The Exchange Function and A Dynamic Exchange Model

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Abstract

In the neoclassical economics the tatonnement process is utilized to explain the price change. However, the tatonnement process assumes that there is no trading before the prices reach the equilibrium prices, which is extremely unrealistic since in the markets of the real world the exchange processes usually occur at disequilibrium prices. In this sense, the tatonnement process fails to describe and explain the price fluctuation and the disequilibrium exchange process in the real world. In this paper we introduce an exchange function to describe the exchange process under fixed prices. Based on the exchange function a discrete-time dynamic exchange model is developed. In the model a disequilibrium exchange process occurs in each period, and the prices vary on the basis of the prices and the exchange outcome in the preceding period.

KEY WORDS: exchange economy; disequilibrium; tatonnement process

1 Introduction

Price fluctuations are regular in markets such as the stock market and the foreign exchange market. And apparently the mismatch of supply and demand is the main cause of price fluctuations. Hence, the prices in those markets usually are not equilibrium prices, and the finance markets usually run in disequilibrium. In other words, the persistent transactions in the markets take place at disequilibrium prices, which may be referred to as disequilibrium exchange.

Besides the finance markets, in some other markets the disequilibrium exchange also prevails. The labor market is another example which is often mentioned. Downwards wage rigidity in the presence of underemployment is common in labor markets, and taking minimum wage laws into consideration some labor markets must be dominated by the disequilibrium exchange.

Since disequilibrium exchange processes are much more common than equilibrium exchange processes, disequilibrium exchange processes should be attached more importance, and which have been elucidated penetratingly by a number of scholars (e.g. Benassy, 1975, 1982; Dreze, 1975; Uzawa, 1962). In this short note we will attempt to present an exchange function to describe the exchange process under fixed prices. Then a dynamic exchange model will be developed based on the exchange function.

The following notations and terms will be used. $\boldsymbol{e}$ denotes the vector $(1, 1, \cdots, 1)^{\top}$. A vector $\boldsymbol{x}$ is called positive (or nonnegative) and we write $\boldsymbol{x} \gg \mathbf{0}$ (or $\boldsymbol{x} \geq \mathbf{0}$) if all its components are positive (or nonnegative). $\boldsymbol{x}$ is called semipositive and we write $\boldsymbol{x} > \mathbf{0}$ if $\boldsymbol{x} \geq \mathbf{0}$ and $\boldsymbol{x} \neq \mathbf{0}$. For vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, we write $\boldsymbol{x} \gg \boldsymbol{y}$, $\boldsymbol{x} > \boldsymbol{y}$ and $\boldsymbol{x} \geq \boldsymbol{y}$ analogously. Such notations and terms are also used for matrices. A semipositive column (or row) vector $\boldsymbol{x}$ is said to be normalized if $\boldsymbol{e}^{\top}\boldsymbol{x} = 1$ (or $\boldsymbol{x}\boldsymbol{e} = 1$) holds. $\hat{\boldsymbol{x}}$ denotes diag($\boldsymbol{x}$), i.e. the diagonal matrix with the vector $\boldsymbol{x}$ as the main diagonal.

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2 The Exchange Function

2.1 The Coffee Problem: An Example of Disequilibrium Exchange Processes

First let's take the following example to illustrate the issue of disequilibrium exchange (see Bapat, Raghavan, 1997).

Suppose there are 3 citizens (namely consumer 1, 2 and 3) on an island. Consumer 1 has 1 unit of coffee powder. Consumer 2 has 1 unit of milk. Consumer 3 has 1 unit of sugar. Hot water is freely available. Consumer 1 likes 0.05 units of coffee powder with 0.1 units of milk and no sugar in every cup of coffee he drinks. Consumer 2 wants 0.05 units of coffee powder and 0.15 units of sugar and no milk in his cup of coffee. Consumer 3 wants 0.1 units of coffee powder, 0.1 units of milk, and 0.05 units of sugar in his cup of coffee.

The equilibrium problem and the disequilibrium problem are as follows:

(i) Equilibrium Coffee Problem: Find a set of market-clearing prices;

(ii) Disequilibrium Coffee Problem: Find the exchange outcome at given market prices, say, all market prices equal to 1.

The solution of the equilibrium coffee problem is well-known. Let's take sugar as numeraire. By some simple computations the market-clearing price vector is found to be $\boldsymbol{p}^* = (0.6, 0.9, 1)^\top$.

And it seems that the disequilibrium coffee problem hasn't been discussed before.

