allowed to be infinite, the sets of rationalizable strategies may be empty.*

Proof. Consider the game discussed in Remark 3 with the strategies ∞ and ∞' removed. For this game,

$$\forall k: \; S_1^k = S_2^k = \{k, k+1, \ldots\}, \tag{11}$$

and thus $S_1^\infty = S_2^\infty = \emptyset$. Then, by applying Theorem 2 one can conclude that $S_1^{\mathsf{R}} = S_2^{\mathsf{R}} = \emptyset$. $\square$

4.4 Effect of additional preference information

Next, we consider the effect of taking into account additional information about the preferences of the players. That is, how the sets of rationalizable strategies change when new preference relations $\succ_i'$ are defined such that $(s_1, \ldots, s_n) \succ_i (s_1', \ldots, s_n') \Rightarrow (s_1, \ldots, s_n) \succ_i' (s_1', \ldots, s_n')$, where $\succ_i$ refers to the original preference relations. We denote the dependence of the rationalizable strategies and mutually nondominated sets on the preference relations by $S^{\mathsf{R}}(\succ_1, \ldots, \succ_n)$, $\mathsf{MND}(\succ_1, \ldots, \succ_n)$.

Theorem 5. *Assume that $\succ$ and $\succ'$ are two preference relations for the players such that $\forall i \in I, \forall s, s' \in S : s \succ_i s' \rightarrow s \succ_i' s'$. Then, $S^{\mathsf{R}}(\succ_1', \ldots, \succ_n') \subseteq S^{\mathsf{R}}(\succ_1, \ldots, \succ_n)$.*

Proof. According to Lemma 1, $S^{\mathsf{R}}(\succ_1', \ldots, \succ_n') \subseteq \mathsf{MND}(\succ_1', \ldots, \succ_n')$, i.e.,

$$\begin{aligned} \forall i \in I : & \forall s_i \in S_i^{\mathsf{R}}(\succ_1', \ldots, \succ_n') : \; \nexists s_i' \in S_i : \\ & \quad \forall s_{-i} \in S_{-i}^{\mathsf{R}}(\succ_1', \ldots, \succ_n') : (s_i', s_{-i}) \succ_i' (s_i, s_{-i}). \end{aligned} \tag{12}$$

Then, the assumption $s \succ_i s' \rightarrow s \succ_i' s'$ implies that the nonexistence in Eq. 12 extends to $\succ$ so that

$$\begin{aligned} \forall i \in I : & \forall s_i \in S_i^{\mathsf{R}}(\succ_1', \ldots, \succ_n') : \; \nexists s_i' \in S_i : \\ & \quad \forall s_{-i} \in S_{-i}^{\mathsf{R}}(\succ_1', \ldots, \succ_n') : (s_i', s_{-i}) \succ_i (s_i, s_{-i}). \end{aligned} \tag{13}$$

Thus, $S^{\mathsf{R}}(\succ_1', \ldots, \succ_n') \in \mathsf{MND}(\succ_1, \ldots, \succ_n)$. Lemma 2 then implies $S^{\mathsf{R}}(\succ_1', \ldots, \succ_n') \subseteq S^{\mathsf{R}}(\succ_1, \ldots, \succ_n)$. $\square$

The above result can be interpreted as follows. The new preference relations $\succ'$ are more accurate than $\succ$. Here, more accurate is understood to mean that while all preference statements composing the original incomplete preference information are correct, the more accurate information contains additional preferences between outcomes that were incomparable according to the original information. Theorem 5 then shows that incomplete preference information will not cause the exclusion of strategies that might be selected with more accurate information about the preferences. On the other hand, more accurate preference information may lead to ruling out more strategies.

However, there may exist rationalizable strategies that are always ruled out when enough additional preference information is added, no matter which additional preference information is added. This shown in the following remark.

