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Abstract

This study examines effects of bottleneck congestion and an optimal time-varying congestion toll on the spatial structure of cities. To this end, we develop a model in which heterogeneous commuters choose departure times from home and residential locations in a monocentric city with a bottleneck located between a central downtown and an adjacent suburb. We then show three properties of our model by analyzing equilibrium with and without congestion tolling. First, commuters with a higher value of travel time choose to live closer to their workplace. Second, congestion tolling causes population to increase in the suburb and generates urban sprawl. Third, commuters with a higher (lower) value of travel time gain (lose) from imposing the congestion toll without toll-revenue redistribution. Our findings are opposite to the standard results of traditional location models, which consider static traffic flow congestion, and differ fundamentally from the results obtained by Arnott (1998), who considers homogeneous commuters.

JEL classification: D62; R21; R41; R48

Keywords: bottleneck congestion; residential location; congestion toll; urban sprawl

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1 Introduction

The traditional residential location model describes spatial structure of cities and its evolution based on the trade-off between land rents and commuting costs (Alonso, 1964; Mills, 1967; Muth, 1969). Those and subsequent studies successfully predict the empirically observed patterns of residential location (e.g., spatial distribution of rich and poor) and the effects of assorted urban policies.[1] However, almost all previous studies describe traffic congestion using static flow congestion models. Their use renders these models inappropriate for dealing with peak-period traffic congestion and for examining the effects of measures intended to alleviate it (e.g., time-varying congestion tolls, flextime, staggered work hours).

The bottleneck model most successfully describes how commuters choose their departure times from home and peak-period congestion (Vickrey, 1969; Hendrickson and Kocur, 1981; Arnott et al., 1990b, 1993). Its simple and effective framework for studying efficacies of various measures to alleviate peak-period congestion has inspired numerous extensions and modifications. However, only Arnott (1998) and Gubins and Verhoef (2014) developed models to describe how commuters choose where they live and when they depart from home. Arnott (1998) considered a (discrete space) monocentric city consisting of two areas—a downtown and a suburb—connected by a single road with a bottleneck. He showed that imposing an optimal congestion toll without redistributing its revenues affects neither commuting costs nor residential locations of commuters. Gubins and Verhoef (2014) considered a (continuous space) monocentric city with a bottleneck at the entrance to its central business district (CBD). Their model introduced an incentive for commuters to spend time at home, which the standard bottleneck model disregards,[2] and assumed that the size of commuters' houses determined their marginal utility of spending time at home. They demonstrated that congestion tolling causes commuters to spend more time at home and to have larger houses, thereby leading to urban sprawl.

Results obtained by Arnott (1998) and Gubins and Verhoef (2014) differ fundamentally from the results of traditional models with static flow congestion, which predict that cities become denser with congestion pricing (Kanemoto, 1980; Wheaton, 1998; Anas et al., 1998). Their models, however, assume that commuters are homogeneous, although it is established that optimal congestion tolling changes commuting costs in bottleneck models with heterogeneous commuters (Arnott et al., 1992, 1994; van den Berg and Verhoef, 2011). That is, the effects of congestion tolling in the bottleneck model with heterogeneous commuters can differ fundamentally from those in models with homogeneous commuters.

This study extends the model developed by Arnott (1998) to consider commuter heterogeneity and a continuous space monocentric city with a bottleneck located between a central downtown and an adjacent suburb.[3] We then systematically analyze our model using the properties of complementarity problems that define equilibrium. Our analysis shows that commuters sort themselves temporally and spatially on the basis of their value of time: commuters with a higher time-based cost per unit schedule delay (marginal schedule delay cost divided by marginal travel

[1] Fujita (1989), Glaeser (2008), and Fujita and Thisse (2013) provide detailed overviews of traditional residential location models. For the effects of congestion pricing, see Kanemoto (1980), Wheaton (1998), and Anas et al. (1998). Recently, Brueckner (2007), Anas and Rhee (2007), Joshi and Kono (2009), Kono et al. (2008, 2012), and Pines and Kono (2012) show the efficiency of urban policies to substitute for congestion pricing (urban growth boundary, floor-to-area ratio regulations).

[2] Vickrey (1973), Tseng and Verhoef (2008), Fosgerau and Engelson (2011), Fosgerau and Lindsey (2013), and Fosgerau and Small (2014) considered the utility of spending time at home.

[3] We do not introduce the utility of spending time at home.

time cost) arrive at work earlier; commuters with higher value of travel time live closer to their workplace.[4] Furthermore, we demonstrate that expanding the bottleneck capacity increases the population of suburban commuters who traverse the bottleneck in our model. Thus, expanding capacity can increase total queuing time at the bottleneck.

This study also investigates the effects of an optimal time-varying congestion toll on spatial distribution of commuters. We show that introducing a congestion toll (with and without redistributing its revenues) changes commuters' commuting costs, thereby altering their spatial distribution. In addition, congestion tolling causes urban sprawl under assumptions common in the literature employing bottleneck models and commuter heterogeneity (Arnott et al., 1992, 1994; van den Berg and Verhoef, 2011). Our finding is not merely opposite to the standard results of traditional location models; it differs substantially from the findings by Arnott (1998). This implies that strategic interactions among heterogeneous commuters change the effects of congestion tolling on spatial structure of cities. Furthermore, we show that the optimal congestion toll leads to an unbalanced distribution of benefits unless toll revenues are redistributed: commuters with a high value of time (rich commuters) gain, while those with a low value of time (poor commuters) lose from tolling.

This study proceeds as follows. Section 2 presents a model in which heterogeneous commuters choose their departure times from home and residential locations in a monocentric city. Section 3 characterizes the equilibrium of our model by using the properties of complementarity problems. Section 4 shows effects of the optimal time-varying congestion toll. To demonstrate properties of our model and the effects of congestion tolling more concretely, Section 5 analyzes it in a simple setting. Section 6 concludes the study.

