False Advertising

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Abstract

There is widespread evidence that some firms use false advertising to overstate the value of their products. Using a model in which a policymaker is able to punish such false claims, we characterize a natural equilibrium in which false advertising actively influences rational buyers. We analyze the effects of policy under different welfare objectives and establish a set of demand and parameter conditions where policy optimally permits a positive level of false advertising. Further analysis considers some wider issues including the implications for product investment and industry self-regulation.

Keywords: Misleading Advertising; Product Quality; Pass-through; Self-Regulation

JEL codes: M37; L15; D83

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1 Introduction

Buyers are often reliant on firms to obtain information about product characteristics. To exploit this, some firms deliberately engage in what we call false advertising - the use of incorrect or exaggerated product claims. They do this in a range of different contexts and despite potential legal penalties[1]. Recent policy cases include Dannon which paid \$21 million to 39 US states after it misled consumers about the health benefits of its Activia yogurt products, and Skechers which paid \$40 million after falsely stating that its toning shoes helped with weight loss. Similarly, in a related example, Volkswagen is now facing a potential multi-billion dollar penalty after cheating tests in order to make false claims about its emission levels[2]. Additional evidence of false advertising also comes from academic research. Such studies carefully document the existence of false advertising and its ability to increase demand[3].

However, despite the existence of false advertising, the theoretical literature has largely restricted attention to truthful advertising. In this paper, we develop an equilibrium model of false advertising where false advertising can actively influence rational buyers. Tougher legal penalties reduce the incidence of false adverts, but also increase their credibility. As a result of the latter effect, we show that stronger penalties can *reduce* buyer and social welfare. In particular, the paper derives conditions on demand and market parameters such that a policymaker optimally uses a low penalty to permit a positive level of false advertising. We then consider several wider issues including investment incentives, and the potential optimality of industry self-regulation.

In more detail, Section 2 introduces our main model where a monopolist is privately informed about its product quality. While we later extend the results to an arbitrary number of quality types, we initially focus on the case where quality is either 'high' or 'low'. The policymaker first commits to a penalty for false advertising. Then having learned its type, the firm chooses a price and makes a (possibly false) claim about its quality. Buyers subsequently update their beliefs and make their purchase decisions, before the policymaker instigates any

[1]In the US, most federal-level regulation is conducted by the FTC which punishes offenses with various public measures, including possible monetary penalties. In Europe, most countries employ varying levels of industry self-regulation alongside statutory regulations. For instance, in the UK, most regulation is conducted by the industry-led Advertising Standards Authority. It is endorsed by various governmental bodies, which have the power to issue fines.

[2]For further details, see http://www.ftc.gov/news-events/press-releases/2010/12/dannon-agrees-drop-exaggerated-health-claims-activia-yogurt, http://www.ftc.gov/news-events/press-releases/2012/05/skechers-will-pay-40-million-settle-ftc-charges-it-deceived, and http://www.bbc.co.uk/news/business-34324772. Accessed 12/14/15.

[3]Zinman and Zitzewitz (2014) and Cawley *et al* (2013) examine ski resorts and over-the-counter weight loss products respectively. See also Mayzlin *et al* (2014) for false advertising in the form of fake user reviews.

penalties. We believe this set-up closely approximates many important markets and contexts where buyers are unable to verify claims, or can only do so after a long time, and where policy plays a key role in regulating advertising.

Section 3 characterizes a natural equilibrium where the high type advertises truthfully but where the low type may engage in false advertising. This equilibrium offers some useful methodological contributions by smoothly unifying several, otherwise separate, cases of advertising. Firstly when the policymaker's punishment is large, there is no false advertising. Here, advertising is akin to verifiable disclosure as assumed within the standard literature. Secondly when the punishment is small, the low type always conducts false advertising. In this case, advertising is effectively cheap talk and is therefore unable to influence buyers' prior beliefs. Finally though, when the punishment is moderate, our equilibrium involves a novel form of partially verifiable advertising. Here, the low type engages in false advertising probabilistically by mixing between i) pooling with the high type, with a false advert and a relatively high price, and ii) advertising truthfully, with a relatively low price. Hence, after observing a high claim, buyers rationally increase their quality expectations beyond their priors even when the claim is false, as consistent with the empirical evidence detailed above.

Section 4 uses this 'smooth' equilibrium to analyze how marginal changes in the level of punishment affect a variety of welfare measures. We first consider buyer surplus. Here, a reduction in the punishment increases the probability of false advertising and generates two opposing effects. The first 'persuasion' effect harms buyers by prompting them to buy too many units of a product at an inflated price. This is akin to a formalization of Dixit and Norman's (1978) classic effect of persuasive advertising - but our effect derives from a change in rational buyers' beliefs, rather than an unmodeled change in their preferences. The second effect derives from the impact of false advertising on damaging the credibility of claims. Understanding the impact on credibility goes back to at least Nelson (1974) and is well-documented empirically (e.g. Darke and Richie 2007). However, instead of viewing this impact as detrimental, we stress its benefits under a novel 'price' effect whereby it counteracts market power by lowering buyers' quality expectations and prompting lower prices.

To compare these two effects, we then utilize some recent methods on demand curvature and cost pass-through (Weyl and Fabinger 2013). However, rather than focusing on cost changes, we consider the impact of changes in (expected) quality on price, which we term as 'quality pass-through'. In many cases, we show how the persuasion effect dominates such that buyer surplus is maximized by eliminating false advertising. However, we also formalize a range of other cases where the price effect dominates, including those where ex-ante product quality is high. Here, the optimal penalty is softer so as to induce a positive level of false advertising in a way that is weakly superior to a blanket ban on false adverts or an outright prohibition of low quality products.

Next, we turn to the effect on profits. Unsurprisingly the low (high) type always prefers smaller (larger) punishments. Interestingly however, ex ante, the monopolist weakly prefers strong punishments to eradicate false advertising so that it can price more effectively under high quality. Hence, if the monopolist could commit, its choice of punishment would coincide with that preferred by buyers in many circumstances. This offers potential support for Europe's use of self-regulation. However, for other circumstances, in contrast to the view that self-regulation may be too soft, the monopolist's preferred punishment is too strong relative to buyers'.

Lastly, we consider total welfare. Here, an increase in the probability of false advertising leads to two different effects. On one hand, false advertising lowers the credibility of any high claim and so prompts any type with such a claim to reduce its output. However, on the other hand, false advertising also allows the low type to expand its output. We then document a range of cases where this latter output expansion is beneficial and dominates the former effect, such that a positive level of false advertising is welfare optimal.

Section 5 extends the main model to consider the additional effects of false advertising when product quality is endogenous. This is an important issue to consider because false advertising may reduce product quality investment by limiting the available returns from high quality products. However, while we confirm that such an 'investment' effect prompts the policymaker to select a weakly higher penalty, we further show how a positive level of false advertising can remain optimal for buyer and social welfare. In addition, once quality is endogenous, we also show how an increase in the penalty can raise the probability of false advertising.

Finally, Section 6 considers some robustness issues. First, we allow for an arbitrary number of quality types. Qualitatively the analysis remains the same as for the two-type case except now the policymaker must also decide which types can engage in false advertising. We show how false advertising can remain optimal for types with 'moderate' quality. Second, we i) let the firm types vary in marginal costs, and ii) allow for more complex forms of punishment. In both cases, the resulting equilibrium and policy results remain qualitatively robust after making stronger requirements on buyer beliefs. Third, we examine a competitive context where an incumbent faces an entrant with private product quality. We demonstrate an equilibrium with false advertising that is qualitatively similar to monopoly, and document parameter regions where a policymaker optimally permits positive levels of false advertising..

Related Literature: The advertising literature typically focuses on truthful advertising (e.g. Anderson and Renault 2006, Johnson and Myatt 2006[4]). In earlier work, Nelson (1974) of-

[4]See also the comprehensive literature reviews by Bagwell (2007) and Renault (2015).

fered a seminal discussion of false advertising and how regulation may increase its credibility. However, since then, false advertising has only been considered by some very recent papers. Some papers assume that buyers are naive and so believe all claims (e.g. Glaeser and Ujhelyi 2010, Hattori and Higashida 2012). Here, false advertising can be socially optimal to offset the output distortion from imperfect competition. A different set of papers, including some within marketing, introduce heterogeneous buyer tastes so that claims can gain credibility by forfeiting revenues from some buyers (e.g. Chakraborty and Harbaugh 2010, 2014). Other papers study credibility from legal penalties in ways more related to our paper. Piccolo *et al* (2015) examine a duopoly where firms have different qualities. They focus only on fully pooling and fully separating equilibria, and find that buyer surplus is maximized with zero penalties to induce full pooling by making the firms non-differentiated. In contrast we consider a richer class of semi-pooling equilibria where false advertising can influence buyers' priors. By establishing a different price mechanism related to the credibility of advertising, we then show how penalties can optimally take a variety of levels depending on demand and market conditions. Like Piccolo *et al*, Corts (2013, 2014a, 2014b) only considers fully pooling and fully separating equilibria with exogenous product quality. However, he takes a different focus by allowing firms to choose whether or not to learn their own quality, and shows that finite penalties can be optimal to induce socially-valuable unsubstantiated claims. Finally, in more distant work, Barigozzi *et al* (2009) study false comparative advertising, and Drugov and Troya-Martinez (2015) analyze false advice where firms can also choose the vagueness of their claims.

Our paper also offers insights to a number of other areas. First, it adds to the growing literature on the economics of consumer protection policy which focuses on other topics, such as high-pressure sales tactics (Armstrong and Zhou 2014), the mis-use of commissions (Inderst and Ottaviani 2009), and refund rights (Inderst and Ottaviani 2013). Second, our model relates to number of communication papers that study equilibrium lying and persuasion under full rationality (e.g. Kartik 2009, and Kamenica and Gentzkow 2011). In contrast, we study policy-related lying costs within a specific advertising context, where a third-party influences not only the amount of information that is communicated to buyers but also indirectly the price that they pay. Third, by allowing for non- or partially verifiable claims, we provide some alternative insights into the literature on verifiable information disclosure (e.g. Dranove and Jin 2010, Daughety and Reinganum 2008, Celik 2014). Even when disclosure costs are zero, our results imply that i) full verifiable disclosure is not always socially optimal, and ii) a firm's ex ante choice of disclosure can be socially excessive.

2 Model

A monopolist sells one product to a unit mass of potential buyers. The monopolist is privately informed about its product quality q. Specifically, the product is of low quality L with probability $x \in (0, 1)$, and of high quality H with probability $1-x$, where $-\infty < L < H < \infty$. Average ex ante quality is then defined as $\bar{q} = xL+(1-x)H$. For our main analysis we assume that marginal costs are independent of quality and normalized to zero. Each buyer has a unit demand and values a given product of quality q at $q + \varepsilon$, where ε is a buyer's privately known match with the product. This match is drawn independently across buyers using a distribution function $G(\varepsilon)$ with support $[a, b]$ where $-\infty \leq a < b \leq \infty$. The associated density $g(\varepsilon)$ is strictly positive, continuously differentiable, and has an increasing hazard rate.

The monopolist sends a publicly observable advertisement or 'report' $r \in \{L, H\}$ at no cost, where a report $r = z$ is equivalent to a claim "Product quality is z". The binary report space is without loss because there are only two firm types and reports are costless. A policymaker is able to verify any advertised claim, and impose a penalty ϕ if it is false.[5] False advertising is defined as the use of a high quality report $r = H$, by a firm with low quality $q = L$. The policymaker can costlessly choose any level of punishment, $\phi \geq 0$, in order to maximize one of three possible objectives: buyer surplus, total profit, or total welfare. Any punishments that involve a fine go straight to the policymaker, and are not used to compensate buyers.

The timing of the game is as follows. At stage 1 the policymaker publicly commits to a penalty ϕ for false advertising. At stage 2 the monopolist privately learns its quality. It then announces a price p and issues a report $r \in \{L, H\}$. At stage 3 buyers decide whether to buy the product, taking into account ϕ as well as the firm's price and report. Finally at stage 4 the policymaker verifies the advertised claim and administers the punishment, ϕ, if it is false. The solution concept is Perfect Bayesian Equilibrium (PBE). All omitted proofs are included in the appendix unless stated otherwise.

3 Equilibrium Analysis

3.1 Benchmark with Known Quality

As a first step, consider a benchmark case in which the firm is known to have quality q. Quality claims are then redundant because it is weakly optimal for the firm to use truthful

[5] We can also interpret ϕ as an expected penalty if claims are only verified probabilistically.

advertising. An individual buyer purchases the product if and only if $\varepsilon \geq p - q$ such that demand equals $D(p-q) = 1 - G(p-q)$. The firm then chooses its price to maximize $p\left[1 - G\left(p-q\right)\right]$, and so:

Lemma 1. *Suppose the firm is known to have quality q, and define $\underset{\sim}{q} = -b$ and $\tilde{q} = -a + 1/g(a)$. The firm's optimal price, $p^*(q)$, is increasing in q and satisfies:*

$$p^*(q) = \begin{cases} 0 & \text{if } q \leq \underset{\sim}{q} \\ \frac{1-G(p^*(q)-q)}{g(p^*(q)-q)} & \text{if } q \in \left(\underset{\sim}{q}, \tilde{q}\right) \\ a + q & \text{if } q \geq \tilde{q} \end{cases} \tag{1}$$

When $q \leq \underset{\sim}{q}$, quality is so low that the firm would make zero sales even if it priced at marginal cost. The market is inactive, and we normalize the firm's price to zero without loss. When instead $q \in (\underset{\sim}{q}, \tilde{q})$, the firm optimally sells to some but not all buyers such that $p^*(q)$ satisfies the usual monopoly first order condition. Finally if $q \geq \tilde{q}$, quality is so high that the firm optimally sells to all potential customers by pricing at the willingness-to-pay of the marginal buyer, $a + q$, such that the market is 'covered'. However, for some distributions, $\tilde{q} = \infty$ and so this final case is redundant. Henceforth to avoid some uninteresting cases, let $\bar{q} > \underset{\sim}{q}$ (or $\bar{q} + b > 0$) such that a product of average quality always has some positive value.