Remark 5. *There exists a game with incomplete preferences that has a rationalizable strategy that is not a rationalizable strategy in any game where the preference relations are completed, that is, where the partial orders $\succ$ are replaced with total orders that are completions of them.*

Proof. Consider a game where $S_1 = \{1, 2, 3\}, S_2 = \{1, 2\}$ and the preferences of player 1 consist of $(1, 2) \succ_1 (1, 1) \succ_1 (3, 1)$, $(2, 1) \succ_1 (2, 2) \succ_1 (3, 2)$, see Fig. 1. Assume that no preferences are defined for player 2 and thus $S_2^{\mathsf{R}} = \{1, 2\}$. All strategies in S_1 are nondominated, and thus, $S_1^{\mathsf{R}} = S_1$. Now, let $\succ_1'$ be an arbitrary completion of $\succ_1$. Either $(3, 1) \succ_1' (2, 1)$ or $(2, 1) \succ_1' (3, 1)$. First, assume that $(3, 1) \succ_1' (2, 1)$. Then, $(1, 1) \succ_1' (3, 1)$ and $(1, 2) \succ_1' (3, 2)$. Therefore, strategy 3 of player 1 is dominated by strategy 1 and thus is not rationalizable. On the other hand, if $(2, 1) \succ_1' (3, 1)$, strategy 3 is dominated by strategy 2. Hence, strategy 3 is not rationalizable with any completion of the preferences of player 1. $\square$

5 Multicriteria Games

Multicriteria games [Shapley, 1959, Blackwell, 1956, Corley, 1985, Borm et al., 1988, Ghose and Prasad, 1989, Zhao, 1991, de Marco and Morgan, 2007, Monroy et al., 2009] are games where players evaluate outcomes according to several criteria and thus the outcomes correspond to vector-valued payoffs.

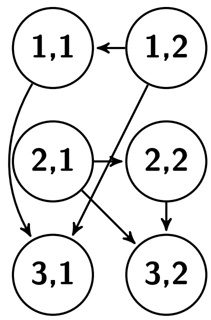

Figure 1: Preferences of player 1 in Remark 5. An arrow from outcome A to outcome B refers to $A \succ_1 B$.

If one outcome is better than another with respect to one criterion but worse with respect to another criterion, a player may not be able to state his/her preference between these outcomes. Therefore, the following multicriteria game is naturally a game with incomplete preferences.

Definition 8. *A multicriteria game is a game with incomplete preferences (c.f. Definition 1) destned so that*

- *For each player $i \in I$, a vector-valued payoff function: $f^i(s): S \to \mathbb{R}^m$, where $s = (s_1, \ldots, s_n)$, is given.s*
- *The preferences of the players are defined so that $s \succ_i s'$ iff*

$$\exists k \in \{1, \ldots, m\} : f^i_k(s) > f^i_k(s'), \tag{14}$$

$$\forall k \in \{1, \ldots, m\} : f^i_k(s) \geq f^i_k(s'). \tag{15}$$

Multicriteria games have been analyzed in the literature mainly from the point of view of equilibrium strategies [e.g. Shapley, 1959, Corley, 1985, Borm et al., 1988, de Marco and Morgan, 2007]. Ghose and Prasad [1989] considered additionally solutions based on so called security strategies, and Zhao [1991] discussed cooperative solutions. Since the multicriteria games are a special case of games with incomplete preferences, rationalizable strategies and the rationality concept elaborated in this paper can be applied to the multicriteria games as well.

5.1 Incomplete preference information

In the existing literature [e.g. Corley, 1985, Borm et al., 1988], information about the relative importance of criteria has been taken into account by introducing weight vectors so that each component of the payoff vectors is weighted according to its relative importance. A multicriteria game is then turned into a scalar game via multiplying the payoff vectors by these weight vectors. However, defining the weights would require complete information about the preferences of the players. Monroy et al. [2009] mentioned the possibility of using incomplete information about the weights, in form of inequality constraints, to obtain a unique bargaining solution. Preference programming [Salo and Hämäläinen, 1995] is a similar idea in the multicriteria decision analysis literature. In preference programming, incomplete preference information of a decision maker is represented by a set of feasible weights. In this paper, we apply the concept of preference programming into multicriteria games as follows. The preferences of player i are described by a set of feasible weights $W_i \subseteq W^0$, where $W^0 = \{w \in \mathbb{R}^m : \ w_k \geq 0, \ \sum w_k = 1\}$. Here, the kth component of a weight vector w describes the relative importance of the kth criterion. The player prefers an outcome to another if it is better with at least some weight vector in the set of feasible weights and at least as good with all weights in the set of feasible weights. This leads to the following game:

