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Abstract: In this paper, a general equilibrium model is developed to analyze the determination of the equilibrium exchange rates. The model can deal with multiple types of moneys and moneys are integrated into the model through demand functions. When the endowments, preferences, production technologies and interest rates are given, the equilibrium exchange rates, equilibrium prices, equilibrium outputs and equilibrium utility levels can be computed by the model. A numerical example of a two-country economy is also presented to illustrate the model.
Keywords: exchange rate; general equilibrium; multi-country economy; CGE model

Ⅰ. Introduction

Due to the particularity of money in contrast with other commodities, it's difficult to integrate money into general equilibrium models, and there is no money in those basic general equilibrium models such as the Arrow–Debreu model[1-3]. Hence some scholars based their analyses of equilibrium exchange rate on some frameworks other than the general equilibrium framework[4,5]. However, in our opinion it's more natural to discuss the determination of the equilibrium exchange rate in the general equilibrium framework.

Though some methods of integrating money into general equilibrium models have been presented, such as the money-in-utility method and the cash-in-advance method, those models adopting those methods are not accepted as widely as those basic general equilibrium models. In this paper we will introduce a new method of integrating money, that is, to integrate money into the demand functions of agents. By this method we will develop a general equilibrium model with multiple types of moneys, which can serve as a tool for the analysis of equilibrium exchange rates. Our model resembles the von Neumann general equilibrium model rather than the Arrow-Debreu model, and can be regarded as a CGE (i.e. computable general equilibrium) model[6,7].

Ⅱ. A General Equilibrium Model with Money

A. Fundamental Concepts and Assumptions

Let's consider an economy containing n types of commodities and m types of agents (i.e. firms or consumers). Those agents may belong to distinct countries. An agent may represent a firm or some homogenous consumers.

For the simplicity of analysis, in this paper the following assumptions are made:

(i) The supply amounts of some commodities are exogenous, which are referred to as primary factors, e.g. labor, land, money etc. The supply amounts of other commodities are endogenous, which are referred to as products. Suppose that all products are supplied by firms, and all primary factors are supplied by consumers. The supply amounts of primary factors are exogenous and fixed. There is no technology progress. Thus the equilibrium growth of the economy is zero;

(ii) Each firm produces one type of product and has a production function with constant returns to scale;

(iii) There are some types of moneys in the economy. All types of moneys are freely convertible. We can choose one type of money arbitrarily as the standard money. The standard money will be taken as the numeraire.

The price of a type of money in terms of the standard money is referred to as the exchange rate of that money, which is equal to the quantity that one unit of that money can exchange for the standard money. Of course, the exchange rate of the standard money is 1. The exchange rates of all types of moneys constitute the exchange rate vector, which is denoted by $\boldsymbol{\varepsilon}$.

The total value of one type of money in terms of the standard money is referred to as its real supply amount. Hence when the supply amount of one type of money is s and its exchange rate is ε, its real supply amount is εs. And in the following analysis all types of moneys will actually be converted into the standard money;

(iv) All money is owned by some consumers (namely money-owners). The supply amount and the interest rate of each type of money are exogenous and fixed. Commodities other than money are referred to as common commodities;

(v) Each agent needs to borrow one type of money from money-owners to buy commodities. When an agent intends to buy v dollars worth of commodities, it needs to hold v dollars of money. Given the interest rate r, it will pay vr dollars of interest to money-owners. Hence when an agent expends w dollars to purchase commodities for its production or consumption, finally it will obtain $\frac{w}{1+r}$ dollars worth of commodity and expend $\frac{rw}{1+r}$ dollars on the interest.

B. The Model

Let $\mathbf{p}$ denote the n-dimensional price vector, which consists of the market prices of n commodities in terms of the standard money. If commodity i is a type of money, then p_i is the interest rate of the money.