2 The model

2.1 Basic assumptions

We consider a long narrow city with a spaceless CBD where all job opportunities are located. The CBD is located at the edge of the city, and a residential location is indexed by distance x from the CBD (Figure 1). Land is uniformly distributed with unit density along a road. The road has a single bottleneck with capacity μ at location $d > 0$. If arrival rates at the bottleneck exceed its capacity, a queue develops. To model queuing congestion, we employ first-in-first-out and a point queue in which vehicles have no physical length as in standard bottleneck models (Vickrey, 1969; Arnott et al., 1993). Free-flow travel time per unit distance is assumed to be constant at $\tau > 0$ (i.e., free-flow speed is $1/\tau$).

There are I types of commuters, each of whom must travel from home to the CBD and who have the same preferred arrival time t^* at work. The number of commuters of type $i \in \mathcal{I} \equiv \{1, 2, \cdots, I\}$, whom we call "commuters i," is fixed and denoted by N_i. Since the bottleneck is located at d, only commuters who reside at $x > d$ pass through the bottleneck, while those who reside at $x \in [0, d]$ do not. Following Arnott (1998), we denote locations $\mathcal{X}^s = \{x \in \mathbb{R}_+ \mid x > d\}$ as "suburb" and locations $\mathcal{X}^d = \{x \in \mathbb{R}_+ \mid x \in [0, d]\}$ as "downtown." We denote the number of commuters i in the suburb and downtown by N_i^s and $N_i^d (= N_i - N_i^s)$, respectively. If d is sufficiently large, all commuters reside downtown and no commuter traverses the bottleneck. Because we are not interested in

[4] This spatial distribution is consistent with observations in a city with heavy traffic congestion (see McCann, 2013).

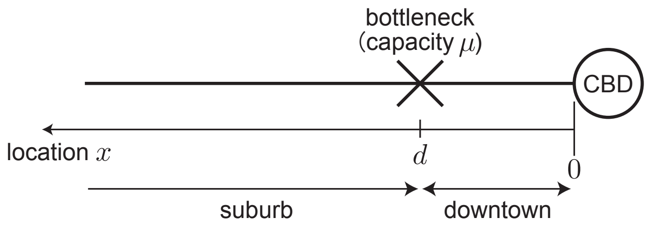

Figure 1: Long narrow city

that case, d is assumed to be small such that $\sum_{k\in\mathcal{I}} N_k^s > 0$.

Commuting cost $c_i(x,t)$ of commuter i who resides at x and arrives at work at time t (travels at t) is expressed as the sum of travel time cost $m_i(x,t)$ and schedule delay cost $s_i(t-t^*)$:

$$c_i(x,t) = m_i(x,t) + s_i(t-t^*), \tag{1a}$$

$$m_i(x,t) = \begin{cases} \alpha_i \tau x & \text{if} \quad x \in \mathcal{X}^d, \\ \alpha_i \{q(t) + \tau x\} & \text{if} \quad x \in \mathcal{X}^s, \end{cases} \tag{1b}$$

$$s_i(t-t^*) = \begin{cases} \beta_i(t^* - t) & \text{if} \quad t \le t^*, \\ \gamma_i(t-t^*) & \text{if} \quad t \ge t^*. \end{cases} \tag{1c}$$

Here $q(t)$ denotes queuing time of a commuter traveling at time t, τx represents free-flow travel time of commuters residing at x, and $\alpha_i > 0$ is the value of travel time of commuters i. $\beta_i > 0$ and $\gamma_i > 0$ are early and late delay costs per unit time, respectively.

The utility of a commuter i who resides at x and travels at time t is given by the logarithmic quasi-linear utility function:[5]

$$u(z_i(x,t), a_i(x)) = z_i(x,t) + \kappa \ln[a_i(x)], \tag{2}$$

where κ is a positive constant, $z_i(x,t)$ denotes consumption of the numéraire goods, and $a_i(x)$ is the lot size at x. The budget constraint is expressed as

$$y_i = z_i(x,t) + \left\{r(x) + r^{\mathrm{A}}\right\} a_i(x) + c_i(x,t), \tag{3}$$

where r^{A} is the exogenous agricultural rent and $r(x) + r^{\mathrm{A}}$ denotes the land rent at x.

The first-order condition of the utility maximization problem ($\max_{z_i(x,t), a_i(x)} u(z_i(x,t), a_i(x))$ s.t. (3)) gives

$$a_i(x)\left\{r(x) + r^{\mathrm{A}}\right\} = \kappa. \tag{4}$$

This implies that lot size $a_i(x)$ is independent of commuters' type i. From (2), (3), and (4), we obtain the indirect utility $v_i(x,t)$ as follows:

$$v_i(x,t) = y_i - c_i(x,t) - \kappa \ln[r(x) + r^{\mathrm{A}}] + \epsilon, \tag{5}$$

where $\epsilon = \kappa \ln[\kappa] - \kappa$.

[5] As Arnott (1998) proved, if commuters are homogeneous, congestion tolling does not affect their spatial distribution under a quasi-linear utility function (2).

2.2 Equilibrium conditions

Similar to models in Peer and Verhoef (2013), Gubins and Verhoef (2014), and Takayama (2015), we assume that commuters make long-run decisions about residential locations and short-run decisions about day-specific trip timing. In the short run, commuters i minimize commuting cost $c_i(x,t)$ by selecting their arrival time t at work (trip timing t) taking their residential location x as given. In the long run, each commuter i chooses a residential location x so as to maximize his/her utility. We, therefore, formalize short-run and long-run equilibrium conditions in turn.