Definition 9. *A multicriteria game with incomplete preference information is a game with incomplete preferences (c.f. Definition 1) defined so that for each player $i \in I$:*

- *A vector valued payoff function: $f^i(s): \ S \to \mathbb{R}^m$ is given.*
- *The set of feasible weights $W_i \subset W^0 = \{w \in \mathbb{R}^m \ : \ w_k \geq 0, \sum w_k = 1\}$ is given.*
- *The preference relation $\succ_i$ is defined so that $s \succ_i s'$ iff*

$$\exists w \in W_i : w^\mathsf{T} f^i(s) > w^\mathsf{T} f^i(s'), \tag{16}$$
$$\forall w \in W_i : w^\mathsf{T} f^i(s) \geq w^\mathsf{T} f^i(s'). \tag{17}$$

Note that $w^\mathsf{T} f^i$ is linear in w and thus W_i can be replaced with the set of the extremal points of W_i in Eqs. (16-17).

Note that the relation defined by Eqs. (16-17) is irreflexive and transitive and thus Definition 9 indeed defines a game with incomplete preferences following Definition 1. Furthermore, the multicriteria game of Definition 8 is a special case of Definition 9 where the sets of feasible weights are equal to the set of all possible weights W^0.

In preference programming, additional preference information is treated by constraining the set of feasible weights, i.e., replacing W_i with $W_i' \subseteq W_i$. A known result [see, e.g. Liesiö et al., 2007] is that limiting the set of the feasible weights tightens the preference relation in the sense of the premise of Theorem 5 under a certain technical condition shown in Lemma 4 below.

Lemma 4. *Assume that a new multicriteria game with incomplete preference information is formed from an original multicriteria game with incomplete preference information so that the original weight sets W_i are replaced with weight sets W_i' so that $W_i' \subseteq W_i$ and $int(W_i) \cup W_i' \neq \emptyset$. Denote by $\succ_i$ the preference relations of the original game and by $\succ_i'$ the preference relations of the new game. Then, for any pair of outcomes such that player i prefers $(s_1, \ldots, s_n)$ over $(s_1', \ldots, s_n')$ in the original game, he/she has the same preference in the new game. That is, $(s_1, \ldots, s_n) \succ_i (s_1', \ldots, s_n') \rightarrow (s_1, \ldots, s_n) \succ_i' (s_1', \ldots, s_n')$.*

Then, Theorem 5 can be applied to multicriteria games with incomplete preference information as follows.

Theorem 6. *If a new multicriteria game with incomplete preference information is formed from an original multicriteria game with incomplete preference information so that the original weight sets W_i are replaced with weight sets W_i' so that $W_i' \subseteq W_i$ and $int(W_i) \cup W_i' \neq \emptyset$, the sets of rationalizable strategies of the new game are subsets of the sets of rationalizable strategies of the original game.*

Proof. The result is directly implied by Lemma 4 and Theorem 5. $\square$

6 Examples

In this section, we present two examples. The first one is a game with a finite number of strategies that is solved by the iterative elimination of dominated strategies. The second example deals with a multicriteria game containing an infinite number of strategies that is solved by deriving equations for the

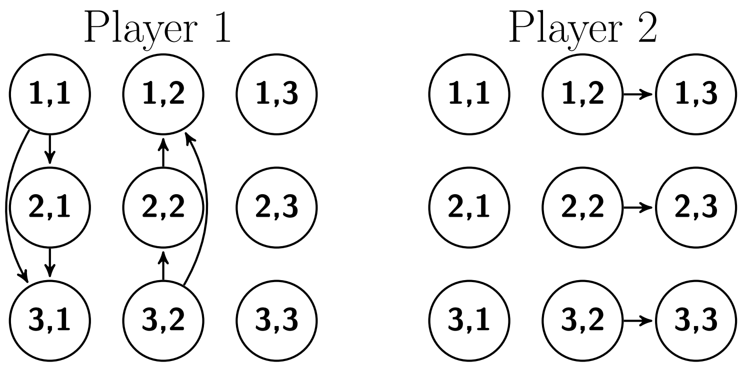

Figure 2: Preferences in the first example. The graph on the left represents the preferences of player 1 and the graph on the right the preferences of player 2, respectively. An arrow from outcome A to outcome B refers to $A \succ B$.

maximal mutually nondominated sets. The effect of adding preference information by limiting the set of feasible weights is also illustrated.

