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# **An Adaptive Successive Over-relaxation Method for Computing the Black-Scholes Implied Volatility**

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# An Adaptive Successive Over-relaxation Method for Computing the Black-Scholes Implied Volatility

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A new successive over-relaxation method to compute the Black-Scholes implied volatility is introduced. Properties of the new method are fully analyzed, including global well-definedness, local convergence, as well as global convergence. Quadratic order of convergence is achieved by either a dynamic relaxation or transformation of sequence technique. The method is further enhanced by introducing a rational approximation on initial values. Numerical implementation shows that uniformly in a very large domain, the new method converges to the true implied volatility with very few iterations. Overall, the new method achieves a very good combination of efficiency, accuracy and robustness.

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**JEL Classifications:** C00, G13

# 1. Introduction

A breakthrough in modern financial theory is the Black-Scholes-Merton theory of option pricing, developed by Black and Scholes (1973), Merton (1973, 1976) and many others. In 1997, the Nobel prize in Economics was awarded to Merton and Scholes for the discovery and extension of this theory. As the press releases described it, the Black-Scholes theory is “among the foremost contributions to economic sciences over the last 25 years.” The ultimate result of the theory is the celebrated Black-Scholes formula, which is being used daily by traders around the world nowadays. Besides being used to price options on stocks, the Black-Scholes theory is also used to value options on futures, options on currencies, options on interest rates, etc. Even models that are out of the Black-Scholes framework, such as stochastic volatility models and more recently Levy-process type models, still refer to the Black-Scholes model as the basic benchmark. See, for example, Heston (1993), and Carr and Wu (2003). The use of the Black-Scholes formula is so pervasive that Nigel Goldenfeld, a Physics professor at the University of Illinois at Urbana-Champaign, claims that the Black-Scholes formula is *the* most frequently used equation by human beings nowadays, beating both Newton’s laws of motion in classical mechanics and Schrödinger’s equation in quantum mechanics.

In practice, traders often do not use the Black-Scholes formula to price options. Rather, they observe the actual price on the market and then use the Black-Scholes formula backwards to compute the volatility parameter, called the implied volatility. Early academic development of this concept and on its applications includes Latané and Rendleman (1976), and Schmalensee and Trippi (1978). The implied volatility is a useful quantity because it is a succinct way to talk about option prices. In fact, traders are usually more comfortable to talk in terms of implied volatilities than prices themselves. Market participants also pay close attention to the implied volatility since it serves as a forward-looking measure of people’s expectation about future market movements. The Black-Scholes volatility is also a useful concept for models outside the Black-Scholes framework. For example, people are constantly interested in whether a new proposed model can produce the Black-Scholes implied volatility curve observed on the market. The importance of the implied volatility is most evident in the classical NatWest Markets case. In 1997, NatWest lost 90.5 million pounds due to consistent underestimating of implied volatilities it used to price interest rate swaptions.

As early researchers have noted in their works, the inversion of the Black-Scholes formula cannot be done in closed form if elementary functions are to be used. Thus, to get the implied volatility, people usually use some solver method, for example, the Dekker-Brent algorithm, or the Newton-Raphson algorithm. Both methods have strengths and weaknesses (see Jäckel 2006, and Li 2008). The Newton-Raphson algorithm has a quadratic convergence order and is thus quite efficient. However, it can suffer from numerical instabilities. For example, each step in the Newton-Raphson algorithm requires the division by the option vega, which can be extremely small for away-from-the-money options. Also, with bad starting values, the Newton-Raphson

algorithm might not converge at all to the true implied volatility. The Dekker-Brent algorithm uses a combination of bisection, secant and inverse quadratic interpolation, and is guaranteed to converge. Some commercial software, such as MATLAB, chooses to use the Dekker-Brent algorithm, possibly because of its robustness. However, the Dekker-Brent algorithm is usually slower than the Newton-Raphson algorithm. For example, as Li (2008) reports, the number of inversions one has to do for the S&P 500 index options for the period January 1996 to December 2004 is about 700,000. On a typical computer, to achieve  $10^{-6}$  accuracy in total volatility (defined as the volatility multiplied by the square root of maturity), it takes the built-in function “blsimpv” in MATLAB hours to finish these inversions, while the Newton-Raphson algorithm without vectorization takes about 960 seconds.

Many research works have focused on inverting the Black-Scholes formula. Manaster and Koehler (1982) show that if one starts the initial estimate for the Newton-Raphson algorithm from the inflection point, then in theory the algorithm always converges. Jäckel (2006) analyzes the Newton-Raphson algorithm in detail and points out various pitfalls one should avoid. He also points out how one can avoid those pitfalls by designing a series of precautionary measures. Researchers have also considered alternatives to the solver methods, such as analytical approximation or quasi-iterative methods. Brenner and Subrahmanyam (1988) develop a simple formula for computing the implied volatility of at-the-money options. Bharadia et al. (1996) derive a simple volatility formula which does not require the option to be exactly at-the-money. Chance (1993, 1996) develops a formula for computing the implied volatility by adding a quadratic adjustment term to the Brenner-Subrahmanyam formula. Chambers and Nawalkha (2001) develop a simplified extension of Chance’s method and obtain a formula that is more accurate than previous methods. One shortcoming of the Chance method and the Chambers-Nawalkha method is that they both require the knowledge of the at-the-money implied volatility. Based on the linearization of the cumulative normal distribution, Corrado and Miller (1996a, 1996b) develop a simple and elegant formula that overcomes this shortcoming. This formula is quite accurate for near-the-money options but unfortunately its performance deteriorates when the option gets away from the money. Kelly (2006) considers inverting the Black-Scholes formula using the implicit function theorem. He also studies the numerical errors of various methods. Charogy-Corona and Ibarra-Valdez (2006) consider an approximation formula for the implied volatility by means of an asymptotic representation of the Black-Scholes formula. However, they do not give a numerical analysis. Li (2008) takes a new approach by using a rational approximation for the implied volatility. Since a rational approximation can have a small uniform error in a large domain, the approximation domain considered in Li (2008) is fairly large, with moneyness ranging from  $-0.5$  to  $0.5$  and total volatility ranging from  $0$  to  $1$ . For most index options or currency options, this domain is large enough. It is found that the rational approximation is the most accurate among all existing closed-form methods, achieving  $10^{-3}$  in terms of uniform error in the total volatility. Also, the rational approximation is able to invert one million options within a few seconds.

In this paper, we introduce a new method to compute the Black-Scholes implied volatility based on successive over-relaxation. Thus, our method is an iterative method and not in closed form. We study this method for two reasons. First, although the approximation domain in Li (2008) is fairly large, in practice, total volatility can be well above 1, especially for energy derivatives, or equity derivatives near earnings announcements. Thus, we use a much larger domain in this paper, with moneyness ranging from  $-3$  to  $3$  and total volatility ranging from  $0$  to  $6$ . This domain should cover almost all practical applications. Second, we try to search for a method which has the best combination of efficiency, accuracy and robustness. We will see that our method can achieve quadratic order of convergence, the same order as the Newton-Raphson algorithm. On the other hand, our method is robust as is the case for the Dekker-Brent algorithm. With just five iterations our method achieves a uniform error in the total volatility in the order of  $10^{-13}$  inside the huge domain. The time it takes to invert one million options is about 10 seconds.

Our first contribution in this paper is that we design a new method to compute the implied volatility. The basic idea of our method is to split the Black-Scholes formula into two parts and to search for the implied volatility iteratively. To improve efficiency, we introduce a relaxation parameter. We call our method with over-relaxation the **SOR** algorithm. Unlike the Newton-Raphson algorithm, the computation of vega is not needed in our method. We analyze the well-definedness, local and global convergence properties of the **SOR** algorithm in detail. In Theorem 3.1, we show that when the relaxation parameter is large enough, the **SOR** algorithm is always globally well-defined. This is a very attractive feature because it gives rise to robustness of our algorithm. Theorem 3.2 shows that the local convergence is dictated by the behavior of the iteration function near the true implied volatility. Theorem 3.3 shows that only two convergence patterns can occur, monotone or oscillating. Theorem 3.4 and 3.5 show the convergence patterns of price errors and implied volatility errors, respectively. The most interesting analysis is on global convergence, which is usually very difficult to study. We obtain many interesting results. Theorem 3.6 and 3.7 study two important special cases where the moneyness or the relaxation parameter is  $0$ , respectively. Theorem 3.8 shows that when the relaxation parameter is larger than some threshold value, the **SOR** algorithm is always globally convergent. The case where the relaxation parameter is smaller than the threshold value is much harder to analyze and we content ourselves by giving three sufficient conditions on global convergence in Theorem 3.9, 3.10 and 3.11.

The second contribution of this paper is that we introduce two convergence acceleration techniques to the **SOR** algorithm. The basic **SOR** algorithm usually has a linear order of convergence, as is shown in Theorem 4.1. The first acceleration technique is based on dynamic relaxation, where we adaptively adjust the relaxation parameter at each iteration. We call this the **SOR-DR** algorithm. Theorem 4.2 shows the global well-definedness property and convergence patterns for the **SOR-DR** algorithm, while Theorem 4.3 shows that it has a quadratic order of convergence. The second acceleration technique is based on a time-honored sequence transformation

technique, where at each iteration step we perform a nonlinear extrapolation, with the weight adaptively adjusted. We call this the **SOR-TS** algorithm. Theorem 4.4 shows the global well-definedness and convergence properties and local convergence patterns for the **SOR-TS** algorithm, while Theorem 4.5 shows that it has a quadratic order of convergence.

Our third contribution is on numerical implementation of our algorithms. To further enhance the efficiency of our algorithms, we introduce a rational approximation on the initial estimates, in the same spirit as Li (2008). We show numerically that uniformly in a very large domain, our accelerated algorithms converge to the true implied volatility with very few iterations. We also briefly compare the **SOR-TS** algorithm with other methods such as the Newton-Raphson algorithm and the Dekker-Brent algorithm, and show that the **SOR-TS** algorithm achieves the best combination of accuracy, efficiency and robustness.

Finally, we extend our successive over-relaxation method to the computation of implied correlation in the Margrabe formula, the normal implied volatility in the Bachelier formula, and the critical stock price in a compound call on call option. The extension to the Margrabe formula is straightforward. The extensions to the Bachelier implied volatility and critical stock price require some analysis, and Theorem 6.1 and 6.2 show that for both applications, the successive over-relaxation method is globally well-defined and convergent with a quadratic order of convergence. These three examples demonstrate that the idea of successive over-relaxation is not limited to the computation of the Black-Scholes implied volatility, but rather applicable in a much wider range of financial problems.

The rest of the paper is organized as follows. Section 2 introduces the problem of computing the Black-Scholes implied volatility. Section 3 introduces the basic successive over-relaxation method and studies its well-definedness, local and global convergence properties. Section 4 considers two acceleration techniques, one based on dynamic relaxation and the other on sequence transformation, and shows that both of them achieve quadratic order of convergence. Section 5 implements all three algorithms with a rational approximation enhancement. Section 6 demonstrates that the successive over-relaxation method is applicable in many other financial problems through three additional examples. Section 7 concludes. Proofs are in the Appendix.

## 2. The Implied Volatility Problem

Let  $C$  be the price of a call option at time  $t$  with expiration date  $T$  and strike price  $K$ . Let  $S$  be the current stock price,  $\sigma$  the volatility of the stock,  $r$  the risk-free interest rate, and  $\delta$  the dividend rate. Then the Black-Scholes formula expresses  $C$  in closed form as follows:

$$C(S, t; r, \sigma, T, K, \delta) = Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\log(Se^{(r-\delta)(T-t)}/K)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}, \quad d_2 = d_1 - \sigma\sqrt{T-t},$$

and  $N(\cdot)$  is the cumulative normal distribution function.

Let us first define the normalized call option price  $c(\cdot, \cdot)$  by

$$C(S, t; r, \sigma, T, K, \delta) = Se^{-\delta(T-t)} c(\log(Se^{(r-\delta)(T-t)}/K), \sigma\sqrt{T-t}).$$

The function  $c(x, v)$  is given by

$$c(x, v) = N\left(\frac{x}{v} + \frac{v}{2}\right) - e^{-x} N\left(\frac{x}{v} - \frac{v}{2}\right), \quad (1)$$

where the moneyness  $x$  and total volatility  $v$  are given by

$$x = \log(Se^{(r-\delta)(T-t)}/K), \quad v = \sigma\sqrt{T-t}. \quad (2)$$

A call option with  $x > 0$ ,  $x = 0$  and  $x < 0$  is said to be in-the-money, at-the-money, and out-of-the-money, respectively. Following Li (2008), we call equation (1) the dimensionless Black-Scholes formula.

The dimensionless Black-Scholes formula implicitly gives  $v$  as a function of  $c$  and  $x$  and is the starting point for our successive over-relaxation method. This dimensionless formula clearly states that the Black-Scholes formula is essentially a relation between three dimensionless quantities, namely the normalized price  $c$ , the integrated volatility  $v$  and the moneyness  $x$ . Given observed values of  $S, t, r, T, K, \delta$  and option price  $C$ , we first calculate  $c$  according to  $c = C/(Se^{-\delta(T-t)})$  and  $x$  according to equation (2), and then compute  $v$  through some algorithm. The implied volatility  $\sigma$  can then be obtained by dividing  $v$  by  $\sqrt{T-t}$ .

Figure 1 gives a surface plot for the normalized call option price  $c$  when the moneyness  $x$  ranges from  $-3$  to  $3$ , and the volatility  $v$  ranges from  $0$  to  $6$ . As we see, when  $|x|/v$  is large,  $c$  is very insensitive to  $v$  and the inversion is not very meaningful because a tiny change in  $c(x, v)$  due to measurement error can give rise to a huge change in  $v$ . Thus we will omit these regions in the numerical implementation of our algorithms in Section 5 by imposing the restriction  $|x|/v \leq 3$ .

We only need to consider inverting call options because by the put-call parity in the Black-Scholes theory (see Stoll 1969), we have  $C = P + Se^{-\delta(T-t)} - Ke^{-r(T-t)}$ . Writing out in normalized call and put prices, we have

$$c(x, v) = p(x, v) + 1 - e^{-x}, \quad (3)$$

where the normalized put price is defined as the ratio of the actual put price  $P$  and the quantity  $Se^{-\delta(T-t)}$ . If we are given a put option with moneyness  $x$  and normalized put price  $p$ , we will simply invert a call option with moneyness  $x$  and normalized call price  $c = p + 1 - e^{-x}$ .

Finally, we only need to consider options with moneyness  $x \leq 0$ . This is because we have the following symmetry, which we call the ‘‘in-out’’ duality:

$$c(-x, v) = e^x c(x, v) + 1 - e^x. \quad (4)$$

If we need to invert the Black-Scholes formula for an option with moneyness  $x \geq 0$  and normalized call price  $c$ , we can as well use the right hand side of the above equation to invert its dual option with moneyness  $x' = -x \leq 0$  and normalized call price  $c' = e^x c + 1 - e^x$ . This is also in line with industry practice of treating out-of-the-money options as more informative than in-the-money options. Another rationale to invert out-of-the-money options is the following. Letting  $\widehat{v}$  be the estimate of the true volatility  $v_*$  that one gets from inverting the dual out-of-the-money option when  $x > 0$ , then the “in-out” duality gives

$$|c(-x, \widehat{v}) - c(-x, v_*)| = e^x |c(x, \widehat{v}) - c(x, v_*)| > |c(x, \widehat{v}) - c(x, v_*)|.$$

That is, if we use “in-out” duality to compute the implied volatility for an in-the-money option, the error in terms of price for the in-the-money option is always smaller than that for its dual out-of-the-money options.

### 3. The Successive Over-relaxation Method

#### 3.1. The SOR algorithm

In this section, we establish a numerical method to compute the implied volatility which is based on the idea of successive over-relaxation. We also analyze the properties of the new method, including global well-definedness, local convergence, as well as global convergence.

We will use  $n(\cdot)$  to denote the standard normal density function. The following functions will play important roles, so we give them specific names in order to shorten expressions:

$$n^+(x, v) = n(x/v + v/2), \tag{5}$$

$$N^+(x, v) = N(x/v + v/2), \tag{6}$$

$$N^-(x, v) = e^{-x} N(x/v - v/2), \tag{7}$$

$$c^+(x, v) = N^+(x, v) + N^-(x, v). \tag{8}$$

Notice that the dimensionless Black-Scholes formula now becomes  $c(x, v) = N^+(x, v) - N^-(x, v)$ . By the discussion in Section 2, we only consider  $x \leq 0$  and  $v > 0$ . The following lemma is very useful later when we analyze the properties of our method. Because of the well-known discontinuity of the Black-Scholes formula, we consider the  $x < 0$  and  $x = 0$  cases separately.

**Lemma 3.1.** *When  $x < 0$ , we have*

- 1)  $N^+(x, v)$  is strictly increasing in  $v$ . Furthermore,  $N^+(x, 0^+) = 0$  and  $N^+(x, +\infty) = 1$ ;
- 2)  $N^-(x, v)$  is strictly increasing in  $v$  for  $v \leq \sqrt{2|x|}$  and decreasing in  $v$  for  $v > \sqrt{2|x|}$ . Furthermore,  $N^-(x, 0^+) = N^-(x, +\infty) = 0$ ;
- 3)  $c(x, v)$  is strictly increasing in  $v$ , strictly convex in  $v$  for  $v < \sqrt{2|x|}$  and strictly concave in  $v$  for  $v > \sqrt{2|x|}$ . Furthermore,  $c(x, 0^+) = 0$  and  $c(x, +\infty) = 1$ ;



- 4)  $c^+(x, v)$  is strictly increasing in  $v$ . Furthermore,  $c^+(x, 0^+) = 0$ , and  $c^+(x, +\infty) = 1$ ;  
 5) Let  $v > 0$ ,  $u > 0$  and  $v \neq u$ . Then

$$\frac{u^2}{2|x|} [c^+(x, v) - c^+(x, u)] < c(x, v) - c(x, u) < \frac{v^2}{2|x|} [c^+(x, v) - c^+(x, u)]. \quad (9)$$

When  $x = 0$ , we have

- 1)  $N^+(0, v)$  is strictly increasing in  $v$ . Furthermore,  $N^+(0, 0^+) = 1/2$  and  $N^+(0, +\infty) = 1$ ;  
 2)  $N^-(0, v)$  is strictly decreasing in  $v$ . Furthermore,  $N^-(0, 0^+) = 1/2$  and  $N^-(0, +\infty) = 0$ ;  
 3)  $c(0, v)$  is strictly increasing in  $v$ , and strictly concave in  $v$ . Furthermore,  $c(0, 0^+) = 0$  and  $c(0, +\infty) = 1$ ;  
 4)  $c^+(0, v) \equiv 1$ .

Hereafter, we will suppress the dependence on  $x$  in the functions  $n^+$ ,  $N^+$ ,  $N^-$ ,  $c$ , and  $c^+$  unless there can be a confusion. That is, we will write  $N^+(v)$  for  $N^+(x, v)$ , etc. To introduce our successive over-relaxation method, we need to define the iteration function. Let  $N^{-1}(\cdot)$  denote the inverse function of the cumulative normal distribution  $N(\cdot)$ . Let the moneyness be  $x$ , the observed option price  $c_*$ , and the true implied volatility  $v_*$ . That is,  $c_* = c(x, v_*)$ . For fixed  $c_* = c(x, v_*)$  and  $x$ , define a function  $F(v; x, v_*, \omega)$  as follows

$$F(v; x, v_*, \omega) \equiv c_* + N^-(v) + \omega N^+(v). \quad (10)$$

The parameter  $\omega$  will play the role of the relaxation parameter in our method. The iteration function is given by

$$G(v; x, v_*, \omega) = N^{-1}\left(\frac{F(v; x, v_*, \omega)}{1 + \omega}\right) + \sqrt{\left[N^{-1}\left(\frac{F(v; x, v_*, \omega)}{1 + \omega}\right)\right]^2 + 2|x|}. \quad (11)$$

Notice that although we write  $G$  and  $F$  as functions of  $v_*$ , they only depend on  $v_*$  through  $c_* = c(x, v_*)$ . In order for  $G(v; x, v_*, \omega)$  to be well-defined, we will require  $\omega > -1$  throughout the paper. We will write  $G(v; x, v_*, \omega)$  as  $G(v)$  unless we want to emphasize the dependence of  $G$  on  $x$  or  $\omega$ . Similarly for  $F(v; x, v_*, \omega)$ . Our problem is to compute the  $v_*$  from the observed  $c_*$  and  $x$ . For a given sequence  $v_k$  of implied volatility estimates, we will write  $c_k$  for the sequence  $c(x, v_k)$ . Our method of finding  $v_*$  is the following:

1. Select an initial point  $v_0$  and a fixed relaxation parameter  $\omega$ ;
2. (SOR) After obtaining  $v_k$ , compute  $v_{k+1}$  from the following equation

$$v_{k+1} = G(v_k); \quad (12)$$

3. Stop when  $|v_k - v_*| < \epsilon$  or  $|c_k - c_*| < \epsilon$ , for some  $\epsilon$  small.

We will call the above algorithm the SOR algorithm. This algorithm is an interesting application of the successive over-relaxation idea to financial problems where the underlying relations

are nonlinear. The successive over-relaxation method was first developed by Young and others around 1950. The most well-known application of successive over-relaxation is to solve systems of linear equations  $\mathbf{Ax} = \mathbf{b}$ . In finance, the PSOR (projected successive over-relaxation) algorithm is widely used in solving partial differential equations for derivative prices. In Nash (1990), Young reviews the historical development of iterative methods. This book also contains an interesting account of how many iterations it took him to finally get his dissertation published in 1954 (Young 1954).

Some simple algebra shows that equation (12) is equivalent to

$$c_* + N^-(x, v_k) + \omega N^+(x, v_k) = (1 + \omega)N^+(x, v_{k+1}), \quad k = 0, 1, \dots \quad (13)$$

By Lemma 3.1,  $c(x, v)$  is strictly increasing in  $v$ , hence  $v_*$  is the unique fixed point of  $G(v)$ . This in turn implies that if the SOR algorithm converges, it will converge to  $v_*$ .

It is helpful to understand the SOR algorithm without relaxation. When  $\omega$  is set to be 0, equation (13) reduces to

$$c_* + N^-(x, v_k) = N^+(x, v_{k+1}), \quad k = 0, 1, \dots \quad (14)$$

That is, we split the Black-Scholes formula into two parts and try to find a fixed point for the above iteration. The equation also shows the need to introduce the relaxation parameter. When  $x$  is very negative or  $v_*$  is very large,  $N^-(x, v_k)$  can be very small so the convergence could be extremely slow. The relaxation parameter  $\omega$  helps in these situations.