2.2.1 Short-run equilibrium conditions

Commuters in the short run determine only their day-specific trip timing t, which implies that the number $N_i(x)$ of commuters i residing at x (i.e., spatial distribution of commuters) is assumed to be given. Since commuting costs are given by (1), short-run equilibrium conditions differ according to commuters' residential locations. We first consider commuters residing in the suburb (suburban commuters), who must traverse the bottleneck. The commuting cost $c_i^s(x,t)$ of suburban commuter i can be divided into two costs: one depends only on trip timing t and the other on residential location x:

$$c_i^s(x,t) = c_i^s(t) + \alpha_i \tau x, \tag{6a}$$

$$c_i^s(t) = \alpha_i q(t) + s_i(t - t^*). \tag{6b}$$

This implies that each suburban commuter chooses a trip timing t so as to minimize $c_i^s(t)$. Therefore, the short-run equilibrium conditions coincide with those in the standard bottleneck model, which are given by three conditions:

$$\begin{cases} n_i^s(t)\left\{c_i^s(t) - c_i^{s*}\right\} = 0 \\ n_i^s(t) \geq 0,\ c_i^s(t) - c_i^{s*} \geq 0 \end{cases} \qquad \forall i \in \mathcal{I}, \tag{7a}$$

$$\begin{cases} q(t)\left\{\mu - \sum_{k\in\mathcal{I}} n_k^s(t)\right\} = 0 \\ q(t) \geq 0,\ \mu - \sum_{k\in\mathcal{I}} n_k^s(t) \geq 0 \end{cases} \qquad \forall t \in \mathbb{R}_+, \tag{7b}$$

$$\int n_i^s(t)\mathrm{d}t = N_i^s \qquad \forall i \in \mathcal{I}, \tag{7c}$$

where $n_i^s(t)$ denotes the number of suburban commuters i who travel at time t (i.e., the arrival rate of suburban commuters i at the CBD) and c_i^{s*} is the short-run equilibrium commuting cost of suburban commuters i.

Condition (7a) represents the no-arbitrage condition for the choice of trip timing. This condition means that, at the short-run equilibrium, no commuter can reduce commuting cost by altering trip timing unilaterally. Condition (7b) is the capacity constraint of the bottleneck, which requires that the total departure rate $\sum_{k\in\mathcal{I}} n_k^s(t)$ at the bottleneck[6] equals capacity μ if there is a queue; otherwise, the total departure rate is (weakly) lower than μ. Condition (7c) is flow conservation for commuting demand. These conditions give $n_i^s(t)$, $q(t)$, and c_i^{s*} at the short-run equilibrium as functions of the number $\boldsymbol{N}^s = [N_i^s]$ of suburban commuters $i \in \mathcal{I}$. Therefore, at the

[6]Note that the departure rate from the bottleneck coincides with the arrival rate of suburban commuters at the CBD.

short-run equilibrium, the commuting cost of a suburban commuter i residing at x is represented as $c_i^{s*}(\boldsymbol{N}^s) + \alpha_i \tau x$.

We next consider commuters who reside downtown (downtown commuters). Since the commuters do not traverse the bottleneck, their commuting cost $c_i^d(x,t)$ is expressed as

$$c_i^d(x,t) = \alpha_i \tau x + s_i(t - t^*). \tag{8}$$

Thus, all downtown commuters must travel at $t = t^*$ and their commuting cost at the short-run equilibrium is given by $\alpha_i \tau x$.

2.2.2 Long-run equilibrium conditions

In the long run, each commuter i chooses a residential location x so as to maximize utility $v_i(x)$, which is expressed as

$$v_i(x) = \begin{cases} y_i - c_i^{s*}(\boldsymbol{N}^s) - \alpha_i \tau x - \kappa \ln[r(x) + r^{\mathrm{A}}] + \epsilon & \text{if} \quad x \in \mathcal{X}^s, \\ y_i - \alpha_i \tau x - \kappa \ln[r(x) + r^{\mathrm{A}}] + \epsilon & \text{if} \quad x \in \mathcal{X}^d. \end{cases} \tag{9}$$

Thus, long-run equilibrium conditions are given by

$$\begin{cases} N_i(x)\left\{v_i^* - v_i(x)\right\} = 0 \\ N_i(x) \geq 0,\ v_i^* - v_i(x) \geq 0 \end{cases} \quad \forall x \in \mathbb{R}_+, \quad \forall i \in \mathcal{I}, \tag{10a}$$

$$\begin{cases} r(x)\left\{1 - \sum_{k \in \mathcal{I}} a_k(x) N_k(x)\right\} = 0 \\ r(x) \geq 0,\ 1 - \sum_{k \in \mathcal{I}} a_k(x) N_k(x) \geq 0 \end{cases} \quad \forall x \in \mathbb{R}_+ \tag{10b}$$

$$\int_0^\infty N_i(x)\,\mathrm{d}x = N_i \qquad \forall i \in \mathcal{I}, \tag{10c}$$

where v_i^* denotes the long-run equilibrium utility of a commuter i.

Condition (10a) is the equilibrium condition for commuters' choice of residential location. This condition implies that, at the long-run equilibrium, each commuter has no incentive to change residential location unilaterally. Condition (10b) is the land market clearing condition. This condition requires that if the total land demand $\sum_{k \in \mathcal{I}} a_k(x) N_k(x)$ for housing at x equals supply 1, land rent $r(x) + r^A$ is (weakly) larger than agricultural rent r^{A}. Condition (10c) expresses the population constraint.