6.1 Game with finite strategy sets

Consider a game with two players having strategy sets $S_1 = S_2 = \{1, 2, 3\}$ and the preference relations

- $\succ_1$: $(1,1) \succ_1 (2,1) \succ_1 (3,1),\ (3,2) \succ_1 (2,2) \succ_1 (1,2),$
- $\succ_2$: $(1,2) \succ_2 (1,3),\ (2,2) \succ_2 (2,3),\ (3,2) \succ_2 (3,3).$

These relations are illustrated in Fig. 2. The strategy sets are finite and thus the rationalizable strategies can be found by the iterative elimination of dominated strategies, as shown in Section 4.2. For player 2, strategy 2 yields a preferred outcome to strategy 3 no matter which strategy player 1 selects. For player 1, no strategies are dominated. Thus, $S_1^1 = \{1, 2, 3\},\ S_2^1 = \{1, 2\}$. Both remaining strategies of player 2 are nondominated with respect to S_1^1 as player 2 has no preferences defined for the relevant outcomes. For player 1, all strategies are nondominated as the preference order of the outcomes is reversed when the strategy of player 2 is switched. Thus, as all remaining strategies of both players are nondominated, the rationalizable strategies of the game are $S_1^{\mathsf{R}} = \{1, 2, 3\}$ and $S_2^{\mathsf{R}} = \{1, 2\}$. Note that strategy 2 of player 1 is not nondominated with respect to any singleton belief among the

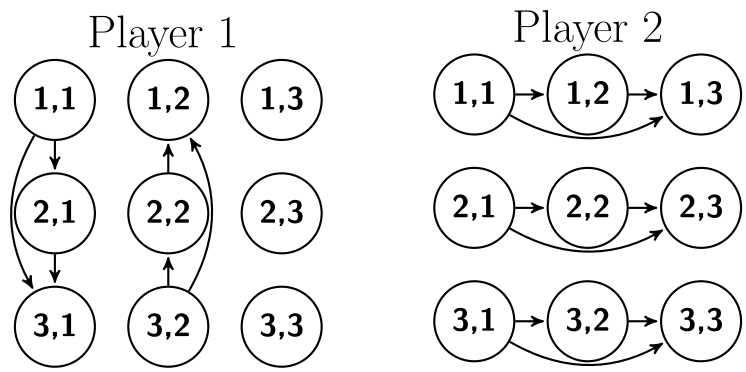

Figure 3: Preferences in the first example with additional preference information. The graph on the left represents the preferences of player 1 and the graph on the right the preferences of player 2, respectively. An arrow from outcome A to outcome B refers to $A \succ B$.

rationalizable strategies of player 2 since it is dominated by strategy 1 if player 2 selects strategy 1 and by strategy 3 if player 2 selects strategy 2. However, strategy 2 of player 1 is rationalized by the belief that player 2 may select either strategy 1 or strategy 2.

Now, let us incorporate additional preference information to the game. Define $\succ_2'$ to consist of $\succ_2$ and the additional preferences $(1,1) \succ_2' (1,2)$, $(1,1) \succ_2' (1,3)$, $(2,1) \succ_2' (2,2)$, $(2,1) \succ_2' (2,3)$, $(3,1) \succ_2' (3,2)$, and $(3,1) \succ_2' (3,3)$. The new preference relations are illustrated in Fig. 3. Now, for player 2, strategy 2 is dominated by strategy 1 and the only nondominated strategy is 1. Therefore, $S_2^{\mathsf{R}} = \{1\}$. When player 2 is known to select strategy 1, the only nondominated response of player 1 is strategy 1. Thus, the rationalizable strategies of the new game are $S_1^{\mathsf{R}} = \{1\}$ and $S_2^{\mathsf{R}} = \{1\}$. Note that the new rationalizable strategies are subsets of the rationalizable strategies of the original game, which is implied by Theorem 5.