Our later analysis will consider how the optimal price varies with quality, and we will sometimes refer to $dp^*(q)/dq$ as 'quality pass-through'. Firstly for $q \in (\underset{\sim}{q}, \tilde{q})$, after differentiating the first order condition:

$$\frac{dp^*(q)}{dq} = \frac{1 - \sigma(p^*(q) - q)}{2 - \sigma(p^*(q) - q))}, \tag{2}$$

where $\sigma(\psi) = -[1 - G(\psi)]g'(\psi)/g(\psi)^2$ is the curvature of demand (see Aguirre *et al* 2010, and Weyl and Fabinger 2013). It then follows that $dp^*(q)/dq \in [0, 1)$ because our assumption of an increasing hazard rate implies that $D(p-q)$ is logconcave in price, such that $\sigma(\psi) \leq 1$. Intuitively, an increase in quality q produces a parallel outward shift in the inverse demand curve, and the firm optimally responds by both charging a higher price and by selling to strictly more buyers.[6] Secondly, where appropriate, when $q \geq \tilde{q}$ quality pass-through is one.

The equilibrium profit earned by a firm of known quality q can be written as

$$\pi^*(q) = p^*(q)\left[1 - G\left(p^*(q) - q\right)\right]. \tag{3}$$

[6] Weyl and Fabinger (2013) note that the optimal price change from any outward unit shift in inverse demand (such as an increase in quality within our model) equals one minus cost pass-through.

It is straightforward to show that $\pi^*(q)$ is increasing and convex in q given that $g(\varepsilon)$ has an increasing hazard rate. Finally, buyer surplus can be expressed as

$$v^*(q) = \int_{p^*(q)}^{b+q} [1 - G(z - q)]\, dz. \tag{4}$$

Observe that $v^*(q) = 0$ when $q \leq \underset{\sim}{q}$ because no buyer purchases the product, but that buyer surplus is positive and weakly increasing in $q > \underset{\sim}{q}$. We further discuss the shape of $v^*(q)$ in Section 4.1 below.

3.2 Privately-Known Quality

Henceforth we assume that the firm is privately informed about its quality. As is typical in signaling games, there exists a large number of PBE because buyers can attribute any off-path claim or price to the low type. However, standard refinements like D1 have little bite in our game.[7] Therefore we approach equilibrium selection as follows. Firstly, we restrict attention to PBE in which the high type always makes a truthful claim. This allows us to focus on the low type's incentives to engage in false advertising. Secondly, we restrict buyer beliefs to depend only on the firm's claim and not on its price. In particular, conditional on observing a high claim, we assume that buyers expect quality $q_H^e \equiv E(q|r = H)$ irrespective of price, such that both types optimally charge $p^*(q_H^e)$. Intuitively, after making a high claim, the payoff functions of the two types differ only by the punishment ϕ due to their common marginal cost. Hence, buyers should expect the two types to price in the same way, and so avoid making any price-based inferences. A more formal justification for this second restriction can also be obtained by using Mailath *et al*'s (1993) Undefeated Equilibrium refinement.[8] We may then state:

[7] In particular there are many equilibria with $r(H) = H$ that are consistent with D1. To see why, note that a high type earns ϕ more than a low type i) on-path in any pooling, semi-pooling, or least-cost separating equilibrium and ii) following a report $r = H$ and any off-path price. Hence D1 does not restrict beliefs following a report $r = H$ and an off-path price.

[8] In particular, suppose buyers believe that on average a firm reporting $r = H$ has quality q_H^e. Consider how buyers should update their belief upon seeing $r = H$ and some price p. It turns out that D1 again has no bite. However the Undefeated Refinement i) is consistent with our second restriction that $E(q|r = H, p) = q_H^e$, and ii) uniquely selects the price $p^*(q_H^e)$ as the price which the firm should charge after reporting $r = H$. Note that by applying the refinement in this way, we assume that buyers interpret any off-path price as a (possible) signal of a firm's type rather than as a signal about $E(q|r = H)$. Other recent uses of this refinement include Gill and Sgroi (2012), Perez-Richet and Prady (2012), Miklos-Thal and Zhang (2013) and Lauermann and Wolinsky (2015).

Proposition 1. *Suppose a high type always reports truthfully, and buyer beliefs depend only on the firm's claim. There exists a unique PBE (up to off-path beliefs[9]), in which:*
i) A high type claims $r = H$ and charges $p^*(q^e_H)$.
ii) A low type randomizes. With probability y^* it claims $r = H$ and charges $p^*(q^e_H)$. With probability $1 - y^*$ it claims $r = L$ and charges $p^*(L)$.
- When $\phi \leq \phi_1 \equiv \pi^*(\bar{q}) - \pi^*(L)$, $y^* = 1$
- When $\phi \geq \phi_0 \equiv \pi^*(H) - \pi^*(L)$, $y^* = 0$
- When $\phi \in (\phi_1, \phi_0)$, $y^* \in (0, 1)$ and uniquely solves

$$\pi^*(q^e_H) - \phi = \pi^*(L), \tag{5}$$

$$\text{where} \quad q^e_H = \frac{xy^*L + (1-x)H}{1 - x + xy^*}. \tag{6}$$

iii) Buyer beliefs are $\Pr(q = H | r = L) = 0$ and $\Pr(q = H | r = H) = \frac{1-x}{1-x+xy^*}$.

Proposition 1 characterizes a natural equilibrium where false advertising can arise. A high quality firm always reports truthfully and charges $p^*(q^e_H)$. With probability y^*, a low quality firm pools with the high type by falsely claiming $r = H$ and charging $p^*(q^e_H)$, and with remaining probability $1 - y^*$ it reports truthfully and charges $p^*(L)$. Therefore when buyers observe a report $r = L$, they correctly infer low quality and demand $D(p^*(L) - L)$ units, such that the firm earns $\pi^*(L)$. On the other hand when buyers observe a report $r = H$ they update their expectations about quality to

$$q^e_H = \frac{xy^*L + (1-x)H}{1 - x + xy^*} \geq \bar{q} > L,$$

where for brevity we omit the dependence of q^e_H on y^*. Consequently by using false advertising, a low quality firm can persuade rational buyers to overestimate its product quality, and earn $\pi^*(q^e_H) - \phi$. However such false advertising never systematically deceives buyers because their beliefs are correct on average due to the additional possibility that the high report comes from a high type.

The precise equilibrium characterization depends smoothly on the size of the punishment ϕ. Firstly if $\phi \leq \pi^*(\bar{q}) - \pi^*(L)$ the equilibrium has full pooling. The punishment is sufficiently low that false advertising is a dominant strategy for the low type, and hence $y^* = 1$. Buyers discount any high advertised claim and maintain their prior such that $q^e_H = \bar{q}$.

Secondly if $\phi \geq \pi^*(H) - \pi^*(L)$ the equilibrium has full separation. The punishment is sufficiently high that truth-telling is a dominant strategy for the low type, and hence $y^* = 0$.

[9]Note that when $\phi < \phi_1$ the claim $r = L$ is off-path, and a range of beliefs $\Pr(q = L | r = L)$ lead to the same equilibrium play.

Advertising is perfectly informative and claims are fully credible such that $q_H^e = H$. The firm earns its full information profit.

Finally and most interestingly, if $\phi \in (\phi_1, \phi_0)$ the equilibrium is semi-pooling. Here, the equilibrium has two novel features. First, the low type makes a false claim with strictly interior probability $y^* \in (0,1)$, where y^* satisfies equation (5) to ensure that the low type is indifferent between lying and telling the truth. Randomization is an essential feature of this equilibrium because the punishment is too high to support full pooling and too low to support full separation. Second, unlike the full pooling equilibrium, false advertising is partially informative and therefore prompts buyers to actively update their beliefs beyond their priors, with $q_H^e \in (\bar{q}, H)$.

4 The Effects of Policy

First consider the effects of policy on the level of false advertising, y^*. By using equations (5) and (6) it follows that:

Lemma 2. *The level of false advertising y^*, is continuous and weakly decreasing in the level of punishment ϕ.*

Stronger policy smoothly increases the informativeness of advertising. When $\phi > \phi_0$ or $\phi < \phi_1$, a low quality firm has a strict preference for truth-telling or lying respectively, and so small changes in ϕ have no effect. However when $\phi \in [\phi_1, \phi_0]$, the probability of false advertising y^* satisfies the indifference condition (5) and is strictly decreasing in ϕ from 1 to 0. Intuitively, to maintain indifference of the low type as ϕ increases, high reports must become more credible. Since buyers are Bayesian, this is only possible if y^* is strictly lower.

4.1 Buyer Surplus

We now consider the effects of policy on a variety of welfare measures, starting with buyer surplus. Using Proposition 1 we can write expected buyer surplus as

$$E(v) = x(1-y^*) \int_{p^*(L)-L}^{b} (L + \varepsilon - p^*(L))\, \mathrm{d}G(\varepsilon) + xy^* \int_{p^*(q_H^e)-q_H^e}^{b} (L + \varepsilon - p^*(q_H^e))\, \mathrm{d}G(\varepsilon)$$
$$+ (1-x) \int_{p^*(q_H^e)-q_H^e}^{b} (H + \varepsilon - p^*(q_H^e))\, \mathrm{d}G(\varepsilon). \tag{7}$$

In words, with probability $x(1-y^*)$ the firm sends a low report and charges $p^*(L)$. Buyers correctly infer low quality, buy if $\varepsilon \geq p^*(L) - L$, and so receive $L + \varepsilon - p^*(L)$. Then with probability $1-x+xy^*$ the firm sends a high report and charges $p^*(q_H^e)$. Buyers update their beliefs according to equation (6), and buy if $\varepsilon \geq p^*(q_H^e) - q_H^e$. With conditional probability $xy^*/(1-x+xy^*)$, the product is low quality, and buyers receive $L+\varepsilon-p^*(q_H^e)$. With conditional probability $(1-x)/(1-x+xy^*)$, the product is high quality, and buyers receive $H+\varepsilon-p^*(q_H^e)$. After collecting terms and using the definition of $v^*(q)$ in equation (4), the above expression simplifies as follows, where $E(v)$ is just a convex combination of $v^*(L)$ and $v^*(q_H^e)$.

$$E(v) = x(1-y^*)v^*(L) + (xy^*+1-x)v^*(q_H^e). \tag{8}$$

We now exploit the smooth feature of our equilibrium to investigate the effect of a marginal increase in the penalty ϕ, under the following regularity condition on demand curvature:

Assumption 1. Let $z(\psi) = -\sigma'(\psi)+[2-\sigma(\psi)]g(\psi)/[1-G(\psi)]$. The demand function satisfies either i) $\tilde{q} < \infty$ and $z(\psi) > 0$ for all $\psi \in (a,b)$, or ii) $\underset{\sim}{q} = -\infty$, $\tilde{q} = \infty$, $z(\psi)$ changes from negative to positive at exactly one value of $\psi \in (a,b)$, and $\lim_{\psi\to a}\sigma(\psi) = -\infty$.

Assumption 1 is satisfied by a wide class of demand functions, and ensures that $v^*(q)$ is s-shaped in quality. For convenience, we denote

$$\hat{q} = \sup\left\{q \in (\underset{\sim}{q}, \tilde{q}) \,:\, z(p^*(q)-q) > 0\right\} \tag{9}$$

as the finite quality level at which $v^*(q)$ changes from being strictly convex to concave. Assumption 1i is satisfied by a rich class of demands that exhibit constant curvature, which includes linear and exponential demand (see Bulow and Pfleiderer 1983). Here $\hat{q} = \tilde{q}$ such that $v^*(q)$ is strictly convex for all $q \in (\underset{\sim}{q}, \tilde{q})$, but independent of quality and equal to $\int_a^b [1-G(z)]\,dz$ when $q > \tilde{q}$. Alternatively, Assumption 1ii is satisfied by many demands with increasing curvature - including those derived from the Normal, Logistic, Type I Extreme Value and Weibull distributions. For these demands $\hat{q}$ solves $z(p^*(q)-q) = 0$, and $v^*(q)$ is strictly convex for $q < \hat{q}$ but strictly concave for $q > \hat{q}$. Further details are provided in Section A of the Supplementary Appendix.[10] We can then state:

Lemma 3. *Consider $\phi \in [\phi_1, \phi_0]$ and suppose that Assumption 1 holds.*
i) If $L < \hat{q}$ expected buyer surplus is quasiconcave in ϕ. For any given L there exists a threshold $\dot{q}(L) \geq \hat{q}$, such that expected buyer surplus is strictly increasing in ϕ if $q_H^e < \dot{q}(L)$,

[10] This appendix may also prove useful for other wider literatures. For instance, a recent literature on the welfare effects of third-degree price discrimination uses a restriction related to Assumption 1 which ensures that buyer surplus is convex with respect to marginal cost (e.g. Cowan 2012 and Chen and Schwartz 2015).

but strictly decreasing in ϕ if $q_H^e > \dot{q}(L)$.
ii) If $L > \hat{q}$ expected buyer surplus is weakly decreasing in ϕ.