For a firm, let the n-dimensional vector $\mathbf{a}(\mathbf{p})$ denote its unit demand bundle, that is, the commodity bundle which is needed by the producer to produce one unit of product under the given market price vector $\mathbf{p}$. The cost to produce one unit product (i.e. the unit cost) is $c(\mathbf{p}) = \mathbf{p}^{\mathrm{T}}\mathbf{a}(\mathbf{p})$, which is referred to as the unit expenditure. As mentioned above, the firm will obtain $\frac{\mathbf{p}^{\mathrm{T}}\mathbf{a}(\mathbf{p})}{1+\mathrm{r}}$ dollars worth of commodity and expend $\frac{r\mathbf{p}^{\mathrm{T}}\mathbf{a}(\mathbf{p})}{1+\mathrm{r}}$ dollars on the interest.

Let the n-dimensional vector $\mathbf{b}$ denote its unit supply bundle, which contains merely one unit of product. Hence given the price vector $\mathbf{p}$, the unit production process of the profit-maximizing producer can be represented by the vector pair $(\mathbf{a}(\mathbf{p}),\mathbf{b})$. The revenue (i.e. the output value) by producing one unit product is $\mathbf{p}^T\mathbf{b}$, which is referred to as the unit revenue.

For a type of consumers, the unit demand bundle is defined as the demand bundle of a consumer under the given price vector $\mathbf{p}$ and a given utility level u, which is denoted by $\mathbf{a}(\mathbf{p},u)$. In other word, the unit demand bundle of one type of consumers is their compensated demand function (i.e. the Hicksian demand function).

For a type of consumers other than money-owners, the unit supply bundle $\mathbf{b}$ is defined as the supply bundle of a consumer, which contains some primary factors. For a type of money-owners, the unit supply bundle $\mathbf{b}(\boldsymbol{\varepsilon})$ is defined as the supply bundle of a consumer, which contains the real supply of the money owned by him.

The unit demand bundles of all agents constitute the n-by-m unit demand matrix $\mathbf{A}(\mathbf{p},\mathbf{u})$, wherein the ith column stands for the unit demand bundle of the ith agent. The unit supply bundles of all agents constitute the n-by-m unit supply matrix $\mathbf{B}(\boldsymbol{\varepsilon})$.

The unit expenditures of all agents constitute the unit expenditure vector, which is equal to $\mathbf{p}^T\mathbf{A}(\mathbf{p},\mathbf{u})$, wherein $\mathbf{u}$ is the utility vector consisting of the utility levels of all types of consumers. The unit revenues of all agents constitutes the unit revenue vector, which is equal to $\mathbf{p}^T\mathbf{B}(\boldsymbol{\varepsilon})$.

In equilibrium the unit expenditure of each agent is no less than its unit revenue[6-8]. Thus we have the following revenue-expenditure equilibrium formula:

$$\mathbf{p}^T\mathbf{A}(\mathbf{p},\mathbf{u}) \geq \mathbf{p}^T\mathbf{B}(\boldsymbol{\varepsilon}) \qquad (1)$$

Now let's turn to the supply-demand side of the equilibrium.

Let $\mathbf{z}$ denoted the n-dimensional activity-level vector consisting of the activity levels of all agents.

For a firm, the activity level z is defined as its output amount. Given the activity level z and the price vector $\mathbf{p}$, the production process of that agent can be represented by a vector pair $(\mathbf{a}(\mathbf{p})z, \mathbf{b}z)$, wherein $\mathbf{a}(\mathbf{p})z$ is the input bundle (i.e. the demand bundle) of the firm and $\mathbf{b}z$ is the output bundle (i.e. the supply bundle) of the firm.

For a type of consumers, the activity level z is defined as the population of that type of consumers. Given the activity level z, the price vector $\mathbf{p}$ and the utility vector $\mathbf{u}$, the demand bundle of that type of consumers is $\mathbf{a}(\mathbf{p})z$ and the supply bundle is $\mathbf{b}z$ or $\mathbf{b}(\boldsymbol{\varepsilon})z$.