Substituting (4) into (10b), we have $r(x)$ as follows:

$$r(x) = \begin{cases} \kappa N(x) - r^{\mathrm{A}} & \text{if} \quad \kappa N(x) \geq r^{\mathrm{A}}, \\ 0 & \text{if} \quad \kappa N(x) \leq r^{\mathrm{A}}, \end{cases} \tag{11}$$

where $N(x) = \sum_{k \in \mathcal{I}} N_k(x)$ represents the total number of commuters residing at x. It follows from (9) and (11) that the indirect utilities of suburban and downtown commuters, $v_i^s(x)$ and $v_i^d(x)$, are expressed as

$$v_i^s(x) = \begin{cases} y_i - c_i^{s*}(\boldsymbol{N}^s) - \alpha_i \tau x - \kappa \ln[\kappa N(x)] + \epsilon & \text{if} \quad \kappa N(x) \geq r^{\mathrm{A}}, \\ y_i - c_i^{s*}(\boldsymbol{N}^s) - \alpha_i \tau x - \kappa \ln[r^{\mathrm{A}}] + \epsilon & \text{if} \quad \kappa N(x) \leq r^{\mathrm{A}}. \end{cases} \tag{12a}$$

$$v_i^d(x) = \begin{cases} y_i - \alpha_i \tau x - \kappa \ln[\kappa N(x)] + \epsilon & \text{if} \quad \kappa N(x) \geq r^{\mathrm{A}}, \\ y_i - \alpha_i \tau x - \kappa \ln[r^{\mathrm{A}}] + \epsilon & \text{if} \quad \kappa N(x) \leq r^{\mathrm{A}}. \end{cases} \tag{12b}$$

Therefore, the long-run equilibrium conditions are rewritten as follows:

$$\begin{cases} N_i(x) \left\{ v_i^{s*}(\boldsymbol{N}^s) - v_i^s(x) \right\} = 0 \\ N_i(x) \geq 0, \ v_i^{s*}(\boldsymbol{N}^s) - v_i^s(x) \geq 0 \end{cases} \quad \forall x \in \mathcal{X}^s, \quad \forall i \in \mathcal{I}, \tag{13a}$$

$$\int_d^{\infty} N_i(x)\, \mathrm{d}x = N_i^s \qquad \forall i \in \mathcal{I}, \tag{13b}$$

$$\begin{cases} N_i(x) \left\{ v_i^{d*}(\boldsymbol{N}^d) - v_i^d(x) \right\} = 0 \\ N_i(x) \geq 0, \ v_i^{d*}(\boldsymbol{N}^d) - v_i^d(x) \geq 0 \end{cases} \quad \forall x \in \mathcal{X}^d, \quad \forall i \in \mathcal{I}, \tag{13c}$$

$$\int_0^{d} N_i(x)\, \mathrm{d}x = N_i^d \qquad \forall i \in \mathcal{I}, \tag{13d}$$

$$\begin{cases} v_i^{d*}(\boldsymbol{N}^d) \geq v_i^{s*}(\boldsymbol{N}^s) & \text{if} \quad N_i^d \geq 0 \\ v_i^{d*}(\boldsymbol{N}^d) \leq v_i^{s*}(\boldsymbol{N}^s) & \text{if} \quad N_i^s \geq 0 \end{cases} \quad \forall i \in \mathcal{I}, \tag{13e}$$

$$N_i^d + N_i^s = N_i \qquad \forall i \in \mathcal{I}, \tag{13f}$$

where $v_i^{s*}(\boldsymbol{N}^s)$ and $v_i^{d*}(\boldsymbol{N}^d)$ denote the utilities that commuters i receive from residing in the suburb and downtown, respectively.

Conditions (13a) and (13b) are the equilibrium conditions for suburban commuters' choice of residential location x. Similarly, conditions (13c) and (13d) are the equilibrium conditions for downtown commuters' choice of residential location x. Conditions (13e) and (13f) are the equilibrium conditions for commuters' choice between residing in the suburb and downtown. We use these conditions for characterizing equilibrium spatial distribution of commuters in Section 3.

3 Equilibrium

3.1 Short-run equilibrium

The short-run equilibrium conditions (7) of suburban commuters coincide with those in the standard bottleneck model, as shown above. Therefore, as proved in Iryo and Yoshii (2007), there is an optimization problem equivalent to the short-run equilibrium conditions.

Proposition 1. *The short-run equilibrium number $[n_i^{s*}(t)]$ of suburban commuters traveling at time t coincides with the solution of the following linear programming problem:*

$$\min_{[n_i^s(t)]} \int \frac{s_i(t - t^*)}{\alpha_i} n_i^s(t) \mathrm{d}t \qquad s.t. \quad \mu - \sum_{k \in \mathcal{I}} n_k^s(t) \geq 0, \quad \int n_i^s(t) \mathrm{d}t = N_i^s. \tag{14}$$

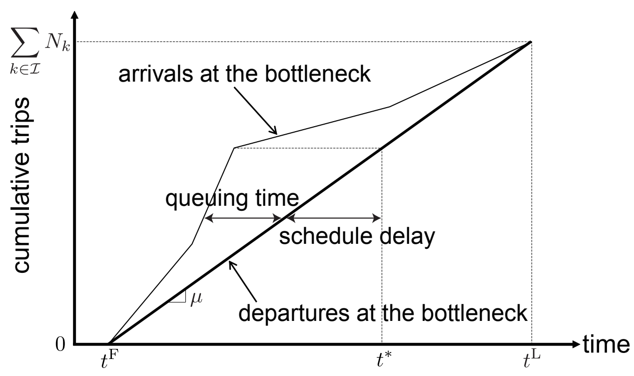

Figure 2: An example of cumulative arrival and departure curves at the short-run equilibrium

Let us define (travel) time-based cost as the cost converted into equivalent travel time. Since that cost for a commuter i is given by dividing the cost by α_i, we say that $s_i(t - t^*)/\alpha_i$ represents the time-based schedule delay cost of a commuter i. Therefore, Proposition 1 shows that, at the short-run equilibrium, the total *time-based* schedule delay cost is minimized, but the total schedule delay cost is not necessarily minimized.