Lemma 3 shows that buyers do not always benefit from tougher penalties. In particular, a well-intentioned policy that increases ϕ may actually reduce expected buyer surplus. To understand why, recall that the level of false advertising y^* is a decreasing function of ϕ, and note that an increase in y^* produces two effects. On the one hand, buyers are more likely to receive a false advert and so be persuaded to buy a low quality product at an inflated price $p^*(q_H^e) > p^*(L)$. On the other hand, the increase in lying damages the credibility of advertising. This reduces buyers' expectations and induces any product with a high claim to set a lower price. In more detail, one can write

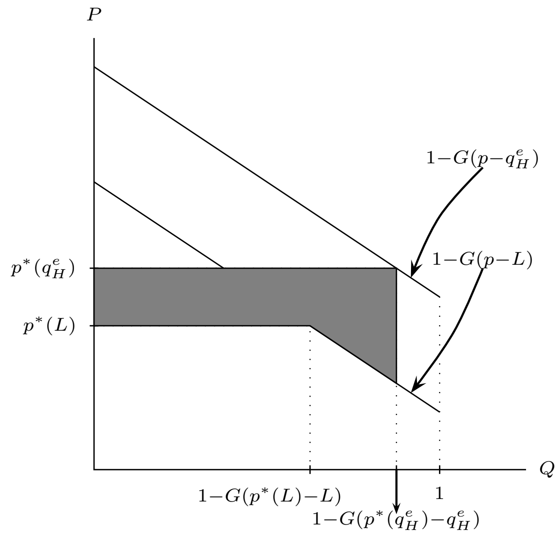
$$\frac{\partial E(v)}{\partial y^*} = \underbrace{x\left[v^*\left(q_H^e\right) - \left(q_H^e - L\right) D\left(p^*(q_H^e) - q_H^e\right) - v^*(L)\right]}_{\text{`Persuasion' effect}} \\ - \underbrace{(1 - x + xy^*) D(p^*(q_H^e) - q_H^e) \frac{\partial p^*(q_H^e)}{\partial q_H^e} \frac{\partial q_H^e}{\partial y^*}}_{\text{`Price' effect}}. \quad (10)$$

The first term is a 'persuasion' effect. Conditional on the firm having low quality (which occurs with probability x), a marginal increase in lying replaces the surplus that the buyer would have received if the firm had told the truth, $v^*(L)$, with the surplus associated with false advertising, $v^*\left(q_H^e\right) - \left(q_H^e - L\right) D\left(p^*(q_H^e) - q_H^e\right)$. To explain this latter surplus, note that after observing a high report, buyers update their beliefs to q_H^e, and expect to receive a surplus $v^*\left(q_H^e\right)$. However since quality is low, each of the $D\left(p^*(q_H^e) - q_H^e\right)$ units bought is worth $q_H^e - L$ less than anticipated. This harms buyers by prompting them to pay too much and to potentially buy too many units of a low quality product, as represented by the shaded area in Figure 1. The effect formalizes the loss in buyer surplus caused by persuasive advertising as identified in the seminal paper by Dixit and Norman (1978). However, our false advertising 'persuasion' effect arises from a change in rational buyers' beliefs, rather than an unmodeled change in their preferences.

The second term in (10) is a 'price' effect. A marginal increase in y^* lowers the probability that a high claim is true. This reduces buyers' confidence in high reports, and lowers their rational expectation of the relevant product quality, $\partial q_H^e/\partial y^* < 0$. While this effect on credibility is typically thought to be detrimental, little attention has been paid to its potential benefits. In particular, with the probability that the firm uses a high claim, $1 - x + xy^*$, the reduction in credibility lowers the firm's market power and prompts a price reduction.

Lemma 3 can now be understood in terms of our two effects. First consider $L < \hat{q}$. Here,

Figure 1: The Persuasion Effect of False Advertising



the persuasion effect dominates when $q^e_H < \dot{q}(L)$, whilst the price effect dominates when $q^e_H > \dot{q}(L)$. Hence a stronger policy benefits buyers in the former situation, but harms them in the latter. The critical threshold

$$\dot{q}(L) = \sup\left\{ q^e_H \; : \; \frac{\partial E(v)}{\partial y^*} < 0 \middle| \, L < \hat{q} \right\} \tag{11}$$

is examined in more detail within the proof. There we show that for demands within Assumption 1i, $\dot{q}(L) = \hat{q}$, while for demands within Assumption 1ii, $\dot{q}(L)$ is strictly decreasing in L and satisfies $\lim_{L \to \hat{q}} \dot{q}(L) = \hat{q}$. Second consider $L > \hat{q}$. If the distribution satisfies Assumption 1i, the price and persuasion effects cancel such that $\partial E(v)/\partial \phi = 0$. Intuitively the market is fully covered irrespective of the firm's claim, and buyers pay either $a + L$ following a low claim, or $a + q^e_H$ following a high claim. The average price paid is therefore $a + \bar{q}$ which is independent of ϕ. However if instead the distribution satisfies Assumption 1ii, the price effect strictly dominates such that $\partial E(v)/\partial \phi < 0$.

We now consider the optimal level of punishment, ϕ^*. To ease exposition, we henceforth focus on the (more interesting) case where $L < \hat{q}$. Recalling Lemma 3, we find that:

Proposition 2. *Fix $L < \hat{q}$ and suppose that Assumption 1 holds. The buyer-optimal penalty, ϕ^*, is characterized as follows:*
i) When $H \leq \dot{q}(L)$, $\phi^* \geq \phi_0$ such that $y^* = 0$.
ii) When $\bar{q} < \dot{q}(L) < H$, $\phi^* = \pi^*(\dot{q}(L)) - \pi^*(L)$ such that $y^* = \frac{(H-\dot{q}(L))(1-x)}{(H-\dot{q}(L))(1-x)+\dot{q}(L)-\bar{q}} \in (0, 1)$.
iii) When $\dot{q}(L) \leq \bar{q}$, $\phi^* \leq \phi_1$ such that $y^* = 1$.

Proposition 2 provides a range of demand and parameter conditions where a buyer-oriented policymaker refrains from eradicating false advertising. Recall from Lemma 3 that for $L < \hat{q}$, a marginal decrease in false advertising increases buyer surplus if and only if buyers are relatively pessimistic about high claims, with $q_H^e < \dot{q}(L)$. Therefore when $H \leq \dot{q}(L)$ buyer surplus is globally decreasing in y^* and the policymaker optimally eliminates false advertising. However when $\bar{q} < \dot{q}(L) < H$ buyer surplus follows an inverted-U and peaks at a $y^* \in (0,1)$, such that the optimal penalty tolerates some false advertising. Finally when $\dot{q}(L) \leq \bar{q}$ buyer surplus is globally increasing in y^* and so the policymaker fully permits false advertising.

The fact that a positive level of false advertising can generate a higher buyer surplus than under full information (where $y^* = 0$) gives several policy implications. First, any instinctive per se implementation of strong punishments or blanket prohibitions on false advertising may actually limit buyer surplus. Second, the optimal use of advertising punishments is superior to an outright ban on low quality products. Such a ban only generates a surplus $E(v) = (1-x)v^*(H)$, which is weakly less than the surplus under full information.

Finally, we further detail the conditions under which positive false advertising is optimal.

Corollary 1. *Given Assumption 1 and $L < \hat{q}$, the buyer-optimal level of false advertising is increasing in L, H, and $(1-x)$.*

When product quality levels are higher, or when the probability of a high type is larger, policy should allow a higher level of false advertising, y^*. Intuitively, when the monopolist's product quality technology is relatively 'healthy', the expected quality from a high claim, q_H^e, is relatively high such that the price effect becomes relatively more powerful. On the contrary, when the product quality technology is less 'healthy', the persuasion effect becomes particularly damaging.

4.2 Profits

We now examine the effect of policy on profits. To begin, consider each individual firm type:

$$E(\pi_L) = \begin{cases} \pi^*(\bar{q}) - \phi & \text{if } \phi < \phi_1 \\ \pi^*(L) & \text{if } \phi \geq \phi_1 \end{cases} \quad \text{and} \quad E(\pi_H) = \begin{cases} \pi^*(\bar{q}) & \text{if } \phi < \phi_1 \\ \pi^*(L) + \phi & \text{if } \phi \in [\phi_1, \phi_0] \\ \pi^*(H) & \text{if } \phi > \phi_0 \end{cases} \tag{12}$$

This is explained as follows. When $\phi < \phi_1$ the equilibrium has full pooling such that each type earns $\pi^*(\bar{q})$, but the low type also incurs a penalty ϕ. When $\phi \in [\phi_1, \phi_0]$ the low type is indifferent between lying and truth-telling, and so earns $\pi^*(L)$. The high type, meanwhile, earns $\pi^*(q^e_H)$ which is equal to $\pi^*(L)+\phi$ from (5). Finally when $\phi > \phi_0$ the equilibrium has full separation, and so each type earns its full information payoff.

Remark 1. An increase in ϕ reduces $E(\pi_L)$, but increases $E(\pi_H)$.

Intuitively, stronger regulation increases the high type's payoff because it leads buyers to update more optimistically upon seeing a high claim. However tougher regulation hurts a low type because it becomes costlier to mimic a high type. Now consider expected equilibrium profit, $E(\Pi) = xE(\pi_L) + (1-x)E(\pi_H)$:

Proposition 3. *Expected profit is quasiconvex in ϕ and minimized at $\phi = \phi_1$. In addition:*
i) If $L < \tilde{q}$, expected profit is maximized by $\phi^* \geq \phi_0$.
ii) If $L \geq \tilde{q}$, expected profit is maximized by either $\phi^* = 0$ or $\phi^* \geq \phi_0$.

A small increase in regulation can either benefit or harm the monopolist, depending upon how existing regulation ϕ compares with ϕ_1. In addition, it is straightforward to see from (12) that $\phi \in (0, \phi_0)$ is strictly dominated under an expected profit objective. This implies that the punishment should never be paid in equilibrium. Then, given the convexity of $\pi^*(q)$, full separation with $\phi^* \geq \phi_0$ is always weakly optimal. Intuitively, strong regulation allows the firm to extract buyer surplus more effectively when it has high quality. Hence, if the monopolist could credibly commit to effective self-regulation, Proposition 3 implies that it would weakly prefer to avoid using false advertising. Such self-regulation might be acceptable to buyers because in some circumstances the monopolist's preferred level of punishment coincides with that of buyers e.g. when $L < \hat{q}$ and $H < \dot{q}(L)$. This may offer some support for Europe's industry-led regulation. However, in other circumstances self-regulation would go against buyers' preferences e.g. when $L < \hat{q}$ and $H > \dot{q}(L)$. Here, contrary to any concerns that self-regulation may be too lax, the monopolist's preferred level of punishment is strictly *higher* than buyers'.

4.3 Total Welfare

We now consider total welfare. Initially, suppose that the punishment, ϕ, is in the form of a fine which is as valuable to the policymaker as it is to the firm. Therefore using Proposition 1, we can write expected total welfare as

$$E(w) = x(1-y^*)\left[v^*(L) + \pi^*(L)\right] + (1-x+xy^*)\left[v^*(q^e_H) + \pi^*(q^e_H)\right]. \qquad (13)$$

Notice that this expression is not just the summation of expected buyer surplus in (8), and weighted firm-type profits in (12), because by assumption the punishment has social value. We now impose a regularity condition that differs slightly to the one used earlier.

Assumption 2. Let $z_w(\psi) = -\sigma'(\psi) + [2 - \sigma(\psi)][3 - \sigma(\psi)]g(\psi)/[1 - G(\psi)]$. The demand function satisfies either i) $\tilde{q} < \infty$ and $z_w(\psi) > 0$ for all $\psi \in (a, b)$, or ii) $\underset{\sim}{q} = -\infty$, $\tilde{q} = \infty$, $z_w(\psi)$ changes from negative to positive at exactly one value of $\psi \in (a, b)$, and $\lim_{\psi \to a} \sigma(\psi) = -\infty$.

Assumption 2 ensures that $w^*(q) \equiv v^*(q) + \pi^*(q)$ is s-shaped in quality, with

$$\hat{q}_w = \sup \left\{ q \in (\underset{\sim}{q}, \tilde{q}) \, : \, z_w(p^*(q) - q) > 0 \right\} \tag{14}$$

denoting the critical quality level at which $w^*(q)$ changes from being strictly convex to concave. The assumption is again satisfied by a wide range of commonly-used distribution functions.[11]

Lemma 4. *Consider $\phi \in [\phi_1, \phi_0]$ and suppose that Assumption 2 holds.*
i) When $q_H^e < \hat{q}_w$ expected total welfare is strictly increasing in ϕ.
ii) When $L < \hat{q}_w < q_H^e$ expected total welfare is quasiconcave in ϕ. There exists a threshold $\dot{L}(q_H^e) < \hat{q}_w$ such that expected total welfare is strictly increasing in ϕ if $L < \dot{L}(q_H^e)$, but strictly decreasing in ϕ if $L > \dot{L}(q_H^e)$.
iii) When $L > \hat{q}_w$ expected total welfare is weakly decreasing in ϕ.

Stronger policy does not necessarily increase expected total welfare. Intuitively a monopolist uses its market power to restrict output below the socially efficient level. An increase in false advertising changes this output distortion in two ways. First, it lowers the credibility of any high claim, and so forces any type with such a claim to further reduce its output below the socially optimal level. Second however, it also induces buyers to over-estimate a low type's quality, thereby causing the low type to increase its output. Under certain circumstances this latter output expansion can raise welfare and dominate the former effect.

[11] Assumption 2i holds for all demands with constant curvature, where $\hat{q}_w = \tilde{q} < \infty$ such that $w^*(q)$ is strictly convex for $q \in (\underset{\sim}{q}, \hat{q}_w)$ and linear for $q \geq \tilde{q}_w$. Assumption 2ii is satisfied by some demands with increasing curvature, including the Normal, Weibull, and Type I Extreme Value, where $\hat{q}_w$ solves $z_w(p^*(q) - q) = 0$ such that $w^*(q)$ is strictly convex for $q < \hat{q}_w$ but strictly concave for $q > \hat{q}_w$. See Section A of the Supplementary Appendix.