A well-known result is that if  $G(v)$  is locally contracting (that is, locally Lipschitz around  $v_*$  with coefficient strictly less than 1), then for  $v_0$  sufficiently close to  $v_*$ , the SOR algorithm will converge to  $v_*$ . See, for example, Isaacson and Keller (1994). In particular, the local Lipschitz condition can be quickly checked by the first-order derivative of  $G(v)$  at  $v_*$ . Lemma 3.4 below gives the expression of  $G'(v_*)$ . If in addition,  $G(v)$  is globally contracting (that is, globally Lipschitz with coefficient strictly less than 1), then starting from any  $v_0$ , the SOR algorithm converges to  $v_*$ .

Unfortunately, in general the function  $G(v)$  does not have these global or even local Lipschitz properties. In fact, it is even possible for  $G(v)$  to be not defined. From the definition of  $G(v)$ , this happens when the following condition is violated

$$0 < F(v; x, v_*, \omega) < 1 + \omega. \quad (15)$$

Because a thorough understanding of the function  $G(v)$  is crucial, we plot the function  $G(v)$  and its derivative in  $v$  in Figure 2 for five different parameter combinations  $(x, v_*, \omega)$ . For ease of exposition, we fix  $x = -0.5$  and vary  $v_*$  and  $\omega$ . Figure 2 shows that all of the following five behavior of  $G(v)$  can occur, corresponding to the five rows of the subplots:

1.  **$G(v)$  is globally well-defined and globally contracting.** This corresponds to the first row of Figure 2, where  $v_* = 0.5$  and  $\omega = 1$ . This is the best case scenario where we can start from any positive  $v_0$  and the SOR algorithm always converges.

2.  $G(v)$  is globally well-defined, not globally contracting, but locally contracting around  $v_*$ . This corresponds to the second row of Figure 2, where  $v_* = 1.2$  and  $\omega = 0$ .
3.  $G(v)$  is globally well-defined, but not locally contracting around  $v_*$ . This corresponds to the third row of Figure 2, where  $v_* = 1.8$ ,  $\omega = -0.28$ , and  $G'(v_*) \doteq -1.32$ .
4.  $G(v)$  is not globally well-defined, but locally contracting around  $v_*$ . This corresponds to the fourth row of Figure 2, where  $v_* = 2.4$ ,  $\omega = -0.1$ , and  $G'(v_*) \doteq -0.89$ .
5.  $G(v)$  is not globally well-defined, and not locally contracting around  $v_*$ . This corresponds to the last row of Figure 2, where  $v_* = 1.2$ ,  $\omega = -0.66$ , and  $G'(v_*) \doteq -2.47$ .

In the rest of this section, we will analyze the global well-definedness property, local convergence properties, as well as the global convergence properties of the SOR algorithm.

### 3.2. Global well-definedness property of the SOR algorithm

We will look at well-definedness first. It is extremely difficult to analyze local well-definedness for a fixed starting point  $v_0$ . The following numerical example illustrates this point. Let  $x = -0.5$ ,  $v_* = 2.5$  and  $\omega = -0.1$ . Then  $G(v)$  is not defined for  $v \in [0.548467, 1.46434]$ . This means that we cannot start the SOR algorithm from a point in this region. However, if  $v_0 = 0.03$ , then the sequence jumps past this bad region to  $v_1 = 2.2285$  and converges to  $v_*$ . Thus, in the following we look for a global well-definedness condition which guarantees that the sequence  $v_k$  is well-defined regardless of the initial point  $v_0$ . We will focus on  $x < 0$  since we show later that the well-definedness property with  $x = 0$  is quite simple to analyze even for a fixed starting point  $v_0$ .

First we define a function  $\tilde{v} = \tilde{v}(\omega; x)$  as follows. For any  $x \leq 0$  and  $-1 < \omega < 1$ , we let

$$\tilde{v} = \tilde{v}(\omega; x) = \sqrt{2|x|} \sqrt{\frac{1+\omega}{1-\omega}}. \quad (16)$$

We will often simply write  $\tilde{v}$  or  $\tilde{v}(\omega)$  for  $\tilde{v}(\omega; x)$ . We first establish two lemmas.

**Lemma 3.2.** *Let  $x < 0$ . Then, we have  $\lim_{v \rightarrow 0^+} F(v) = c_*$  and  $\lim_{v \rightarrow +\infty} F(v) = c_* + \omega$ . Also, if  $\omega \in (-1, 1)$ , then  $F(v)$  strictly increases on  $(0, \tilde{v})$  and strictly decreases on  $(\tilde{v}, \infty)$ . If  $\omega \geq 1$ , then  $F(v)$  strictly increases for all  $v > 0$ .*

**Lemma 3.3.** *Let  $x < 0$ . Suppose  $\omega \in (-1, 1)$ . Define*

$$H(\omega) = H(\omega; x, v_*) \equiv c_* + N^-(\tilde{v}(\omega)) + \omega N^+(\tilde{v}(\omega)) - (1 + \omega). \quad (17)$$

*Then,  $H(\omega)$  has a unique root  $\hat{\omega}$  in  $(-1, 1)$ . Furthermore, if  $\omega \leq \hat{\omega}$ , then  $H(\omega) \geq 0$ , and if  $\omega > \hat{\omega}$ , then  $H(\omega) < 0$ .*

Notice that  $\hat{\omega}$  is implicitly a function of  $x$  and  $v_*$ . For each fixed  $\omega \in (-1, 1)$ , the function  $F(v)$  takes its maximum at the point  $\tilde{v}$  and for global well-definedness we would need  $H(\omega) < 0$ .

The following theorem gives a necessary and sufficient condition for global well-definedness of the SOR algorithm when  $x < 0$ . The  $x = 0$  case is deferred to Theorem 3.6 because a separate analysis is needed for this simpler case.

**Theorem 3.1. (global well-definedness)** *Let  $x < 0$ . When  $\omega \geq 1$ ,  $\{v_k\}$  in the SOR algorithm is well-defined for any  $v_0 \in \mathbb{R}_+$ . When  $-1 < \omega < 1$ ,  $\{v_k\}$  is well-defined for any  $v_0 \in \mathbb{R}_+$  if and only if  $\omega > \hat{\omega}$  and  $\omega \geq -c_*$ .*

While the global well-definedness condition guarantees that the SOR algorithm produces a well-defined sequence, it does not guarantee that such an  $\omega$  will make the SOR algorithm convergent. In the following, we study the local and global convergence properties of the SOR algorithm.

### 3.3. Local convergence properties of the SOR algorithm

We will first analyze the local convergence properties. We say that the SOR algorithm **converges locally** if there exists a neighborhood  $V_\epsilon$  of  $v_*$ , such that for any  $v_0 \in V_\epsilon$ , the SOR algorithm converges. Let us introduce two functions  $\Phi(v; x)$  and  $\Psi(v; x)$  defined as follows

$$\Phi(v; x) = \frac{v^2 - 2|x|}{v^2 + 2|x|}, \quad \text{and} \quad \Psi(v; x) = \frac{-2|x|}{v^2 + 2|x|}. \quad (18)$$

We will write  $\Phi(v)$  instead of  $\Phi(v; x)$  unless we want to emphasize the dependence on  $x$ . Similarly for  $\Psi(v; x)$ . The two functions are related through  $\Phi(v) = 1 + 2\Psi(v)$ . Notice that when  $x < 0$ ,  $-1 < \Phi(v) < 1$ ,  $-1 < \Psi(v) < 0$ , and both  $\Phi(v)$  and  $\Psi(v)$  are strictly increasing in  $v$ . When  $x = 0$ ,  $\Phi(v) \equiv 1$  and  $\Psi(v) \equiv 0$ . These two functions  $\Phi(v)$  and  $\Psi(v)$  play extremely important roles in the analysis that follows.

The following lemma gives properties of the iteration function  $G(v)$ . In particular, it says that  $G(v)$  is strictly increasing on any connected open subsets of  $\text{Dom}G \cap \{v : \omega > \Phi(v)\}$ .

**Lemma 3.4.** *Let  $x \leq 0$ . Let  $\text{Dom}G$  denote the open set of  $v$  for which  $G(v)$  is well-defined in a neighborhood of  $v$ . For  $v \in \text{Dom}G$ , we have*

$$\frac{dG(v)}{dv} = \frac{\omega - \Phi(v)}{1 + \omega} \frac{G(v)^2 n^+(v)}{v^2 n^+(G(v))} \frac{v^2 + 2|x|}{G(v)^2 + 2|x|}. \quad (19)$$

Furthermore,  $v_* \in \text{Dom}G$ , and

$$G'(v_*) = \frac{\omega - \Phi(v_*)}{1 + \omega}. \quad (20)$$

We need another technical lemma to establish the local convergence property. Define a new function  $Q(v; x, v_*, \omega) : \text{Dom}G \mapsto \mathbb{R}$  as follows:

$$Q(v) = Q(v; x, v_*, \omega) \equiv c(v) + c(G(v)) - 2c_*. \quad (21)$$

Notice that  $Q(v_*) = 0$ . Suppose  $G(v)$  and  $G(G(v))$  are both defined. Taking the sum of two consecutive iteration equations, we have

$$Q(v) = (1 + \omega)[N^+(v) - N^+(G(G(v)))]. \quad (22)$$

Thus, the sign of  $Q(v_k)$  controls whether  $v_{k+2}$  is larger than  $v_k$  or not. In particular, to help convergence, we would like to see  $Q(v) > 0$  if  $v > v_*$  and  $Q(v) < 0$  if  $v < v_*$ . The following lemma gives the derivatives of  $Q(v)$ . It is useful in proving both Theorem 3.2 on local convergence and Theorem 3.9 on global convergence.

**Lemma 3.5.** *For any  $v \in \text{Dom}G$ , the derivative of  $Q(v)$  is given by*

$$Q'(v) = \frac{n^+(v)}{\frac{|x|}{G(v)^2} + \frac{1}{2}} \left[ \left( \frac{|x|}{v^2} + \frac{|x|}{G(v)^2} \right) + \frac{\omega}{1 + \omega} \right]. \quad (23)$$

*In particular, when  $v = v_*$ , we have*

$$Q'(v_*) = \frac{2n^+(v_*)}{1 + \omega} [\omega - \Psi(v_*)]. \quad (24)$$

*Furthermore, when  $\omega = \Psi(v_*)$ , we have  $Q'(v_*) = Q''(v_*) = 0$  and*

$$Q'''(v_*) = \frac{2n^+(v_*)|x|}{v_*^4(v_*^2 + 2|x|)} (12v_*^2 - v_*^4 + 4x^2).$$

The above two lemmas give the following theorem, which establishes conditions for the SOR algorithm to have local convergence.

**Theorem 3.2. (conditions for local convergence)** *Let  $\omega > -1$ .*

*1) When  $x < 0$ , a necessary condition for the SOR algorithm to converge locally is  $\omega \geq \Psi(v_*)$  and a sufficient condition is  $\omega > \Psi(v_*)$ . If  $\omega = \Psi(v_*)$ , a further necessary condition for local convergence is  $12v_*^2 - v_*^4 + 4x^2 \geq 0$  while a further sufficient condition is  $12v_*^2 - v_*^4 + 4x^2 > 0$ .*

*2) When  $x = 0$ , the necessary and sufficient condition for the SOR algorithm to converge locally is  $\omega > 0$ .*

Ordinarily, to check these conditions, we would need to know the value of  $v_*$ , which is the goal of the SOR algorithm. However, in the next section, we will introduce two acceleration techniques which employ the above theorem but avoid the knowledge of  $v_*$ . In one of the techniques, we dynamically vary  $\omega$  in each iteration  $k$ , such that  $\omega_k$  is eventually larger than  $\Psi(v_*)$ . In the other technique, we set  $\omega$  to be the constant 1 in each iteration so  $\omega \geq \Psi(v_*)$  is trivially satisfied.

Although we give a detailed proof of Theorem 3.2 in the Appendix, Theorem 3.2 (except for the borderline case  $\omega = \Psi(v_*)$ ) is actually a special application of a well-known result in numerical analysis. See, for example, Isaacson and Keller (1994) for a more general statement. Indeed, in our proof we make little use of the special structure of the SOR algorithm. Notice that

when  $x < 0$ , the condition  $\omega > \Psi(v_*)$  is only sufficient but not necessary for local convergence, and the condition  $\omega \geq \Psi(v_*)$  is necessary but not sufficient. This is because the borderline case  $\omega = \Psi(v_*)$  is a little bit complicated. The SOR algorithm can either locally converge or diverge in this case. When both  $\omega = \Psi(v_*)$  and  $12v_*^2 - v_*^4 + 4x^2 = 0$  hold, it can be shown that  $Q^{(4)}(v_*) = 0$  since in this case it is proportional to  $[3v_*^6 - 6v_*^4x + 24x^3 - 4v_*^2x(8 + 3x)](12v_*^2 - v_*^4 + 4x^2)$ . Thus, a further condition on  $Q^{(5)}(v_*)$  is needed to guarantee local convergence. We do not analyze this double borderline case in more detail because we will never encounter it in our actual implementation.

The following function will play a pivotal role in the analysis of the SOR algorithm:

$$\phi(u, v) = \phi(u, v; x) \equiv \begin{cases} \frac{N^-(u) - N^-(v)}{N^+(v) - N^+(u)} & \text{if } u \neq v, \\ \Phi(u; x) & \text{if } u = v. \end{cases} \quad (25)$$

**Lemma 3.6.** *The function  $\phi(u, v)$  is symmetric in  $u$  and  $v$  and continuous on  $\mathbb{R}_+^2$ . Furthermore, if  $x < 0$ , then  $|\phi(u, v)| < 1$ , and for any fixed  $v$ ,  $\phi(u, v)$  is continuously differentiable and strictly increasing in  $u$ , with the derivative given by*

$$\phi_1(u, v) = \begin{cases} \frac{n^+(u)}{[N^+(u) - N^+(v)]^2} \left\{ \frac{x}{u^2} [c(u) - c(v)] + \frac{1}{2} [c^+(u) - c^+(v)] \right\} & \text{if } u \neq v, \\ \frac{4|x|v}{(v^2 + 2|x|)^2} & \text{if } u = v. \end{cases} \quad (26)$$

Similarly, for any fixed  $u$ ,  $\phi(u, v)$  is continuously differentiable and strictly increasing in  $v$ , with  $\phi_2(u, v) = \phi_1(v, u)$ . If  $x = 0$ , then  $\phi(u, v) \equiv 1$  on  $\mathbb{R}_+^2$ .

The following three lemmas are useful in analyzing both the local and global behavior of the sequence  $\{v_k\}$  from the SOR algorithm.

**Lemma 3.7.** *Let  $\{v_k\}$  be a well-defined SOR sequence and  $v_k \neq v_*$  for all  $k \in \mathbb{N}$ .*

- 1) *If  $v_k < v_*$ , then  $v_{k+1} > v_k$ ;*
- 2) *If  $v_k > v_*$ , then  $v_{k+1} < v_k$ .*

**Lemma 3.8.** *Let  $\{v_k\}$  be a well-defined SOR sequence and  $v_k \neq v_*$  for all  $k \in \mathbb{N}$ . Then for any  $k$ , we have  $\omega \neq \phi(v_k, v_*)$ , and*

- 1) *If  $\omega > \phi(v_k, v_*)$ , then  $v_k$  and  $v_{k+1}$  are on the same side of  $v_*$ ;*
- 2) *If  $\omega < \phi(v_k, v_*)$ , then  $v_k$  and  $v_{k+1}$  are on the opposite side of  $v_*$ .*

**Lemma 3.9.** *Let  $\{v_k\}$  be a well-defined SOR sequence and  $v_k \neq v_*$  for all  $k \in \mathbb{N}$ .*

- 1) *If  $1 + 2\omega > \phi(v_k, v_{k+1})$ , then  $v_{k+1}$  and  $v_{k+2}$  are on the same side of  $v_k$ ;*
- 2) *If  $1 + 2\omega < \phi(v_k, v_{k+1})$ , then  $v_{k+1}$  and  $v_{k+2}$  are on the opposite side of  $v_k$ ;*
- 3) *If  $1 + 2\omega = \phi(v_k, v_{k+1})$ , then  $v_{k+2} = v_k$ .*

The next theorem states that if the SOR algorithm converges, then it is either eventually monotone or eventually oscillating around  $v_*$ .

**Theorem 3.3. (local convergence pattern)** *Let  $x \leq 0$ . Suppose that  $v_k$  from the SOR algorithm converges to  $v_*$  and  $v_k \neq v_*$  for all  $k \in \mathbb{N}$ . Then there exists a  $k_0 \in \mathbb{N}$ , such that if  $k > k_0$ , the following is true:*

- 1) *If  $\omega \geq \Phi(v_*)$ , then  $v_k$  approaches  $v_*$  monotonically;*
- 2) *If  $\omega < \Phi(v_*)$ , then  $v_k$  is oscillating around  $v_*$ . More specifically, for any  $m \in \mathbb{N}$ , we have  $v_{k_0+2m} < v_{k_0+2m+2} < v_*$  and  $v_{k_0+2m-1} > v_{k_0+2m+1} > v_*$ .*

We give two more results on the patterns of the SOR algorithm below.

**Theorem 3.4. (local pattern for  $|c_k - c_*|$ )** *Suppose that the SOR algorithm converges to  $v_*$  with  $v_k \neq v_*$  for all  $k \in \mathbb{N}$ . Then the sequence  $|c_k - c_*|$  eventually monotonically decreases to 0.*

**Theorem 3.5. (local pattern for  $|v_k - v_*|$ )** *Suppose that the SOR algorithm converges to  $v_*$  and  $v_k \neq v_*$  for all  $k \in \mathbb{N}$ . When  $\omega > \Psi(v_*)$ ,  $|v_k - v_*|$  is eventually monotonically decreasing. When  $\omega = \Psi(v_*)$ ,  $|v_k - v_*|$  is eventually monotonically decreasing if and only if  $v_* = \sqrt{2|x|}$ .*

When  $\omega = \Psi(v_*)$  and  $v_* \neq \sqrt{2|x|}$ ,  $|v_k - v_*|$  no longer monotonically decreases to 0. However, in this case,  $v_k$  oscillates around  $v_*$  and the two subsequences above and below  $v_*$  approach  $v_*$  monotonically.

### 3.4. Global convergence properties of the SOR algorithm

In the actual implementation it is hard to know whether the initial  $v_0$  is close enough to  $v_*$  or not. Thus, in the following we study the global convergence properties. That is, in the analysis below, we do not assume that  $v_0$  is close to  $v_*$ . Furthermore, in the majority of the analysis below, we also do not assume that the global well-definedness condition is satisfied.

The following theorem completely characterizes the well-definedness and convergence patterns for the SOR algorithm when  $x = 0$ . Recall from Theorem 3.2 that a necessary condition for the algorithm to converge when  $x = 0$  is  $\omega > 0$ .

**Theorem 3.6. (SOR algorithm when  $x = 0$ )** *Assume  $x = 0$  and  $v_0 \neq v_*$ . We have three cases:*

- 1)  $\omega > 1$ . *In this case, the SOR algorithm is globally well-defined and convergent. Furthermore, if  $v_0 > v_*$ , then  $v_k$  strictly decreases to  $v_*$ . If  $v_0 < v_*$ , then  $v_k$  strictly increases to  $v_*$ .*
- 2)  $\omega = 1$ . *In this case, the SOR algorithm is globally well-defined and convergent. In fact,  $v_k = v_*$  for any  $k \geq 1$ .*
- 3)  $0 < \omega < 1$ . *In this case, the SOR algorithm is well-defined if and only if  $G(v_0)$  is defined. Furthermore, if  $G(v_0)$  is defined, then the SOR algorithm is convergent and  $\{v_k\}$  is immediately oscillating around  $v_*$ .*



The following analysis will focus on  $x < 0$ . We first consider the baseline case  $\omega = 0$  in the SOR algorithm. The following theorem shows that in this case, the SOR algorithm is always globally well-defined and convergent provided that  $v_1$  is defined.

**Theorem 3.7. (SOR algorithm when  $\omega = 0$ )** *Let  $x < 0$  and  $\omega = 0$ . Suppose  $c_* + N^-(v_0) < 1$ , then  $\{v_k\}$  in the SOR algorithm is globally well-defined and converges to  $v_*$ . Furthermore, the condition  $c_* + N^-(v_0) < 1$  is always satisfied when  $v_* \leq \sqrt{2|x|}$ , and satisfied if  $v_0 \geq v_*$  or  $v_0 \leq 2|x|/v_*$  when  $v_* > \sqrt{2|x|}$ .*

We will now consider a general relaxation parameter  $\omega$ . Since a necessary condition for the SOR algorithm to converge locally is  $\omega \geq \Psi(v_*)$ , we only consider such  $\omega$ 's.

It turns out that when  $\omega \geq \Phi(v_*)$ , the SOR algorithm is globally convergent if the sequence is well-defined. By Theorem 3.3, the SOR sequence eventually monotonically approaches  $v_*$  if it converges in this case.

**Theorem 3.8. (SOR algorithm when  $\omega \geq \Phi(v_*)$ )** *Let  $x < 0$  and  $v_0 \neq v_*$ . Assume that  $\omega \geq \Phi(v_*)$ . When  $v_0 < v_*$ ,  $\{v_k\}$  from the SOR algorithm is globally well-defined and monotonically increases to  $v_*$ . When  $v_0 > v_*$ , we have the following three cases:*

- 1) *If  $\omega > \phi(v_0, v_*)$ , then  $\{v_k\}$  is globally well-defined and monotonically decreases to  $v_*$ .*
- 2) *If  $\omega = \phi(v_0, v_*)$ , then  $\{v_k\}$  is globally well-defined and  $v_k = v_*$  for all  $k \geq 1$ .*
- 3) *If  $\Phi(v_*) \leq \omega < \phi(v_0, v_*)$ , then  $\{v_k\}$  is globally well-defined if  $v_1$  is defined. Furthermore, if  $v_1$  is defined, then  $v_1 < v_*$  and the sequence monotonically increases to  $v_*$ .*

While the above theorem completely characterizes the behavior of the SOR algorithm when  $\omega \geq \Phi(v_*)$ , we do not have a complete characterization for the  $\omega < \Phi(v_*)$  case. Notice that by Theorem 3.3, the SOR sequence is eventually oscillating around  $v_*$  if it converges when  $\omega < \Phi(v_*)$ .

Nevertheless, we have obtained three sufficient conditions to guarantee convergence that are possibly weaker than  $\omega \geq \Phi(v_*)$ . These are given in the three theorems that follow. They show that to have global convergence, we only need to slightly strengthen the local convergence condition  $\omega \geq \Psi(v_*)$ , which is the same as  $1 + 2\omega \geq \Phi(v_*)$ .