6.2 Multicriteria game with infinite strategy sets

A bicriteria Cournot game is discussed by Bade [2005] from the point of view of equilibrium strategies. In this game setting, there are n players representing firms that select produced quantities simultaneously. Denote the quantity selected by player i by s_i. Assume that the market clearing price is $2 - \sum s_i$ and that the unit cost of production is 1. The profit for player i is $s_i\,(1-\sum_{k\neq i} s_k) - s_i^2$. Besides profits, the firms desire to maximize sales as long as profits are nonnegative. This can be expressed as a bicriteria game where the strategy sets are $\forall i \in I = \{1,\ldots,n\}:\ S_i = [0,\infty)$, and the payoff vectors are

$$\forall i \in I:\ f_i(s_1,\ldots,s_n) = \begin{pmatrix} s_i\,(1-\sum_{k\neq i} s_k) - s_i^2 \\ \min(s_i,\ 1-\sum_{k\neq i} s_k) \end{pmatrix}. \tag{18}$$

6.2.1 Two players

Here, we obtain the rationalizable stategies of the bicriteria Cournot game with 2 players by deriving the maximal mutually nondominated sets of the game. We consider the game first as a multicriteria game in the sense of Definition 8. Then, we consider additional incomplete preference information in the sense of Definition 9.

If the opponent selects strategy s_j, the payoff vector of player i is

$$f_i(s_i,s_j) = \left(s_i\,(1-s_j) - s_i^2,\quad \min(s_i, 1-s_j)\right). \tag{19}$$

For player i, any strategy $s_i > 1$ is dominated by $s_i = 1$ as the profits will be less than and the sales at most equal to what is obtained by selecting $s_i = 1$, no matter which strategy the opponent selects. Hence, no strategies $s_i > 1$ belong to any mutually nondominated sets. When player i believes the opponent may select $s_j \in [0,1]$, $1-s_j$ is a nondominated strategy since the sales criterion is lower with any strategy $s_i < 1-s_j$ and the profit criterion is lower with any strategy $s_i > 1-s_j$. This implies that $S_1' = [0,1]$ and $S_2' = [0,1]$ are mutually nondominated. As no strategies $s_i > 1$ belong to any mutually nondominated sets, it has been shown that $S_1^{\mathsf{R}} = [0,1]$ and $S_2^{\mathsf{R}} = [0,1]$.

Now, let us add preference information in the sense of Definition 9. Assume that one unit of profits is known to be at least as valuable as α units of sales for both firms. Thus, the sets of feasible weights are $W_1 = W_2 = \{w \in W^0 \mid w_2 \le \frac{1}{1+\alpha}\}$, where $W^0 = \{(w_1,w_2) \mid w_1, w_2 \ge 0, w_1+w_2 = 1\}$. The extreme points of these sets are $(1,0)$ and $(\frac{\alpha}{1+\alpha},\ \frac{1}{1+\alpha})$. Thus, the payoffs at the extreme points are

$$(1,\ 0)^{\mathsf{T}} f_i(s_i,s_j) = s_i\,(1-s_j) - s_i^2, \tag{20}$$

$$\left(\frac{\alpha}{1+\alpha},\ \frac{1}{1+\alpha}\right)^{\mathsf{T}} f_i(s_i,s_j) = \frac{\alpha\,(s_i\,(1-s_j)-s_i^2) + \min(s_i,\ 1-s_j)}{1+\alpha}. \tag{21}$$