In more detail:

$$\frac{\partial E(w)}{\partial y^*} = \underbrace{x\Big[v^*\left(q_H^e\right) - v^*\left(L\right) - (q_H^e - L)D(p^*(q_H^e) - q_H^e) + \pi^*(q_H^e) - \pi^*\left(L\right)\Big]}_{\text{Output expansion by a firm with } q=L}$$
$$+ \underbrace{(1 - x + xy)D(p^*(q_H^e) - q_H^e)\left(1 - \frac{\partial p^*(q_H^e)}{\partial q_H^e}\right)\frac{\partial q_H^e}{\partial y^*}}_{\text{Output contraction by a firm with } r=H} \quad (15)$$

The first term in (15) represents the change in welfare when a low type moves from reporting $r = L$ and generating a total surplus of $v^*\left(L\right) + \pi^*\left(L\right)$, to claiming $r = H$ and generating a surplus of $v^*\left(q_H^e\right) - \left(q_H^e - L\right) D(p^*(q_H^e) - q_H^e) + \pi^*\left(q_H^e\right)$. This term is positive if and only if L is above a certain threshold. Intuitively, the low type's socially optimal output level $D(-L)$ is increasing in L. Moreover when a low type engages in false advertising, its output increases from $D\left(p^*(L) - L\right) \leq D(-L)$ to $D(p^*(q_H^e) - q_H^e)$. Therefore if L is relatively small, this 'output expansion effect' goes far beyond the efficient level and so is bad for welfare. However if L is relatively large, the output expansion effect brings the low type closer to the efficient level, and so is good for welfare. The second term in (15) represents the change in surplus generated by a firm that claims to have high quality, following a small increase in y^*. As explained above, this is unambiguously negative because an increase in y^* reduces the credibility (and hence output) of a firm that reports $r = H$. Ceteris paribus, this 'output contraction effect' is smaller when quality pass-through $\partial p^*(q_H^e)/\partial q_H^e$ is larger since in that case the firm's output is less sensitive to buyers' belief about its quality.

Lemma 4 can then be understood as follows. When $q_H^e < \hat{q}_w$ quality pass-through is relatively small, such that the output contraction effect dominates, and so $E(w)$ decreases in the level of false advertising y^*. When $L < \hat{q}_w < q_H^e$ quality pass-through is relatively stronger, and so the output contraction effect is weaker. A small increase in y^* therefore raises welfare provided L is sufficiently large, such that the expansion in the low type's output is not (too) excessive. In the appendix we show that the critical threshold

$$\dot{L}(q_H^e) = \sup\left\{ L \,:\, \frac{\partial E(v)}{\partial y^*} < 0 \,\middle|\, q_H^e > \hat{q}_w \right\} \quad (16)$$

is strictly below $\hat{q}_w$ and is also (weakly) decreasing in q_H^e. Finally when L is large with $L > \hat{q}_w$, an increase in the penalty can never raise welfare as the output expansion always weakly dominates.[12]

[12] Glaeser and Ujhelyi (2010) show in a model with one firm type and naive buyers that some false advertising always improves welfare by increasing output. Our result differs in two ways. First, in our

Now consider the implications for the optimal penalty. To ease exposition, we focus on the (more interesting) case where $L < \hat{q}_w$. First, note that the policymaker will always eliminate false advertising with $\phi^* \geq \phi_0$ when $H < \hat{q}_w$. This follows from Lemma 4 because $q_H^e < \hat{q}_w$ for all $y^* \in [0, 1]$. For the remaining cases:

Proposition 4. *Fix $L < \hat{q}_w < H$ and suppose that Assumption 2 holds. The welfare-optimal penalty, ϕ^*, is characterized as follows:*
i) When $L \leq \dot{L}(H)$, $\phi^* \geq \phi_0$ such that $y^* = 0$.
ii) When $L \in (\dot{L}(H), \dot{L}(\bar{q}))$, ϕ^* induces $y^* \in (0, 1)$ such that $q_H^e = q^{**}$ where $L = \dot{L}(q^{**})$.
iii) When $L \in [\dot{L}(\bar{q}), \hat{q}_w)$, $\phi^* \leq \phi_1$ such that $y^* = 1$.

Proposition 4 shows that a positive level of false advertising is welfare-optimal for a non-empty set of parameters. In line with intuition, one can also show that the optimal level of false advertising is weakly lower than that under a buyer surplus objective. However, the optimal level of false advertising remains increasing in the 'healthiness' of the market (e.g. L, H, and $(1 - x)$).[13]

5 Endogenous Quality Investment

We now extend the main model to examine some additional effects of false advertising in a market with endogenous product quality. These effects are important to consider because the existence of false advertising may reduce the incentives to invest in product quality by limiting the credibility of advertising. Suppose that the firm is initially endowed with low quality L, but can upgrade to high quality H by paying an investment cost C. This cost is drawn privately from a distribution $F(C)$ on $(0, \infty)$, with corresponding density $f(C) > 0$. The move order is then as follows. At stage 1 the policymaker commits to a penalty ϕ. At stage 2 the firm learns its investment cost C, and privately chooses whether to upgrade. It also announces its report and price. The game then proceeds as in the main model, with buyers making their purchase decisions, and the policymaker instigating any potential punishments. Let $x^*(\phi)$ denote the endogenous probability that the firm has low quality.

model the low type's output expansion is endogenous, and hence may be so large that it reduces welfare. Second, since our buyers are Bayesian, false advertising by the low type decreases the high type's output, which also reduces welfare. We also note that the way in which false advertising distorts the two types' output is reminiscent of the output effects in third-degree price discrimination (e.g. Aguirre et al 2010).

[13] Finally, we note that false advertising can remain optimal even when a fraction τ of the punishment is 'lost' and does not contribute to total welfare. For instance when $\dot{L}(\bar{q}) < L < \hat{q}_w \leq \bar{q}$ the optimal penalty induces $y^* = 1$ for all $\tau \in [0, 1]$. The only difference is that when $\tau = 0$ any $\phi \in [0, \phi_1]$ maximizes total welfare, whereas when $\tau > 0$ it becomes more attractive to reduce the penalties incurred, and so $\phi^* = 0$ is the unique optimum.

There always exists a trivial equilibrium in which $x^*(\phi) = 1$. If buyers believe that product quality is low for all reports and prices, the firm has no incentive to invest. However, in general, there also exist other alternative PBE. Henceforth, we restrict attention to PBE where, as before, buyer beliefs do not depend on price. Moreover whenever possible, we select an equilibrium where the firm invests with positive probability.

Lemma 5. *i) When $\phi = 0$ all equilibria have $x^*(\phi) = 1$. ii) When $\phi \in (0, \phi_0]$ there is a unique equilibrium (up to off-path beliefs) satisfying our restrictions, with $x^* = 1 - F(\phi) \in (0, 1)$ and $r(H) = H$.*

Intuitively, an increase in ϕ induces investment by widening the gap in profits earned by high and low quality firms. In more detail, when $\phi = 0$ buyers cannot distinguish between high and low quality. The firm earns the same profit regardless and therefore chooses not to invest. Alternatively when $\phi \geq \phi_0$, claims are fully credible. A low quality firm reports $r = L$ and earns $\pi^*(L)$, whilst a high quality firm reports $r = H$ and earns $\pi^*(H)$. Since the gains from investing are $\pi^*(H) - \pi^*(L) \equiv \phi_0$, the firm upgrades if and only if $C \leq \phi_0$. Finally when $\phi \in (0, \phi_0)$, the level of false advertising is necessarily positive for the same reason as in the main model. This further implies that a high quality product earns ϕ more than a low quality product such that the firm invests with probability $F(\phi)$. However unlike the main model, the probability of false advertising y^* is not necessarily decreasing everywhere in ϕ. Recall the definition $q_H^e \equiv E(q|r = H)$. Intuitively, an increase in ϕ can enhance advertising credibility and cause investment to increase by so much that, ceteris paribus, the net gains from false advertising, $\pi^*(q_H^e) - \phi$, actually rise, and prompt a higher y^*. In his seminal discussion, Nelson (1974) suggested that advertising policy may increase the credibility of false advertising. Here, we formalize an even stronger relationship - policy can provide so much credibility that parameters exist where the probability of false advertising is increasing in the level of penalty. Nevertheless, despite any potential increase in y^*, stronger penalties still always induce a larger expected quality, q_H^e. Now consider the optimal penalty:

Proposition 5. *Suppose Assumption 1 holds and that $L < \hat{q}$. A buyer-orientated policy-maker i) always sets $\phi > 0$, and ii) sets $\phi < \phi_0$ such that $y^* > 0$ provided $H > \dot{q}(L)$ and $f(\phi_0)/F(\phi_0)$ is sufficiently small.*

To understand this result, rewrite (8) from earlier using $x \equiv x^*(\phi)$ as

$$E(v) = v^*(L) + \underbrace{(H - L)\frac{v^*(q_H^e) - v^*(L)}{q_H^e - L}}_{\text{Price/Persuasion terms}} \times \underbrace{(1 - x^*(\phi))}_{\text{Investment term}} . \tag{17}$$

The second term captures the tradeoff between the price and persuasion effects. As in the main model, Assumption 1 ensures that this term is increasing in ϕ if and only if $q_H^e < \dot{q}(L)$. The third term relates to a new 'investment effect'. A high quality product generates more buyer surplus than a low quality product. Therefore ceteris paribus, an increase in ϕ is beneficial since it prompts a higher level of investment. Proposition 5 is then explained as follows. Firstly, unlike in the main model, $\phi = 0$ is never optimal because the firm then never invests and so buyers get only $v^*(L)$. Alternatively, for any $\phi > 0$, the firm invests with positive probability and so from (17) buyer surplus strictly exceeds $v^*(L)$. Secondly though, despite this new investment effect, policy may still refrain from completely eliminating false advertising. In particular, this is the case when $H > \dot{q}(L)$ and $f(\phi_0)/F(\phi_0)$ is relatively small. Intuitively the latter restriction on $f(C)$ implies that starting from strong regulation, $\phi = \phi_0$, a small decrease in ϕ only has a small effect on the investment probability, such that the combined price and persuasion effects dominate. Consequently, as in the main model, false advertising can sometimes benefit buyers.[14]

Finally, one can compare the optimal penalty with that under exogenous quality. In particular, let ϕ_{en}^* denote the optimal penalty with endogenous quality, and impose a technical condition $f(\phi)(H-L) < 1$ to ensure it is unique. Then, to make a comparison, let $x^*(\phi_{en}^*)$ be the proportion of low types under exogenous quality, and denote the associated optimal penalty by ϕ_{ex}^*. One can then prove that $\phi_{ex}^* \leq \phi_{en}^*$, such that the optimal penalty is stronger when quality is endogenous due to the existence of the investment effect.

6 Robustness

This final section shows how the results of the main model are robust to i) an arbitrary number of quality types, ii) asymmetric costs, iii) more complex forms of punishments, and iv) competition.

6.1 An Arbitrary Number of Types

Suppose there are now $n > 2$ quality levels, denoted by $q_1 < ... < q_n$, and that the firm has quality q_i with probability $x_i \in (0,1)$. To simplify the exposition, let $q_2 > \underset{\sim}{q}$ and $q_{n-1} < \hat{q}$ (relaxing these assumptions is straightforward, but adds no new insights). Marginal cost is the same for all types and normalized to zero, while ex ante expected quality is again denoted by $\bar{q} = \sum x_i q_i$. The firm may send any report from the set $Q = \{q_1, ..., q_n\}$, and

[14] Similarly, one can show that a welfare-maximizing policymaker may also refrain from completely eradicating false advertising.

the policymaker can commit to a richer punishment $\phi(q, r) \geq 0$, which depends on both the firm's actual and reported qualities. We assume that the firm can only be fined if it over-reports its quality i.e. $\phi(q, r) = 0$ for all $r \leq q$. The game and move order are otherwise unchanged.

As usual, for any particular punishment $\phi(q, r)$ there may exist a large number of PBE. Therefore, for reasons analogous to the main model, we continue to restrict attention to PBE in which i) $r(q) \geq q\ \forall q$ such that no type under-reports its quality, and ii) buyer beliefs depend on the firm's claim but not its price. Notice that in any PBE satisfying these restrictions, the punishment function $\phi(q, r)$ induces a mapping from quality types into reports. It is then convenient to let $y^*_{i,j}$ be the probability that a firm of type i claims to have quality j; hence $y^*_{i,i}$ denotes the probability that firm type i sends a truthful report. Letting $\mathbf{y}^*$ be the (triangular) matrix of such probabilities, we may then state:

Lemma 6. *The optimal penalty can be derived in two steps:*
i) First, choose the matrix of probabilities $\mathbf{y}^*$ *which maximizes the policymaker's objective.*
ii) Second, there exists a punishment function $\phi(q, r)$ *which induces the policymaker's optimal* $\mathbf{y}^*$ *as the unique equilibrium outcome of the game.*

Thus conceptually the problem is similar to the two-type case. In particular, analogous to Lemma 2, we can work with the matrix of report probabilities $\mathbf{y}^*$, and be sure that at least one punishment function can implement the desired $\mathbf{y}^*$. Now consider optimal penalties. Given our main model, it is not surprising that under certain conditions the policymaker will permit some false advertising. However once there is an arbitrary number of types, the policymaker also has to decide *which* quality types will be allowed to engage in false advertising, and which quality level(s) they will mimic. To simplify the exposition we now focus on distributions satisfying Assumptions 1i and 2i for which $\hat{q} = \hat{q}_w = \tilde{q}$[15]:

Proposition 6. *Suppose the match distribution satisfies Assumptions 1i and 2i. The optimal report probabilities are as follows:*
i) Buyer surplus. (a) When $q_n \leq \tilde{q}$ *it is maximized by* $y^*_{i,i} = 1$ *for all* i. *(b) When* $q_n > \tilde{q} > \bar{q}$ *there exists a critical type* i^* *satisfying* $E(q|q \geq q_{i^*}) \leq \tilde{q} < E(q|q \geq q_{i^*+1})$, *such that the optimal solution has* $y^*_{i,i} = 1$ *for all* $i < i^*$, $y^*_{i,n} = 1$ *for all* $i > i^*$, *and* $y^*_{i^*,i^*} = 1 - y^*_{i^*,n}$ *where* $y^*_{i^*,n}$ *satisfies:*

$$\frac{x_{i^*} y^*_{i^*,n} q_{i^*} + \sum_{i=i^*+1}^{n} x_i q_i}{x_{i^*} y^*_{i^*,n} + \sum_{i=i^*+1}^{n} x_i} = \tilde{q}.$$

[15]The optimal pattern of false advertising is qualitatively the same for distributions satisfying the alternative Assumptions 1ii and 2ii. Further details are available on request.