2.2 The Exchange Function

Let's consider an pure exchange economy including m traders and n commodities under a given price vector $\boldsymbol{p}$.

Let $\boldsymbol{S}$ denote the $(n \times m)$ supply matrix, whose (i, j) entry denotes trader j's supply amount of commodity i. Let $\boldsymbol{s} := \boldsymbol{S}\boldsymbol{e}$ denote the supply vector, which is supposed to be positive.

For example, in the coffee economy the supply matrix is

$$\boldsymbol{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

and the supply vector is $\boldsymbol{s} = (1, 1, 1)^\top$.

Demand structures of traders are represented by an *standard demand matrix* $\boldsymbol{A}$ and each trader intends to purchase some commodity bundles indicated by $\boldsymbol{A}$. Each column of $\boldsymbol{A}$ is called a *standard demand bundle*. That is, in the exchange process the bundle purchased by trader i must be $z_i \boldsymbol{a}_{\bullet i}$, where z_i is a nonnegative real number and $\boldsymbol{a}_{\bullet i}$ is the ith column of $\boldsymbol{A}$. z_i is called the *purchase amount* or *exchange amount* of trader i. Let $\boldsymbol{z}$ denote the vector consisting of purchase amounts of m traders, and $\boldsymbol{z}$ is called the *purchase vector* or *exchange vector* (of standard demand bundles), and $\boldsymbol{A}\boldsymbol{z}$ is called the *sales vector of commodities*.

For example, in the coffee economy the standard demand matrix is

$$\boldsymbol{A} = [\boldsymbol{a}_{\bullet 1}\ \boldsymbol{a}_{\bullet 2}\ \boldsymbol{a}_{\bullet 3}] = \begin{bmatrix} 0.05 & 0.05 & 0.1 \\ 0.1 & 0 & 0.1 \\ 0 & 0.15 & 0.05 \end{bmatrix} \tag{2}$$

The bundles purchased by consumer 1, 2 and 3 must be $\boldsymbol{a}_{\bullet 1} z_1$, $\boldsymbol{a}_{\bullet 2} z_2$ and $\boldsymbol{a}_{\bullet 3} z_3$ respectively, where z_1, z_2 and z_3 are nonnegative real numbers. The exchange vector is $\boldsymbol{z} = (z_1, z_2, z_3)^\top$, which indicates the purchase amounts of standard demand bundles, and the corresponding sales vector of commodities is

$$\boldsymbol{A}\boldsymbol{z} = (0.05z_1 + 0.05z_2 + 0.1z_3, 0.1z_1 + 0.1z_3, 0.15z_2 + 0.05z_3)^\top \tag{3}$$

which indicates the sales amounts of three commodities.

Let $\hat{\boldsymbol{x}}$ denote diag($\boldsymbol{x}$), i.e. the diagonal matrix with the vector $\boldsymbol{x}$ as the main diagonal. For example, for the supply vector $\boldsymbol{s} = (1,1,1)^\top$ and the supply matrix in (1), we have $\boldsymbol{S} = \hat{\boldsymbol{s}}$.

The *sales rate* of a commodity refers to the proportion of its sales amount to its supply amount. Suppose for one commodity all its suppliers share the same sales rate, and let $\boldsymbol{q}$ be the n-dimensional *sales rate vector* indicating the sales rates of n commodities, that is,

$$\boldsymbol{q} := \hat{\boldsymbol{s}}^{-1}\boldsymbol{A}\boldsymbol{z} \tag{4}$$

Obviously, the matrices $\boldsymbol{A}\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{q}}\boldsymbol{S}$ indicate each trader's purchase and sales amounts of commodities respectively. Under the given price vector $\boldsymbol{p}$, the purchase and sales values of m traders are $\boldsymbol{p}^\top\boldsymbol{A}\hat{\boldsymbol{z}}$ and $\boldsymbol{p}^\top\hat{\boldsymbol{q}}\boldsymbol{S}$ respectively. Suppose the value each trader purchases must equal the value he sells, that is,

$$\boldsymbol{p}^\top\boldsymbol{A}\hat{\boldsymbol{z}} = \boldsymbol{p}^\top\hat{\boldsymbol{q}}\boldsymbol{S} \equiv \boldsymbol{p}^\top\widehat{\hat{\boldsymbol{s}}^{-1}\boldsymbol{A}\boldsymbol{z}}\boldsymbol{S} \tag{5}$$

Eq. (5) is the equivalent exchange condition. When Eq. (5) holds and $\boldsymbol{S}^\top\boldsymbol{A}$ is indecomposable the following proposition shows that there exists a unique normalized exchange vector. Here the indecomposability of $\boldsymbol{S}^\top\boldsymbol{A}$ implies that traders cannot be divided into two groups, namely group 1 and group 2, such that each trader in group 1 has no demand for the supply of any trader in group 2.