When s_j is fixed, the value of Eq. 20 as a function of s_i is increasing when $s_i \in [0, \frac{1-s_j}{2}]$ and decreasing when $s_i > \frac{1-s_j}{2}$. The value of Eq. 21 as a function of s_i is increasing when $s_i \in \left[0, \min\left(\frac{1+\frac{1}{\alpha}-s_j}{2},\ 1-s_j\right)\right]$. Next, we argue that if the infimum of S'_j is $s_{\min}$ and the supremum of S'_j is $s_{\max}$, the set of nondominated strategies with respect to S'_j is

$$\mathsf{ND}(i, S'_j) = \left[\max\left(0, \frac{1-s_{\max}}{2}\right),\ \ \max\left(0, \min\left(\frac{1+\frac{1}{\alpha}-s_{\min}}{2},\ 1-s_{\min}\right)\right)\right]. \tag{22}$$

First, any $s_i < \max(0, \frac{1-s_{\max}}{2})$ is dominated by $\max(0, \frac{1-s_{\max}}{2})$ and any $s_i > \min\left(\frac{1+\frac{1}{\alpha}-s_{\min}}{2},\ 1-s_{\min}\right)$ is dominated by $\max\left(0, \min\left(\frac{1+\frac{1}{\alpha}-s_{\min}}{2},\ 1-s_{\min}\right)\right)$. Thus, any strategy outside the interval is dominated. On the other hand, for any s_i in the interval, all $s'_i < s_i$ lead to a lower value of Eq. 21 with some $s_j \in S'_j$ and all $s'_i > s_i$ result in a lower value of Eq. 20 with some $s_j \in S'_j$. Therefore, the nondominated strategies indeed consist of the interval.

For symmetry reasons, $S_1^{\mathsf{R}} = S_2^{\mathsf{R}}$ and thus it suffices to search for the maximal symmetric mutually nondominated sets. Eq. 22 implies that the maximal symmetric mutually nondominated sets are intervals $[s_{\min}, s_{\max}]$ that satisfy

$$\begin{cases} s_{\min} = \max\left(0, \frac{1-s_{\max}}{2}\right), \\ s_{\max} = \min\left(\frac{1+\frac{1}{\alpha}-s_{\min}}{2},\ 1-s_{\min}\right), \end{cases} \tag{23}$$

whose solution is

$$\begin{cases} s_{\min} = \max\left(\frac{1}{3} - \frac{1}{3\,\alpha},\ 0\right), \\ s_{\max} = \min\left(\frac{1}{3} + \frac{2}{3\,\alpha},\ 1\right). \end{cases} \tag{24}$$

To summarize, with no information about the relative importance of the criteria, all strategies between $[0, 1]$ are rationalizable. For example, $s_1 = 1,\ s_2 = 1$ is not an equilibrium [Bade, 2005], but in the absence of any equilibrium selection mechanism, it is possible that both players select strategy 1 unaware that also the opponent will select strategy 1. When the game is considered with the additional preference information that the players consider one unit of profits at least as valuable as α units of sales, if $\alpha \le 1$, the rationalizable strategies given by Eq. 24 are $S_1^{\mathsf{R}} = S_2^{\mathsf{R}} = [0, 1]$, i.e., the preference information does not change the rationalizable strategies. However, when $\alpha > 1$, the rationalizable strategies given by Eq. 24 are $S_1^{\mathsf{R}} = S_2^{\mathsf{R}} = [\frac{1}{3} - \frac{1}{3\,\alpha}, \frac{1}{3} + \frac{2}{3\,\alpha}]$. Increasing the value of the bound α corresponds to adding preference information to the game and as we have shown in Section 4.4, Theorem 5, no new rationalizable strategies appear. When $\alpha \to \infty$, i.e., profits become more important, the rationalizable strategies approach $S_1^{\mathsf{R}} = S_2^{\mathsf{R}} = \{\frac{1}{3}\}$, i.e., the equilibrium [Bade, 2005] and the rationalizable strategies [Bernheim, 1984] of the single-criterion profit-maximizing Cournot game.