**Theorem 3.9. (first sufficient condition for global convergence)** *Let  $x < 0$ . Suppose that  $\omega \geq \Psi(\sqrt{2}v_*)$ . Let  $\{v_k\}$  be the SOR sequence, possibly only defined for finitely many  $k$ . Let  $v_{k_0}$  be well-defined and the first point in the sequence  $\{v_k\}$  such that  $\omega < \phi(v_k, v_*)$ . Then the whole sequence  $\{v_k\}$  is well-defined if and only if  $G(v_{k_0})$  is defined. Furthermore, if  $G(v_{k_0})$  is defined, then the SOR algorithm converges to  $v_*$ .*

Whether the sequence is eventually monotone or oscillating in Theorem 3.9 depends on whether  $\omega \geq \Phi(v_*)$ . Also, if  $k_0$  in the theorem does not exist, then the SOR algorithm is globally well-defined and converges monotonically to  $v_*$ . Another consequence of the above theorem is that since  $\Psi(\sqrt{2}v_*) < 0$  when  $x < 0$ , the SOR algorithm always converges when  $\omega \geq 0$  provided that  $\{v_k\}$  is well-defined.



**Theorem 3.10. (second sufficient condition for global convergence)** *Let  $x \leq 0$ . Suppose that the SOR sequence  $\{v_k\}$  is well-defined and bounded above by some  $\bar{v} \geq v_*$ . Then the sequence converges to  $v_*$  if  $1 + 2\omega > \phi(v_*, \bar{v})$ .*

For the last theorem, we first need to establish some properties of  $\phi(u, G(u))$ .

**Lemma 3.10.** *Let  $x < 0$ ,  $-1 < \omega < 1$ , and  $u \in \text{Dom}G$ . If  $u < G(u)$ , there exists  $K > 0$ , such that*

$$\phi(u, G(u))' > K(\omega - \Phi(u)).$$

*As a result, when  $\omega < \Phi(v_*)$ , we have  $\phi(u, G(u))' > 0$  if  $u < \tilde{v}$ , where  $\tilde{v}$  is the unique point satisfying  $\omega = \Phi(\tilde{v})$ .*

**Theorem 3.11. (third sufficient condition for global convergence)** *Let  $x \leq 0$ . Suppose that  $\omega$  satisfies the global well-definedness condition. Then for any  $v_0 \in \mathbb{R}_+$ , the SOR algorithm converges to  $v_*$  if*

$$1 + 2\omega > \sup_{\tilde{v} < u < v_*} \phi(u, G(u)). \quad (27)$$

Figure 3 summarizes the results of this section by giving all the possible convergence patterns for the SOR algorithm. The top half of Figure 3 plots all the possible convergence patterns when  $x = 0$ . Notice that by Theorem 3.6 all the conditions above each subplot are both sufficient and necessary for that particular convergence pattern to occur. The bottom half of Figure 3 plots all the possible convergence patterns when  $x < 0$ . Notice that Theorem 3.8 only guarantees convergence when  $\omega \geq \Phi(v_*)$ . We box two conditions in two of the subplots to indicate that these conditions are only necessary and not sufficient for these two particular global convergence patterns.

## 4. Convergence acceleration methods

### 4.1. The convergence order of the SOR algorithm

We next take a look at the speed of convergence. We make use of the following common notions of convergence speed. Suppose that the sequence  $v_k$  converges to  $v_*$ . We say that this sequence converges linearly if there exists a number  $\mu \in (0, 1)$  such that

$$\lim_{k \rightarrow +\infty} \frac{|v_{k+1} - v_*|}{|v_k - v_*|} = \mu. \quad (28)$$

If the above limit exists with  $\mu = 0$ , we say that the sequence converges superlinearly. If  $\mu = 1$ , we say that the sequence converges sublinearly. The number  $\mu$  is usually called the convergence

rate. A smaller  $\mu$  means faster convergence. We say that the sequence converges with order  $q$  if for some  $q \in \mathbb{N}$ ,

$$\lim_{k \rightarrow +\infty} \frac{|v_{k+1} - v_*|}{|v_k - v_*|^q} = \mu \text{ with } \mu > 0. \quad (29)$$

In particular, convergence with order 2 is called quadratic convergence. The following proposition shows that the SOR algorithm in the last section usually has a linear order of convergence.

**Theorem 4.1. (convergence order of the SOR algorithm)** *Let  $x \leq 0$ . Suppose that  $v_k$  from the SOR algorithm converges to  $v_*$  and  $v_k \neq v_*$  for all  $k \in \mathbb{N}$ . Then,*

1) *If  $\omega \neq \Phi(v_*)$  and  $\omega > \Psi(v_*)$ , the sequence  $\{v_k\}$  has a linear order of convergence, with convergence rate  $\mu = |G'(v_*)|$ .*

2) *If  $\omega = \Phi(v_*)$ , then the sequence converge superlinearly.*

3) *If  $\omega = \Psi(v_*)$ , then the sequence converges sublinearly.*

The above theorem immediately implies the following. Consider a fixed option with money-ness  $x \leq 0$  and implied volatility  $v_*$ . Let  $v_k^A$  and  $v_k^B$  be two convergent SOR sequences associated with initial values  $v_0^A$  and  $v_0^B$ , and relaxation parameters  $\omega^A$  and  $\omega^B$ , respectively. Suppose  $v_k^i \neq v_*$  for all  $k \in \mathbb{N}$ ,  $i = A, B$ . If  $\mu^A < \mu^B$ , then Theorem 4.1 implies that regardless of the relative accuracy of  $v_0^A$  and  $v_0^B$ , there exists  $k_0 \in \mathbb{N}$ , such that for all  $k > k_0$ ,  $|v_k^A - v_*| < |v_k^B - v_*|$ . Thus, in selecting  $\omega$ , we should try to select it such that it is as close to  $\Phi(v_*)$  as possible. In particular, if  $\omega$  happens to be  $\Phi(v_*)$ , then we have superlinear convergence.

A linear convergence order is not very efficient. The rest of this section improves the convergence order of the SOR algorithm through two convergence acceleration techniques: dynamic relaxation and transformation of sequence.

## 4.2. Dynamic relaxation (SOR-DR)

Although the theory tells us that we should select  $\omega$  to be  $\Phi(v_*)$ , in practice we do not know the value of  $v_*$ , because after all, the precise value of  $v_*$  is the goal of the SOR algorithm. This difficulty can be overcome by a dynamic relaxation technique which approximates  $\Phi(v_*)$  adaptively with  $\Phi(v_k)$ . More specifically, we modify the SOR algorithm to the following, which we label as the SOR-DR algorithm.

1. Select an initial point  $v_0$  and set  $\omega_0 = \Phi(v_0)$ ;
2. (SOR-DR) After obtaining  $v_k$ , compute  $v_{k+1}$  from the following equation

$$v_{k+1} = G(v_k; x, v_*, \omega_k), \quad (30)$$

and set  $\omega_{k+1} = \Phi(v_{k+1})$ ;

3. Stop when  $|v_k - v_*| < \epsilon$  or  $|c_k - c_*| < \epsilon$ , for some  $\epsilon$  small.

The next theorem shows that if  $v_1$  is defined, then the SOR-DR algorithm always converges monotonically to  $v_*$ .

**Theorem 4.2. (convergence properties of the SOR-DR algorithm)** *When  $x = 0$ , the SOR-DR algorithm is globally well-defined and  $v_k = v_*$  for all  $k \in \mathbb{N}$ . When  $x < 0$  and  $v_0 > v_*$ ,  $v_k$  is globally well-defined and monotonically decreases to  $v_*$ . When  $x < 0$  and  $v_0 < v_*$ , the SOR-DR algorithm is globally well-defined if  $v_1$  is defined. If  $v_1$  is defined, then  $v_1 > v_*$ , and  $v_k$  monotonically decreases to  $v_*$  afterwards.*

In addition to having nice global well-definedness and convergence properties, the SOR-DR algorithm has at least quadratic order of convergence.

**Theorem 4.3. (convergence order of the SOR-DR algorithm)** *Assuming  $v_k \neq v_*$  for all  $k$ , the SOR-DR algorithm has at least a quadratic order of convergence if  $v_1$  is defined.*

Theorem 4.3 is a very interesting result, because while the Newton-Raphson algorithm uses derivative information of  $c(x, v)$ , the SOR-DR algorithm never requires the calculation of the derivatives. The derivative of  $c(x, v)$  with respect to  $v$  is given by  $n^+(v)$ . For deeply away-from-the-money options,  $n^+(v)$  could be extremely small, often resulting in overflow problems in implementing the Newton-Raphson algorithm. That the SOR-DR algorithm works well for away-from-the-money options is one major advantage over the Newton-Raphson method.

### 4.3. Transformation of sequence (SOR-TS)

While we can obtain quadratic order of convergence through a dynamic relaxation technique, quadratic convergence can also be obtained through a classical transformation of sequence technique. Well-known examples of sequence transformation include Aitken's delta-squared method and Richardson extrapolation. A very good up-to-date reference to this technique is Sidi (2002). We will label the following the SOR-TS algorithm.

1. Select an initial point  $v_0$  and a relaxation parameter  $\omega$ ;
2. (SOR-TS) After obtaining  $v_k$ , compute  $v_{k+1}$  from the following equation

$$v_{k+1} = \alpha_k G(v_k; x, v_*, \omega) + (1 - \alpha_k)v_k, \quad (31)$$

where

$$\alpha_k = \frac{1 + \omega}{1 + \Phi(v_k)}; \quad (32)$$

3. Stop when  $|v_k - v_*| < \epsilon$  or  $|c_k - c_*| < \epsilon$ , for some  $\epsilon$  small.

The iteration step in the SOR-TS algorithm can be more succinctly written as

$$v_{k+1} = M(v_k; x, v_*, \omega), \quad (33)$$

where the iteration function  $M(v; x, v_*, \omega)$  is given by

$$M(v) = M(v; x, v_*, \omega) \equiv \frac{1 + \omega}{1 + \Phi(v; x)} G(v; x, v_*, \omega) + \left(1 - \frac{1 + \omega}{1 + \Phi(v; x)}\right) v. \quad (34)$$

We will use the common practice in numerical analysis of using the term extrapolation generally to include interpolation. That is, we treat extrapolation as a synonym for sequence transformation. The transformation of sequence technique above is a simple extrapolation using only two points  $v_k$  and  $G(v_k)$ . The weight  $\alpha_k$  is chosen carefully so that  $M'(v_*; x, v_*, \omega) = 0$ , which guarantees at least quadratic order of convergence if the SOR-TS algorithm converges. Using the notions in Sidi (2002), the extrapolation is nonlinear, in that the parameter  $\alpha_k$  is not a constant, but rather depends on the sequence  $\{v_k\}$  itself. As Sidi (2002) points out, nonlinear extrapolation is usually needed to improve order of convergence. Also, the sequence transformation is iterative in that for each  $k$ , the extrapolated value  $v_k$  is used to perform the next extrapolation to get  $v_{k+1}$ .

While  $\omega$  is a free parameter in the SOR-TS algorithm, there is some guidance on how to choose a good  $\omega$ . First,  $\omega$  should be chosen such that the sequence  $\{v_k\}$  is well-defined. Thus,  $\omega$  can not be too small. For example, if  $\omega$  is close to  $\Psi(v_*)$ ,  $G(v)$  fails to be globally well-defined. Also,  $\omega$  can not be too large because a too large  $\omega$  might send the extrapolated point  $v_{k+1}$  below 0. The following numerical example illustrates this point. Let  $x = -0.1$ ,  $v_* = 0.1$  and  $\omega = 5$ . Let  $v_0 = 2$ . Then  $G(v_0) = 1.3137$  and the extrapolated point  $v_1 = -0.1617 < 0$ . Another consideration in selecting  $\omega$  is stability. Let  $\hat{v}_k$  be the computed numerical value for  $v_k$ . As Sidi (2002) points out, when  $v_k$  is sufficiently close to  $v_*$ , the round-off error  $|\hat{v}_k - v_k|$  might start to dominate the total error  $|\hat{v}_k - v_*|$ . Furthermore, the round-off error might propagate and make the algorithm unstable if more than necessary many iterations are performed. There are a few possible sources of round-off errors. The first source is the iteration equation. This is due to the inaccuracy in computing the cumulative normal distribution and its inverse, and the effect is largely controlled by the coefficients  $\omega$  and  $1 + \omega$ . For example, a very large  $\omega$  will amplify the errors. This source of round-off error is also present in the SOR-DR algorithm. However, the SOR-TS algorithm has another source of round-off error coming from the sequence transformation. The magnitude of this source is largely controlled by the coefficients  $\alpha_k$  and  $1 - \alpha_k$ .

Because of the above considerations, we recommend setting  $\omega = 1$  always in the SOR-TS algorithm. There are several reasons for this choice. First, the choice of  $\omega = 1$  is good from stability considerations. Second, if  $x = 0$  and  $\omega = 1$ , we immediately have  $G(v_1) = v_*$ , and since  $\alpha_1 = 1$ , we have  $v_1 = v_*$ . That is, the sequence immediately lands on  $v_*$ . Finally, if  $x < 0$  and  $\omega = 1$ , Theorem 3.1 guarantees the global well-definedness of  $G(v)$ , and Theorem 4.4 below in turn guarantees the global well-definedness of  $M(v)$ . The global well-definedness property is extremely useful because it brings in the robustness of the SOR-TS algorithm.

We first establish some useful results for the extrapolated iteration function  $M(v; x, v_*, \omega)$  in Lemma 4.1 below when  $\omega = 1$ . These results will be used in Theorem 4.4.

**Lemma 4.1.** *Let  $x < 0$  and  $\omega = 1$ . The derivative of  $M(v)$  with respect to  $v$  is given by*

$$M'(v) = -\frac{4G|x|}{v^3} + \frac{2|x|}{v^2} + \frac{n^+(v)}{n^+(G)} \frac{1}{\frac{1}{2} + \frac{|x|}{G^2}} \left( \frac{|x|}{v^2} + \frac{2x^2}{v^4} \right), \quad (35)$$

where we have written  $G$  for  $G(v; x, v_*, 1)$ . Furthermore,  $M(v) > 0$  for all  $v \in \mathbb{R}_+$ . Also, we have  $M'(v_*) = 0$ , and

$$M''(v_*) = -\frac{|x|}{2v_*^3(v_*^2 + 2|x|)^2}m(v_*, x)$$

where the function  $m(v, x)$  is given by  $m(v, x) = v^6 - 2(4+x)v^4 - 4x^2v^2 + 8x^3$ . For any fixed  $x < 0$ , there exists a unique  $v_r = v_r(x) \in \mathbb{R}_+$  such that  $m(v_r, x) = 0$ . Furthermore,  $m(v, x) < 0$  when  $v < v_r$ , and  $m(v, x) > 0$  when  $v > v_r$ .

Since  $M'(v_*) = 0$ , we immediately have local convergence. Also, the sign of  $M''(v_*)$  controls the convergence pattern. For global convergence, we need to define two more quantities:

$$M_{\min}(x, v_*) \equiv \inf\{M(v; x, v_*, 1) : v \geq v_*\}, \quad (36)$$

$$B(x, v_*) \equiv \sup\{|M'(v; x, v_*, 1)| : v \geq M_{\min}(x, v_*)\}. \quad (37)$$

Notice that since  $M(v_*) = v_*$ , we have  $M_{\min}(x, v_*) \leq v_*$ . We have the following theorem:

**Theorem 4.4. (convergence properties of the SOR-TS algorithm)** *Let  $x \leq 0$ . Then the SOR-TS algorithm with  $\omega = 1$  is globally well-defined and  $\{v_k\}$  converges to  $v_*$  locally. For any option with  $(x, v_*)$ , the SOR-TS algorithm converges globally for any  $v_0 \in \mathbb{R}_+$  if  $B(x, v_*) < 1$ . Furthermore, if  $x = 0$ , then  $v_k = v_*$  for all  $k \in \mathbb{N}$ . If  $x < 0$  and  $\{v_k\}$  converges to  $v_*$  with  $v_k \neq v_*$  for all  $k \in \mathbb{N}$ , we have*

- 1) *The sequence  $\{v_k\}$  eventually decreases to  $v_*$  if  $v_* < v_r(x)$ .*
- 2) *The sequence  $\{v_k\}$  eventually increases to  $v_*$  if  $v_* > v_r(x)$ .*
- 3) *If  $v_* = v_r(x)$ , the convergence can be either eventually monotone or eventually oscillating.*

The condition  $B(x, v_*) < 1$  is only sufficient for global convergence and not necessary. Through extensive numerical analysis, we conjecture that the SOR-TS sequence converges globally without this condition, but so far we are not able to prove it analytically. In any case, we have numerically verified (details are available upon request) that the condition  $B(x, v_*) < 1$  is satisfied inside a very large domain  $D^-$  on which we will implement our algorithms. That is, inside the domain  $D^-$ , the SOR-TS algorithm with  $\omega = 1$  is globally convergent.

**Theorem 4.5. (convergence order of the SOR-TS algorithm)** *Let  $x \leq 0$ . Assuming the sequence  $\{v_k\}$  from the SOR-TS algorithm with  $\omega = 1$  converges to  $v_*$  with  $v_k \neq v_*$  for all  $k \in \mathbb{N}$ , then the sequence has at least a quadratic order of convergence.*

Figure 4 gives all the convergence patterns for the SOR-DR and SOR-TS ( $\omega = 1$ ) algorithms. Notice that the two algorithms are always globally well-defined, except that when  $v_0 < v_*$  and  $x < 0$ , we need  $G(v_0)$  to be well-defined to guarantee that  $\{v_k\}$  is defined in the SOR-DR algorithm. When  $x = 0$ , the convergence patterns of the two algorithms are exactly the same. They both land on  $v_*$  after just one iteration. When  $x < 0$ , the SOR-DR algorithm always

eventually converges to  $v_*$  monotonically from above. We classify the local convergence behavior of the SOR-TS algorithm according to the sign of  $M''(v_*)$ . The three conditions listed should be interpreted as necessary conditions. For example, a necessary condition for the SOR-TS sequence to eventually decrease to  $v_*$  is that  $M''(v_*) \geq 0$ .

While the above discussion has focused on improving the asymptotic convergence speed of the SOR algorithm by modifying the iteration step, in the actual implementation the initial estimate  $v_0$  is often of crucial importance because in practice only a finite number of iterations can be performed. A good estimate  $v_0$  usually also helps with numerical stability. Therefore, for each option with observed moneyness  $x$  and option price  $c_*$ , we use a rational approximation  $v_0 = v_0(x, c_*)$  for the initial estimate. This will be discussed in detail in the next section.

## 5. Numerical implementation and performance

### 5.1. Numerical implementation with rational approximation enhancement

First, we will describe the domain of inversion. The domain of inversion consists of all options we consider, with different values of moneyness  $x$  and implied total volatility  $v_*$ . We will restrict  $v_* \geq 0.0005$  since volatilities lower than this bound are extremely rare in real financial applications. Options with values  $c_*$  extremely close to 0 or 1 are also excluded. The final inversion domain  $D$  we consider is as follows:

$$D = \{ |x| \leq 3, 0.0005 \leq v_* \leq 6, 0.0005 \leq c_* \leq 0.9995, |x|/v_* \leq 3 \}. \quad (38)$$

Notice that this domain is much larger than those considered by most authors. For example, in Li (2008),  $v_*$  is bounded above by 1,  $|x|$  bounded by 0.5, and  $x/|v_*|$  bounded by 2. Figure 5 plots the domain  $D$  in the two-dimensional  $(c_*, v_*)$ -space. Recall that we will only consider options with  $x \leq 0$  by the “in-and-out” duality. We denote the left half of  $D$  by  $D^-$ , where  $x \leq 0$ . Notice we have also plotted the curve  $M''(v_*) = 0$ , which is the same as  $v_* = v_r(x)$ . By Lemma 4.1, this curve separates the left domain  $D^-$  further into two parts.

Before we look at the SOR-DR and SOR-TS algorithms in the whole domain  $D^-$ , let us look at their performance on a particular option. The option used has moneyness  $x = -0.5$  and volatility  $v_* = 1$ . For the SOR and SOR-TS algorithms, we set  $\omega = 1$  for all the iterations. Table 1 shows the effect of introducing dynamic relaxation or sequence transformation to the SOR algorithm. All the numbers are computed using Mathematica 6.0 with  $(10^{-16})$ -precision arithmetic, except for the rows labeled as 4' and 5', where all the numbers are computed with  $(10^{-50})$ -precision arithmetic. For all three algorithms, the initial estimate  $v_0$  is set to be 0.6 in Panel A and 1.4 in Panel B. While all three algorithms converge to the true  $v_*$ , the SOR-DR and SOR-TS algorithms converge much faster. In each of the two latter algorithms, the number of correct digits roughly double after each iteration, indicating quadratic convergence. For the SOR-DR algorithm, we also give the value of  $\omega_k$  for each iteration. Notice that  $\omega_k$  approaches

$\Phi(v_*) = 0$  quickly, giving rise to quadratic convergence. For the SOR-TS algorithm, we also give the value of  $\alpha_k$  for each iteration. As we see,  $\alpha_k$  approaches a value of 2 quickly. Thus for each iteration, after getting  $G(v_k)$ , the SOR-TS algorithm will perform an extrapolation, which is the source of the quadratic convergence. Also notice that the round-off error kicks in and dominates the total error when  $v_k$  is extremely close to  $v_*$  if we use  $(10^{-16})$ -precision arithmetic.