Proposition 1. Let $\boldsymbol{A}$ and $\boldsymbol{S}$ be $(n \times m)$ semipositive matrices such that $\boldsymbol{s} := \boldsymbol{S}\boldsymbol{e}$ is positive and $\boldsymbol{S}^\top\boldsymbol{A}$ is indecomposable. Let $\boldsymbol{p}$ be an n-dimensional positive vector and $\boldsymbol{z}$ be an m-dimensional semipositive vector. Then:

(i) $\boldsymbol{Z} := \widehat{\boldsymbol{A}^\top\boldsymbol{p}}^{-1}\boldsymbol{S}^\top\hat{\boldsymbol{s}}^{-1}\hat{\boldsymbol{p}}\boldsymbol{A}$ is an indecomposable nonnegative matrix possessing the P-F (i.e. Perron-Frobenius) eigenvalue 1;

(ii) $\boldsymbol{z}$ satisfies $\boldsymbol{p}^\top\boldsymbol{A}\hat{\boldsymbol{z}} = \boldsymbol{p}^\top\widehat{\hat{\boldsymbol{s}}^{-1}\boldsymbol{A}\boldsymbol{z}}\boldsymbol{S}$ if and only if $\boldsymbol{z}$ is a right P-F eigenvector of $\boldsymbol{Z}$, i.e. $\boldsymbol{Z}\boldsymbol{z} = \boldsymbol{z}$ holds; moreover, if $\boldsymbol{z}$ satisfies $\boldsymbol{p}^\top\boldsymbol{A}\hat{\boldsymbol{z}} = \boldsymbol{p}^\top\widehat{\hat{\boldsymbol{s}}^{-1}\boldsymbol{A}\boldsymbol{z}}\boldsymbol{S}$ then $\boldsymbol{z}$ is positive.

Proof (i) Because $\boldsymbol{S}^\top\boldsymbol{A}$ is indecomposable, each column of $\boldsymbol{A}$ must be semipositive. Then $\boldsymbol{A}^\top\boldsymbol{p}$ is a positive vector, and all entries on the main diagonals of $\widehat{\boldsymbol{A}^\top\boldsymbol{p}}^{-1}$, $\hat{\boldsymbol{s}}^{-1}$ and $\hat{\boldsymbol{p}}$ are positive. Hence if the (i,j) entry of $\boldsymbol{S}^\top\boldsymbol{A}$ is positive then the (i,j) entry of $\boldsymbol{Z}$ is also positive. Therefore $\boldsymbol{Z}$ is indecomposable.

And it can be readily verified that $\boldsymbol{p}^\top\boldsymbol{A}\boldsymbol{Z} = \boldsymbol{p}^\top\boldsymbol{A}$ holds. By Perron-Frobenius theorem, the P-F eigenvalue of $\boldsymbol{Z}$ equals 1 and $\boldsymbol{p}^\top\boldsymbol{A}$ is a left P-F eigenvector of $\boldsymbol{Z}$.

(ii) We have

$$\boldsymbol{p}^\top\boldsymbol{A}\hat{\boldsymbol{z}} = \boldsymbol{p}^\top\widehat{\hat{\boldsymbol{s}}^{-1}\boldsymbol{A}\boldsymbol{z}}\boldsymbol{S} \Leftrightarrow \boldsymbol{p}^\top\boldsymbol{A}\hat{\boldsymbol{z}} = \boldsymbol{p}^\top\widehat{\boldsymbol{A}\boldsymbol{z}}\hat{\boldsymbol{s}}^{-1}\boldsymbol{S} \Leftrightarrow \widehat{\boldsymbol{A}^\top\boldsymbol{p}}\boldsymbol{z} = \boldsymbol{S}^\top\hat{\boldsymbol{s}}^{-1}\hat{\boldsymbol{p}}\boldsymbol{A}\boldsymbol{z}$$

$$\Leftrightarrow \widehat{\boldsymbol{A}^\top\boldsymbol{p}}^{-1}\boldsymbol{S}^\top\hat{\boldsymbol{s}}^{-1}\hat{\boldsymbol{p}}\boldsymbol{A}\boldsymbol{z} = \boldsymbol{z} \Leftrightarrow \boldsymbol{Z}\boldsymbol{z} = \boldsymbol{z}$$