We let $\mathrm{supp}\,(n_i^{s*}) = \{t \in \mathbb{R}_+ \mid n_i^{s*}(t) > 0\}$ be the support of the short-run equilibrium number $n_i^{s*}(t)$ of suburban commuters i who travel at t. Then, from Proposition 1, we have

$$\mathrm{supp}\,(\textstyle\sum_{i\in\mathcal{I}} n_i^{s*}) = [t^{\mathrm{F}}, t^{\mathrm{L}}], \tag{15}$$

where t^{F} and t^{L} denote the fastest and latest arrival times at the CBD of commuters, which satisfy

$$t^{\mathrm{L}} = t^{\mathrm{F}} + \frac{\sum_{i\in\mathcal{I}} N_i}{\mu}. \tag{16}$$

This indicates that, at the short-run equilibrium, the rush hour in which queuing congestion occurs must be a single time interval (Figure 2).

Furthermore, by using short-run equilibrium condition (7a), we obtain

$$c_i^s(t_i) + c_j^s(t_j) \leq c_i^s(t_j) + c_j^s(t_i) \qquad \forall t_i \in \mathrm{supp}\,(n_i^{s*}),\ t_j \in \mathrm{supp}\,(n_j^{s*}). \tag{17}$$

Substituting (6b) into this yields the following proposition:

Proposition 2. *For any $t_i \in \mathrm{supp}\,(n_i^{s*})$, $t_j \in \mathrm{supp}\,(n_j^{s*})$, and $i, j \in \mathcal{I}$, the following conditions hold:*

$$\left(\frac{\beta_i}{\alpha_i} - \frac{\beta_j}{\alpha_j}\right)\left(t_i - t_j\right) \geq 0 \qquad \textit{if} \quad \max\{t_i, t_j\} \leq t^*, \tag{18a}$$

$$\left(\frac{\gamma_i}{\alpha_i} - \frac{\gamma_j}{\alpha_j}\right)\left(t_i - t_j\right) \leq 0 \qquad \textit{if} \quad \min\{t_i, t_j\} \geq t^*. \tag{18b}$$

This proposition indicates that the short-run equilibrium has the following properties: if $\beta_i/\alpha_i < \beta_j/\alpha_j$, early-arriving commuters i (commuters i arriving at the CBD earlier than the preferred arrival time t^*) arrive at the CBD earlier than early-arriving commuters j; if $\gamma_i/\alpha_i < \gamma_j/\alpha_j$, late-arriving commuters i (commuters i arriving at the CBD later than t^*) arrive at the CBD later than late-arriving commuters j. That is, at the short-run equilibrium, commuters sort themselves temporally on the basis of their marginal time-based schedule delay cost.

3.2 Long-run equilibrium

3.2.1 Suburban and downtown spatial structures

We first characterize long-run equilibrium spatial distribution of suburban commuters and that of downtown commuters by using properties of complementarity problems (13a), (13b) (13c), and (13d). We show that these problems are equivalent to the following optimization problem:

Proposition 3. *The long-run equilibrium number* $[N_i^*(x)]$ $(x \in \mathcal{X}^s)$ *of suburban commuters residing at x coincides with the solution of the following optimization problem:*

$$\max_{[N_i(x)]} \sum_{k \in \mathcal{I}} \int_d^\infty v_k^s(x) N_k(x) \mathrm{d}x \qquad s.t. \quad \int_d^\infty N_i(x) \mathrm{d}x = N_i^s \quad \forall i \in \mathcal{I}. \tag{19}$$

Furthermore, the long-run equilibrium number $[N_i^*(x)]$ $(x \in \mathcal{X}^d)$ *of downtown commuters residing at x coincides with the solution of the following optimization problem:*

$$\max_{[N_i(x)]} \sum_{k \in \mathcal{I}} \int_0^d v_k^d(x) N_k(x) \mathrm{d}x \qquad s.t. \quad \int_0^d N_i(x) \mathrm{d}x = N_i^d \quad \forall i \in \mathcal{I}. \tag{20}$$

Proof. The Karush–Kuhn–Tucker (KKT) conditions of problem (19) are equivalent to equilibrium conditions (13a) and (13b). Additionally, KKT conditions of problem (20) are equivalent to equilibrium conditions (13c) and (13d). These, together with the monotonicity of the indirect utility functions (12) with respect to $N_i(x)$, give Proposition 3. □

This proposition shows that the total utility of suburban commuters and that of downtown commuters are maximized at the long-run equilibrium.

Equilibrium conditions (13a) and (13c) yield the following lemma.

Lemma 1. *The long-run equilibrium number* $N^*(x)$ *of commuters residing at x has the following properties:*

(a) The support of $N^*(x)$ *is given by*

$$\mathrm{supp}\,(N^*) = [0, X^{\mathrm{B}}], \tag{21}$$

where X^{B} *denotes the residential location for commuters farthest from the CBD (i.e., the city boundary).*

(b) $N^*(x)$ *satisfies*

$$\kappa N^*(x) > r^{\mathrm{A}} \qquad \forall x \in \mathrm{supp}\,(N^*) \backslash \{X^{\mathrm{B}}\}, \tag{22a}$$

$$\kappa N^*(X^{\mathrm{B}}) = r^{\mathrm{A}}. \tag{22b}$$

Proof. See Appendix A. □

Let $N_i^{s*}(x)$ and $N_i^{d*}(x)$ be the respective long-run equilibrium number of suburban and downtown commuters i residing at x. Then, it follows from Lemma 1 that, for any $x_i^s \in \mathrm{supp}\,(N_i^s)$ and