We now consider the choice of  $v_0 = v_0(x, c_*)$ . For each option characterized by  $(x, c_*)$ , we use the following third-order rational approximation for the initial estimate  $v_0$ :

$$v_0 = \frac{\sum_{i+j \leq 3} m_{ij} x^i c_*^j}{\sum_{i+j \leq 3} n_{ij} x^i c_*^j}, \quad (39)$$

where  $m_{ij}$  and  $n_{ij}$  are coefficients and we set  $n_{00} = 1$  for normalization. Thus there are a total of 19 parameters. We search for the optimal values for  $\{m_{ij}\}$  and  $\{n_{ij}\}$  numerically by minimizing the uniform error in  $v_*$  from both the SOR-DR and SOR-TS algorithms. Specifically, let  $v_k^{\text{SOR-DR}}$  and  $v_k^{\text{SOR-TS}}$  denote the numerical estimate of  $v_*$  after  $k$  iterations for a fixed set of parameters  $\{m_{ij}\}$  and  $\{n_{ij}\}$  from the SOR-DR and SOR-TS algorithms, respectively. Our objective is to search for the optimal parameters through the following problem:

$$G_D = \min_{\{m_{ij}\}, \{n_{ij}\}} \max_{(x, v_*) \in D^-} g(x, v_*, \{m_{ij}\}, \{n_{ij}\}), \quad (40)$$

where  $g$  is the error function for each fixed option  $(x, v_*)$  with parameters  $\{m_{ij}\}$  and  $\{n_{ij}\}$ :

$$g(x, v_*, \{m_{ij}\}, \{n_{ij}\}) = |v_k^{\text{SOR-DR}} - v_*| + |v_k^{\text{SOR-TS}} - v_*|.$$

We have set  $k$  to be the same for all options in domain  $D^-$  for vectorization consideration. Vectorization increases the efficiency of our algorithms. To compute the max part of the above objective function, we populate the domain  $D^-$  densely with roughly one million options, with the boundary populated denser than the inner region. Each of these options is characterized by a different pair of value  $(x, v_*)$ . For each fixed set of parameters  $\{m_{ij}\}$  and  $\{n_{ij}\}$ , we compute  $v_k^{\text{SOR-DR}}$  and  $v_k^{\text{SOR-TS}}$ , and then approximate the uniform error with the maximum error in  $v_*$  of all the options. If the domain  $D^-$  is populated sufficiently densely, this is a very accurate approximation. For the min part, we use a downhill simplex method of Nelder and Mead (1965), which is described in detail in Press et al. (1992). The choice of using the simplex method is dictated by the fact that we do not have any derivative information of the objective function with respect to the parameters  $\{m_{ij}\}$  and  $\{n_{ij}\}$ .

A few details are worth brief mentioning. First, it is extremely difficult to minimize over uniform errors using simplex methods because uniform errors are prone to problems such as local minimums, slow convergence, etc. Thus, we actually minimize the following penalized objective function

$$G_D^\lambda = \min_{\{m_{ij}\}, \{n_{ij}\}} \left( \max_{(x, v_*) \in D^-} g(x, v_*, \{m_{ij}\}, \{n_{ij}\}) + \lambda \sum_{(x, v_*) \in D^-} g(x, v_*, \{m_{ij}\}, \{n_{ij}\}) \right), \quad (41)$$



where  $\lambda$  controls the strength of penalty. When  $\lambda$  is larger, the objective function becomes smoother but deviates more from the uniform error. We dynamically adjust the value  $\lambda$  so that we put more weight on the uniform error as we move closer to the optimal values for  $\{m_{ij}\}$  and  $\{n_{ij}\}$ . Second, we set  $k = 5$ . Numerically, we find that setting  $k = 5$  uniformly for all options achieves the best combination of accuracy and efficiency. Finally, by design, the rational approximation estimate  $v_0$  is not necessarily close to  $v_*$  for a given option  $(x, v_*)$ . This is because  $v_k$  in our algorithms can “jump” and globally it is not always the case that a closer  $v_0$  would result in a quicker convergence to  $v_*$ .

Our final choice for the  $\{m_{ij}\}$  and  $\{n_{ij}\}$  from the above numerical procedure is:

$$\begin{aligned}
m_{00} &= -0.00006103098165; & n_{00} &= 1; \\
m_{01} &= 5.33967643357688; & n_{01} &= 22.96302109010794; \\
m_{10} &= -0.40661990365427; & n_{10} &= -0.48466536361620; \\
m_{02} &= 3.25023425332360; & n_{02} &= -0.77268824532468; \\
m_{11} &= -36.19405221599028; & n_{11} &= -1.34102279982050; \\
m_{20} &= 0.08975394404851; & n_{20} &= 0.43027619553168; \\
m_{03} &= 83.84593224417796; & n_{03} &= -5.70531500645109; \\
m_{12} &= 41.21772632732834; & n_{12} &= 2.45782574294244; \\
m_{21} &= 3.83815885394565; & n_{21} &= -0.04763802358853; \\
m_{30} &= -0.21619763215668; & n_{30} &= -0.03326944290044.
\end{aligned} \tag{42}$$

Plugging the above parameters values into equation (39) gives us the initial starting point  $v_0$  for the successive over-relaxation algorithms.

## 5.2. Numerical performance

Table 2 gives the performance of the three algorithms **SOR**, **SOR-DR**, **SOR-TS** inside the domain  $D^-$ . The accuracy is measured in terms of both  $|v_k - v_*|$  and  $|c_k - c_*|$ . All three algorithms are implemented with  $(10^{-16})$ -precision arithmetic in MATLAB 7.1 on a Dell Dimension 4600 desktop computer (2.8 GHz, 1G RAM). We report the accuracy of the rational approximation ( $k = 0$ ), and of each of the three algorithms when  $k = 4$  and  $k = 5$ . The means, medians and maximums of  $|v_k - v_*|$  and  $|c_k - c_*|$  are calculated by uniformly and densely populating the domain  $D^-$  with roughly 1 million options. The computing time for all options of each algorithm is also reported for  $k = 0, 4$  and  $5$ . As we see, the rational approximation has a uniform error of 0.783 for  $|v_k - v_*|$  and a uniform error of 0.29 for  $|c_k - c_*|$  in domain  $D^-$ . Thus, the rational approximation by itself does not give accurate enough estimates for  $v_*$ . This is because we have not tried to obtain the best possible rational approximation per se in our implementation, but rather we have tried to obtain the best possible rational approximation conditioning on that we are going to perform five more successive over-relaxation iterations. The rational approximation



takes only 0.65 seconds for all roughly one million options. When  $k = 4$ , both the accelerated algorithms **SOR-DR** and **SOR-TS** achieve a uniform error in  $v_*$  in the order of  $10^{-8}$  while that of the **SOR** algorithm is 0.078. The medians of the errors in  $v_*$  and  $c_*$  are quite small for all three algorithms. The larger errors tend to occur near the boundary of the domain  $D^-$ . All three algorithms take around 12 seconds. For  $k = 5$ , both accelerated algorithms **SOR-DR** and **SOR-TS** achieve a uniform error in  $v_*$  in the order of  $10^{-13}$  while that of the **SOR** algorithm is 0.058. We do not recommend using more than 5 iterations because the round-off error can start to dominate the total error in  $v_*$ . The computing times are still only about 12 seconds for all roughly one million options. Overall, we see that the performance of the **SOR-DR** and **SOR-TS** algorithms are quite good in terms of both accuracy and speed. Also, numerically we find that our algorithms work for a much larger domain with only a slight decrease in accuracy.

Most existing methods fail to work in our large domain  $D^-$ . This is the case, for example, for the Corrado-Miller method and the rational approximation of Li (2008). Thus we cannot perform a comparison between these methods and ours. One exception is the Dekker-Brent algorithm. Table 3 compares the performance of the **SOR-TS** algorithm, the Newton-Raphson algorithm, and the Dekker-Brent algorithm for a particular option  $(x, v_*)$ , where we let  $x = -1$  and  $v_* = 2$ . We do not consider the **SOR-DR** algorithm in this comparison, because its performance is very similar to that of the **SOR-TS** algorithm, except for the fact that the **SOR-DR** algorithm is not always globally well-defined. We start all three algorithms from three different initial values, namely, 0.1, 4, and 20 and report the first 5 iterations (Panel A) together with the errors (Panel B). As we see, a naive implementation of the Newton-Raphson algorithm fails in all three cases. The reason is that in each Newton-Raphson iteration, one needs to divide the price error by the vega in each step. For away-from-the-money options, vega can be extremely small, leading to numerical instability. Although not reported, for this particular option, if we start from the the guess  $v_0 = \sqrt{2|x|}$  as was suggested in Manaster and Koehler (1982), then the Newton-Raphson algorithm converges. However, examples can be easily given where even this choice of  $v_0 = \sqrt{2|x|}$  leads to failure in the Newton-Raphson algorithm. This happens for large values of  $|x|$  and  $v_*$  in our domain  $D^-$ . On the other hand, both the **SOR-TS** and Dekker-Brent algorithms converge in all three cases. The quadratic order of convergence of the **SOR-TS** algorithm is evident after only one or two iterations. The Dekker-Brent algorithm uses a combination of bisection and interpolation and some of the bisection steps are evident in the table. Overall, the **SOR-TS** algorithm achieves the best combination of robustness, efficiency and accuracy. One particular attractive feature of this algorithm is that it is globally convergent for any positive  $v_0$ .

## 6. Further Applications

The idea of using successive over-relaxation to compute the implied volatility is useful in other financial applications. Below we give three examples to demonstrate this point. In our first example, we consider the implied correlation for an exchange option in the Margrabe framework.

In the second example, we consider the implied volatility for call options in the Bachelier model. The last example considers the critical stock price level in a compound call on call option.

### 6.1. The Margrabe implied correlation

The most commonly used formula for the price of an exchange option is the Margrabe formula, which was independently discovered by Fischer (1978) and Margrabe (1978). Margrabe (1978) considers the price of an option to exchange one asset for another, while Fischer (1978) considers the price of a call option when the exercise price is uncertain.

Under the Margrabe formula setup, the dynamics of the two stock prices  $S_1(t)$  and  $S_2(t)$  under the risk-neutral measure  $\mathbb{Q}$  are given by

$$dS_i(t) = (r - \delta_i)S_i(t)dt + \sigma_i S_i(t)dW_i(t), \quad (i = 1, 2)$$

where the two Brownian motions  $W_1(t)$  and  $W_2(t)$  are correlated with constant coefficient  $\rho$ . The Margrabe formula gives the time-0 price  $C$  of a European option to exchange stock 2 for stock 1 at time  $T$  as follows:

$$C = S_1(0)e^{-\delta_1 T} N(d_1) - S_2(0)e^{-\delta_2 T} N(d_2),$$

where

$$d_1 = \frac{\log[(S_1(0)e^{-\delta_1 T})/(S_2(0)e^{-\delta_2 T})] + \sigma^2 T/2}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},$$

and  $\sigma \equiv \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$ . Li (2007) contains three different methods to prove the Margrabe formula.

In practice, people often use the Margrabe formula to compute the implied correlation  $\rho$  from the observed option price  $C$ , with  $\sigma_1$  and  $\sigma_2$  estimated using some other methods. Our methods can be applied directly to compute the implied correlation. This is because if we define

$$c = C/(S_1 e^{-\delta_1 T}), \quad x = \log[(S_1(0)e^{-\delta_1 T})/(S_2(0)e^{-\delta_2 T})], \quad v = \sigma\sqrt{T}, \quad (43)$$

then the Margrabe formula reduces to the dimensionless Black-Scholes formula in (1). The parameter  $v$  can then be computed using either the SOR-DR or the SOR-TS algorithm. Once  $v$  is computed, the implied correlation can be computed by

$$\rho = \frac{\sigma_1^2 T + \sigma_2^2 T - v^2}{2\sigma_1\sigma_2 T}. \quad (44)$$

### 6.2. The Bachelier implied volatility

We now consider the Bachelier formula (Bachelier, 1900) for a European call option. This formula was discovered by L. Bachelier, who was among the first to analyze Brownian motion

mathematically. Schachermayer and Teichmann (2007) contain an interesting comparison of Bachelier’s formula with that of Black and Scholes. The original Bachelier formula considers interest rate  $r = 0$ . There are a few approaches to extend it to nonzero interest rates. We adopt the following one which models the risk-neutral dynamics of the stock price process as:

$$dS_t = rS_t dt + \sigma e^{(r-\delta)t} dW_t.$$

Notice that we have introduced a nonzero dividend rate  $\delta$  too. In this case,  $S_T$  has the explicit solution  $S_T = e^{(r-\delta)T}(S_0 + \sigma W_T)$ , and the Bachelier formula for the time-0 price of a call option is given by

$$C = \sigma\sqrt{T}e^{-\delta T}n(d) + (S_0e^{-\delta T} - Ke^{-rT})N(d), \quad (45)$$

where

$$d = \frac{S_0e^{-\delta T} - Ke^{-rT}}{\sigma\sqrt{T}e^{-\delta T}}.$$

In practice, people observe the actual price  $C_*$  and compute the volatility parameter  $\sigma_*$  which satisfies the Bachelier formula. This parameter (after sometimes normalizing it by  $S_0$ ) is called the Bachelier implied volatility. It is also often called the normal volatility by practitioners because future stock price is assumed to be normally distributed in the Bachelier model.

Recently, Choi, Kim and Kwak (2007) develop a closed-form numerical approximation for the Bachelier implied volatility, which is found to be very accurate. In this paper, we will apply our successive over-relaxation method to compute the Bachelier implied volatility. We first make the following parameter reduction. Define the normalized call price  $c$ , moneyness  $x$  and volatility  $v$  as follows:

$$c = C/(S_0e^{-\delta T}), \quad x = 1 - \frac{Ke^{-rT}}{S_0e^{-\delta T}}, \quad v = \sigma\sqrt{T}/S_0. \quad (46)$$

Then Bachelier’s formula in equation (45) reduces to the following dimensionless Bachelier’s formula

$$c = c^B(x, v) = v n(x/v) + xN(x/v). \quad (47)$$

Notice that like the Black-Scholes formula, the Bachelier option price is also a strictly increasing function of  $v$  because the Bachelier vega is given by  $\partial c^B(x, v)/\partial v = n(x/v) > 0$ .

Like the dimensionless Black-Scholes formula, we only need to consider call options because put-call parity is also satisfied in the Bachelier framework. Also, we only need to consider out-of-the-money call options because of the following “in-out” duality in the Bachelier formula:

$$c^B(x, v) - c^B(-x, v) = x.$$

That is, to compute the Bachelier implied volatility for an option with positive moneyness  $x$  and normalized price  $c_*$ , we could just compute the volatility for its dual option with moneyness  $-x$  and normalized option price  $c_* - x$ .

Let  $v_*$  be the true Bachelier implied volatility. That is,  $c_* = c^B(x, v_*)$ . Define the iteration function  $G^B(v; x, v_*, \omega)$  as follows:

$$G^B(v; x, v_*, \omega) = \frac{c_* - xN(x/v) + \omega vn(x/v)}{(1 + \omega)n(x/v)}. \quad (48)$$

We will sometimes just write  $G^B(v)$ . One possible successive over-relaxation method is the following, which we label as the SOR-B algorithm:

1. Select an initial point  $v_0$ ;
2. (SOR-B) After obtaining  $v_k$ , select  $\omega_k$  and compute  $v_{k+1}$  by

$$v_{k+1} = G^B(v_k; x, v_*, \omega_k), \quad (49)$$

3. Stop when  $|v_k - v_*| < \epsilon$  or  $|c_k - c_*| < \epsilon$ , for some  $\epsilon$  small.

It turns out that with the choice  $\omega_k = 0$ , the SOR-B algorithm is globally well-defined and converges to  $v_*$  with a quadratic order of convergence.

**Theorem 6.1. (convergence properties of the SOR-B algorithm)** *Let  $\omega_k \equiv 0$  in the SOR-B algorithm. Then the sequence  $\{v_k\}$  is globally well-defined. If  $x = 0$ , then  $v_k = v_*$  for all  $k \geq 1$ . If  $x < 0$  and  $v_0 \neq v_*$ , then  $v_1 > v_*$  and  $v_k$  monotonically decreases to  $v_*$  afterwards with a quadratic order of convergence.*

The efficiency of the SOR-B algorithm can be improved by introducing a rational approximation like the one we did for the Black-Scholes implied volatility case. We omit the details here. Another way to extend the Bachelier model to nonzero interest rate is to model the stock price process as  $dS_t = (r - \delta)S_t dt + \sigma dW_t$ . In this case, a different formula from equation (45) will be obtained. See, Musiela and Rutkowski (2005). However, our method of using successive over-relaxation to compute the Bachelier implied volatility is still applicable after some minor modifications.

### 6.3. The critical stock price in a compound option

Compound options are options on options. Their pricing and various applications are considered by Geske (1977), Geske (1979), Hodges and Selby (1987), and Chandrasekhar and Gukhal (2004), among others. Below we consider a call on call option in the standard Black-Scholes framework. We will design a successive over-relaxation method to compute the critical stock price. For other types of compound options, such as call on put, put on put and put on call, our method still applies with minor modifications.

Let the current date be 0, and  $T_1$  and  $T_2$  be two future dates with  $T_2 > T_1 > 0$ . We will let  $\tau = T_2 - T_1$ . Let the current stock price be  $S_0$ , the constant interest rate be  $r$ , the dividend rate be  $\delta$ , and the volatility of the stock be  $\sigma$ . At time  $T_1$ , the holder of the compound option is entitled to receive either cash  $K_1$ , or a call option with strike price  $K_2$  and maturity date  $T_2$ . The time-0 value  $C^c$  of the compound option is given by

$$C^c = S_0 e^{-qT_2} N_2(a_1, b_1; \sqrt{T_1/T_2}) - K_2 e^{-rT_2} N_2(a_2, b_2; \sqrt{T_1/T_2}) - e^{-rT_1} K_1 N(a_2),$$

where  $N_2(\cdot, \cdot; \rho)$  is the cumulative bivariate normal distribution function with correlation  $\rho$ , and

$$\begin{aligned} a_1 &= \frac{\log(S_0/S_*) + (r - q + \sigma^2/2)T_1}{\sigma\sqrt{T_1}}, & a_2 &= a_1 - \sigma\sqrt{T_1}, \\ b_1 &= \frac{\log(S_0/K_2) + (r - q + \sigma^2/2)T_2}{\sigma\sqrt{T_2}}, & b_2 &= b_1 - \sigma\sqrt{T_2}. \end{aligned}$$

The formula for  $C^c$  is in closed form except that the critical stock price level  $S_*$  at time  $T_1$  to exercise the compound option is given implicitly by

$$K_1 = S_* e^{-q\tau} N\left(\frac{1}{\sigma\sqrt{\tau}} \log \frac{S_* e^{-q\tau}}{K_2 e^{-r\tau}} + \frac{\sigma\sqrt{\tau}}{2}\right) - K_2 e^{-r\tau} N\left(\frac{1}{\sigma\sqrt{\tau}} \log \frac{S_* e^{-q\tau}}{K_2 e^{-r\tau}} - \frac{\sigma\sqrt{\tau}}{2}\right).$$

A Newton-Raphson algorithm can be used to compute  $S_*$ , but it is subject to the same numerical instability we encountered before. To implement a successive over-relaxation method, we first perform a dimension reduction. Defining modified strike  $\kappa$ , moneyness  $x$ , and total volatility  $v$  by

$$\kappa = \frac{K_1}{K_2 e^{-r\tau}}, \quad x = \log \frac{S_* e^{-q\tau}}{K_2 e^{-r\tau}}, \quad v = \sigma\sqrt{\tau}, \quad (50)$$

the critical stock price equation becomes

$$\kappa = \kappa(x, v) = e^x N(x/v + v/2) - N(x/v - v/2). \quad (51)$$

The problem is then to compute the moneyness  $x_*$  that satisfies the above equation with observed modified strike  $\kappa_*$  and assumed value of the total volatility  $v$ .

Notice that  $\kappa_* = \kappa(x_*, v)$ . Define the iteration function  $G^{CS}(x; v, x_*, \omega)$  by

$$G^{CS}(x; v, x_*, \omega) = \log \left( \frac{\kappa(x_*, v) + N(x/v - v/2) + \omega e^x N(x/v + v/2)}{(1 + \omega)N(x/v + v/2)} \right). \quad (52)$$

One possible successive over-relaxation method is the following **SOR-CS** algorithm:

1. **Select an initial point  $x_0$ ;**
2. **(SOR-CS) After obtaining  $x_k$ , select  $\omega_k$  and compute  $x_{k+1}$  by**

$$x_{k+1} = G^{CS}(x_k; v, x_*, \omega_k); \quad (53)$$

3. **Stop when  $|x_k - x_*| < \epsilon$  or  $|\kappa_k - \kappa_*| < \epsilon$ , for some  $\epsilon$  small.**

Notice that unlike the Newton-Raphson algorithm, in the SOR-CS algorithm, no evaluation of the standard normal density function  $n(\cdot)$  is required. The following theorem shows that the SOR-CS algorithm is globally well-defined and converges to  $x_*$  with a quadratic order of convergence.

**Theorem 6.2. (convergence properties of the SOR-CS algorithm)** *Let  $x_* \in \mathbb{R}$  and  $v > 0$ . Let  $\omega_k \equiv 0$  in the SOR-CS algorithm. Then the sequence  $\{x_k\}$  is globally well-defined. Furthermore, if  $x_0 \neq x_*$ , then  $x_1 > x_*$  and  $x_k$  monotonically decreases to  $x_*$  afterwards with a quadratic order of convergence.*

Although we do not give detailed report here, numerically we find that the above algorithm converges extremely fast and is very robust. One critical advantage of the above method versus Newton-Raphson is that it is not very sensitive to the initial guess  $x_0$  for all reasonable values of  $x_*$  and  $v$ . Also, a rational approximation can be employed on the initial value  $x_0$  to enhance convergence. We omit the details here.

## 7. Conclusion

The Black-Scholes formula is one of the most frequently used formula in finance. In most of the applications, it is used backwards to compute the implied volatility. Some traditional methods, such as the Newton-Raphson algorithm, use derivative information and are thus subject to numerical instability. Numerically more stable methods, such as the Dekker-Brent algorithm, are usually slow and often not able to handle the real-time computation of the implied volatility in bulk volume.

In this paper, we design a successive over-relaxation (SOR) algorithm which does not use the derivative information. We analyze the well-definedness, local and global convergence properties of the SOR algorithm in detail. With mild conditions, the algorithm is globally well-defined and converges to the true implied volatility.

The SOR algorithm generally has a linear order of convergence. By introducing dynamic relaxation or sequence transformation techniques, both new algorithms SOR-DR and SOR-TS achieve quadratic order of convergence, the same order as the Newton-Raphson algorithm. The efficiency of the algorithms are further enhanced by introducing a rational approximation on the initial estimates. It is shown that uniformly in a very large inversion domain, our accelerated algorithms converge to the true implied volatility with very few iterations. Thus, the accelerated algorithms serve as good alternatives to the traditional methods because of their good performance in terms of speed, accuracy and robustness.

Finally, we extend our successive over-relaxation method to the computation of implied correlation in the Margrabe formula, the normal implied volatility in the Bachelier formula, and the critical stock price level in a compound option. These three examples demonstrate that the idea of successive over-relaxation is applicable in a much wider range of financial problems.