Hence by Perron-Frobenius theorem the statement holds. ■

Let $\boldsymbol{x}$ denote the normalized right P-F eigenvector of $\boldsymbol{Z}$. Then by Proposition 1(ii) we have $\boldsymbol{z} = \xi\boldsymbol{x}$, where ξ is a nonnegative real number. Since the sales amount of each commodity is no more than its supply amount, we find $\boldsymbol{A}\boldsymbol{z} \le \boldsymbol{s}$ holds, that is, $\xi\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{s}$. Hence ξ is no greater than the minimal component of $\widehat{\boldsymbol{A}\boldsymbol{x}}^{-1}\boldsymbol{s}$. Suppose all traders attempt to obtain maximal exchange amounts. The unique maximal exchange vector can be found by following steps, which stands for the outcome of the exchange process:

Step 1. Compute the matrix $\boldsymbol{Z} := \widehat{\boldsymbol{A}^\top\boldsymbol{p}}^{-1}\boldsymbol{S}^\top\hat{\boldsymbol{s}}^{-1}\hat{\boldsymbol{p}}\boldsymbol{A}$;
Step 2. Find the normalized right P-F eigenvector of $\boldsymbol{Z}$, denoted by $\boldsymbol{x}$;
Step 3. Find the minimal component of $\widehat{\boldsymbol{A}\boldsymbol{x}}^{-1}\boldsymbol{s}$, denoted by ξ;
Step 4. Compute the exchange vector $\boldsymbol{z} := \xi\boldsymbol{x}$.

Thus the exchange process can be represented by the following *exchange function*:

$$(\boldsymbol{q},\boldsymbol{z}) = Z(\boldsymbol{A},\boldsymbol{p},\boldsymbol{S}) \tag{6}$$

wherein $\boldsymbol{A}$, $\boldsymbol{S}$ and $\boldsymbol{p}$ satisfy those assumptions in Proposition 1, $\boldsymbol{z}$ is computed by steps above and $\boldsymbol{q}$ equals $\hat{\boldsymbol{s}}^{-1}\boldsymbol{A}\boldsymbol{z}$. When the demand structures of traders vary with prices, the exchange function may be written

$$(\boldsymbol{q},\boldsymbol{z}) = Z\big(\boldsymbol{A}(\boldsymbol{p}),\boldsymbol{p},\boldsymbol{S}\big) \tag{7}$$

Sometimes $\boldsymbol{S}$ is an $(n\times n)$ diagonal matrix. In such a case $\boldsymbol{S}^\top\hat{\boldsymbol{s}}^{-1} = \boldsymbol{I}$ holds and the matrix $\boldsymbol{Z}$ becomes

$$\boldsymbol{Z} = \widehat{\boldsymbol{A}^\top\boldsymbol{p}}^{-1}\hat{\boldsymbol{p}}\boldsymbol{A} \tag{8}$$

2.3 The Disequilibrium Exchange Outcome in the Coffee Economy

By the exchange function we can find the disequilibrium exchange outcome in the coffee economy. Given the price vector $(1,1,1)^\top$, the exchange vector is computed to be $\boldsymbol{z} = (6.25,5,3.75)^T$. That is, consumer 1, 2 and 3 will drink 6.25, 5 and 3.75 cups of coffee respectively. The sales rate vector is $\boldsymbol{q} = (0.9375,1,0.9375)^T$, which indicates that only milk is sold out. And we have

$$\big((0.9375,1,0.9375)^\top,(6.25,5,3.75)^\top\big) = Z\big(\boldsymbol{A},(1,1,1)^\top,\boldsymbol{S}\big)$$

The disequilibrium allocation is shown in Table 1.

Table 1. Disequilibrium Allocation under the Price Vector $(1,1,1)^\top$

	Consumer 1	Consumer 2	Consumer 3	Total Purchase
Coffee Powder Purchase	0.3125	0.25	0.375	0.9375
Milk Purchase	0.625	0	0.375	1
Sugar Purchase	0	0.75	0.1875	0.9375
Supply	1	1	1	

Since all prices are 1, the price-measured amount and the physical amount are the same.

In the next exchange process, the prices of the three commodities may change based on the exchange outcome above. In that case the price of milk will rise relatively, and the prices of coffee powder and sugar will fall relatively.