6.2.2 Multiple players

With $n > 2$, the analysis of the bicriteria Cournot game is similar to the case with two players. When the opponents are believed to select strategies from $[s_{\min},\ s_{\max}]$, the total quantity produced by the opponents varies in $[(n-1)\, s_{\min}, (n-1)\, s_{\max}]$. Hence, the rationalizable strategies are obtained by solving the equations

$$\begin{cases} s_{\min} = \max\left(0,\ \frac{1-(n-1)s_{\max}}{2}\right), \\ s_{\max} = \min\left(\frac{1+\frac{1}{\alpha}-(n-1)\, s_{\min}}{2},\ 1-(n-1)\, s_{\min}\right). \end{cases} \tag{25}$$

With some values of n and α, these equations have multiple solutions. Since the rationalizable strategies are the largest mutually nondominated sets, the solution corresponding to the rationalizable strategies is the one that produces an interval that contains the intervals produced by possible other solutions. That is,

$$\begin{cases} s_{\min} = 0, \\ s_{\max} = \min\left(\frac{1+\frac{1}{\alpha}}{2},\ 1\right). \end{cases} \tag{26}$$

When $\alpha \leq 1$, all strategies in $[0, 1]$ are rationalizable as in the two-player case, wheras with $\alpha > 1$, the rationalizable strategies are $[0, \frac{1+\frac{1}{\alpha}}{2}]$. When $\alpha \to \infty$, the rationalizable strategies approach the rationalizable strategies of the single-criterion profit-maximizing Cournot game, i.e., $[0, \frac{1}{2}]$ [Bernheim, 1984].

7 Conclusions

In this paper, we considered normal-form games with incomplete preferences [Bade, 2005]. In terms of epistemic game theory [Perea, 2014], the approach of this paper was to consider the implications of common knowledge of rationality with a relaxed definition such that preferences over outcomes may be incomplete and the players do not possess probability distributions.

Rationalizable strategies [Bernheim, 1984, Pearce, 1984] have been proposed as an alternative solution concept to Nash equilibrium for situations where no equilibrium-driving process exists. We revised the concept of rationalizable strategies so that players are not required to have subjective probability distributions over the opponents' strategies nor utility functions over the outcomes of the game. Instead, the players are assumed only to have set-valued beliefs and to select nondominated strategies given these beliefs. In the case of incomplete preferences, we argue that these assumptions are easier to accept than the framework of expected utility maximization because information needed to define subjective probabilities and utilities may not exist.

We showed that no new rationalizable strategies appear in a game with incomplete preferences when preference information is added in the sense of adding new preferences over pairs of outcomes into the preference relations. Another interpretation of this result is that no rationalizable strategies disappear when preferences are relaxed in the sense of removing preferences over pairs of outcomes from the preference relations. Thus, the game can be analyzed using only such preference information that one is definitely willing to assume, with confidence that no strategies are wrongly ruled out.

We considered multicriteria games as a special case of games with incomplete preferences and introduced a way of adding preference information to the multicriteria games by modeling incomplete preferences with sets of feasible weights of the criteria, following the treatment in the literature on multicriteria decision analysis [e.g. Salo and Hämäläinen, 1995]. While the idea of constraining the set of feasible weights has previously been considered by Monroy et al. [2009] in a multicriteria game, they used it only in a cooperative context and assumed that all players have the same feasible weights. Besides multicriteria games, the game and solution concept developed in this paper could be applied to, for example, interval games [Levin, 1999], where payoffs are not known exactly, but only as intervals of possible values.

We showed that the sets of rationalizable strategies as defined in this paper are nonempty in the case of finite strategy sets. In the infinite case, nonemptiness is not guaranteed. However, the nonexistence of rationalizable strategies might be due to unreasonable structure of the preference relations. A possible topic for future research would be to define intuitively reasonable conditions on the preference relations that guarantee the existence of rationalizable strategies.

We assumed that the players themselves have incomplete preferences and thus they are allowed to select strategies that would be dominated with any completion of their preferences. Alternatively, if the incompleteness is only due to limited information of an external analyst, one could define rationality so that the players are required to select strategies that are nondominated with respect to some completion of their preferences.

Recently, Li [2015] generalized the nonmonetized game so that players may be indifferent between outcomes. The definition of games with incomplete preferences could similarly be extended so that the preference relations allow for indifference. This would necessitate a refinement of our rationality concept to take weak dominance into account [see Pearce, 1984]. Another possible extension would be to consider extensive-form games with incomplete preferences, which would require introducing appropriate restrictions on belief updating.

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