(c) When $\bar{q} \geq \tilde{q}$ it is maximized by $y^*_{i,n} = 1$ for all i.
ii) Profit is maximized by $y^*_{i,i} = 1$ for all i.
iii) Total welfare. There exists a threshold $\dot{L}$ such that: (a) When $q_n \leq \tilde{q}$ it is maximized by $y^*_{i,i} = 1$ for all i. (b) When $q_n > \tilde{q}$ and $q_{i^*} \geq \dot{L}$ it is maximized by the buyer-optimal matrix. (c) When $q_n > \tilde{q}$ and $q_{i^*} < \dot{L}$, it is maximized by $y^*_{i,i} = 1$ for all i with $q_i < \dot{L}$, and $y^*_{i,n} = 1$ for all i with $q_i \geq \dot{L}$.

The policymaker induces each firm type to either report truthfully or to claim to have the highest possible quality q_n. Similar to the two-type model, in many cases one firm type is required to randomize over its report. Whether buyers gain from false advertising depends upon how the highest quality type q_n compares with $\tilde{q}$. If $q_n \leq \tilde{q}$ the persuasion effect dominates, such that buyers are better off if the firm truthfully reveals its quality. However if $q_n > \tilde{q}$, the highest type has a lot of market power, and so lower types are pooled with it to generate a beneficial price effect. In order to minimize the negative persuasion effect, this pooling is done from the top i.e. first the q_{n-1} type is pooled, then the q_{n-2} type, and so forth, until either $E(q|r = q_n) = \tilde{q}$ or no more types are left to pool. Hence the optimum has full pooling when $\bar{q} > \tilde{q}$, and semi-pooling when $\bar{q} < \tilde{q} < H$. In the latter case, the policymaker permits 'small' lies by types close to q_n, whilst forbidding 'large' lies by types at or close to q_1.

Policy under a total welfare objective also depends upon whether $q_n \gtrless \tilde{q}$. When $q_n \leq \tilde{q}$ a welfare-maximizing policy involves truthful advertising and so coincides with what is optimal for both buyers and the firm. However when $q_n > \tilde{q}$ a welfare-oriented policymaker may allow some lower types to use false advertising in order to raise their output. As with buyer surplus, types with quality closer to q_n are more likely to be allowed to use false advertising since their socially-optimal output levels are highest. Overall, the main insights from the two-type model carry over into this richer multi-type environment.

6.2 Asymmetric Costs

Returning to the two-type case, we now permit the types to differ in marginal costs. This may allow the types to separate more easily by facilitating price signaling. Hence, we first characterize the least-cost separating equilibrium where the firm can signal by using both its report and its price. However, as such separating equilibria necessarily involve truthful advertising, they cannot account for false advertising. Consequently, we then use a different selection approach to study a version of our previous semi-pooling equilibrium with false advertising.[16]

Suppose that a product of quality q now has constant marginal cost $c(q)$ with $c'(q) \in (0,1)$ and $c''(q) = 0$. Let $\pi(p, q^e; i) = (p - c(i))[1 - G(p - q^e)]$ be the profit earned with price p, expected quality q^e, and actual quality $i \in \{L, H\}$, and denote $p^*(q^e; i) = \arg\max_p \pi(p, q^e; i)$ and $\pi^*(i) = \pi(p^*(i; i), i; i)$. First, consider the least-cost separating equilibrium. Without advertising regulation, standard results show that the low type charges its full information price $p^*(L; L)$, whilst the high type charges $p^*(H; H)$ if $\pi^*(L) \geq \pi(p^*(H; H), H; L)$, and otherwise distorts its price above $p^*(H; H)$. To make our problem with advertising regulation interesting, we henceforth assume $\pi^*(L) < \pi(p^*(H; H), H; L)$:

Remark 2. Suppose $\phi > 0$. At the least-cost separating equilibrium, the low type claims $r = L$ and charges $p^*(L; L)$, while the high type claims $r = H$ and charges $p^s(\phi)$ where
i) If $\phi \geq \phi_0' \equiv \pi(p^*(H; H), H; L) - \pi^*(L)$, $p^s(\phi) = p^*(H; H)$.
ii) If $\phi \in (0, \phi_0')$, $p^s(\phi) > p^*(H; H)$ and is the largest solution to $\pi(p, H; L) - \phi = \pi^*(L)$.

When $\phi = 0$, the high type separates only by distorting its price. However when $\phi > 0$, the high type optimally issues a high report in order to directly reduce the low type's incentive to mimic - now, if the high type charges p, the low type will not mimic if $\pi(p, H; L) - \phi \leq \pi^*(L)$. Therefore when $\phi \geq \phi_0'$ the high type can separate without distorting its price, but when $\phi \in (0, \phi_0')$ some upward price distortion is still required. Nevertheless the high type's separating price $p^s(\phi)$, and the size of the resulting distortion, are decreasing in ϕ.[17]

To study false advertising, we now characterize an alternative semi-pooling equilibrium in a similar style to the main model. Notice that if both types send the same report with positive probability, they must charge the same price when sending that report. If not, buyers would be able to infer the firm's type and the low type would wish to deviate. However, in contrast to the main model, after making a high claim with price p and subsequent belief q^e, the payoff functions of the two types now differ by more than just a constant. Therefore, the two types no longer have the same pricing incentives, and so we require a stronger equilibrium selection approach.

Firstly, we still focus on PBE where the high type always sends a high report. Secondly, however, we now restrict attention to PBE where given a high report, the firm always charges the high type's sequentially optimal price. In particular, if buyers believe that on average a firm with $r = H$ has quality q_H^e, then after reporting $r = H$ both types set a pooling price $p^*(q_H^e; H)$. Intuitively, since the high type is the one being mimicked, it should have

[16] See also Mailath *et al* (1993) for a number of wider arguments against the de facto selection of least-cost separating equilibria.

[17] Corts (2013) makes a related point in an extension of his main model where firms are imperfectly informed about their own product quality.

some ‘leadership’ in choosing its preferred pooling price. A more formal justification for this second restriction follows with the use of a stronger version of the Undefeated Equilibrium refinement, as proposed by Mezzetti and Tsoulouhas (2000).[18] We may then state:

Lemma 7. *Suppose $c(H) - c(L)$ is not too large. There is a unique semi-pooling PBE (up to off-path beliefs) satisfying our restrictions, in which:*
i) A high type firm claims $r = H$ and charges $p^*(q_H^e; H)$.
ii) A low type firm randomizes. With probability y^* it claims $r = H$ and charges $p^*(q_H^e; H)$. With probability $1 - y^*$ it claims $r = L$ and charges $p^*(L; L)$.
- When $\phi \leq \phi_1' \equiv \pi^*(p^*(\bar{q}; H), \bar{q}; L) - \pi^*(L)$, $y^* = 1$.
- When $\phi \geq \phi_0' = \pi^*(p^*(H; H), H; L) - \pi^*(L)$, $y^* = 0$.
- When $\phi \in (\phi_1', \phi_0')$, $y^* \in (0, 1)$ and uniquely solves

$$\pi^*(p^*(q_H^e; H), q_H^e; L) - \phi = \pi^*(L). \tag{18}$$

iii) q_H^e is given by (6). Buyer beliefs are such that $\Pr\left(q = H | \{r, p\} = \{H, p^*(q_H^e; H)\}\right) = \frac{1-x}{1-x+xy^*}$ *and* $\Pr\left(q = H | \{r, p\} \neq \{H, p^*(q_H^e; H)\}\right) = 0$.

The equilibrium is qualitatively similar to the main model, although pessimistic off-path beliefs are now required to prevent the low type from reporting $r = H$ but deviating to a price $p \neq p^*(q_H^e; H)$. Note that equilibrium play varies smoothly with $c(H) - c(L)$, and converges to that of the main model as $c(H) \to c(L)$.

Finally, given the similar equilibrium, we now briefly comment on the implications for policy.[19] Suppose that $c(H) - c(L)$ is not too large. Relative to symmetric costs, there is now an additional reason to eradicate false advertising. Under asymmetric costs, a lying low type must distort its price upwards at $p^*\left(q_H^e; H\right)$ instead of its preferred (lower) price $p^*\left(q_H^e; L\right)$. This distortion provides a further loss to buyer surplus, total welfare, and ex ante profits. Nevertheless under Assumptions 1 and 2, it remains true that for certain values of L, H, and x both a buyer- and a welfare-oriented policymaker would permit a strictly positive level of false advertising.

6.3 More Complex Punishments

In practice, contrary to our assumption of a fixed punishment ϕ, the punishment might for example depend on how many units the firm sold, or on the degree to which buyers were

[18] In particular, fix buyers’ belief that on average a firm reporting $r = H$ has quality q_H^e. As in the main model, suppose that buyers interpret any off-path price as a (possible) signal of the firm’s type rather than a signal about $E(q|r = H)$. Then provided $c(H) - c(L)$ is not too large, the Strongly Undefeated refinement uniquely selects $p^*\left(q_H^e; H\right)$ as the price which the firm should charge after reporting $r = H$.

[19] Full details are available on request.

harmed. To capture these and other possibilities, we now consider a general punishment $\phi(q,r,p,q^e) \geq 0$, which depends on both the firm's actual and reported qualities, as well as its price, and buyers' expectations. We assume that the firm can only be fined if it over-reports its quality i.e. $\phi(q,r,p,q^e) = 0$ if $r \leq q$. Hence it is sufficient to work with $\phi(p,q^e) \equiv \phi(L,H,p,q^e)$, which we assume to be strictly positive and continuous in both p and q^e. For similar reasoning to the previous subsection, we restrict attention to PBE where the high type reports truthfully, and where conditional on reporting $r = H$ the firm charges the high type's sequentially optimal price.[20]

Lemma 8. *There exists a PBE satisfying our restrictions which is similar to the main model except:*
i) $y^* = 1$ *is an equilibrium if* $\pi^*(L) \leq \pi^*(\bar{q}) - \phi(p^*(\bar{q}), \bar{q})$,
ii) $y^* = 0$ *is an equilibrium if* $\pi^*(L) \geq \pi^*(H) - \phi(p^*(H), H)$,
iii) If some $y^* \in (0,1)$ *solves* $\pi^*(L) = \pi^*(q_H^e) - \phi(p^*(q_H^e), q_H^e)$, *then it is also an equilibrium,*
iv) Buyer beliefs are

$$\Pr(q = H | \{r,p\} = \{H, p^*(q_H^e)\}) = \frac{1-x}{1-x+xy^*} \quad \text{and} \quad \Pr(q = H | \{r,p\} \neq \{H, p^*(q_H^e)\}) = 0$$

When a firm reports $r = H$ and is believed by buyers to have quality q_H^e, it still charges the same price as in the main model, $p^*(q_H^e)$, despite the more general penalty, $\phi(p,q^e)$. Intuitively, since the high type never incurs the penalty, its sequentially optimal price is not affected by the precise form of $\phi(p,q^e)$. Consequently equilibrium play is almost identical to that in Proposition 1, with the only difference that for a general $\phi(p,q^e)$, y^* is not necessarily unique. Nevertheless uniqueness can be guaranteed with a regularity condition that $\pi^*(z) - \phi(p^*(z), z)$ strictly increases in z. Overall, we can again view the policymaker as choosing a lying probability y^* to maximize its objective function. The desired y^* can then be implemented by using a fixed fine ϕ or a more ornate fine $\phi(p,q^e)$ with no difference in final outcome.

6.4 Competition

This final subsection introduces competition into the main model. Suppose an established incumbent, I, with quality q_I, competes against an entrant, E, with quality, q_E. Product differentiation is modeled using a Hotelling line such that a buyer with location $z \in [0,1]$

[20] As before with asymmetric costs, the types do not necessarily have the same pricing incentives when $r = H$, but the high type's sequentially optimal price is again uniquely selected by Mezzetti and Tsoulouhas's (2000) Strongly Undefeated Equilibrium refinement provided that $\phi(p,q^e) \geq \phi(p^*(L), L)$ and that $\phi(p,q^e)$ is not too sensitive to changes in p and q^e.

can gain $U_I(z) = q_I - p_I - tz$ or $U_E(z) = q_E - p_E - t(1-z)$ from trading with the respective firms. While the incumbent's product quality is known, the entrant's quality is private information. Specifically, the entrant's product quality equals L with probability $x \in (0,1)$ and $H > L$ with probability $1-x$, such that the entrant's ex ante average quality level equals $\bar{q} = xL+(1-x)H$. Let all marginal costs be zero, and the buyers' outside option be sufficiently poor such that buyers always buy. The game then proceeds with i) the policymaker publicly selecting ϕ, ii) the entrant learning its quality and issuing a report $r \in \{L, H\}$, iii) the entrant and incumbent simultaneously selecting their prices, p_E and p_I, iv) buyers making their purchase decisions, and v) the policymaker administering any potential punishments.

To begin, consider a benchmark case where q_E is public information. The Nash equilibrium price charged by firm $i \in \{I, E\}$ is then

$$p_i^*(q_i, q_{-i}) = \begin{cases} 0 & \text{if } q_i \leq \underset{\sim}{q_i} \\ t + \left(\frac{q_i - q_{-i}}{3}\right) & \text{if } q_i \in \left(\underset{\sim}{q_i}, \tilde{q}_i\right) \\ q_i - q_{-i} - t & \text{if } q_i \geq \tilde{q}_i \end{cases} \tag{19}$$

where $\underset{\sim}{q_i} = q_{-i} - 3t$ and $\tilde{q}_i = q_{-i} + 3t$. Intuitively, when $q_i \leq \underset{\sim}{q_i}$ firm i is uncompetitive so its price is driven down to marginal cost. When instead $q_i \in (\underset{\sim}{q_i}, \tilde{q}_i)$, both firms are active. Here, an increase in q_i shifts out firm i's demand curve at the expense of its rival, prompting firm i to charge more and its rival to charge less. Finally, when $q_i \geq \tilde{q}_i$, firm i's product is so strong that it monopolizes the whole market; firm i then sets its price such that the marginal buyer is indifferent about buying from it. In addition, one can show that firm i's equilibrium profit, $\pi_i^*(q_i, q_{-i})$, is increasing in its own quality q_i, and decreasing in that of its rival q_{-i}.

Now let the entrant's product quality be private information. As consistent with the main model, we restrict attention to PBE where i) the high entrant type always issues a high report, and ii) buyer beliefs do not depend on price.