## Appendix

### Proof of Lemma 3.1:

For  $x < 0$ , we have

$$\begin{aligned}\frac{\partial N^+(x, v)}{\partial v} &= n^+(v) \left( \frac{|x|}{v^2} + \frac{1}{2} \right), \\ \frac{\partial N^-(x, v)}{\partial v} &= n^+(v) \left( \frac{|x|}{v^2} - \frac{1}{2} \right), \\ \frac{\partial c(x, v)}{\partial v} &= n^+(v), \quad \frac{\partial^2 c(x, v)}{\partial v^2} = \frac{n^+(v)}{v} \left( \frac{|x|^2}{v^2} - \frac{v^2}{4} \right), \\ \frac{\partial c^+(x, v)}{\partial v} &= n^+(v) \frac{2|x|}{v^2},\end{aligned}$$

where we have used the identity  $e^{-x}n(x/v - v/2) = n(x/v + v/2)$  in the second equation. These prove statements 1 through 4. For statement 5, assume  $u < v$  without loss of generality. Thus

$$c(v) - c(u) = \int_u^v \frac{\partial c^+(x, \xi)}{\partial \xi} \frac{\xi^2}{2|x|} d\xi < \frac{v^2}{2|x|} \int_u^v \frac{\partial c^+(x, \xi)}{\partial \xi} d\xi = \frac{v^2}{2|x|} [c^+(v) - c^+(u)].$$

Similarly for the left-hand inequality in equation (9).

For  $x = 0$ , all the statements can be quickly verified.  $\square$

### Proof of Lemma 3.2:

We have  $F'(v) = n^+(v) \left( \frac{|x|}{v^2} + \frac{1}{2} \right) [\omega - \Phi(v)]$ , and  $\omega = \Phi(v)$  if and only if  $v = \tilde{v}(\omega)$ . Thus, if  $\omega \geq 1$ , then  $F'(v) > 0$ . If  $\omega \in (-1, 1)$ , then  $\tilde{v}$  satisfies  $\omega = \Phi(\tilde{v})$ . Since  $\Phi(v)$  is strictly increasing in  $v$ , we have  $F'(v) > 0$  for  $v < \tilde{v}$  and  $F'(v) < 0$  for  $v > \tilde{v}$ .  $\square$

### Proof of Lemma 3.3:

Since  $\omega = \Phi(\tilde{v})$ , we have  $H'(\omega) = N^+(\tilde{v}) - 1$ . Hence,  $H(\omega)$  is strictly decreasing in  $\omega$  for  $\omega \in (-1, 1)$ . By Lemma 3.1, we have  $\lim_{\omega \rightarrow -1} H(\omega) = c_* > 0$ , and  $\lim_{\omega \rightarrow +1} H(\omega) = c_* - 1 < 0$ . Therefore, there exists a unique  $\hat{\omega}$  such that  $H(\omega) \geq 0$  for  $\omega \leq \hat{\omega}$  and  $H(\omega) < 0$  for  $\omega > \hat{\omega}$ .  $\square$

### Proof of Theorem 3.1:

Note that  $v_{k+1}$  is well-defined if and only if  $0 < F(v_k) < 1 + \omega$ .

By Lemma 3.2, when  $\omega \geq 1$ ,  $F(v)$  is positive, strictly increasing in  $v$ , and bounded above by  $F(+\infty) = c_* + \omega$ , which is less than  $1 + \omega$ . Hence,  $\{v_k\}$  is well-defined for any  $v_0 \in \mathbb{R}_+$ .

When  $\omega \in (-1, 1)$ , by Lemma 3.2 and 3.3, we know that if  $\omega > \hat{\omega}$ , then for any  $v \in \mathbb{R}_+$ ,

$$c_* + N^-(v) + \omega N^+(v) - (1 + \omega) \leq H(\omega) < 0.$$

By Lemma 3.2, for all  $v \in \mathbb{R}_+$ , we also have

$$F(v) = c_* + N^-(v) + \omega N^+(v) > \min \{F(0^+), F(+\infty)\} = \min \{c_*, c_* + \omega\}.$$

Thus, if  $\omega \geq -c_*$  additionally, we have  $0 < F(v) < 1 + \omega$ . This guarantees that  $\{v_k\}$  is well-defined for any  $v_0 \in \mathbb{R}_+$ .

Conversely, if  $\omega < -c_*$ , then  $F(+\infty) < 0$ , so for sufficiently large  $v_0$ , we have  $F(v_0) < 0$  and  $v_1 = G(v_0)$  is not defined. If  $\omega \leq \hat{\omega}$ , then if  $v_0 = \tilde{v}(\omega)$ , we have  $F(v) - (1 + \omega) = H(\omega) \geq 0$ . So  $v_1 = G(v_0)$  is not defined.  $\square$

**Proof of Lemma 3.4:**

Take any  $v \in \text{Dom}G$ . Equation (19) can be obtained by differentiating

$$c_* + N^-(v) + \omega N^+(v) = (1 + \omega)N^+(G(v))$$

with respect to  $v$  and using Lemma 3.1. Since  $0 < F(v_*) < 1 + \omega$ , for  $v$  sufficiently close to  $v_*$ , we also have  $0 < F(v) < 1 + \omega$ , thus  $v_* \in \text{Dom}G$ . For equation (20), notice that  $G(v_*) = v_*$ . All the other statements can be quickly verified.  $\square$

**Proof of Lemma 3.5:**

Since  $Q'(v) = n^+(v) + n^+(G(v))G'(v)$  for  $v \in \text{Dom}G$ , substituting the expression for  $G'(v)$  in Lemma 3.4 gives us the formula for  $Q'(v)$ . The higher-order derivatives of  $Q(v)$  when  $\omega = \Psi(v_*)$  can be obtained by recursively differentiating  $Q'(v)$  and using Lemma 3.4.  $\square$

**Proof of Theorem 3.2:**

First consider the case  $x < 0$ . Since  $v_* \in \text{Dom}G$  by Lemma 3.4, there exists a neighborhood  $V_\epsilon = (v_* - \epsilon, v_* + \epsilon)$  of  $v_*$  such that  $G(v)$  is smooth on  $V_\epsilon$  and  $M \equiv \sup \{|G''(v)|/2 : v \in V_\epsilon\} < \infty$ . Suppose  $\omega < \Psi(v_*)$  and  $v_k$  converges to  $v_*$ . Then, since  $G'(v_*) = [\omega - \Phi(v_*)]/(1 + \omega)$  is strictly increasing in  $\omega$ , we have  $G'(v_*) < (\Psi(v_*) - \Phi(v_*))/(1 + \Psi(v_*)) = -1$ . Pick a  $k_0 \in \mathbb{N}$  large such that  $|v_k - v_*| < \min \{\epsilon, \delta\}$ , where  $\delta = (|G'(v_*)| - 1)/(2M)$ . Then, since  $v_{k+1} - v_* = G(v_k) - G(v_*)$ , a Taylor expansion gives  $|(v_{k+1} - v_k) - G'(v_*)(v_k - v_*)| \leq M(v_k - v_*)^2$ . That is,

$$\begin{aligned} |v_{k+1} - v_*| &\geq |G'(v_*)| |v_k - v_*| - M(v_k - v_*)^2 \\ &\geq |v_k - v_*| + (|G'(v_*)| - 1 - M\delta) |v_k - v_*| > |v_k - v_*|. \end{aligned}$$

This contradicts that  $v_k$  converges to  $v_*$ . Thus, a necessary for local convergence is  $\omega \geq \Psi(v_*)$ .

Now, suppose  $\omega = \Psi(v_*)$ ,  $12v_*^2 - v_*^4 + 4x^2 < 0$ , and  $v_k$  converges to  $v_*$ . Then, by Lemma 3.5,

$$Q'(v_*) = Q''(v_*) = 0 \quad \text{and} \quad Q'''(v_*) < 0.$$

Thus, there exists a neighborhood  $V_\epsilon = (v_* - \epsilon, v_* + \epsilon)$  of  $v_*$  such that  $Q(v) \geq 0$  on  $(v_* - \epsilon, v_*]$  and  $Q(v) \leq 0$  on  $[v_*, v_* + \epsilon)$ . Since  $Q(v_k) = (1 + \omega)[N^+(v_k) - N^+(v_{k+2})]$ , we have that



$v_{k+2} < v_k$  if  $v_k < v_*$ , and  $v_{k+2} > v_k$  if  $v_k > v_*$ . This contradicts that  $v_k$  converges to  $v_*$ . So, a further necessary condition  $12v_*^2 - v_*^4 + 4x^2 \geq 0$  is needed for local convergence. The proof that  $12v_*^2 - v_*^4 + 4x^2 > 0$  is sufficient is similar.

We now show that  $\omega > \Psi(v_*)$  is sufficient for local convergence. Notice that  $|G'(v_*)| < 1$ . Pick any number  $\lambda \in (|G'(v_*)|, 1)$ . Let  $\delta = \frac{\lambda - |G'(v_*)|}{2M}$ . Then, if  $|v_k - v_*| < \min\{\epsilon, \delta\}$ , we have

$$\begin{aligned} |v_{k+1} - v_*| &\leq |G'(v_*)| |v_k - v_*| + M |v_k - v_*|^2 \\ &\leq (|G'(v_*)| + M\delta) |v_k - v_*| < \lambda |v_k - v_*|. \end{aligned}$$

So,  $v_k$  converges to  $v_*$ .

Next, we consider the case  $x = 0$ . A very similar proof to the above shows that  $\omega \geq 0$  is necessary and  $\omega > 0$  is sufficient for local convergence. However, when  $\omega = 0$  and  $v_0 \neq v_*$ , we always have  $v_1 \neq v_*$ , and the sequence oscillates on the set  $\{v_0, v_1\}$ . Thus, if  $v_0 \neq v_*$ , the sequence  $\{v_k\}$  will never converge when  $\omega = 0$ .  $\square$

### Proof of Lemma 3.6:

First consider the case  $x < 0$ . Before proving that  $\phi(u, v)$  is continuous on  $\mathbb{R}_+^2$ , we will first prove that for any fixed  $v$ ,  $\phi(u, v)$  is continuous and strictly increasing in  $u$  and  $\phi_1(u, v)$  is also continuous in  $u$ . Take any  $v \in \mathbb{R}_+$ . When  $u \neq v$ , differentiating  $\phi(u, v)$  with respect to  $u$  gives

$$\phi_1(u, v) = \frac{n^+(u)}{[N^+(u) - N^+(v)]^2} \left\{ \frac{x}{u^2} [c(u) - c(v)] + \frac{1}{2} [c^+(u) - c^+(v)] \right\}. \quad (54)$$

By Lemma 3.1,  $\phi_1(u, v) > 0$  when  $u \neq v$ . When  $u = v$ , by L'Hospital's rule, we have

$$\phi_1(v, v) \equiv \lim_{w \rightarrow v} \frac{\phi(w, v) - \Phi(v)}{w - v} = \lim_{w \rightarrow v} \phi_1(w, v).$$

Thus  $\phi_1(u, v)$  is continuous in  $u$  for all  $u \in \mathbb{R}_+$ . Applying L'Hospital's rule two more times to equation (54), we have

$$\begin{aligned} \phi_1(v, v) &= n^+(v) \lim_{w \rightarrow v} \frac{\frac{x}{w^2} [c(w) - c(v)] + \frac{1}{2} [c^+(w) - c^+(v)]}{[N^+(w) - N^+(v)]^2} \\ &= \frac{\frac{|x|}{v^3}}{\frac{|x|}{v^2} + \frac{1}{2}} \lim_{w \rightarrow v} \frac{c(w) - c(v)}{N^+(w) - N^+(v)} = \frac{4|x|v}{(v^2 + 2|x|)^2} > 0. \end{aligned}$$

The partial derivative of  $\phi(u, v)$  with respect to  $v$  can be easily obtained by noticing that  $\phi(u, v) = \phi(v, u)$ . Since  $\phi(u, v)$  is strictly increasing in both  $u$  and  $v$ , we have

$$-1 < \Phi(\min(u, v)) \leq \phi(u, v) \leq \Phi(\max(u, v)) < 1.$$

Finally, we show that  $\phi(u, v)$  is continuous on  $\mathbb{R}_+^2$ . If  $u \neq v$ , the continuity is obvious. So we assume  $u = v$ . Note first that  $\Phi(v) = \phi(v, v)$  is continuous, strictly increasing in  $v$ , so for any

$\epsilon > 0$ , there exists a  $\delta > 0$  such that  $|a - v| < \delta$  implies  $|\phi(a, a) - \phi(v, v)| < \epsilon/4$ . Consider an open disc that centers at  $(v, v)$  and has a diameter of  $\delta$ . Then, by triangular inequality, for any  $(a, b)$  in the disc, we have

$$\begin{aligned} |\phi(a, b) - \phi(v, v)| &\leq |\phi(a, b) - \phi(a, v)| + |\phi(a, v) - \phi(v, v)| \\ &< 2|\phi(v + \delta/2, v + \delta/2) - \phi(v - \delta/2, v - \delta/2)| < \epsilon. \end{aligned}$$

For the case  $x = 0$ , notice that  $c^+(u) \equiv 1$  on  $\mathbb{R}_+$  by Lemma 3.1, so  $\phi(u, v) \equiv 1$ . □

**Proof of Lemma 3.7:**

Rearranging the iteration equation  $c_* + N^-(v_k) + \omega N^+(v_k) = (1 + \omega)N^+(v_{k+1})$ , we have

$$c(v_*) - v(v_k) = (1 + \omega) [N^+(v_{k+1}) - N^+(v_k)].$$

Since both  $c(v)$  and  $N^+(v)$  are strictly increasing in  $v$  by Lemma 3.1,  $v_* - v_k$  always have the same sign as  $v_{k+1} - v_k$  when  $v_k \neq v_*$ . □

**Proof of Lemma 3.8:**

Rearranging the iteration equation  $c_* + N^-(v_k) + \omega N^+(v_k) = (1 + \omega)N^+(v_{k+1})$ , we have

$$(N^+(v_k) - N^+(v_*)) [\omega - \phi(v_k, v_*)] = (1 + \omega) [N^+(v_{k+1}) - N^+(v_*)].$$

Since  $v_{k+1} \neq v_*$ , we have  $\omega \neq \phi(v_k, v_*)$ . Furthermore, if  $\omega > \phi(v_k, v_*)$ , then  $v_k$  and  $v_{k+1}$  are on the same side of  $v_*$ . If  $\omega < \phi(v_k, v_*)$ , then  $v_k$  and  $v_{k+1}$  are on the opposite side of  $v_*$ . □

**Proof of Lemma 3.9:**

Subtracting two consecutive iteration equations, we get

$$(N^+(v_{k+1}) - N^+(v_k)) \left[ (1 + 2\omega) - \phi(v_k, v_{k+1}) \right] = (1 + \omega) [N^+(v_{k+2}) - N^+(v_k)].$$

Hence, if  $1 + 2\omega > \phi(v_k, v_{k+1})$ , then  $v_{k+1}$  and  $v_{k+2}$  are on the same side of  $v_k$ . If  $1 + 2\omega < \phi(v_k, v_{k+1})$ , then  $v_{k+1}$  and  $v_{k+2}$  are on the opposite side of  $v_k$ . If  $1 + 2\omega = \phi(v_k, v_{k+1})$ , then  $v_{k+2} = v_k$ . □

**Proof of Theorem 3.3:**

We only consider the case  $x < 0$ . The case  $x = 0$  can be similarly shown by noticing that the necessary and sufficient condition for local convergence when  $x = 0$  is  $\omega > 0$ .

First we consider  $\omega > \Phi(v_*)$ . By Lemma 3.6, there exists an  $\epsilon > 0$  such that  $\omega > \phi(v, v_*)$  for all  $v \in (v_* - \epsilon, v_* + \epsilon)$ . Since  $v_k$  converges to  $v_*$ , there exists a  $k_0 \in \mathbb{N}$  such that  $v_k \in (v_* - \epsilon, v_* + \epsilon)$  for all  $k \geq k_0$ . Suppose  $v_{k_0} < v_*$ . Then, by Lemmas 3.8 and 3.9,  $v_k < v_{k+1} < v_*$  for all  $k \geq k_0$ . Thus,  $v_k$  ( $k \geq k_0$ ) strictly increases to  $v_*$ . Similarly, if  $v_{k_0} > v_*$ , then  $v_k$  ( $k \geq k_0$ ) strictly decreases to  $v_*$ .

Next, suppose  $\omega = \Phi(v_*)$ . Since  $\phi(v, v_*)$  is strictly increasing in  $v$  and  $\phi(v_*, v_*) = \Phi(v_*)$  by Lemma 3.6, there exists an  $\epsilon > 0$  such that  $\omega > \phi(v, v_*)$  for all  $v \in (v_* - \epsilon, v_*)$  and  $\omega < \phi(v, v_*)$  for all  $v \in (v_*, v_* + \epsilon)$ . If  $v_{k_0} < v_*$ , then we get  $v_k < v_{k+1} < v_*$  for all  $k \geq k_0$ , so the sequence is eventually monotonically increasing. If  $v_{k_0} > v_*$ , then we have  $v_{k_0+1} < v_*$  by Lemma 3.8 and the sequence is eventually monotonically increasing too.

Now suppose  $\omega < \Phi(v_*)$ . By Lemma 3.6, there exists an  $\epsilon > 0$  such that  $\omega < \phi(v, v_*)$  for all  $v \in V_\epsilon \equiv (v_* - \epsilon, v_* + \epsilon)$ . Since  $v_k$  converges to  $v_*$ , by Lemma 3.7 and 3.8, there exists  $k_1 \in \mathbb{N}$ , such that  $v_{k_1+m} \in V_\epsilon$ , with  $v_{k_1+2m} < v_*$  and  $v_{k_1+2m+1} > v_*$  for all  $m \in \mathbb{N}$ . We are left to show that the subsequences  $\{v_{k_1+2m}\}$  and  $\{v_{k_1+2m+1}\}$  indexed by  $m$  are both eventually monotone. For this purpose, consider the function  $Q(v)$  defined in equation (21). Recall  $Q(v_k) = (1 + \omega)[N^+(v_k) - N^+(v_{k+2})]$ . Notice that for any  $\delta > 0$ ,  $Q(v)$  cannot be constant on any interval  $[v_*, v_* + \delta)$  or  $(v_* - \delta, v_*]$ , because then we will have  $v_k = v_{k+2m}$  for all  $m \in \mathbb{N}$  for some  $k$  sufficiently large and this contradicts that  $v_k$  converges to  $v_*$ . Since the function  $Q(v)$  is continuous and  $Q(v_*) = 0$ , there exists a  $\delta > 0$ , such that  $Q(v)$  has determinate signs on  $(v_* - \delta, v_*)$  and  $(v_*, v_* + \delta)$ . There are four possible cases but the only one that does not contradict convergence is  $Q(v) > 0$  if  $v \in (v_*, v_* + \delta)$  and  $Q(v) < 0$  if  $v \in (v_* - \delta, v_*)$ . All other three cases will result in a subsequence of  $v_k$  which does not converge to  $v_*$ . This shows that the subsequences  $\{v_{k_1+2m}\}$  and  $\{v_{k_1+2m+1}\}$  are both eventually monotone. The proof is now complete by selecting  $k_0 > k_1$  sufficiently large and  $v_{k_0} < v_*$ .  $\square$

**Proof of Theorem 3.4:**

By Theorem 3.3, if  $\{v_k\}$  converges, the sequence is either eventually monotone or eventually oscillating around  $v_*$ . It is obvious that  $|c_k - c_*|$  eventually decreases to zero monotonically if  $\{v_k\}$  is eventually monotone. In the case of oscillating convergence, consider the function  $Q(v)$ . The proof of Theorem 3.3 establishes that to guarantee convergence, there exists a  $\delta > 0$ , such that  $Q(v) > 0$  on  $(v_*, v_* + \delta)$  and  $Q(v) < 0$  on  $(v_* - \delta, v_*)$ . Thus, for sufficiently large  $k$ , if  $v_k > v_*$ , then  $|c_{k+1} - c_*| - |c_k - c_*| = -Q(v_k) < 0$ . If  $v_k < v_*$ , then  $|c_{k+1} - c_*| - |c_k - c_*| = Q(v_k) < 0$ . In any case,  $|c_k - c_*|$  is eventually strictly decreasing.  $\square$

**Proof of Theorem 3.5:**

By Theorem 3.3, if  $\{v_k\}$  converges, the sequence is either eventually monotone or eventually oscillating around  $v_*$ . It is obvious that  $|v_k - v_*|$  eventually decreases to zero monotonically if  $\{v_k\}$  is eventually monotone. Thus we only consider the case  $\omega < \Phi(v_*)$ , where  $\{v_k\}$  is oscillating around  $v_*$  if  $k \geq k_0$ . Without loss of generality, we assume  $v_{k_0} < v_*$ . Thus, we have  $v_{k_0} < v_{k_0+2} < \dots < v_* < \dots < v_{k_0+3} < v_{k_0+1}$ .

Define a function  $J(v)$  as

$$J(v) = c_* + N^-(v) + \omega N^+(v) - (1 + \omega)N^+(2v_* - v). \quad (55)$$

Then,  $J(v_*) = 0$ , and

$$J'(v_*) = n^+(v_*) \frac{v_*^2 + 2|x|}{v_*^2} (\omega - \Psi(v_*)).$$

First consider the case  $\omega > \Psi(v_*)$ . In this case,  $J'(v_*) > 0$ . Hence, there exists an  $\epsilon > 0$  such that  $J(v) < 0$  for  $v \in (v_* - \epsilon, v_*)$  and  $J(v) > 0$  for  $v \in (v_*, v_* + \epsilon)$ . Suppose  $v_k \in (v_* - \epsilon, v_* + \epsilon)$  for  $k \geq k_0$ . Then, for points to the right of  $v_*$ , we have

$$\begin{aligned} & (1 + \omega) [N^+(v_{k_0+2m+1}) - N^+(2v_* - v_{k_0+2m})] \\ &= c_* + N^-(v_{k_0+2m}) + \omega N^+(v_{k_0+2m}) - (1 + \omega) N^+(2v_* - v_{k_0+2m}) \\ &= J(v_{k_0+2m}) < 0, \end{aligned} \tag{56}$$

and hence  $v_{k_0+2m+1} < 2v_* - v_{k_0+2m}$ . That is,  $|v_{k_0+2m+1} - v_*| < |v_{k_0+2m} - v_*|$ . Similarly, for points to the left of  $v_*$ ,

$$(1 + \omega) [N^+(v_{k_0+2m+2}) - N^+(2v_* - v_{k_0+2m+1})] = J(v_{k_0+2m+1}) > 0, \tag{57}$$

and hence  $|v_{k_0+2m+2} - v_*| < |v_{k_0+2m+1} - v_*|$ . Therefore,  $|v_k - v_*|$  strictly decreases to 0 after  $k \geq k_0$ .