$x_i^d \in \mathrm{supp}\,(N_i^d)$, the indirect utilities $v_i^s(x_i^s)$ and $v_i^d(x_i^d)$ are expressed as

$$v_i^s(x_i^s) = y_i - c_i^{s*}(\boldsymbol{N}^s) - \alpha_i \tau x_i^s - \kappa \ln[\kappa N(x_i^s)] + \epsilon, \tag{23a}$$
$$v_i^d(x_i^d) = y_i - \alpha_i \tau x_i^d - \kappa \ln[\kappa N(x_i^d)] + \epsilon. \tag{23b}$$

In addition, equilibrium conditions (13a) and (13c) give the following conditions for $N_i^{s*}(x)$ and $N_i^{d*}(x)$:

$$v_i^s(x_i^s) + v_j^s(x_j^s) \geq v_i^s(x_j^s) + v_j^s(x_i^s) \qquad \forall x_i^s \in \mathrm{supp}\,(N_i^{s*}),\ \forall x_j^s \in \mathrm{supp}\,(N_j^{s*}),\ \forall i, j \in \mathcal{I}, \tag{24a}$$
$$v_i^d(x_i^d) + v_j^d(x_j^d) \geq v_i^d(x_j^d) + v_j^d(x_i^d) \qquad \forall x_i^d \in \mathrm{supp}\,(N_i^{d*}),\ \forall x_j^d \in \mathrm{supp}\,(N_j^{d*}),\ \forall i, j \in \mathcal{I}. \tag{24b}$$

Substituting (23) into (24) yields the following proposition.

Proposition 4. *For any $x_i \in \mathrm{supp}\,(N_i^{s*})$, $x_j \in \mathrm{supp}\,(N_j^{s*})$, and $i, j \in \mathcal{I}$, the following condition holds at the long-run equilibrium:*

$$\left(\alpha_i - \alpha_j\right)\left(x_i - x_j\right) \geq 0. \tag{25}$$

This condition also holds for any $x_i \in \mathrm{supp}\,(N_i^{d*})$, $x_j \in \mathrm{supp}\,(N_j^{d*})$, and $i, j \in \mathcal{I}$.

This proposition states that in the suburb and downtown, commuters with a higher value of travel time reside closer to the CBD. That is, commuters sort themselves spatially on the basis of their value of travel time. Furthermore, spatial distribution of suburban commuters and that of downtown commuters are unaffected by short-run equilibrium commuting cost $c_i^{s*}(\boldsymbol{N}^s)$.

We explicitly obtain $v_i^{d*}(\boldsymbol{N}^d)$ and $v_i^{s*}(\boldsymbol{N}^s)$ using Proposition 4. We assume, without loss of generality, that

$$\alpha_{i-1} > \alpha_i \qquad \forall i \in \mathcal{I} \backslash \{1\}. \tag{26}$$

Let X_i^s and X_i^d denote the respective locations for suburban and downtown commuters i residing nearest the CBD. It follows from Proposition 4 that suburban and downtown commuters i reside in $[X_i^s, X_{i+1}^s]$ and $[X_i^d, X_{i+1}^d]$, respectively (i.e., $\mathrm{supp}\,(N_i^{s*}) = [X_i^s, X_{i+1}^s]$ and $\mathrm{supp}\,(N_i^{d*}) = [X_i^d, X_{i+1}^d]$ for all $i \in \mathcal{I}$). By using X_i^d and X_i^s, the utility differences $v_i^{s*}(\boldsymbol{N}^s) - v_{i-1}^{s*}(\boldsymbol{N}^s)$ and $v_i^{d*}(\boldsymbol{N}^d) - v_{i-1}^{d*}(\boldsymbol{N}^d)$ are represented as

$$v_i^{s*}(\boldsymbol{N}^s) - v_{i-1}^{s*}(\boldsymbol{N}^s) = y_i - y_{i-1} - \left\{c_i^{s*}(\boldsymbol{N}^s) - c_{i-1}^{s*}(\boldsymbol{N}^s)\right\} - (\alpha_i - \alpha_{i-1})\,\tau X_i^s, \tag{27a}$$
$$v_i^{d*}(\boldsymbol{N}^d) - v_{i-1}^{d*}(\boldsymbol{N}^d) = y_i - y_{i-1} - (\alpha_i - \alpha_{i-1})\,\tau X_i^d. \tag{27b}$$

Therefore, we have the following indirect utilities of suburban and downtown commuters:

$$v_i^{s*}(\boldsymbol{N}^s) = y_i - c_i^{s*}(\boldsymbol{N}^s) - \kappa \ln[r^{\mathrm{A}}] - \alpha_i \tau X_i^s - \sum_{k=i}^{I} \alpha_k \tau (X_{k+1}^s - X_k^s) + \epsilon. \tag{28a}$$

$$v_i^{d*}(\boldsymbol{N}^d) = y_i - \kappa \ln[r(d)] - \alpha_i \tau X_i^d - \sum_{k=i}^{I} \alpha_k \tau (X_{k+1}^d - X_k^d) + \epsilon, \tag{28b}$$

where X_k^s and X_k^d are given by

$$X_1^s = d, \qquad X_{i+1}^s = X_i^s + \frac{\kappa}{\alpha_i \tau} \ln\left[\frac{\alpha_i \tau N_i^s}{\phi_i} + 1\right], \qquad \phi_i = r^{\mathrm{A}} + \sum_{k=i+1}^{I} \alpha_k \tau N_k^s, \tag{29a}$$

$$X_1^d = 0, \qquad X_{i+1}^d = X_i^d + \frac{\kappa}{\alpha_i \tau} \ln\left[\frac{\alpha_i \tau N_i^d}{\psi_i} + 1\right], \qquad \psi_i = r(d) + \sum_{k=i+1}^{I} \alpha_k \tau N_k^d. \tag{29b}$$

Land rent $r(d)$ at location d is determined from the condition $X_{I+1}^d = d$.