Lemma 9. *There exists a unique semi-pooling equilibrium (up to off-path beliefs) satisfying our restrictions, in which:*
i) The probability with which a low quality entrant reports $r = H$, y^*, is the same as in Proposition 1ii) after replacing $\pi^*(z)$ with $\pi_E^*(z, q_I)$.
ii) The firms charge $p_E^*(L, q_I)$ and $p_I^*(q_I, L)$ respectively if the entrant reports $r = L$; and $p_E^*(q_H^e, q_I)$ and $p_I^*(q_I, q_H^e)$ if the entrant reports $r = H$.
iii) Buyer beliefs are similar to Proposition 1iii) i.e. following a report $r = H$, buyers expect the entrant to have quality $q_H^e = \frac{xy^*L+(1-x)H}{1-x+xy^*}$.

There exists a natural semi-pooling equilibrium which is qualitatively the same as under monopoly where a low quality entrant randomizes between lying and reporting truthfully. Buyers update their beliefs about entrant quality accordingly, and conditional on those beliefs, the two firms charge Nash equilibrium prices.

Now consider the optimal penalty, starting with an industry profits objective. One can verify that both the entrant and the incumbent weakly prefer $y^* = 0$, with a strict preference whenever $L < \tilde{q}_E$. Hence, like the monopoly case, an industry self-regulator would choose to completely eliminate false advertising with the use of a tough policy $\phi^* \geq \phi_0 \equiv \pi_E^*(H, q_I) - \pi_E^*(L, q_I)$.

We now consider buyer surplus and total welfare. Here, in order to demonstrate that a policymaker may still permit a positive level of false advertising, it is sufficient to focus on the case where the entrant always has positive market share, with $L \geq \underset{\sim}{q_E}$. First, consider buyer surplus.

Proposition 7. *When $L \geq \underset{\sim}{q_E}$ the buyer-optimal level of false advertising y^* is the same as in Proposition 2 after replacing $\hat{q}$ and $\dot{q}(L)$ with $\tilde{q}_E$.*

To understand this result, write

$$\frac{\partial E(v)}{\partial y^*} = x\left[v^*(q_I, q_H^e) - (q_H^e - L)D_E^*(q_H^e, q_I) - v^*(q_I, L)\right] \\ - (1 - x + xy^*)\frac{\partial q_H^e}{\partial y^*}\left[D_I^*(q_I, q_H^e)\frac{\partial p_I^*(q_I, q_H^e)}{\partial q_H^e} + D_E^*(q_H^e, q_I)\frac{\partial p_E^*(q_H^e, q_I)}{\partial q_H^e}\right], \quad (20)$$

where $D_I^*(q_I, q_H^e)$ and $D_E^*(q_H^e, q_I)$ are the respective equilibrium demands. The first term is a revised 'persuasion' effect which measures the change in buyer surplus generated by a low quality entrant when it changes its report from $r = L$ to $r = H$. As usual, a low type entrant uses false advertising to induce buyers to buy too many units, at an inflated price. However, false advertising now also allows the low type entrant to compete more effectively, which reduces the incumbent's price from $p_I^*(q_I, L)$ to $p_I^*(q_I, q_H^e)$. The second term is a revised 'price' effect. Conditional on the entrant using a high claim, an increase in lying reduces q_H^e and prompts the entrant to charge a (weakly) lower price, but allows the incumbent to select a (weakly) higher price. This net price effect need no longer benefit buyers - it is beneficial if and only if the entrant's market share exceeds 1/2, which is equivalent to $q_H^e \geq q_I$. However, in aggregate, the optimal penalty remains qualitatively similar to that under monopoly. In particular, false advertising remains optimal when $H > \tilde{q}_E$ in order to weaken the high type entrant's market power.

Now consider total welfare. From above, it is optimal to set $\phi^* \geq \phi_0$ to induce $y^* = 0$ when $H \leq \tilde{q}_E$ as this is preferred by all parties. For the remaining cases, we can state:

Proposition 8. *When $L \geq \underaccent{\sim}{q}_E$ the welfare-optimal y^* is the same as in Proposition 4 after substituting $\tilde{q}_E$ for $\hat{q}_w$, and $\dot{L}_E = q_I + 3t/5$ for $\dot{L}(H)$ and $\dot{L}(\bar{q})$.*

This can be understood as follows. Firstly, an increase in y^* expands the output of a low type entrant. As with monopoly, this can either increase or decrease welfare depending on the level of L. Secondly, an increase in y^* reduces q_H^e, and therefore decreases the output of an entrant who reports $r = H$. However unlike monopoly, this second effect can actually increase welfare because, under competition, the firm with the highest (expected) quality uses its market power to restrict its output below the socially efficient level. Hence when $q_H^e \in (\underaccent{\sim}{q}_E, q_I)$ an entrant who reports $r = H$ actually overproduces, and so a small reduction in its output is socially beneficial. The proposition then shows that the aggregate of these two effects is qualitatively similar to monopoly. False advertising can remain optimal. In particular, false advertising is used if and only if $H > \tilde{q}_E$ and L is relatively large with $L \geq \dot{L}_E$ in order to raise the output of a low type entrant.

7 Conclusions

Despite its prevalence and importance, false advertising has previously remained understudied. However, this paper shows how it can influence rational buyers in equilibrium. Moreover, the paper has provided conditions under which buyers and society benefit from a positive level of false advertising due its effects in counteracting market power. This finding remains robust to the possibility of endogenous quality, arbitrary quality types, asymmetric costs, different forms of policy, and competition.

We hope that our paper will prompt a new research agenda on false advertising and advertising policy in a number of directions. First, further work should extend our analysis to understand more complex issues. For instance, we have assumed that other contractual and reputational sources of credibility are unavailable, as consistent with buyers only being able to assess a product's value with sufficient delay. Future work to consider such issues would be valuable. Second, such a dynamic understanding would also help better transfer our findings to empirical work. However, even our static model presents a rich set of empirical predictions for how changes in policy should affect the use of advertising, and market prices. Finally, much work remains in building on our analysis to study other types of false advertising and other forms of advertising policy.

Appendix

Proof of Lemma 1. i) If $q \leq \underset{\sim}{q}$ demand is zero for all $p \geq 0$, so profit is weakly maximized at $p^* = 0$. ii) If $q > \underset{\sim}{q}$ profit is strictly increasing in $p < a + q$, therefore the optimal price must satisfy $p^* \geq a + q$. At an interior solution, the first order condition is

$$1 - pg\,(p - q) / \left[1 - G\,(p - q)\right] = 0. \tag{21}$$

a) When $q \in \left(\underset{\sim}{q}, \tilde{q}\right)$ the left-hand side of (21) is strictly positive at $p \to a+q$, strictly negative as $p \to b + q$, and strictly decreasing in p because $1 - G(\varepsilon)$ is logconcave. Hence a unique p^* solves equation (21). Define $\sigma(\psi) = -[1 - G(\psi)]g'(\psi)/g(\psi)^2$. Differentiating (21) gives $\partial p^*(q)/\partial q = (1 - \sigma(p^*(q) - q))/(2 - \sigma(p^*(q) - q))$, which lies in $[0, 1)$ because logconcavity of $1 - G(\varepsilon)$ implies $\sigma(\psi) \leq 1$. b). When $q \geq \tilde{q}$ the lefthand side of (21) is strictly negative at all $p > a + q$ and hence $p^* = a + q$. □

Proof of Proposition 1. The proof proceeds in several steps. a) Beliefs depend only on the firm's report, so define $\beta_i^e = \Pr\left(q = H | r = i\right)$ and $q_i^e = \left(1 - \beta_i^e\right) L + \beta_i^e H$ for $i \in \{L, H\}$. b) Conditional on its report and buyer beliefs, the firm's price must maximize its profit. So given a report $r = i$ for $i \in \{L, H\}$, the firm charges $p^*(q_i^e)$. c) As $y^* = \Pr\left(r(L) = H\right)$, Bayes' rule implies $\beta_H^e = (1 - x)/(1 - x + xy^*)$, and $\beta_L^e = 0$ if $y^* < 1$. However, Bayes' rule places no restriction on β_L^e if $y^* = 1$. d) y^* must be consistent with the low type behaving optimally. Firstly given $y^* = 0$, $r = L$ is weakly dominant iff $\phi \geq \phi_0$. Secondly given $y^* = 1$, reporting $r = H$ is weakly dominant iff $\phi \leq \pi^*(\bar{q}) - \pi^*\left(q_L^e\right)$ i.e. for any $\phi \leq \phi_1$ given an appropriate off-path belief β_L^e. Thirdly given $y^* \in (0, 1)$, the low type must be indifferent between $r = L$ and $r = H$ i.e. (5) must hold. Moreover $y^* \in (0, 1)$ implies $q_H^e \in (\bar{q}, H)$, such that equation (5) cannot hold for $\phi \notin (\phi_1, \phi_0)$, but has a unique solution for any $\phi \in (\phi_1, \phi_0)$. e) Finally, given buyer beliefs, it is optimal for the high type to report $r = H$. □

Proof of Lemma 3. Recall that y^* strictly decreases in $\phi \in [\phi_1, \phi_0]$. Using (6) and (8):

$$\frac{\partial E(v)}{\partial y^*} = x\left[v^*\left(q_H^e\right) - v^*(L) - \frac{dv^*\left(q_H^e\right)}{dq} \times \left(q_H^e - L\right)\right], \tag{22}$$

i) Consider $L < \hat{q}$. a) Under Assumption 1i $dv^*(q)/dq$, $d^2v^*(q)/dq^2 > 0$ for $q \in (\underset{\sim}{q}, \tilde{q})$, and $dv^*(q)/dq = 0$ for $q > \tilde{q}$. Hence (22) is strictly negative (positive) for q_H^e below (above) $\dot{q}(L) = \tilde{q}$. b) Under Assumption 1ii we have the following results. First, (22) is strictly negative when

$q_H^e \leq \hat{q}$ because by assumption $d^2v^*(q)/dq^2 < 0$ for all $q \in (\underset{\sim}{q}, \hat{q})$. Second, (22) is strictly increasing in $q_H^e > \hat{q}$ because in that region $d^2v^*(q_H^e)/dq^2 < 0$. Third, (22) is strictly positive for sufficiently high q_H^e. To see this, note that $dv^*(q)/dq = [1 - G(p^*(q) - q)]/[2 - \sigma(p^*(q) - q)]$, hence $dv^*(L)/dq > 0$, and also $\lim_{q\to\infty} dv^*(q)/dq = 0$ because $\lim_{q\to\infty} p^*(q) - q = a$ and by assumption $\lim_{\psi\to a} \sigma(\psi) = -\infty$. Therefore since $v^*(q)$ is strictly convex for $q < \hat{q}$ and strictly concave for $q > \hat{q}$, we infer that for sufficiently high q_H^e we have $dv^*(q_H^e)/dq < dv^*(z)/dq$ for all $z \in (L, q_H^e)$. Rewriting (22) as $x \int_L^{q_H^e} ((dv^*(z)/dq) - (dv^*(q_H^e)/dq))\, dz$ shows that (22) is strictly positive for sufficiently high q_H^e. Fourth then, (22) has a unique root which we denote by $\dot{q}(L)$, and is strictly negative (positive) for q_H^e below (above) $\dot{q}(L)$. Fifth, note that $\dot{q}(L)$ is strictly decreasing in L because $dv^*(\dot{q}(L))/dq > dv^*(L)/dq$ and so (22) is strictly increasing in L. Also note that $\lim_{L\to\hat{q}} \dot{q}(L) = \hat{q}$. Finally since q_H^e increases in ϕ, it is immediate that under Assumption 1 $E(v)$ is quasiconcave in ϕ. ii) Consider $L > \hat{q}$. Given Assumption 1 $v^*(q)$ is weakly increasing and concave in $q \geq \hat{q}$, so (22) is weakly positive. □

Proof of Proposition 2. i) Note that $q_H^e \leq \dot{q}(L)$ for all ϕ, so by Lemma 3 $E(v)$ is maximized at $\phi^* \geq \phi_0$. ii) Note that $q_H^e < \dot{q}(L)$ when $\phi < \pi^*(\dot{q}(L)) - \pi^*(L)$, and $q_H^e > \dot{q}(L)$ when $\phi > \pi^*(\dot{q}(L)) - \pi^*(L)$. Hence from Lemma, 3 $E(v)$ is maximized at $\phi^* = \pi^*(\dot{q}(L)) - \pi^*(L)$ such that $q_H^e = \dot{q}(L)$. iii) Note that $q_H^e \geq \dot{q}(L)$ for all ϕ, hence by Lemma 3 $E(v)$ is maximized at $\phi^* \leq \phi_1$. Finally, Proposition 1 gives the associated optimal y^* for each case. □

Proof of Corollary 1. Using Proposition 2 optimal false advertising is

$$y^* = \min\left\{\max\left\{\frac{(H - \dot{q}(L))(1-x)}{(H - \dot{q}(L))(1-x) + \dot{q}(L) - \bar{q}}, 0\right\}, 1\right\}. \tag{23}$$

Recall from the proof of Lemma 3 that $\dot{q}(L)$ is weakly decreasing in L. Hence (23) is weakly increasing in L, H, and $(1-x)$. □

Proof of Proposition 3. Given $E(\Pi) = xE(\pi_L) + (1-x)E(\pi_H)$, it is immediate from (12) that a) $E(\Pi) = \pi^*(\bar{q}) - x\phi$ when $\phi < \phi_1$, b) $E(\Pi) = \pi^*(L) + (1-x)\phi$ when $\phi \in [\phi_1, \phi_0]$, and c) $E(\Pi) = x\pi^*(L) + (1-x)\pi^*(H)$ when $\phi > \phi_0$. Hence $E(\Pi)$ is quasiconvex, minimized at ϕ_1, and cannot be maximized at any $\phi \in (0, \phi_0)$. Then for part i), $\phi = \phi_0$ strictly dominates $\phi = 0$ because $\pi^*(q)$ is convex everywhere and strictly convex for $q \in \left(\underset{\sim}{q}, \tilde{q}\right)$. For part ii) note that $\pi^*(q) = a + q$ for all $q \geq \tilde{q}$, and hence $E(\Pi) = a + \bar{q}$ for any $\phi \in \{0\} \cup [\phi_0, \infty)$. □