Now consider the case  $\omega = \Psi(v_*)$ . In this case,  $J(v_*) = J'(v_*) = 0$ , and

$$J''(v_*) = \frac{n^+(v_*)}{4v_*^3} (v_*^4 - 4x^2).$$

If  $v_* \neq \sqrt{2|x|}$ , then either  $J''(v_*) < 0$  or  $J''(v_*) > 0$ . Thus, on sufficiently small neighborhood of  $v_*$ , we have either  $J(v) \geq 0$ , or  $J(v) \leq 0$ . Equations (56) and (57) tell us that in either case,  $|v_k - v_*|$  fails to be monotonically decreasing. If  $v_* = \sqrt{2|x|}$ , we need the third-order derivative of  $J(v)$ :

$$J'''(v_*) = \frac{3}{\sqrt{2\pi}v_*^2} > 0.$$

Hence, there exists an  $\epsilon > 0$  such that  $J(v) < 0$  for  $v \in (v_* - \epsilon, v_*)$  and  $J(v) > 0$  for  $v \in (v_*, v_* + \epsilon)$ , and equations (56) and (57) again hold. Thus when  $\omega = \Psi(v_*)$ ,  $|v_k - v_*|$  is eventually monotonically decreasing if and only if  $v_* = \sqrt{2|x|}$ .  $\square$

### Proof of Theorem 3.6:

When  $x = 0$ , we have  $\Phi(v) = 1$  and  $\phi(u, v) = 1$  for all  $(u, v) \in \mathbb{R}_+^2$ .

1) If  $\omega > 1$ , then the SOR algorithm is globally well-defined by Theorem 3.1. Furthermore, since  $\omega > \phi(u, v)$  for all  $(u, v) \in \mathbb{R}_+^2$ , by Lemmas 3.7 and 3.8, if  $v_0 > v_*$ , then  $v_k$  strictly decreases, and if  $v_0 < v_*$ ,  $v_k$  strictly increases. By Lemma 3.1, it is easy to show that the limit points have to be  $v_*$ .

2) If  $\omega = 1$ , then the SOR algorithm is globally well-defined by Theorem 3.1. In fact, since  $\omega = \phi(v_0, v_*)$ , by Lemma 3.8, we have  $v_k = v_*$  for all  $k \geq 1$ .

3) If  $0 < \omega < 1$ , then we have  $\omega < \phi(u, v)$  and  $1 + 2\omega > \phi(u, v)$  for all  $(u, v) \in \mathbb{R}_+^2$ . Hence, as long as  $G(v_0)$  is defined, the SOR algorithm is well-defined. By Lemma 3.9, the two subsequences above and below  $v_*$  are both monotone. Thus, they converge to two limit points, say  $v_L$  and  $v_H$ , with  $v_L \leq v_*$  and  $v_H \geq v_*$ . We need to show that  $v_L = v_H = v_*$ . Suppose  $v_L < v_H$ . Taking a limit on the iteration equation for odd and even  $k$ , we have

$$\begin{aligned} c_* + N^-(v_L) + \omega N^+(v_L) &= (1 + \omega)N^+(v_H), \\ c_* + N^-(v_H) + \omega N^+(v_H) &= (1 + \omega)N^+(v_L). \end{aligned}$$

Subtracting the above two equations gives us  $1 + 2\omega = \phi(v_L, v_H) = 1$ . This contradicts that  $\omega > 0$ . Thus, the sequence  $\{v_k\}$  is convergent and eventually oscillating around  $v_*$ .  $\square$

### Proof of Theorem 3.7:

We first focus on the well-definedness. Suppose  $v_* \leq \sqrt{2|x|}$ . If  $v \geq v_*$ , then

$$c_* + N^-(v) = N^+(v_*) - N^-(v_*) + N^-(v) \leq c^+(v) < 1,$$

and if  $v < v_*$ , then

$$c_* + N^-(v) = N^+(v_*) - N^-(v_*) + N^-(v) < N^+(v_*) < 1.$$

Hence,  $0 < c_* + N^-(v) < 1$  for all  $v \in \mathbb{R}_+$ .

Suppose  $v_* > \sqrt{2|x|}$  and  $c_* + N^-(v_0) < 1$ . Then,

$$0 < c_* + N^-(v_1) = N^+(v_1) - N^-(v_0) + N^-(v_1) < c^+(v_1) < 1.$$

Hence,  $v_2$  is well-defined. By induction, the whole sequence  $\{v_k\}$  in the SOR algorithm is well-defined. Furthermore, if either  $v_0 \geq v_*$  or  $v_0 \leq 2|x|/v_*$ , then  $N^-(v) \leq N^-(v_*)$  and we have  $0 < c_* + N^-(v) \leq N^+(v_*) < 1$ .

Let us turn to the proof of convergence, assuming that the sequence  $\{v_k\}$  is well-defined. Note that  $\omega = 0$  in this case, and thus  $1 + 2\omega > \phi(u, v)$  for all  $(u, v) \in \mathbb{R}_+^2$  by Lemma 3.6. We consider three cases.

1)  $v_* < \sqrt{2|x|}$ . Then, we have  $0 = \omega > \Phi(v_*)$ . Suppose first  $v_0 > v_*$ . If  $\omega > \phi(v_0, v_*)$ , then, by Lemma 3.8, we have a strictly decreasing, convergent sequence. If  $\omega < \phi(v_0, v_*)$ , then we have  $v_1 < v_*$ . From here, since  $\omega > \Phi(v_*) > \phi(v_1, v_*)$ , we have a strictly increasing, convergent sequence. Suppose now  $v_0 < v_*$ . Since  $\omega > \Phi(v_*) > \phi(v_0, v_*)$ , we have a strictly increasing, convergent sequence.

2)  $v_* = \sqrt{2|x|}$ . Then, we have  $0 = \omega = \Phi(v_*)$ . Since  $\phi(v, v_*)$  is strictly increasing in  $v$  and  $\phi(v_*, v_*) = \Phi(v_*)$  by Lemma 3.6, we have  $\omega > \phi(v, v_*)$  for all  $v < v_*$  and  $\omega < \phi(v, v_*)$  for all

$v > v_*$ . This guarantees that  $\{v_k\}$  is eventually monotonically increasing. It is easy to show that the limit point is  $v_*$ .

3)  $v_* > \sqrt{2|x|}$ . Then, since  $0 = \omega < \Phi(v_*)$ , there exists an  $\epsilon > 0$  such that  $\omega < \phi(v, v_*)$  for all  $v \in (v_* - \epsilon, v_* + \epsilon)$ . First suppose  $v_0 < v_*$ . Let  $v_{k_0}$  be the first point in  $\{v_k\}$  such that  $\omega < \phi(v_k, v_*)$ . Such a  $k_0$  exists since  $\omega < \Phi(v_*)$  and if  $\omega > \phi(v, v_*)$ , then we have  $v < G(v) < v_*$ . Then, since we always have  $1 + 2\omega > \phi(u, v)$ , by Lemma 3.9, we have

$$v_{k_0} < v_{k_0+2} < \cdots < v_* < \cdots < v_{k_0+3} < v_{k_0+1}.$$

Let  $v_L = \lim_m v_{k_0+2m}$  and  $v_H = \lim_m v_{k_0+2m+1}$ . Suppose  $v_L < v_H$ . Then, from the iteration equation, we have  $c_* + N^-(v_L) = N^+(v_H)$  and  $c_* + N^-(v_H) = N^+(v_L)$ . Subtracting these two equations leads to  $c^+(v_L) = c^+(v_H)$ . This is a contradiction since  $c^+$  is strictly increasing. Thus,  $v_L = v_H = v_*$ . That is,  $v_k$  converges to  $v_*$ . Now suppose  $v_0 > v_*$ . Then, since  $0 = \omega < \phi(v_0, v_*)$ , we immediately have  $v_1 < v_*$ . Thus, the proof reduces to the above with the role of  $v_0$  replaced by  $v_1$ .  $\square$

### Proof of Theorem 3.8:

When  $v_0 < v_*$ , we have  $\omega > \phi(v_0, v_*)$  and hence we have  $v_0 < v_1 < v_*$ . We still have  $\omega > \phi(v_1, v_*)$ , and thus  $v_1 < v_2 < v_*$ . Hence, by induction,  $v_k$  strictly increases to  $v_*$ .

When  $v_0 > v_*$ ,

1) If  $\omega > \phi(v_0, v_*)$ , then we have  $v_* < v_1 < v_0$ . At the second step, we still have  $\omega > \phi(v_1, v_*)$ , and thus  $v_* < v_2 < v_1$ . Hence,  $v_k$  strictly decreases to  $v_*$ .

2) If  $\omega = \phi(v_0, v_*)$ , then, by Lemma 3.8,  $v_1 = v_*$  and thus  $v_k = v_*$  for all  $k \geq 1$ .

3) If  $\Phi(v_*) \leq \omega < \phi(v_0, v_*)$ , then we have  $v_1 < v_*$  if  $v_1$  is defined. Since  $\omega > \phi(v_1, v_*)$ , we have  $v_1 < v_2 < v_*$  and we still have  $\omega > \phi(v_2, v_*)$ , and thus  $v_* < v_2 < v_3$ . Hence,  $v_k$  strictly increases to  $v_*$ .  $\square$

### Proof of Theorem 3.9:

By Theorem 3.8, we know that when  $\omega \geq \Phi(v_*)$ , as long as the first jump over  $v_*$  (in this case always  $G(v_0)$ ) is well-defined, the SOR algorithm is well-defined and converges to  $v_*$ . Hence, we will consider only the case  $\omega < \Phi(v_*)$  in the following proof.

We first establish a very useful result. When  $\omega \geq \Psi(\sqrt{2}v_*)$ , each iteration of the successive over-relaxation will decrease the distance of  $c_k$  to  $c_*$ , provided that  $v_{k+1}$  is well-defined. To show this, take any  $v \in \text{Dom}G$ . Suppose that  $v$  and  $G(v)$  are on the opposite side of  $v_*$ . If  $\omega \geq \Psi(\sqrt{2}v_*)$ , then

$$\left( \frac{|x|}{v^2} + \frac{|x|}{G(v)^2} \right) + \frac{\omega}{1 + \omega} > \frac{|x|}{v_*^2} + \frac{\omega}{1 + \omega} > 0.$$

Thus,  $Q'(v) > 0$  by Lemma 3.5. Since  $Q(v_*) = 0$ ,  $Q(v) > 0$  if  $v > v_*$  and  $Q(v) < 0$  if  $v < v_*$ , provided that  $v$  and  $G(v)$  are on the opposite side of  $v_*$ . Regardless of whether  $v > v_*$  or  $v < v_*$ ,

by the definition of  $Q(v)$ , this guarantees

$$|c(G(v)) - c_*| < |c(v) - c_*| \quad (58)$$

whenever  $(v - v_*)(G(v) - v_*) < 0$ . Suppose now that  $v$  and  $G(v)$  are on the same side of  $v_*$ . Then by Lemma 3.7,  $|G(v) - v_*| < |v - v_*|$ , so equation (58) is still true.

We now consider the well-defined of the sequence  $\{v_k\}$ . For any  $v$  with  $\omega > \phi(v, v_*)$ , it is easy to show that  $G(v)$  is always well-defined and lies in the region between  $v$  and  $v_*$ . If  $\omega = \phi(v, v_*)$ , then  $G(v)$  is also well-defined and equals  $v_*$ . Now let  $v_{k_0}$  be the first point in  $\{v_k\}$  such that  $\omega < \phi(v_k, v_*)$  and suppose  $G(v_{k_0})$  is defined. We need to show that  $v_k$  is also well-defined for any  $k > k_0 + 1$ . Notice that by Lemma 3.8,  $(v_{k_0} - v_*)(v_{k_0+1} - v_*) < 0$ . Suppose first  $v_{k_0} < v_*$ . Then  $v_{k_0+1} > v_*$ . From  $\omega < \phi(v_{k_0+1}, v_*)$ , we have

$$F(v_{k_0+1}) = c_* + N^-(v_{k_0+1}) + \omega N^+(v_{k_0+1}) < (1 + \omega)N^+(v_*) < 1 + \omega.$$

From equation (58), we have  $2c_* - c_{k_0} - c_{k_0+1} > 0$ . Thus,

$$\begin{aligned} F(v_{k_0+1}) &= c_* + N^-(v_{k_0+1}) + \omega N^+(v_{k_0+1}) = c_* - c_{k_0+1} + (c_* + N^-(v_{k_0}) + \omega N^+(v_{k_0})) \\ &= 2c_* - c_{k_0} - c_{k_0+1} + (1 + \omega)N^+(v_{k_0}) > (1 + \omega)N^+(v_{k_0}) > 0. \end{aligned}$$

Thus,  $v_{k_0+2} = G(v_{k_0+1})$  is well-defined and  $v_{k_0} < v_{k_0+2} < v_*$ . Similarly, if  $v_{k_0} > v_*$ , then  $v_{k_0+2} = G(v_{k_0+1})$  is well-defined and  $v_{k_0} > v_{k_0+2} > v_*$ . Exactly the same proof now shows that  $v_k$  is well-defined for any  $k > k_0 + 2$  and lies in between  $v_{k-1}$  and  $v_*$ . This shows that the whole sequence  $\{v_k\}$  is well-defined if and only if  $G(v_{k_0})$  is defined.

Now assume that  $\{v_k\}$  is well-defined. We know the two subsequences consisting of points above  $v_*$  and below  $v_*$  respectively are both monotone. Thus they converge to two limit points, say  $v_H$  and  $v_L$ . Taking the limit on the iteration equation for odd and even  $k$  then gives  $v_L = G(v_H)$  and  $v_H = G(v_L)$ . Suppose  $v_H > v_L$ . Notice  $v_L = G(G(v_L))$ . This is a contradiction, however, because since  $|G(G(v_L)) - v_*|$  has to be smaller than  $|v_L - v_*|$  by equation (58), we must have  $G(G(v_L)) > v_L$ . Thus  $v_H = v_L$  and the sequence  $\{v_k\}$  converges.  $\square$

### Proof of Theorem 3.10:

By Theorem 3.6, we only consider  $x < 0$ . Furthermore, by Theorem 3.8, we need only to consider the case  $\omega < \Phi(v_*)$ .

If  $v_0 > v_*$ , then  $v_1 < v_*$  since  $\omega < \Phi(v_*) < \phi(v_0, v_*)$ . Thus without loss of generality, we assume  $v_0 < v_*$ . We consider three cases.

1) If  $\omega < \phi(v_0, v_*)$ , then we have  $v_1 > v_*$ . Clearly, we still have  $\omega < \phi(v_1, v_*)$  and thus  $v_2 < v_*$ . By the assumption  $1 + 2\omega > \phi(v_*, \bar{v})$ , we have  $1 + 2\omega > \phi(v_0, v_1)$  since  $\phi(u, v)$  is increasing in both  $u$  and  $v$  by Lemma 3.6. This guarantees  $v_0 < v_2 < v_*$  by Lemma 3.9. Hence, we still have  $\omega < \phi(v_2, v_*)$  and thus  $v_3 > v_*$ . Again, we have  $1 + 2\omega > \phi(v_1, v_2)$  and hence  $v_* < v_3 < v_1$ . Repeating this process gives us  $v_0 < v_2 < \dots < v_* < \dots < v_3 < v_1$ . Let

$v_L = \lim_k v_{2k}$  and  $v_H = \lim_k v_{2k+1}$ . Suppose  $v_L < v_H$ . Then, taking the limit on the iteration equation for both odd and even  $k$ , we have

$$\begin{aligned} c_* + N^-(v_L) + \omega N^+(v_L) &= (1 + \omega)N^+(v_H), \\ c_* + N^-(v_H) + \omega N^+(v_H) &= (1 + \omega)N^+(v_L). \end{aligned}$$

Subtracting these two equations leads to  $1 + 2\omega = \phi(v_L, v_H)$ . This contradicts the assumption  $1 + 2\omega > \phi(v_*, \bar{v})$ . Hence, we have  $v_L = v_H = v_*$ .

2) If  $\omega = \phi(v_0, v_*)$ , then we have  $v_k = v_*$  for all  $k \geq 1$ .

3) If  $\omega > \phi(v_0, v_*)$ , then we have  $v_0 < v_1 < v_*$ . Let  $k_0$  be the first  $k$  such that  $\omega < \phi(v_k, v_*)$ . Then, starting from  $v_{k_0}$ , the proof reduces to that of case 1.  $\square$

### Proof of Lemma 3.10:

We will write  $v$  for  $G(u)$ . Suppose  $u < v$ . By Lemma 3.4 and 3.6, we have

$$\begin{aligned} \phi(u, G(u))' &= \phi_1(u, G(u)) + \phi_2(u, G(u))G'(u) \\ &= \frac{n^+(u)}{[N^+(u) - N^+(v)]^2} \times \left\{ (c(v) - c(u)) \left[ -\frac{|x|}{v^2} \frac{\left(\frac{|x|}{u^2} - \frac{1}{2}\right) + \omega \left(\frac{|x|}{u^2} + \frac{1}{2}\right)}{(1 + \omega) \left(\frac{|x|}{v^2} + \frac{1}{2}\right)} + \frac{|x|}{u^2} \right] \right. \\ &\quad \left. + \frac{1}{2}(c^+(v) - c^+(u)) \left[ -1 + \frac{\left(\frac{|x|}{u^2} - \frac{1}{2}\right) + \omega \left(\frac{|x|}{u^2} + \frac{1}{2}\right)}{(1 + \omega) \left(\frac{|x|}{v^2} + \frac{1}{2}\right)} \right] \right\} \\ &= M \left\{ (c(v) - c(u)) \left[ -\frac{|x|}{v^2} A + \frac{|x|}{u^2} \right] + \frac{1}{2} (c^+(v) - c^+(u)) [-1 + A] \right\}, \end{aligned}$$

where  $M = n^+(u)/[N^+(u) - N^+(v)]^2 > 0$ , and

$$A = \frac{\left(\frac{|x|}{u^2} + \frac{1}{2}\right)}{(1 + \omega) \left(\frac{|x|}{v^2} + \frac{1}{2}\right)} [\omega - \Phi(u)].$$

Since  $u < v$  and  $1 + \omega > 0$ , it can be easily shown that  $-\frac{|x|}{v^2} A + \frac{|x|}{u^2} > 0$ . Using the inequality between  $c$  and  $c^+$  in Lemma 3.1, we have

$$\begin{aligned} \phi(u, G(u))' &\geq M \left\{ \frac{u^2}{2|x|} (c^+(v) - c^+(u)) \left[ -\frac{|x|}{v^2} A + \frac{|x|}{u^2} \right] + \frac{1}{2} (c^+(v) - c^+(u)) [-1 + A] \right\} \\ &= \frac{M}{2} (c^+(v) - c^+(u)) \left[ 1 - \frac{u^2}{v^2} \right] A \geq K (\omega - \Phi(u)). \end{aligned}$$

Notice that  $\tilde{v}$  is finite when  $\omega < \Phi(v_*)$ . Furthermore,  $\tilde{v} < v_*$  since  $\Phi(v)$  is strictly increasing. Since  $u < \tilde{v} < v_*$ , by Lemma 3.7, we have  $G(u) > u$ . Also, we have  $\omega = \Phi(\tilde{v}) > \Phi(u)$ . Thus,  $\phi(u, G(u))' > 0$ .  $\square$



**Proof of Theorem 3.11:**

When  $x = 0$ , the condition in the theorem becomes  $\omega > 0$  and guarantees convergence by Theorem 3.6. Thus we assume  $x < 0$ . Furthermore, we assume  $\omega < \Phi(v_*)$  since if  $\omega \geq \Phi(v_*)$ , then Theorem 3.8 guarantees convergence.

By Lemma 3.10,  $\phi(u, G(u))$  is increasing in  $u$  when  $u < \tilde{v}$ . Thus,  $\sup_{u < \tilde{v}} \phi(u, G(u)) = \phi(\tilde{v}, G(\tilde{v}))$ . The condition in the theorem then implies that  $1 + 2\omega > \sup_{u < v_*} \phi(u, G(u))$ . By Lemma 3.2, we have  $G(\tilde{v}) = \max_{v \in \text{Dom}G} G(v)$ .

If  $v_0 > v_*$ , then  $v_1 < v_*$  since  $\omega < \Phi(v_*) < \phi(v_0, v_*)$ . Thus without loss of generality, we assume  $v_0 < v_*$ . Since the sequence obviously converges if  $v_k = v_*$  for some  $k$ , we will assume  $v_k \neq v_*$  for all  $k$ . This in particular implies that  $v_k \neq v_{k+1}$  for all  $k$ , and  $\omega \neq \phi(v_k, v_*)$  for any  $k$ . We consider two cases.

1)  $\omega < \phi(v_0, v_*)$ . We have  $v_1 > v_*$ . Since  $\omega < \phi(v_1, v_*)$ ,  $v_2 < v_*$ . By equation (7), we have  $1 + 2\omega > \phi(v_0, v_1)$ , so  $v_0 < v_2 < v_*$  by Lemma 3.9. Now since  $\omega < \phi(v_2, v_*)$ , we have  $v_3 > v_*$ . Since  $\omega < \phi(v_3, v_*)$ , we get  $v_4 < v_*$ . Again, because  $1 + 2\omega > \phi(v_2, v_3)$ ,  $v_2 < v_4 < v_*$ . Repeating this process gives us  $v_0 < v_2 < v_4 < \dots < v_*$ , and  $v_* < v_{2k+1} < G(\tilde{v})$  for all  $k \geq 0$ . We now show  $v_{2k+1}$  is decreasing. Suppose  $v_3 > v_1$ . Since  $G(v)$  is strictly decreasing for  $v > \tilde{v}$  by Lemma 3.4, we get  $v_4 = G(v_3) < G(v_1) = v_2$ . This is a contradiction. Hence, we also have  $v_* < \dots < v_5 < v_3 < v_1$ . Let  $v_L = \lim_k v_{2k}$  and  $v_H = \lim_k v_{2k+1}$ . Suppose  $v_L < v_H$ . Similar argument as in the proof of Theorem 3.6 gives that when  $v_L \neq v_H$ , we have  $1 + 2\omega = \phi(v_L, v_H)$ . This contradicts equation (7). Hence, we have  $v_L = v_H = v_*$ .