The city boundary $X^{\mathrm{B}} = X_{I+1}^s$ is obtained from (29a) as

$$X^{\mathrm{B}} = d + \kappa \sum_{k \in \mathcal{I}} \frac{1}{\alpha_k \tau} \ln\left[\frac{\alpha_k \tau N_k^s}{\phi_k} + 1\right]. \tag{30}$$

This indicates that the city boundary is affected by the number $\boldsymbol{N}^s$ of suburban commuters i.

3.2.2 Equilibrium population of suburban and downtown commuters

We next characterize the long-run equilibrium number $\boldsymbol{N}^{s*} = [N_i^{s*}]$ and $\boldsymbol{N}^{d*} = [N_i^{d*}]$ of suburban and downtown commuters i. From (27), we have the following proposition:

Proposition 5. *The difference $v_i^{d*}(\boldsymbol{N}^d) - v_i^{s*}(\boldsymbol{N}^s)$ between utilities from residing in the suburb and downtown satisfies*

$$\begin{aligned} &\left(v_i^{d*}(\boldsymbol{N}^d) - v_i^{s*}(\boldsymbol{N}^s)\right) - \left(v_{i-1}^{d*}(\boldsymbol{N}^d) - v_{i-1}^{s*}(\boldsymbol{N}^s)\right) \\ &\qquad = c_i^{s*}(\boldsymbol{N}^s) - c_{i-1}^{s*}(\boldsymbol{N}^s) - (\alpha_{i-1} - \alpha_i)\, \tau \left(X_i^s - X_i^d\right) \qquad \forall i \in \mathcal{I} \backslash \{1\}. \end{aligned} \tag{31}$$

From (26), (29), and Proposition 5, if $c_i^{s*}(\boldsymbol{N}^s) < c_{i-1}^{s*}(\boldsymbol{N}^s)$ for all $i \in \mathcal{I} \backslash \{1\}$,[7] we have

$$v_i^{d*}(\boldsymbol{N}^d) - v_i^{s*}(\boldsymbol{N}^s) < v_{i-1}^{d*}(\boldsymbol{N}^d) - v_{i-1}^{s*}(\boldsymbol{N}^s) \qquad \forall i \in \mathcal{I} \backslash \{1\}. \tag{32}$$

Note that from equilibrium condition (13e), every commuter i resides downtown if $v_i^{d*}(\boldsymbol{N}^{d*}) - v_i^{s*}(\boldsymbol{N}^{s*}) > 0$ and every commuter i resides in the suburb if $v_i^{d*}(\boldsymbol{N}^{d*}) - v_i^{s*}(\boldsymbol{N}^{s*}) < 0$. Therefore, (32) shows that commuters with a high (low) value of travel time reside downtown (in the suburb).

Furthermore, (28) gives us the utility difference $v_i^{d*}(\boldsymbol{N}^d) - v_i^{s*}(\boldsymbol{N}^s)$ as

$$v_i^{d*}(\boldsymbol{N}^d) - v_i^{s*}(\boldsymbol{N}^s) = c_i^{s*}(\boldsymbol{N}^s) - \kappa \ln\left[\frac{r(d)}{r^{\mathrm{A}}}\right] + \alpha_i \tau \left(X_i^s - X_i^d\right) + \sum_{k=i}^{I} \alpha_k \tau \left\{\left(X_{k+1}^s - X_k^s\right) - \left(X_{k+1}^d - X_k^d\right)\right\}. \tag{33}$$

4 Optimal congestion toll

Studies on the standard bottleneck model show that queuing time is a pure deadweight loss. Hence, there is no queue at the social optimum. In our model, the social optimum can be achieved by imposing an optimal time-varying congestion toll that eliminates queuing congestion. This

[7] Note that, as shown in Section 5, $c_{i-1}^{s*}(\boldsymbol{N}^s)$ is greater than $c_i^{s*}(\boldsymbol{N}^s)$ in many cases since $\alpha_{i-1} > \alpha_i$.

section considers the introduction of an optimal congestion toll $p(t)$. That is, the commuting cost $c_i^t(x,t)$ of a commuter i is given by

$$c_i^t(x,t) = \begin{cases} c_i^{dt}(t) + \alpha_i \tau x & \text{if} \quad x \in \mathcal{X}^d, \\ c_i^{st}(t) + \alpha_i \tau x & \text{if} \quad x \in \mathcal{X}^s, \end{cases} \tag{34a}$$

$$c_i^{dt}(t) = s_i(t - t^*), \tag{34b}$$

$$c_i^{st}(t) = p(t) + s_i(t - t^*). \tag{34c}$$

We then characterize equilibrium under the optimal congestion toll $p(t)$ and demonstrate that this pricing policy alters spatial distribution of commuters. Our result differs fundamentally from that obtained by Arnott (1998), who considers homogeneous commuters.

4.1 Short-run equilibrium

Congestion toll $p(t)$ eliminates queuing congestion.[8] Note that since we consider heterogeneous commuters, congestion toll $p(t)$ does not equal travel time cost $\alpha_i q(t)$ at the no-toll equilibrium and is set so that travel demand $\sum_{i\in\mathcal{I}} n_i^s(t)$ at the bottleneck equals the supply (i.e., capacity) μ. Therefore, short-run equilibrium conditions for suburban commuters are expressed as

$$\begin{cases} c_i^{st}(t) = c_i^{st*} & \text{if} \quad n_i^s(t) > 0 \\ c_i^{st}(t) \geq c_i^{st*} & \text{if} \quad n_i^s(t) = 0 \end{cases} \quad \forall i \in \mathcal{I},\ \forall t \in \mathbb{R}, \tag{35a}$$

$$\begin{cases} \sum_{i\in\mathcal{I}} n_i^s(t) = \mu & \text{if} \quad p(t) > 0 \\ \sum_{i\in\mathcal{I}} n_i^s(t) \leq \mu & \text{if} \quad p(t) = 0 \end{cases} \quad \forall t \in \mathbb{R}, \tag{35b}$$

$$\int n_i^s(t)\,\mathrm{d}t = N_i^s \quad \forall i \in \mathcal{I}, \tag{35c}$$

where c_i^{st*} denotes the short-run equilibrium commuting cost of suburban commuters i under the congestion toll.