Hence it's clear that given the activity level z, the price vector $\mathbf{p}$ and the utility vector $\mathbf{u}$, the demand vector consisting of the total demands for n types of commodities is $\mathbf{A}(\mathbf{p},\mathbf{u})\mathbf{z}$, and the supply vector is $\mathbf{B}(\boldsymbol{\varepsilon})\mathbf{z}$.

In equilibrium the demand for each commodity is no more than its supply. Thus we have the following supply-demand equilibrium formula:

$$\mathbf{A}(\mathbf{p},\mathbf{u})\mathbf{z} \le \mathbf{B}(\boldsymbol{\varepsilon})\mathbf{z} \tag{2}$$

(1) and (2) constitute a general equilibrium model with money. In the model the activity level of each type of consumers (i.e. the population of that type of consumers) is exogenous. The activity levels of firms (i.e. the output amounts of firms) are endogenous. By (1) and (2) the equilibrium price vector $\mathbf{p}^*$, the equilibrium exchange rate vector $\boldsymbol{\varepsilon}^*$, the equilibrium utility vector $\mathbf{u}^*$ and the equilibrium activity-level vector $\mathbf{z}^*$ can be solved.

Ⅲ. A Numerical Example

Let's consider a two-country economy containing 6 types of commodities and 6 types of agents.

Country 1 has 3 types of commodities (namely, wheat, labor 1 and money 1) and 3 types of agents (namely, wheat producer, laborer 1 and money-owner 1).

Country 2 also has 3 types of commodities (namely, iron, labor 2 and money 2) and 3 types of agents (namely, iron producer, laborer 2 and money-owner 2).

Hence the 6 types of commodities in the economy are wheat, labor 1, money 1, iron, labor 2 and money 2.

Without loss of generality, let the money of country 1 (i.e. money 1) be the standard money and the numeraire. Let's make the following assumptions:

(i) The interest rates of both types of moneys are 0.1;

(ii) Both the wheat producer and the iron producer need iron and labor for their production. Consumers consume wheat only. That is, iron is a capital commodity and wheat is a consumption commodity;

(iii) The firm of one country must use the labor of that country, and the firm and consumers of one country must use the money of that country;

(iv) The production functions and the utility functions are as follows:

	Production Functions or Utility Functions
Wheat Producer	$x_2^{0.5} x_4^{0.5}$
Laborer 1	x_1
Money-owner 1	x_1
Iron Producer	$x_4^{0.5} x_5^{0.5}$
Laborer 2	x_1
Money-owner 2	x_1

(v) For either country, the population of laborers and the supply amount of labor are 100 all the time. Either country has a money-owner. The supply amount of money 1 is 600 units. The supply amount of money 2 is 100 units.

Let ε be the exchange rate of money 2. Then the value of one unit of money 2 is equal to ε units of money 1, and the real supply amount of money 2 (i.e. the supply amount in terms of the standard money) is 100ε.

For the convenience of the analysis, all prices will be measured by the standard money, and the quantity of money 2 will be measured by the standard money.

The unit demand matrix is computed to be

$$\mathbf{A}(\mathbf{p},\mathbf{u}) = \begin{bmatrix} 0 & u_1 & u_2 & 0 & u_3 & u_4 \\ \left(\frac{p_4}{p_2}\right)^{0.5} & 0 & 0 & 0 & 0 & 0 \\ 2(p_2p_4)^{0.5} & p_1u_1 & p_1u_2 & 0 & 0 & 0 \\ \left(\frac{p_2}{p_4}\right)^{0.5} & 0 & 0 & \left(\frac{p_5}{p_4}\right)^{0.5} & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{p_4}{p_5}\right)^{0.5} & 0 & 0 \\ 0 & 0 & 0 & 2(p_4p_5)^{0.5} & p_1u_3 & p_1u_4 \end{bmatrix}$$