3 A Dynamic Exchange Model

3.1 The Model

The coffee economy can be extended into a dynamic economy when the three consumers conduct an exchange process in each day. And the following assumptions will be taken:

(i) The supply of each trader is fixed in all days, that is, the supply matrix $\boldsymbol{S}$ is fixed;

(ii) In each day the price adjustment occurs before the exchange process, which is based on the prices and sales rates of the preceding day. And in the exchange process the prices keep fixed.

The price adjustment process can be denoted by a price adjustment function as follows:

$$\boldsymbol{p}^{(t+1)} = P\big(\boldsymbol{p}^{(t)},\boldsymbol{q}^{(t)}\big) \tag{9}$$

In the simulation we will use the following price adjustment function:

$$\boldsymbol{p}^{(t+1)} = (1-\theta)\boldsymbol{p}^{(t)} + \theta\widehat{\boldsymbol{q}^{(t)}}\boldsymbol{p}^{(t)} \tag{10}$$

Here $\theta \in (0,1)$ indicates the velocity of the price adjustment. In the following simulation θ will be set to 0.15.

The dynamic exchange model can be written as

$$\boldsymbol{p}^{(t+1)} = P\big(\boldsymbol{p}^{(t)}, \boldsymbol{q}^{(t)}\big) \tag{11}$$

$$\big(\boldsymbol{q}^{(t+1)}, \boldsymbol{z}^{(t+1)}\big) = Z\big(\boldsymbol{A}(\boldsymbol{p}^{(t+1)}), \boldsymbol{p}^{(t+1)}, \boldsymbol{S}\big) \tag{12}$$

3.2 A Simulation of the Dynamic Coffee Economy

By the dynamic exchange model (11)-(12) the market-clearing prices of the coffee economy can be computed.

Let's set all the initial prices of all commodities to 1. Let sugar be the numeraire. The price dynamics of coffee powder and milk are shown in the left panel of Fig. 1, and the exchange amounts of each consumer are shown in the right panel in Fig. 1.

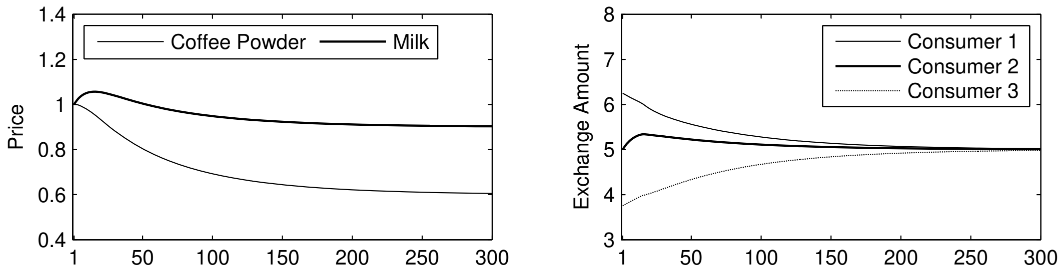

Figure 1. Prices and Exchange Amounts in the Dynamic Coffee Economy

We can see that the market prices are approaching the market-clearing prices, and in equilibrium each consumer can drink 5 cups of coffee.

4 Concluding Remarks

In reality economies run in disequilibrium paths and economic fluctuations can be seen everyday. Post-Keynesian economics is concerned with disequilibrium, non-market clearing analysis and change over time (Arestis, 1996). In the viewpoint of post-Keynesian economics, deterministic dynamic disequilibrium models are indispensable to describe and explain the reality in a complete advanced economic theory. Though a variety of such models have been developed (e.g. see Benetti, Bidard, Klimovsky, Rebeyrol, 2012; Song, 2003), none such model has been accepted as widely as various equilibrium models. And maybe a consensus cannot be reached on deterministic dynamic disequilibrium models unless a consensus can be reached on a disequilibrium exchange process. Hence its crucial to develop new disequilibrium exchange models.

In this paper, we developed a relatively simple disequilibrium exchange model, which is different from the neoclassical tatonnement process in essence. Both the tatonnement process and the dynamic exchange model in this paper can describe price fluctuations. However, no transactions take place at disequilibrium prices in the tatonnement process, and this point makes the tatonnement process far from the exchange process in the real world. Hence the tatonnement process cannot be referred to as a dynamic exchange model. On the contrary, the dynamic exchange model in this paper allows for disequilibrium exchange, and prices are adjusted based on the preceding exchange outcome instead of the somewhat fictitious excess demand. Moreover, the model can be readily extended to a dynamic economic model including production, exchange and consumption.

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