2)  $\omega > \phi(v_0, v_*)$ . We have  $v_0 < v_1 < v_*$ . Let  $k_0$  be the first  $k$  such that  $\omega < \phi(v_k, v_*)$ . Notice  $k_0$  is finite by Lemma 3.7 and the fact that  $\omega < \Phi(v_*)$ . For otherwise,  $v_k$  would strictly increase to a limit point other than  $v_*$ , which is impossible since  $v_*$  is the only fixed point of  $v = G(v)$ . The proof now reduces exactly to the previous case, with  $v_{k_0}$  playing the role of  $v_0$ .  $\square$

**Proof of Theorem 4.1:**

From the proof of Theorem 3.2, we have

$$v_{k+1} - v_* = G(v_k) - G(v_*) = G'(v_*)(v_k - v_*) + \mathcal{O}(v_k - v_*)^2.$$

Since  $v_k$  converges to  $v_*$ , we have

$$\lim_{k \rightarrow +\infty} \frac{|v_{k+1} - v_*|}{|v_k - v_*|} = |G'(v_*)|.$$

Notice that  $\omega \geq \Psi(v_*)$  implies that  $|G'(v_*)| \leq 1$ . By Lemma 3.4, we have  $G'(v_*) = 0$  if and only if  $\omega = \Phi(v_*)$ . Also,  $|G'(v_*)| = 1$  if and only if  $\omega = \Psi(v_*)$ . Thus,  $\{v_k\}$  has sublinear convergence when  $\omega = \Psi(v_*)$ , superlinear convergence when  $\omega = \Phi(v_*)$ , and linear convergence in all other cases.  $\square$

**Proof of Theorem 4.2:**

When  $x = 0$ , by Lemma 3.6, we have  $\omega_0 = \Phi(v_0) = \Phi(v_*) = \phi(v_0, v_*) = 1$  for any  $v_0 \in \mathbb{R}_+$ . Hence, by using a similar proof as in Lemma 3.8, we have that  $v_k = v_*$  for all  $k \in \mathbb{N}$ .

Now suppose  $x < 0$ . When  $v_0 > v_*$ , by Lemma 3.7 and 3.8, we have that  $v_1$  is well-defined and  $v_* < v_1 < v_0$  since  $\omega_0 = \Phi(v_0) > \phi(v_0, v_*)$ . Again, since  $\omega_1 = \Phi(v_1) > \phi(v_1, v_*)$ , we have that  $v_2$  is well-defined and  $v_* < v_2 < v_1$ . Hence,  $v_k$  strictly decreases to  $v_*$ .

When  $v_0 < v_*$ , by Lemma 3.8 we have  $v_1 > v_*$  if  $v_1$  is well-defined, since  $\omega_0 = \Phi(v_0) < \phi(v_0, v_*)$ . The proof now reduces to the previous case, with  $v_1$  playing the role of  $v_0$ .  $\square$

### Proof of Theorem 4.3:

By Theorem 4.2,  $v_k$  converges to  $v_*$  from above if  $v_1$  is well-defined. Thus  $\omega_k = \Phi(v_k)$  converges to  $\Phi(v_*)$ . Let us write  $G(v_k; \omega_k)$  for  $G(v_k; x, v_*, \omega_k)$ . Since  $v_* \in \text{Dom}G$  by Lemma 3.4, there exists a neighborhood  $V_\epsilon = (v_* - \epsilon, v_* + \epsilon)$  of  $v_*$  such that  $G(v; \omega)$  is smooth on  $V_\epsilon$  and

$$M_1(\omega) \equiv \sup \{|G''(v; \omega)|/2 : v \in V_\epsilon\} < \infty,$$

for each fixed  $\omega$ . Also, since  $G(v; \omega)$  is smooth in  $\omega$ , we can assume

$$M_2 = \sup \{|G''(v; \omega)|/2 : v \in V_\epsilon, \omega \in \Omega_\epsilon\} < \infty,$$

where  $\Omega_\epsilon = (\Phi(v_*) - \epsilon, \Phi(v_*) + \epsilon)$ .

Now, Taylor expansion gives us

$$v_{k+1} - v_* = G(v_k; \omega_k) - G(v_*; \omega_k) = G'(v_k; \omega_k)(v_k - v_*) + \frac{1}{2}G''(\xi_k, \omega_k)(v_k - v_*)^2,$$

where  $\xi_k \in (v_k, v_*)$ . Hence, using L'Hospital's rule, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{|v_{k+1} - v_*|}{|v_k - v_*|^2} &\leq \lim_{k \rightarrow \infty} \frac{|G'(v_k; \omega_k)|}{|v_k - v_*|} + M_2 = \lim_{k \rightarrow \infty} \frac{|\omega_k - \Phi(v_*)|/(1 + \omega_k)}{|v_k - v_*|} + M_2 \\ &= \frac{|\Phi'(v_*)|}{1 + \Phi(v_*)} + M_2 = \frac{4|x|}{v_*(v_*^2 + 2|x|)} + M_2 < \infty. \end{aligned}$$

This shows that the convergence is of at least quadratic order.  $\square$

### Proof of Lemma 4.1:

Notice that when  $\omega = 1$ , we have

$$M(v) = \frac{2}{1 + \Phi(v)}G(v) + \left(1 - \frac{2}{1 + \Phi(v)}\right)v = \frac{L(v)}{1 + \Phi(v)}, \quad (59)$$

where  $L(v) = 2G(v) + (\Phi(v) - 1)v$ . The derivative of  $M(v)$  with respect to  $v$  follows directly from Lemma 3.4.

We now show that  $M(v) > 0$  for all  $v \in \mathbb{R}_+$ . Fix  $v > 0$ . Note that  $G'(u) > 0$  for all  $u \in \mathbb{R}_+$ , and  $L'(u) > 0$  for all  $u \geq \sqrt{2|x|}$  since

$$L'(u) = 2G'(u) + \frac{4|x|(u^2 - 2|x|)}{(u^2 + 2|x|)^2}. \quad (60)$$

Notice also  $M(v_*) = v_* > 0$  and  $L(v_*) > 0$ . Thus, we assume  $v \neq v_*$ .

Suppose first  $v < v_*$ . Then, we have  $v < G(v) < v_*$  and thus  $M(v) > G(v) > 0$ .

Next, we consider  $v > v_*$ . We prove  $M(v) > 0$  by dividing the proof into two cases:  $\sqrt{2|x|} \leq v_*$  and  $v_* < \sqrt{2|x|}$ .

**Case 1:**  $\sqrt{2|x|} \leq v_*$ . Since  $L'(u) > 0$  for all  $u \geq \sqrt{2|x|}$ , we have

$$M(v) = \frac{L(v)}{1 + \Phi(v)} \geq \frac{L(v_*)}{1 + \Phi(v)} > 0.$$

**Case 2:**  $v_* < \sqrt{2|x|}$ . There are two subcases to consider. First, suppose  $v_* < v \leq \sqrt{2|x|}$ . By equation (35), we have

$$M'(u) = K \left[ -2G^3u + G^2u^2 + \frac{n^+(u)}{n^+(G)}G^2u^2 + 2\frac{n^+(u)}{n^+(G)}G^2|x| - 4Gu|x| + 2u^2|x| \right],$$

where  $G = G(u)$  and  $K > 0$ . Since  $u > G(u)$  and  $n^+(u) > n^+(G)$  for all  $u \in (v_*, \sqrt{2|x|}]$ , we have

$$\begin{aligned} M'(u) &\geq K \left[ -2G^3u + 2G^2u^2 + 2G^2|x| - 4Gu|x| + 2u^2|x| \right] \\ &= K \left[ 2G^2u(u - G) + 2|x|(u - G)^2 \right] > 0 \end{aligned}$$

for all  $u \in (v_*, \sqrt{2|x|}]$ . Therefore, we have  $M(v) > M(v_*) > 0$  for all  $v_* < v \leq \sqrt{2|x|}$ .

Second, suppose  $v_* < \sqrt{2|x|} < v$ . Since  $L'(u) > 0$  for all  $u > \sqrt{2|x|}$  by equation (60), we have  $L(v) > L(\sqrt{2|x|})$ . Hence,

$$M(v) \geq \frac{L(\sqrt{2|x|})}{1 + \Phi(v)} = \frac{M(\sqrt{2|x|})}{1 + \Phi(v)} > \frac{M(v_*)}{1 + \Phi(v)} > 0,$$

where we have used the fact that  $M(\sqrt{2|x|}) > M(v_*)$  in the proof of the previous case.

That  $M'(v_*) = 0$  follows directly from equation (35) and the fact that  $G(v_*) = v_*$ . The second order derivative  $M''(v_*)$  follows from direct computation and Lemma 3.4.

We now examine the function  $m(v, x)$ . Notice first that

$$\lim_{v \rightarrow 0^+} m(v, x) = 8x^3 < 0, \quad \lim_{v \rightarrow +\infty} m(v, x) = +\infty.$$

Furthermore, we have

$$\frac{\partial m(v, x)}{\partial v} = 2v(-4x^2 - 4(4+x)v^2 + 3v^4) = 6v(v^2 - z_1)(v^2 - z_2),$$

where

$$z_1 = \frac{2}{3} \left[ 4 + x + 2\sqrt{(x+1)^2 + 3} \right] > 0, \quad z_2 = \frac{2}{3} \left[ 4 + x - 2\sqrt{(x+1)^2 + 3} \right] < 0.$$

Thus, for fixed  $x < 0$ ,  $m(v, x)$  as a function of  $v$  is strictly decreasing on  $(0, \sqrt{z_1})$ , and strictly increasing on  $(\sqrt{z_1}, +\infty)$ . By taking into account the limiting behavior of  $m(x, v)$ , we see that  $m(v, x) = 0$  has a unique root  $v_r = v_r(x) > \sqrt{z_1}$ . Furthermore,  $m(v, x) < 0$  when  $v < v_r$ , and  $m(v, x) > 0$  when  $v > v_r$ .  $\square$

**Proof of Theorem 4.4:**

The case with  $x = 0$  can be easily verified, so we assume  $x < 0$  below.

By Lemma 4.1, when  $\omega = 1$ , for any  $v \in \mathbb{R}_+$ ,  $M(v)$  is well-defined and  $M(v) > 0$ . Thus the SOR-TS algorithm is globally well-defined. Since  $M'(v_*) = 0$ ,  $M(v)$  is locally contracting near  $v_*$  since  $M(v)$  is smooth function of  $v$ . Hence, if  $v_0$  is sufficiently close to  $v_*$  then the sequence  $\{v_k\}$  converges to  $v_*$ .

We now show that the SOR-TS algorithm converges globally when  $B(x, v_*) < 1$ . Assume  $v_k \neq v_*$  for all  $k \in \mathbb{N}$ . We consider the following two cases.

**Case 1:**  $v_0 > v_*$ . In this case, the sequence  $\{v_k\}$  is bounded below by  $M_{\min}(x, v_*)$ . But then the assumption  $B(x, v_*)$  implies that  $M(v)$  is globally contracting on  $[M_{\min}(x, v_*), \infty)$ , since for any  $v \in [M_{\min}(x, v_*), \infty)$ , there exists  $\xi$  between  $v$  and  $v_*$ , such that

$$|M(v) - v_*| = |M'(\xi)| \cdot |v - v_*| < |v - v_*|.$$

Thus,  $\{v_k\}$  converges to  $v_*$ .

**Case 2:**  $v_0 < v_*$ . In this case, we have  $M(v_0) > G(v_0) > v_0$ . If the sequence  $\{v_k\}$  is bounded above by  $v_*$ , then  $v_k < v_{k+1} < v_*$  for all  $k \in \mathbb{N}$ . Thus  $v_k$  monotonically increases. It is easy to show that it converges to  $v_*$ . The other case is when there exists  $k_0 \in \mathbb{N}$  such that  $v_{k_0} > v_*$ , which reduces to Case 1 above.

The convergence pattern follows from the following Taylor expansion

$$v_{k+1} - v_* = \frac{1}{2}M''(v_*)(v_k - v_*)^2 + \mathcal{O}(v_k - v_*)^3. \quad (61)$$

Notice that when  $v_* < v_r(x)$ , by Lemma 4.1,  $M''(v_*) > 0$ . This implies that for any  $v_k$  sufficiently close to  $v_*$  and  $v_k \neq v_*$ , we have  $v_* < M(v_k) < v_k$ . By induction, the sequence  $\{v_k\}$  monotonically decreases to  $v_*$ . The case  $v_* > v_r(x)$  can be proven similarly. Finally, notice that when  $v_* = v_r(x)$ ,  $M''(v_*) = 0$ . In this case, the local behavior of the algorithm around  $v_*$  depends on  $M'''(v_*)$ , which can be either nonnegative or nonpositive.  $\square$

**Proof of Theorem 4.5:**

If  $v_k$  converges to  $v_*$ , by equation (61) we have

$$\lim_{k \rightarrow \infty} \frac{|v_{k+1} - v_*|}{|v_k - v_*|^2} = \frac{1}{2} |M''(v_*)| < \infty.$$

Hence, the SOR-TS algorithm with  $\omega = 1$  has at least a quadratic order of convergence. In particular, when  $v_* = v_r(x)$ , the convergence has at least a cubic order.  $\square$

**Proof of Theorem 6.1:**

The SOR-B algorithm with  $\omega_k \equiv 0$  is always well-defined since for any  $v \in \mathbb{R}_+$ ,  $G^B(v)$  is well-defined.

We now show that the SOR-B algorithm always converges. First consider the case  $x = 0$ . Notice  $G^B(v; 0, v_*, 0) \equiv \sqrt{2\pi}c_* = v_*$  for  $\forall v \in \mathbb{R}_+$ . Thus the SOR-B algorithm lands on  $v_*$  after one iteration.

Now let  $x < 0$ . Define two function  $f(v)$  and  $g(v)$  as follows:

$$\begin{aligned} f(v) &= |x| \left[ N\left(\frac{x}{v}\right) - N\left(\frac{x}{v_*}\right) \right] - v_* \left[ n\left(\frac{x}{v}\right) - n\left(\frac{x}{v_*}\right) \right], \\ g(v) &= |x| \left[ N\left(\frac{x}{v}\right) - N\left(\frac{x}{v_*}\right) \right] - \left[ vn\left(\frac{x}{v}\right) - v_*n\left(\frac{x}{v_*}\right) \right]. \end{aligned}$$

Then  $f(v_*) = g(v_*) = 0$ , and

$$f'(v) = \frac{|x|^2 n(x/v)}{v^3} (v - v_*), \quad g'(v) = -n(x/v) < 0.$$

Thus,  $f(v) \geq 0$  with equality if and only if  $v = v_*$ , and  $g(v) < 0$  if  $v > v_*$ . Since  $G^B(v) > v_*$  if and only if  $f(v) > 0$  when  $\omega \equiv 0$ , we have  $v_1 > v_*$  for any  $v_0 \in \mathbb{R}_+$ . Since for  $v > v_*$ ,  $G^B(v) < v$  if and only if  $g(v) < 0$ , we have  $v_* < v_{k+1} < v_k$  for all  $k \in \mathbb{N}$ . Thus the sequence  $\{v_k\}$  ( $k \geq 1$ ) monotonically decreases to a limit point  $v_\infty$ . Taking a limit on the iteration equation then gives  $c_* = c^B(x, v_\infty)$ . Since vega is strictly positive in the Bachelier formula, we have  $v_* = v_\infty$ .

To see that the convergence is of quadratic order, notice that when  $v_k$  is sufficiently close to  $v_*$ , a Taylor expansion of the iteration equation gives

$$v_{k+1} - v_* = \frac{x^2}{2v_*^3} (v_k - v_*)^2 + \mathcal{O}(v_k - v_*)^3,$$

and thus

$$\lim_{k \rightarrow +\infty} \frac{|v_{k+1} - v_*|}{|v_k - v_*|^2} = \frac{x^2}{2v_*^3} > 0.$$

□

**Proof of Theorem 6.2:**

Since  $\omega_k = 0$  and  $\kappa(x, v) > 0$  for all  $x \in \mathbb{R}$  and  $v > 0$ ,  $G^{CS}(x; v, x_*, 0)$  is well-defined for any  $x \in \mathbb{R}$ . Thus, the sequence  $\{x_k\}$  is well-defined.

We now show the global convergence of the SOR-CS algorithm. We will simply write  $G(x)$  for  $G^{CS}(x; v, x_*, 0)$ . Define two functions  $f(x)$  and  $g(x)$  by

$$\begin{aligned} f(x) &= e^{x_*} N\left(\frac{x_*}{v} + \frac{v}{2}\right) - e^{x_*} N\left(\frac{x}{v} + \frac{v}{2}\right) - N\left(\frac{x_*}{v} - \frac{v}{2}\right) + N\left(\frac{x}{v} - \frac{v}{2}\right), \\ g(x) &= e^{x_*} N\left(\frac{x_*}{v} + \frac{v}{2}\right) - e^x N\left(\frac{x}{v} + \frac{v}{2}\right) - N\left(\frac{x_*}{v} - \frac{v}{2}\right) + N\left(\frac{x}{v} - \frac{v}{2}\right). \end{aligned}$$

By the definition of  $G(x) = G(x; v, x_*, \omega)$ , we have

$$f(x) = \left( e^{G(x)} - e^{x_*} \right) N\left( \frac{x}{v} + \frac{v}{2} \right), \quad g(x) = \left( e^{G(x)} - e^x \right) N\left( \frac{x}{v} + \frac{v}{2} \right).$$

To examine the signs of  $f(x)$  and  $g(x)$ , notice  $f(x_*) = g(x_*) = 0$ , and

$$f'(x) = \frac{1}{v} n\left( \frac{x}{v} + \frac{v}{2} \right) (e^x - e^{x_*}), \quad g'(x) = -e^x N\left( \frac{x}{v} + \frac{v}{2} \right),$$

where we have used the identity  $n\left(\frac{x}{v} + \frac{v}{2}\right)e^x = n\left(\frac{x}{v} - \frac{v}{2}\right)$  in deriving the above derivatives. Since  $f'(x) > 0$  when  $x > x_*$ , and  $f'(x) < 0$  when  $x < x_*$ ,  $f(x)$  achieves minimum at  $x = x_*$ . Thus  $f(x) \geq 0$  on  $\mathbb{R}$  with  $f(x) = 0$  if and only if  $x = x_*$ . On the other hand, the function  $g(x)$  is monotonically decreasing on  $\mathbb{R}$ . In particular, for any  $x > x_*$ , we have  $f(x) > 0$  and  $g(x) < 0$ .

Now let  $x_0 \neq x_*$  in the **SOR-CS** algorithm. We have  $f(x_0) > 0$  by the above analysis. That is,  $x_1 = G(x) > x_*$ . The above analysis then shows that  $x_* < x_{k+1} < x_k$  for any  $k \in \mathbb{N}$ . Thus  $x_k$  monotonically decreases to a limit  $x_\infty$ . It can be easily shown that  $x_\infty = x_*$  since  $x_*$  is the only fixed point of  $G(x)$ .

To see the convergence is of quadratic order, notice that when  $x_k$  is sufficiently close to  $x_*$ , a Taylor expansion of the iteration equation gives

$$x_{k+1} - x_* = \frac{n\left(\frac{x}{v} + \frac{v}{2}\right)}{2vN\left(\frac{x}{v} + \frac{v}{2}\right)} (x_k - x_*)^2 + \mathcal{O}(x_k - x_*)^3,$$

and thus

$$\lim_{k \rightarrow +\infty} \frac{|x_{k+1} - x_*|}{|x_k - x_*|^2} = \frac{n\left(\frac{x}{v} + \frac{v}{2}\right)}{2vN\left(\frac{x}{v} + \frac{v}{2}\right)} > 0.$$

□

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**Table 1**  
**The SOR, SOR-DR and SOR-TS algorithms**

This table gives the values of  $v_k$  for the first five iterations in the SOR, SOR-DR and SOR-TS algorithms. The option used has moneyness  $x = -0.5$  and volatility  $v_* = 1$  so that  $\Phi(v_*) = 0$ . For the SOR and SOR-TS algorithms, we set  $\omega = 1$  for all the iterations. The initial estimate is chosen to be  $v_0 = 0.6$  or  $1.4$  for all three algorithms. All the rows are computed with  $(10^{-16})$ -precision arithmetic, except that the rows  $4'$  and  $5'$  are computed with  $(10^{-50})$ -precision arithmetic.

**Panel A:  $v_0 < v_*$**

$k$	SOR		SOR-DR			SOR-TS		
	$v_k$	$ v_k - v_* $	$v_k$	$ v_k - v_* $	$\omega_k$	$v_k$	$ v_k - v_* $	$\alpha_k$
0	0.6000	0.4000	0.6000	0.4000	-0.4706	0.6000	0.4000	3.7778
1	0.7284	0.2716	1.2429	0.2429	0.2141	1.0850	0.0850	1.8495
2	0.8327	0.1673	1.0192	0.0192	0.0190	1.0016	0.0016	1.9968
3	0.9050	0.0950	1.0002	$2 \times 10^{-4}$	$2 \times 10^{-4}$	1.0000	$6 \times 10^{-7}$	2.0000
4	0.9489	0.0511	1.0000	$2 \times 10^{-8}$	$2 \times 10^{-8}$	1.0000	$1 \times 10^{-13}$	2.0000
5	0.9735	0.0265	1.0000	$7 \times 10^{-16}$		1.0000	$4 \times 10^{-16}$	
$5'$	0.9735	0.0265	1.0000	$1 \times 10^{-16}$		1.0000	$3 \times 10^{-27}$	

**Panel B:  $v_0 > v_*$**

$k$	SOR		SOR-DR			SOR-TS		
	$v_k$	$ v_k - v_* $	$v_k$	$ v_k - v_* $	$\omega_k$	$v_k$	$ v_k - v_* $	$\alpha_k$
0	1.4000	0.4000	1.4000	0.4000	0.3243	1.4000	0.4000	1.5102
1	1.1507	0.1507	1.0413	0.0413	0.0404	1.0235	0.0235	1.9546
2	1.0675	0.0675	1.0008	$8 \times 10^{-4}$	$8 \times 10^{-4}$	1.0001	$1 \times 10^{-4}$	1.9997
3	1.0321	0.0321	1.0000	$3 \times 10^{-7}$	$3 \times 10^{-7}$	1.0000	$4 \times 10^{-9}$	2.0000
4	1.0157	0.0157	1.0000	$5 \times 10^{-14}$	$5 \times 10^{-14}$	1.0000	$4 \times 10^{-16}$	2.0000
5	1.0077	0.0077	1.0000	$1 \times 10^{-16}$		1.0000	$4 \times 10^{-16}$	
$4'$	1.0157	0.0157	1.0000	$5 \times 10^{-14}$	$5 \times 10^{-14}$	1.0000	$5 \times 10^{-18}$	2.0000
$5'$	1.0077	0.0077	1.0000	$1 \times 10^{-27}$		1.0000	$6 \times 10^{-36}$	

**Table 2**  
**Accuracy of the SOR, SOR-DR and SOR-TS algorithms in  $D^-$**

This table gives the accuracy of the SOR , SOR-DR and SOR-TS algorithms in domain  $D^-$ . All three algorithms are implemented with  $(10^{-16})$ -precision arithmetic in MATLAB 7.1 on a Dell Dimension 4600 desktop computer (2.8 GHz, 1G RAM). The relaxation parameter  $\omega$  is chosen to be 1 in both SOR and SOR-TS algorithms. The result for  $k = 0$  represents the rational approximation, which is the same for all three algorithms. The means, medians and maximums are calculated by uniformly and densely populating the domain  $D^-$  with roughly 1 million options. An asterisk (\*) indicates that the number is in the order of or smaller than the machine accuracy ( $10^{-16}$ ). For a given  $k$ , the computing times reported are for all the options in the domain  $D^-$ .