The unit supply matrix is

$$\mathbf{B}(\varepsilon) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 600 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 100\varepsilon \end{bmatrix}$$

It's obvious that in the equilibrium of this economy the demand for each commodity will equal the supply of that commodity, and the unit expenditure of each agent will equal its unit revenue. In other words, there are no free commodity and no inactive firm in the equilibrium of this economy. Hence the equilibrium formulas can also be written as

$$\mathbf{p}^T\mathbf{A}(\mathbf{p},\mathbf{u}) = \mathbf{p}^T\mathbf{B}(\varepsilon) \tag{3}$$

$$\mathbf{A}(\mathbf{p},\mathbf{u})\mathbf{z} = \mathbf{B}(\varepsilon)\mathbf{z} \tag{4}$$

Wherein the price vector is

$$\mathbf{p} = (p_1, p_2, 0.1, p_4, p_5, 0.1)^T$$

and the activity-level vector is

$$\mathbf{z} = (z_1, 100, 1, z_4, 100, 1)^T$$

The equilibrium price vector is computed to be $\mathbf{p}^* = (8.284, 1.875, 0.1, 7.563, 1.563, 0.1)^T$.

The equilibrium exchange rate is computed to be $\varepsilon^* = 5$. That is, the real supply amount of money 2 is 500 units.

The equilibrium utility vector and the equilibrium activity-level vector are computed to be $\mathbf{u}^* = (0.2058, 6.584, 0.1715, 5.487)^T$ and $\mathbf{z}^* = (49.79, 100, 1, 45.45, 100, 1)^T$.

The equilibrium allocation is shown in Table 1. The equilibrium value allocation is shown in Table 2.

Ⅳ. Concluding Remarks

In this paper we present a general equilibrium model with multiple types of moneys. Like the von Neumann general equilibrium model[8,9], the model consists of two inequalities, i.e. a revenue-expenditure equilibrium formula and a supply-demand equilibrium formula.

The model assumes that each firm has a production function with constant returns to

Table 1. The Equilibrium Allocation in the Two-country Economy

	Wheat Producer	*Laborer 1*	*Money Owner 1*	*Iron Producer*	*Laborer 2*	*Money Owner 2*	*Total Demand*
Wheat Demand	0	20.58	6.584	0	17.15	5.487	49.79
Labor 1 Demand	100	0	0	0	0	0	100
Money 1 Demand	375	170.5	54.55	0	0	0	600
Iron Demand	24.79	0	0	20.66	0	0	45.45
Labor 2 Demand	0	0	0	100	0	0	100
Money 2 Demand	0	0	0	312.5	142	45.45	500
Supply	49.79	100	600	45.45	100	500	

Table 2. The Equilibrium Value Allocation in the Two-country Economy in Terms of Money 1

	Wheat Producer	*Laborer 1*	*Money Owner 1*	*Iron Producer*	*Laborer 2*	*Money Owner 2*	*Total Expenditure*
Wheat Expenditure	0	170.5	54.55	0	142	45.45	412.5
Labor 1 Expenditure	187.5	0	0	0	0	0	187.5
Interest Expenditure on Money 1	37.5	17.05	5.455	0	0	0	60
Iron Expenditure	187.5	0	0	156.2	0	0	343.7
Labor 2 Expenditure	0	0	0	156.2	0	0	156.2
Interest Expenditure on Money 2	0	0	0	31.25	14.2	4.545	50
Supply Value	412.5	187.5	600	343.7	156.2	500	
[Interest Income]			[60]			[50]	

scale. It's well known that a production function with decreasing returns to scale can be transformed into a production function with constant returns to scale by introducing an additional primary factor. Hence in fact the model also allows for decreasing returns to scale.

As the numerical example shows, the model can be utilized to analyze a multi-country monetary economy and to compute the equilibrium exchange rates.

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