Proof of Lemma 4. Recall that y^* strictly decreases in $\phi \in [\phi_1, \phi_0]$. Using equation (13):

$$\frac{\partial E(w)}{\partial y^*} = x\left[w^*\left(q_H^e\right) - w^*(L) - \frac{dw^*\left(q_H^e\right)}{dq} \times \left(q_H^e - L\right)\right]. \tag{24}$$

i) When $q_H^e < \hat{q}_w$ (24) is strictly negative because $w^*(q)$ is strictly convex for all $q \in \left(\underset{\sim}{q}, \hat{q}_w\right)$. ii) Consider $L < \hat{q}_w < q_H^e$, and define $\check{L} < \hat{q}_w$ as the unique solution to $dw^*(\check{L})/dq = dw^*(q_H^e)/dq$. First, (24) is strictly increasing in $L < \check{L}$ and strictly decreasing in $L > \check{L}$. Second, (24) is continuous in L around $\hat{q}_w$, and (weakly) positive at $L = \hat{q}_w$ because by Assumption 2 $w^*(q)$ is weakly concave for $q > \hat{q}_w$. Third, (24) is strictly negative for sufficiently low L. To prove this, note that (24) is proportional to $\frac{w^*\left(q_H^e\right) - w^*(L)}{q_H^e - L} - \frac{dw^*\left(q_H^e\right)}{dq}$. Fixing q_H^e, there exists a $\delta > 0$ such that $\frac{dw^*\left(q_H^e\right)}{dq} > \delta$. Moreover $\frac{w^*\left(q_H^e\right) - w^*(L)}{q_H^e - L}$ is weakly less than $\frac{w^*\left(q_H^e\right)}{q_H^e - L}$, which in turn is strictly less than δ for sufficiently low L. Fourth then, (24) has a unique root $\dot{L}(q_H^e) < \check{L}$, and is strictly negative (positive) for L below (above) $\dot{L}(q_H^e)$. Therefore, since q_H^e increases in ϕ, $E(w)$ is quasiconcave in ϕ under Assumption 2. iii) When $L > \hat{q}_w$ (24) is weakly positive because $w^*(q)$ is weakly concave for all $q > \hat{q}_w$. □

Proof of Proposition 4. This follows directly from Lemma 4 and its proof. Note that $\dot{L}(q_H^e)$ is weakly decreasing in q_H^e.[21] i) Since $L \leq \dot{L}(q_H^e)$ for all $y^* \in [0, 1]$, $E(w)$ is maximized at $y^* = 0$. ii) Since $L > \dot{L}(q_H^e)$ for $q_H^e > q^{**}$, and $L < \dot{L}(q_H^e)$ for $q_H^e < q^{**}$, $E(w)$ is maximized by the unique y^* such that $q_H^e = q^{**}$. iii) Since $L \geq \dot{L}(q_H^e)$ for all $y^* \in [0, 1]$, $E(w)$ is maximized at $y^* = 1$. □

Proof of Lemma 5. Part i) follows from arguments in the text. For part ii) look for an equilibrium in which a positive measure of types invest. Since $\pi^*(H) - \pi^*(L) < \infty$ not all types invest, hence $x^*(\phi) \in (0, 1)$. Since the firm's payoff following $r = L$ is independent of q, any firm reporting $r = L$ must have low quality; equivalently, $r(H) = H$. Firstly, in any equilibrium with $y^* = 0$ a firm with $q = L$ earns ϕ_0 less than a firm with $q = H$. Secondly, in any equilibrium with $y^* > 0$ a firm with $q = L$ earns ϕ less than a firm with $q = H$, such that $x^*(\phi) = 1 - F(\phi)$. a) Consider $\phi = \phi_0$. There is clearly an equilibrium with $y^* = 0$. There is no equilibrium with $y^* > 0$, since $\pi^*(q_H^e) - \phi_0 < \pi^*(H) - \phi_0 = \pi^*(L)$, such that a firm with $q = L$ would deviate and report $r = L$. b) Consider $\phi \in (0, \phi_0)$. There is no equilibrium with $y^* = 0$, since $\pi^*(q_H^e) - \phi = \pi^*(H) - \phi > \pi^*(L)$, such that a firm with $q = L$

[21] Under Assumption 2i $w^*(q)$ is linear in $q > \hat{q}_w = \tilde{q}$ such that $\dot{L}(q_H^e)$ in invariant to q_H^e and solves $v^*(\tilde{q}) - v^*(L) + a - \pi^*(L) + L = 0$. Under Assumption 2ii $\dot{L}(q_H^e)$ is strictly decreasing in q_H^e because the righthand side of (24) is strictly increasing in both $L = \dot{L}(q_H^e)$ and q_H^e.

would deviate and report $r = H$. Therefore look for an equilibrium with $y^* > 0$: the gain to a firm with $q = L$ from reporting $r = H$ instead of $r = L$ is $\pi^*(q_H^e) - \phi - \pi^*(L)$. Using $x^*(\phi) = 1 - F(\phi)$, this equals:

$$\pi^* \left(L + (H - L) \frac{F(\phi)}{F(\phi) + y^* (1 - F(\phi))} \right) - \phi - \pi^*(L). \tag{25}$$

This is continuous and strictly decreasing in y^*, and is strictly positive at $y^* = 0$. If (25) is weakly positive at $y^* = 1$ it is strictly positive at all $y^* \in [0, 1)$, hence there is a unique equilibrium with $y^* = 1$. If (25) is strictly negative at $y^* = 1$, there exists a unique equilibrium with $y^* \in (0, 1)$ which makes (25) equal to zero. $\square$

Proof of Proposition 5. i) The proof that $\phi = 0$ is never optimal is given in the text after the proposition. ii) It is enough to show that $\partial E(v)/\partial\phi|_{\phi=\phi_0} < 0$. Note that for $\phi > 0$,

$$\pi^* (q_H^e(\phi)) = \max \{\pi^*(L) + \phi, \pi^* (L + (H - L)F(\phi))\}, \tag{26}$$

where the first part applies when $y^* \in (0, 1)$, and the second part applies when $y^* = 1$. Equation (26) implies that for some small $\delta > 0$, $\pi^* (q_H^e(\phi)) = \pi^*(L)+\phi$ for all $\phi \in [\phi_0-\delta, \phi_0]$. Using $dq_H^e/d\phi = 1/ (d\pi^*(q_H^e)/dq)$ and equation (17), $\partial E(v)/\partial\phi|_{\phi=\phi_0}$ is proportional to

$$\frac{(H - L) (dv^*(H)/dq) - (v^*(H) - v^*(L))}{(d\pi^*(H)/dq) (v^*(H) - v^*(L)) (H - L)} + \frac{f(\phi_0)}{F(\phi_0)}.$$

The first term is strictly negative since $H > \dot{q}(L)$, and dominates the second term provided $f(\phi_0)/F(\phi_0)$ is sufficiently small. $\square$

All remaining proofs for the paper are in Section B of the Supplementary Appendix.

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Supplementary Appendix

Section A: Further Information on Assumptions 1 and 2

This section provides further details on Assumptions 1 and 2.

Claim 1. Assumption 1 (resp. Assumption 2) ensures that buyer surplus (resp. total welfare) is strictly convex for $q \in (\underset{\sim}{q}, \hat{q})$ (resp. $q \in (\underset{\sim}{q}, \hat{q}_w)$), and weakly concave for q above $\hat{q}$ (resp. $\hat{q}_w$).

Proof. Using the definitions of $p^*(q)$ and $v^*(q)$ in equations (1) and (4), and also the definition $w^*(q) = v^*(q) + \pi^*(q)$, we have that $d^2v^*(q)/dq^2 \propto z\,(p^*(q) - q)$ and $d^2w^*(q)/dq^2 \propto z_w\,(p^*(q) - q)$ for all $q \in (\underset{\sim}{q}, \tilde{q})$. Then note that since $1 - G(\varepsilon)$ is logconcave, $p^*(q) - q$ is strictly decreasing in q, with $\lim_{q \to \underset{\sim}{q}} p^*(q) - q = b$ and $\lim_{q \to \tilde{q}} p^*(q) - q = a$. Finally note that for $\tilde{q} < \infty$, $v^*(q)$ and $w^*(q)$ are both linear (and so weakly concave) for all $q > \tilde{q}$. □

Now consider the following generalized setting in which demand equals $s\left[1 - G\left(\frac{p-q-\mu}{m}\right)\right]$, where μ is a location parameter and $m, s \in (0, \infty)$ are stretch parameters (Weyl and Tirole 2012). This corresponds to a setting in which a mass $s > 0$ of buyers have unit demand, and each buyer's valuation is given by $q + \mu + m\varepsilon$ with ε distributed according to $G(\varepsilon)$. In the main text we focus on the case $\mu = 0$ and $m = s = 1$. However in fact:

Claim 2. If Assumptions 1 and 2 hold for a demand $1 - G(p - q)$, they also hold for any generalized demand of the form $s\left[1 - G\left(\frac{p-q-\mu}{m}\right)\right]$.

Proof. Consider Assumption 1. The market coverage point for this generalized demand is $\tilde{q}(s, m, \mu) = \mu + m\,(-a + 1/g(a))$, hence $\tilde{q}(s, m, \mu) < \infty$ if and only if $\tilde{q} < \infty$. Also the other threshold $\underset{\sim}{q}(s, m, \mu)$ satisfies $\underset{\sim}{q}(s, m, \mu) = -\infty$ if and only if $\underset{\sim}{q} = -\infty$. Let $\sigma(\psi; s, m, \mu)$ be the curvature of the generalized demand form. We may then write the analogue of $z(\psi)$ for this new demand as

$$z(\psi; s, m, \mu) = -\frac{d\sigma(\psi; s, m, \mu)}{d\psi} + [2 - \sigma(\psi; s, m, \mu)]\left[\frac{dsG\left(\frac{\psi-\mu}{m}\right)}{d\psi} \Bigg/ s\left[1 - G\left(\frac{\psi - \mu}{m}\right)\right]\right].$$

After solving for $\sigma(\psi; s, m, \mu)$ and substituting it in, then canceling terms:

$$z(\psi; s, m, \mu) = \frac{1}{m}\left[-\sigma'\left(\frac{\psi - \mu}{m}\right) + \left[2 - \sigma\left(\frac{\psi - \mu}{m}\right)\right]\frac{g\left(\frac{\psi-\mu}{m}\right)}{1 - G\left(\frac{\psi-\mu}{m}\right)}\right] \propto z\left(\frac{\psi - \mu}{m}\right).$$

Hence $z(\psi; s, m, \mu)$ satisfies Assumption 1 if and only if $z(\psi)$ satisfies it. The proof for Assumption 2 is very similar and so is omitted. □

Specific Examples

We now show that Assumptions 1 and 2 are satisfied by a wide range of common demand curves. In light of Claim 2 it is sufficient to focus on the case $s = m = 1$ and $\mu = 0$. For further related background material, including a proof that demands with distributions 2-6 below have increasing curvature, see Fabinger and Weyl (2015) and their associated online appendix.

1. Generalized Pareto Distribution: $G(\psi) = 1 - \left(1 - \frac{(1-\sigma)\psi}{(2-\sigma)}\right)^{\frac{1}{1-\sigma}}$ on $\left[0, \frac{2-\sigma}{1-\sigma}\right)$ for $\sigma < 1$, and $G(\psi) = 1 - e^{-\psi}$ on $[0, \infty)$ for $\sigma = 1$. Special cases include the Uniform ($\sigma = 0$) and Exponential ($\sigma = 1$) distributions. Note that $\tilde{q} = (2 - \sigma) < \infty$ and $\sigma(\psi) = \sigma$. Hence Assumptions 1i and 2i are satisfied, because $z(\psi) = (2-\sigma) > 0$ and $z_w(\psi) = (3-\sigma)(2-\sigma) > 0$.

2. Normal: $G(\psi) = \int_{-\infty}^{\psi} \frac{e^{-\psi^2/2}}{\sqrt{2\pi}} dx$ on $(-\infty, \infty)$. Note that $\underset{\sim}{q} = -\infty$, $\tilde{q} = \infty$, and $\sigma(\psi) = \frac{\psi[1-G(\psi)]}{g(\psi)}$ because $g'(\psi) = -\psi g(\psi)$. Hence $\lim_{\psi \to a} \sigma(\psi) = -\infty$. Moreover

$$z(\psi) \propto 2\left(\frac{g(\psi)}{1 - G(\psi)}\right)^2 - 1 - \psi^2. \tag{27}$$

Assumption 1ii is satisfied because (27) is negative as $\psi \to -\infty$, is strictly increasing in $\psi \leq 0$ since $\frac{g(\psi)}{1-G(\psi)}$ is strictly increasing, and is strictly positive for all $\psi \geq 0$. To prove the latter, note that for all $\psi \geq 0$ we have the lower bound $\frac{g(\psi)}{1-G(\psi)} \geq \frac{\psi + \sqrt{\psi^2 + 8/\pi}}{2}$ (see Duembgen 2010). In addition

$$z_w(\psi) \propto 6\left(\frac{g(\psi)}{1 - G(\psi)}\right)^2 - 4\psi\frac{g(\psi)}{1 - G(\psi)} - 1 = 6\left(\frac{g(\psi)}{1 - G(\psi)}\right)^2 + 4\frac{g'(\psi)}{1 - G(\psi)} - 1. \tag{28}$$

Assumption 2ii is satisfied. Firstly as $\psi \to -\infty$, (28) tends to -1. Secondly (28) is strictly increasing in $\psi < -1$, because $\frac{g(\psi)}{1-G(\psi)}$ and $g'(\psi) > 0$ are both strictly increasing. Thirdly (28) is strictly positive for all $\psi \in [-1, 0]$. This can be proved by noting that on this interval, we have the lower bound $g(\psi) \geq \left(1 - \frac{\psi^2}{2}\right)/\sqrt{2\pi}$, and the upper bound $1 - G(\psi) \leq \frac{1}{2} - xg(0)$. Fourthly (28) is also strictly positive for all $\psi > 0$. This can be proved by noting that $\frac{g(\psi)}{1-G(\psi)}$ strictly increasing implies $2\left(\frac{g(\psi)}{1-G(\psi)}\right)^2 > 2\left(\frac{g(0)}{1-G(0)}\right)^2 > 1$, and also $4\left[\left(\frac{g(\psi)}{1-G(\psi)}\right)^2 + \frac{g'(\psi)}{1-G(\psi)}\right] > 0$.