	SOR			SOR-DR			SOR-TS		
	mean	median	max	mean	median	max	mean	median	max
$ v_0 - v_* $	0.356	0.319	0.783						
$ c_0 - c_* $	0.038	0.022	0.290						
Time (s)		0.65							
$ v_4 - v_* $	0.003	$2 \times 10^{-6}$	0.078	$2 \times 10^{-11}$	$1 \times 10^{-15}$	$2 \times 10^{-8}$	$5 \times 10^{-11}$	*	$2 \times 10^{-8}$
$ c_4 - c_* $	$9 \times 10^{-4}$	$2 \times 10^{-7}$	0.024	$6 \times 10^{-12}$	*	$6 \times 10^{-9}$	$4 \times 10^{-12}$	*	$5 \times 10^{-10}$
Time (s)		9.43			9.70			10.43	
$ v_5 - v_* $	0.002	$8 \times 10^{-8}$	0.058	$4 \times 10^{-15}$	*	$1 \times 10^{-13}$	$3 \times 10^{-15}$	*	$1 \times 10^{-13}$
$ c_5 - c_* $	$6 \times 10^{-4}$	$8 \times 10^{-9}$	0.015	*	*	$4 \times 10^{-14}$	*	*	$2 \times 10^{-14}$
Time (s)		11.58			12.60			13.17	

**Table 3**  
**Comparing the Newton-Raphson, Dekker-Brent, and SOR-TS algorithms**

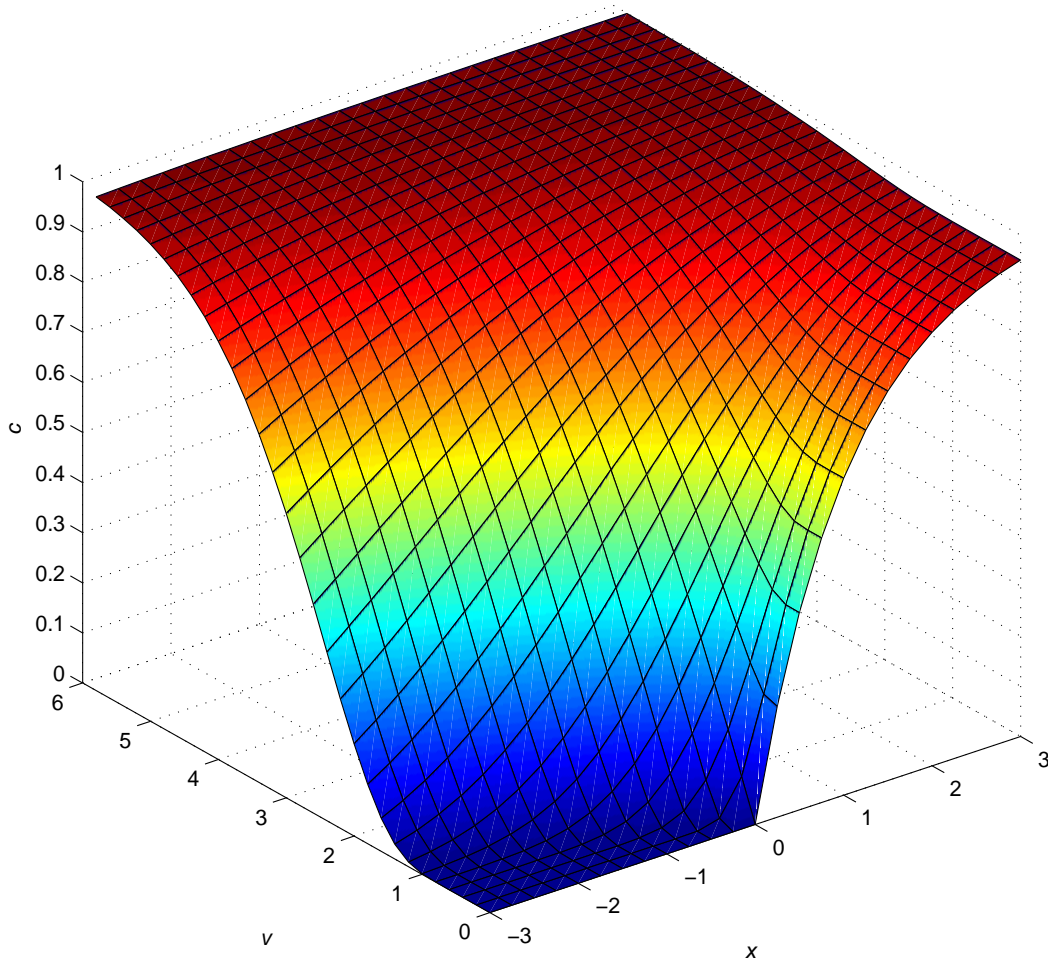
This table compares the performance of the Newton-Raphson (NR) algorithm, the Dekker-Brent (DB) algorithm and the SOR-TS algorithm for a particular option  $(x, v_*)$ , where we let  $x = -1$  and  $v_* = 2$ . We start all three algorithms from three different initial values, namely, 0.1, 4, and 20. The Newton-Raphson algorithm is implemented naively, that is, without any safeguarding feature such as checking for positivity. The initial Dekker-Brent bracketed region is taken to be  $[0.0005, \max(v_0, 10)]$ . Panel A reports the values of each iteration while Panel B reports the errors. **Inf** is the IEEE arithmetic representation for positive infinity, while **NaN** is the IEEE arithmetic representation for Not-a-Number. A **NaN** is obtained as a result of mathematically undefined operations like  $0.0/0.0$  and **Inf**–**Inf**. An asterisk (\*) indicates that the number is in the order of or smaller than the machine accuracy ( $10^{-16}$ ).

**Panel A:  $v_k$**

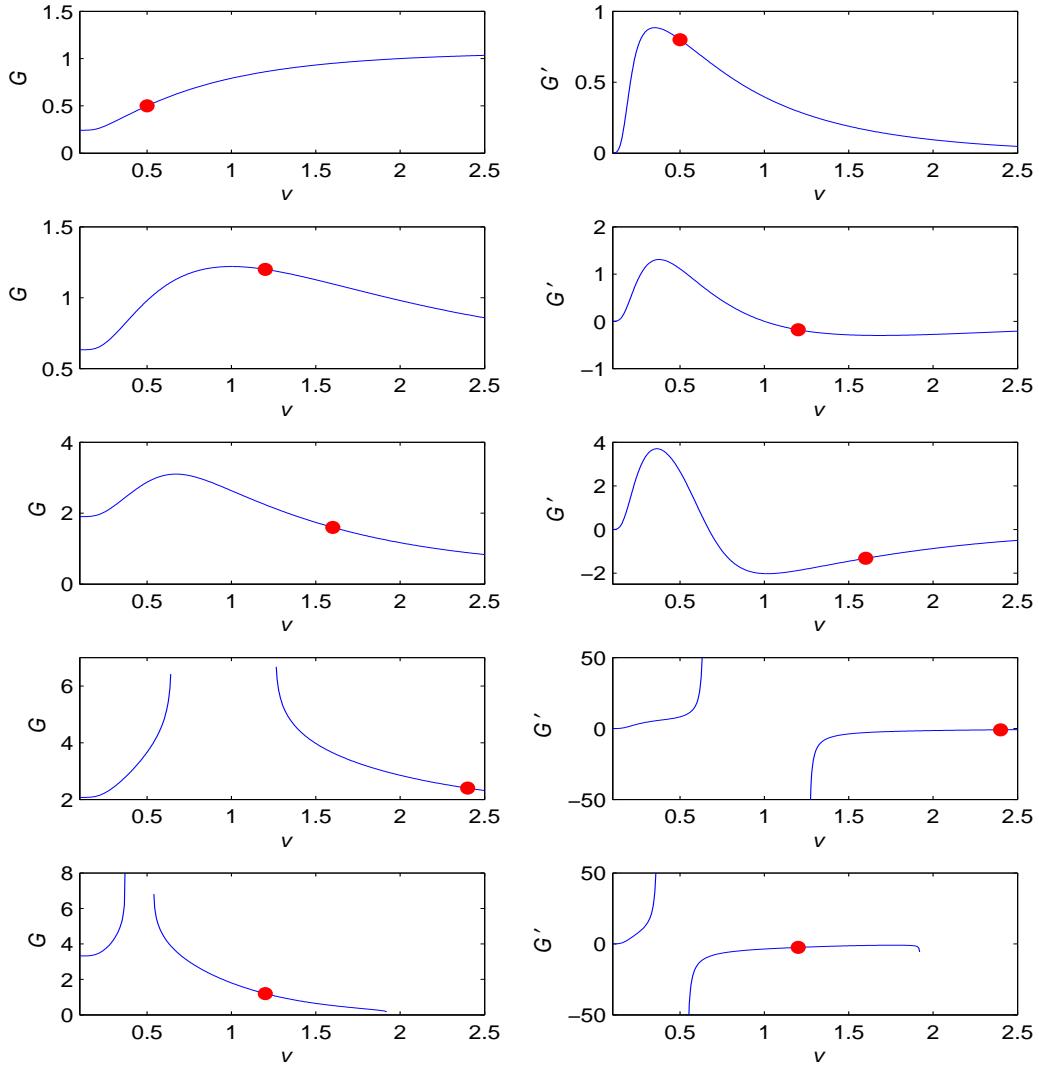
$k$	NR			DB			SOR-TS		
	0.1	4	20	0.1	4	20	0.1	4	20
0	0.1	4	20	0.1	4	20	0.1	4	20
1	$4 \times 10^{21}$	-0.8315	$-4 \times 10^{21}$	0.0005	0.0005	0.0005	161.14	2.0292	2.1750
2	<b>Inf</b>	7.0268	<b>Inf</b>	10	2.2010	10.1975	2.2515	2.0000	2.0011
3	<b>NaN</b>	-353.15	<b>NaN</b>	5.1476	1.8917	5.0990	2.0022	2.0000	2.0000
4	<b>NaN</b>	<b>Inf</b>	<b>NaN</b>	2.6238	2.0042	2.5497	2.0000	2.0000	2.0000
5	<b>NaN</b>	<b>Inf</b>	<b>NaN</b>	2.0668	2.0001	2.0518	2.0000	2.0000	2.0000

**Panel B:  $|v_k - v_*|$**

$k$	NR			DB			SOR-TS		
	1.9	2	18	1.9	2	18	1.9	2	18
0	1.9	2	18	1.9	2	18	1.9	2	18
1	$4 \times 10^{21}$	2.8315	$-4 \times 10^{21}$	1.9995	1.9995	1.9995	159.14	0.0292	0.1750
2	<b>Inf</b>	5.0268	<b>Inf</b>	8	0.2010	8.1975	0.2515	$3 \times 10^{-5}$	0.0011
3	<b>NaN</b>	355.15	<b>NaN</b>	3.1476	0.1083	3.0990	0.0022	$7 \times 10^{-11}$	$7 \times 10^{-8}$
4	<b>NaN</b>	<b>Inf</b>	<b>NaN</b>	0.6238	0.0042	0.5497	$2 \times 10^{-7}$	*	*
5	<b>NaN</b>	<b>Inf</b>	<b>NaN</b>	0.0668	0.0001	0.0518	$3 \times 10^{-15}$	*	*



**Figure 1. Normalized Black-Scholes call price  $c$  as a function of moneyness  $x$  and total volatility  $v$ .** This surface implicitly gives  $v$  as a function of  $x$  and  $c$ . In the regions where  $x/v \leq -3$  and  $x/v \geq 3$ , option prices are very insensitive to changes in  $v$ , implying that the inversion of Black-Scholes formula is not very meaningful in these regions.

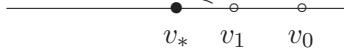


**Figure 2.** The function  $G(v; x, v_*, \omega)$  and its derivative  $G'(v; x, v_*, \omega)$ . Each row corresponds to a different combination of parameters  $(x, v_*, \omega)$ . The dots indicate the positions of  $v_*$ . As the graphs show, the function  $G(v)$  can fail to be globally well-defined, globally contracting, or locally contracting.

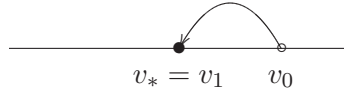
$x = 0$

1.  $v_0 > v_*$

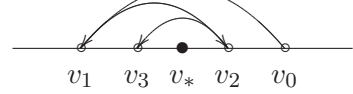
$\omega > 1$



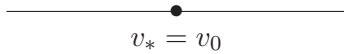
$\omega = 1$



$0 < \omega < 1$

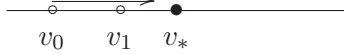


2.  $v_0 = v_*$

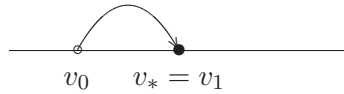


3.  $v_0 < v_*$

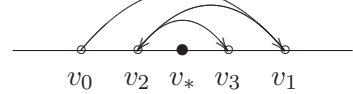
$\omega > 1$



$\omega = 1$



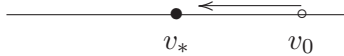
$0 < \omega < 1$



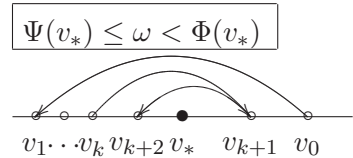
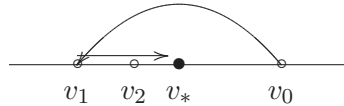
$x < 0$

1.  $v_0 > v_*$

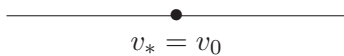
$\omega \geq \phi(v_0, v_*)$



$\Phi(v_*) \leq \omega < \phi(v_0, v_*)$

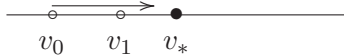


2.  $v_0 = v_*$

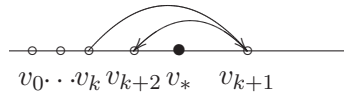


3.  $v_0 < v_*$

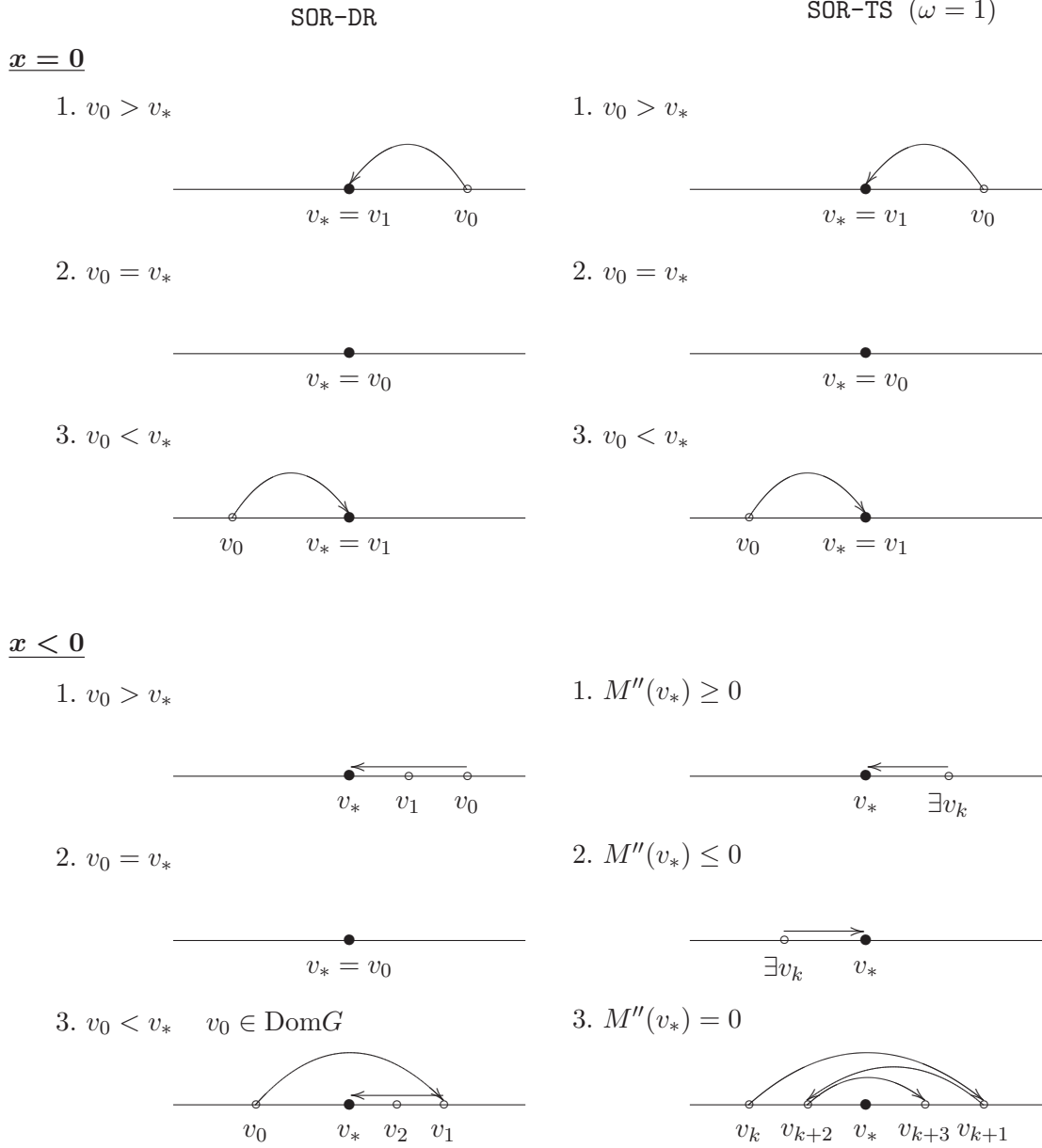
$\omega \geq \Phi(v_*)$



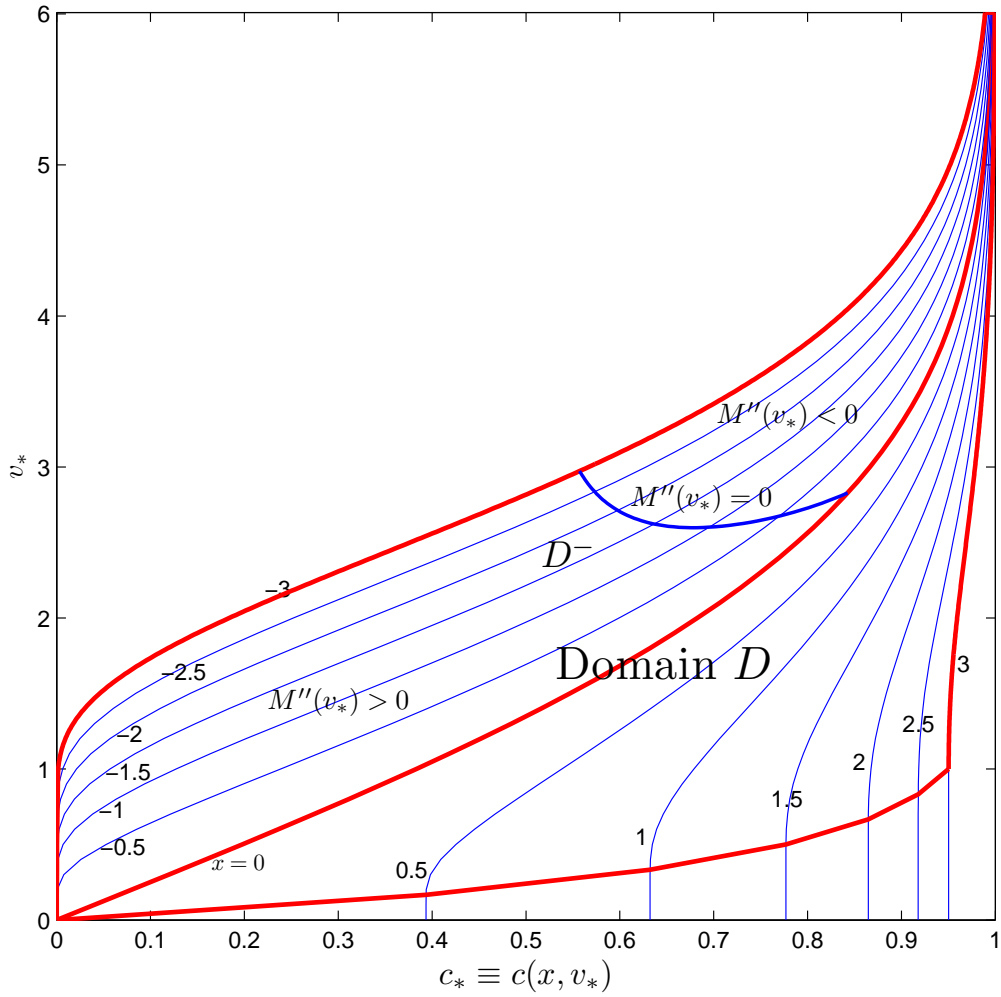
$\Psi(v_*) \leq \omega < \Phi(v_*)$



**Figure 3. Convergence patterns for the SOR algorithm.** This figure shows all the possible convergence patterns for the SOR algorithm. The two cases  $x = 0$  and  $x < 0$  are listed separately. Except for the two cases whose conditions are boxed, all the conditions above each subplots are both sufficient and necessary for each particular convergence pattern to occur. The two boxed conditions are only necessary conditions.



**Figure 4. Convergence patterns for the SOR-DR and SOR-TS ( $\omega = 1$ ) algorithms.** The left and right half of the figure shows all the possible convergence patterns for the SOR-DR and SOR-TS  $\omega = 1$  algorithms, respectively. The two cases  $x = 0$  and  $x < 0$  are listed separately. The two algorithms are always globally well-defined, except that when  $v_0 < v_*$  and  $x < 0$ , we need  $G(v_0)$  to be well-defined to guarantee that  $\{v_k\}$  is defined.



**Figure 5. Domain of inversion  $D$ .**  $D$  is constructed using four criteria, namely,  $0.0005 \leq v_* \leq 6$ ,  $|x| \leq 3$ ,  $0.0005 \leq c_* \leq 0.9995$  and  $|x|/v_* \leq 3$ . Because of the “in-and-out” duality, only options with  $x \leq 0$  will be considered. The curve  $M''(v_*) = 0$  divides the left half domain  $D^-$  further into two parts.