3. Weibull: $G(\psi) = 1 - e^{-\psi^\alpha}$ on $[0, \infty)$ where $\alpha > 1$. Note that $\underset{\sim}{q} = -\infty$, $\tilde{q} = \infty$, $\sigma(\psi) = 1 - \left(\frac{\alpha-1}{\alpha\psi^\alpha}\right)$ and $\lim_{\psi \to a} \sigma(\psi) = -\infty$. Moreover

$$z(\psi) \propto (\alpha - 1)(\psi^\alpha - 1) + \alpha\psi^{2\alpha} \text{ and } z_w(\psi) \propto 2\alpha^2\psi^{2\alpha} + 3\alpha(\alpha - 1)\psi^\alpha - (\alpha - 1) \tag{29}$$

Assumptions 1ii and 2ii are both satisfied, since both expressions in (29) are strictly negative as $\psi \to 0$, strictly increasing in ψ and strictly positive as $\psi \to \infty$.

4. Type I Extreme Value (Max version): $G(\psi) = e^{-e^{-\psi}}$ on $(-\infty, \infty)$. Note $\underset{\sim}{q} = -\infty$, $\tilde{q} = \infty$, $\sigma(\psi) = (e^{\psi} - 1)(e^{e^{-\psi}} - 1)$ and $\lim_{\psi \to a} \sigma(\psi) = -\infty$. Numerical simulations show that Assumptions 1ii and 2ii are both satisfied.

5. Logistic: $G(\psi) = \frac{e^{\psi}}{1+e^{\psi}}$ on $(-\infty, \infty)$. Note that $\underset{\sim}{q} = -\infty$, $\tilde{q} = \infty$, $\sigma(\psi) = 1 - e^{-\psi}$ and $\lim_{\psi \to a} \sigma(\psi) = -\infty$. Assumption 1ii is satisfied because $z(\psi) \propto e^{2\psi} - 1$, which is single-crossing from negative to positive at $\psi = 0$. However Assumption 2ii is not satisfied since $z_w(\psi) \propto 2 + 2e^{-\psi}$, which is strictly positive everywhere.[22]

6. Type I Extreme Value (Min version): $G(\psi) = 1 - e^{-e^{\psi}}$ on $(-\infty, \infty)$. Note that $\underset{\sim}{q} = -\infty$, $\tilde{q} = \infty$, $\sigma(\psi) = 1 - e^{-\psi}$ and $\lim_{\psi \to a} \sigma(\psi) = -\infty$. Assumption 1ii is satisfied because $z(\psi) \propto e^{-\psi}(1 - e^{-\psi}) - 1$, which is single-crossing from negative to positive at $\psi = \ln\left(\frac{-1+\sqrt{5}}{2}\right)$. However Assumption 2ii is not satisfied since $z_w(\psi) \propto 2 + 3e^{-\psi}$, which is strictly positive everywhere.

Section B: Remaining Proofs

Proof of Lemma 6 and Proposition 6. We prove Lemma 6 and Proposition 6 together, in several steps.

1) Given that beliefs are price-independent, $E(q|r)$ fully determines prices. Hence $\mathbf{y}^*$ is necessary and sufficient to write down expected buyer surplus, total welfare, and profit (before punishments are deducted). Lemma 6i then follows (we return to 6ii later).

2) Buyer surplus. Firstly, buyer surplus is not maximized if any report $r = q_{i<n}$ is sent by more than one type. To see why, consider a new triangular matrix with $y'_{i,i} = \sum_{j=1}^{n-1} y^*_{i,j}$ and $y'_{i,n} = y^*_{i,n}$ for all $i < n$. Strict convexity of $v^*(q) \in (\underset{\sim}{q}, \tilde{q})$ implies that buyer surplus is strictly higher, by Jensen's inequality. Secondly, buyer surplus is not maximized if $E(q|r = q_n) > \tilde{q}$ and $y^*_{i,n} < 1$ for some $i < n$. This is because the derivative of expected buyer surplus with respect to $y^*_{i,n}$ is $x_i \left[v^*(\tilde{q}) - v^*(q_i)\right] > 0$. Thirdly, buyer surplus is not maximized if $E(q|r = q_n) = \tilde{q}$, and there exists some $j < k$ such that $y^*_{k,n} < 1$ but $y^*_{j,n} > 0$. To see this, note that $\left.\frac{\partial y^*_{j,n}}{\partial y^*_{k,n}}\right|_{E(q|r=q_n)=\tilde{q}} = -\frac{x_k(\tilde{q}-q_k)}{x_j(\tilde{q}-q_j)}$. The derivative of $E(v)$ with respect to $y^*_{k,n}$, whilst adjusting $y^*_{j,n}$ to ensure $E(q|r = q_n) = \tilde{q}$, is proportional to

$$(\tilde{q} - q_j)\left[v^*(\tilde{q}) - v^*(q_k)\right] - (\tilde{q} - q_k)\left[v^*(\tilde{q}) - v^*(q_j)\right],$$

[22] Consequently a welfare-maximizing policymaker always optimally induces $y^* = 0$. This is also true for the next distribution.

which is strictly positive since $v^*(q)$ is strictly convex. Proposition 6i then follows.
3) Profit. Since $\pi^*(q)$ is convex, and strictly so for $q \in (\underset{\sim}{q}, \tilde{q})$, a similar approach to the first part of the previous step shows that expected profit (before punishments are deducted) is maximized by $y_{i,i}^* = 1$ for all i. Hence expected profit once punishments are deducted, is also maximized by $y_{i,i}^* = 1$ for all i, and Proposition 6ii follows.
4) Total welfare. Firstly, total welfare is not maximized if any report $r = q_i$ for $i < n$ is sent by more than one type, and the proof is similar to that for buyer surplus. Secondly, if $E(q|r = q_n) > \tilde{q}$ and there exists some $i < n$ with $y_{i,n}^* < 1$, total welfare is increasing in $y_{i,n}^*$ if and only if $q_i \geq \dot{L}$. To see this, the derivative of $E(TW)$ with respect to $y_{i,n}^*$ is $v^*(\tilde{q}) + a + q_i - v^*(q_i) - \pi^*(q_i)$, which is positive if and only if q_i exceeds a threshold (which we call $\dot{L}$). Thirdly, total welfare is not maximized if $E(q|r = q_n) = \tilde{q}$, and there exists some $j < k$ such that $y_{k,n}^* < 1$ but $y_{j,n}^* > 0$. The proof closely follows the same arguments for buyer surplus. Proposition 6iii then follows.
5) Implementation. Note that the maximum gain from false advertising is $\bar{\phi} = \pi^*(q_n) - \pi^*(q_1)$. First, set $\phi(q_i, q_j) = \bar{\phi}$ for all $j \notin \{q_i, q_n\}$ so that in any equilibrium, each firm either reports truthfully or reports $r = q_n$. Second, for any type i for whom $y_{i,i}^* = 1$, also set $\phi(q_i, q_n) = \bar{\phi}$. Third, for any type i for whom $y_{i,n}^* = 1$, set $\phi(q_i, q_n) = 0$. Fourth, let $q_n^e = (\sum_{j=1}^n x_j y_{j,n}^* q_j)/(\sum_{j=1}^n x_j y_{j,n}^*)$. For any type i for whom $y_{i,i}^* = 1 - y_{i,n}^*$ and $y_{i,n}^* \in (0, 1)$ (there is at most one such i) set $\phi(q_i, q_n) = \pi^*(q_n^e) - \pi^*(q_i)$. Fifth, it is easy to see there is a unique equilibrium outcome in which $\mathbf{y}^*$ is played, and so Lemma 6ii follows. □

Proof of Remark 2. a) In any separating equilibrium, buyers perfectly infer each type. Therefore the low type optimally reports $r = L$ and charges $p^*(L, L)$. b) The high type chooses p and r to maximize $\pi(p, H; H)$ subject to the no-mimicking constraint $\pi(p, H; L) - \phi I_{r(H)=H} \leq \pi^*(L)$. Clearly the optimum has $r(H) = H$. Then i) if $\phi \geq \phi_0'$ the optimum has $p = p^*(H; H)$. ii) If $\phi \in (0, \phi_0')$, given the quasiconcavity of $\pi(p, H; H)$ and $\pi(p, H; L)$, the no-mimicking constraint should bind. This gives two possible prices p_l and p_h satisfying $p_l < p^*(H; H) < p_h$. Since $c(H) > c(L)$ it is easy to show that $p = p_h$ is optimal. □

Proof of Lemma 7. The proof closely follows that of Proposition 1.
a) As usual let $q_H^e = E(q|r = H)$ and $y^* = \Pr(r(L) = H)$. The second restriction implies that following $r = H$ the firm charges $p^*(q_H^e; H)$. Bayes' rule implies that following $r = H$ and $p = p^*(q_H^e; H)$ the firm is believed to have high quality with probability $(1-x)/(1-x+xy^*)$.
b) Suppose $r = L$ is on-path. Firstly if a firm reports $r = L$ its price must maximize profit given buyer beliefs. Secondly buyer beliefs must satisfy Bayes' rule following $r = L$ and any on-path price(s). Hence given the first restriction, a firm that reports $r = L$ must charge

$p^*(L;L)$, and be believed to have low quality with probability 1.
c) Necessary conditions for optimality of the low type's behavior: Firstly given $y^* = 0$, reporting $r = L$ is weakly dominant only if $\phi \geq \phi'_0$. Secondly given $y^* = 1$, reporting $r = H$ is weakly dominant only if $\phi \leq \phi'_1$. Thirdly given $y^* \in (0,1)$, the low type is indifferent between $r = L$ and $r = H$ iff (18) holds. Note that for $c(H) - c(L)$ small, $\phi'_1 < \phi'_0$, and that (18) has a unique solution $y^* \in (0,1)$ if and only if $\phi \in (\phi'_1, \phi'_0)$.
d) The conditions given in the previous step are also sufficient for optimality of the low type's behavior, given appropriate off-path beliefs such as those in the lemma.
e) Clearly the high type strictly prefers to report $r = H$ and charge $p^*(q^e_H; H)$ for appropriate off-path beliefs, such as those in the lemma. □

Proof of Lemma 8. By inspection this is a valid PBE. a) Given an expectation $q^e_H = E(q|r = H)$, $p^*(q^e_H; H)$ is the high-type's sequentially optimal price (as defined before the lemma) because it never incurs the penalty. b) We now prove existence. Clearly if $\pi^*(\bar{q}) - \phi(p^*(\bar{q}), \bar{q}) \geq \pi^*(L)$ and/or $\pi^*(H) - \phi(p^*(H), H) \leq \pi^*(L)$ we have an equilibrium. If neither holds, by continuity there exists a $y^* \in (0,1)$ and hence a $q^e_H \in (\bar{q}, H)$ such that $\pi^*(q^e_H) - \phi(p^*(q^e_H), q^e_H) = \pi^*(L)$, therefore an equilibrium exists. □

Proof of Lemma 9. We can simply repeat all the steps used in the proof of Proposition 1. The only difference is that in the second step, each firm's price maximizes its profits given buyer beliefs and its conjecture about the other firm's price. Hence following a report $r = i$ for $i \in \{L, H\}$, the firms play Nash equilibrium prices $p^*_I(q_I, E(q_E|r = i))$ and $p^*_E(E(q_E|r = i), q_I)$. □

Proof of Proposition 7. Under full information:

$$v^*(q_I, q_E) = \begin{cases} -\frac{5t}{4} + \frac{q_I + q_E}{2} + \frac{(q_E - q_I)^2}{36t} & \text{if } q_E \in \left(\underset{\sim}{q_E}, \tilde{q}_E\right) \\ q_I + \frac{t}{2} & \text{if } q_E \geq \tilde{q}_E \end{cases}$$

Expected buyer surplus is $E(v) = x(1-y^*)v^*(q_I, L) + (1 - x + xy^*)v^*(q_I, q^e_H)$. Given $L \geq \underset{\sim}{q_E}$, $v^*(q_I, q_E)$ has the same shape as $v^*(q)$ in the monopoly problem under Assumption 1i with $\hat{q} = \tilde{q}_E$. Hence the proposition is proved in a similar way to Proposition 2, just with $\tilde{q}_E$ replacing $\dot{q}(L)$ and $\hat{q}$. □

Proof of Proposition 8. Under full information, $w^*(q_I, q_E) = v^*(q_I, q_E) + \pi_I^*(q_I, q_E) + \pi_E^*(q_I, q_E)$ equals:

$$w^*(q_I, q_E) = \begin{cases} -\frac{t}{4} + \frac{q_I + q_E}{2} + \frac{5(q_E - q_I)^2}{36t} & \text{if } q_E \in \left(\underset{\sim}{q}_E, \tilde{q}_E\right) \\ q_E - \frac{t}{2} & \text{if } q_E \geq \tilde{q}_E \end{cases}$$

Expected total welfare is $E(w) = x(1 - y^*)w^*(q_I, L) + (1 - x + xy^*)w^*(q_I, q_H^e)$. At $L \in [\underset{\sim}{q}_E, \tilde{q}_E)$ direct computation reveals that a) $\partial E(w)/\partial y^* < 0$ when $q_H^e \leq \tilde{q}_E$, and b) for $q_H^e > \tilde{q}_E$, $\partial E(w)/\partial y^* < 0$ if and only if $L < \hat{L} = q_I + 3t/5$. Hence the claim can be proved using a similar approach as in Proposition 4. $\square$

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