Condition (35a) is the no-arbitrage condition for suburban commuters' trip timing choices. Condition (35b) denotes the bottleneck's capacity constraints, which assure that queuing congestion is eliminated at the equilibrium. Condition (35c) provides the flow conservation for commuting demand. From these conditions, we have $n_i^s(t)$, $p(t)$, and c_i^{st*} at the short-run equilibrium as functions of the number $\boldsymbol{N}^s$ of suburban commuters $i \in \mathcal{I}$.

As in the case without the congestion toll, there is an optimization problem equivalent to (35):

Proposition 6. *The short-run equilibrium number $[n_i^{st*}(t)]$ of suburban commuters traveling at time t under the optimal congestion toll coincides with the solution of the following linear programing problem:*

$$\min_{[n_i^s(t)]} \sum_{i\in\mathcal{I}} \int s_i(t-t^*)\, n_i^s(t)\,\mathrm{d}t \tag{36a}$$

$$\text{s.t.} \quad \sum_{i\in\mathcal{I}} n_i^s(t) \leq \mu \quad \forall t \in \mathbb{R}, \quad \int n_i^s(t)\,\mathrm{d}t = N_i^s \quad \forall i \in \mathcal{I}, \quad n_i^s(t) \geq 0 \quad \forall i \in \mathcal{I},\ \forall t \in \mathbb{R}. \tag{36b}$$

[8]The tradable network permit scheme proposed by Akamatsu (2007) and Wada and Akamatsu (2013) has the same effect as the optimal congestion toll.

This proposition suggests that the total schedule delay cost is minimized at the short-run equilibrium under the congestion toll. Note that the total schedule delay cost equals total commuting cost minus total toll revenue. Hence, Proposition 6 indicates that, in the short run, the optimal congestion toll minimizes the social cost of commuting. Furthermore, Propositions 1 and 6 show that the equilibrium commuting cost $c_i^{st*}(\boldsymbol{N}^s)$ under the congestion toll generally differs from the no-toll equilibrium commuting cost $c_i^{s*}(\boldsymbol{N}^s)$.

From equilibrium condition (35a), we obtain

$$c_i^{st*}(t_i) + c_j^{st*}(t_j) \leq c_i^{st*}(t_j) + c_j^{st*}(t_i) \quad \forall t_i \in \operatorname{supp}(n_i^{st*}),\ \forall t_j \in \operatorname{supp}(n_j^{st*}),\ \forall i, j \in \mathcal{I}. \tag{37}$$

Substituting (34c) into (37), we have the following proposition.

Proposition 7. *Consider the short-run equilibrium under the optimal congestion toll. Then, for any* $t_i \in \operatorname{supp}(n_i^{st*})$, $t_j \in \operatorname{supp}(n_j^{st*})$, *and* $i, j \in \mathcal{I}$, *the following conditions hold:*

$$\left(\beta_i - \beta_j\right)\left(t_i - t_j\right) \geq 0 \qquad \textit{if} \quad \max\{t_i, t_j\} \leq t^*, \tag{38a}$$

$$\left(\gamma_i - \gamma_j\right)\left(t_i - t_j\right) \leq 0 \qquad \textit{if} \quad \min\{t_i, t_j\} \geq t^*. \tag{38b}$$

This proposition indicates that early-arriving commuters travel in order of increasing β_i and late-arriving commuters travel in order of decreasing γ_i. Furthermore, the value of travel time (i.e., α_i) does not affect the short-run equilibrium distribution of trip timing under the congestion toll.

Downtown commuters travel at $t = t^*$ as they need not traverse the bottleneck. That is, the commuting cost of downtown commuters i at the short-run equilibrium under the congestion toll is given by $\alpha_i \tau x$.

4.2 Long-run equilibrium

We characterize long-run equilibrium spatial distribution of commuters by using the short-run equilibrium commuting cost. The difference between cases with and without tolling appears only in the indirect utility $v_i^s(x)$ of suburban commuters. Specifically, under the congestion toll, the indirect utility $v_i^s(x)$ of suburban commuters i is expressed as

$$v_i^s(x) = \begin{cases} y_i - c_i^{st*}(\boldsymbol{N}^s) - \alpha_i \tau x - \kappa \ln[\kappa N(x)] + \epsilon & \text{if} \quad \kappa N(x) \geq r^{\mathrm{A}}, \\ y_i - c_i^{st*}(\boldsymbol{N}^s) - \alpha_i \tau x - \kappa \ln[r^{\mathrm{A}}] + \epsilon & \text{if} \quad \kappa N(x) \leq r^{\mathrm{A}}, \end{cases} \tag{39}$$

where $c_i^{st*}(\boldsymbol{N}^s)$ is the short-run equilibrium commuting cost. Following the same procedure as in Section 3.2 reveals that the spatial distribution of suburban commuters and that of downtown commuters at the long-run equilibrium under the optimal congestion toll have the same properties as those without tolling (i.e., Proposition 4).

Proposition 8. *Consider the long-run equilibrium under the optimal congestion toll. Then, the spatial distribution of suburban commuters and that of downtown commuters have the following properties:*

(a) The long-run equilibrium number $[N_i^{st*}(x)]$ $(x \in \mathcal{X}^s)$ *of suburban commuters at x coincides with the*

solution of the following optimization problem:

$$\max_{[N_i(x)]} \sum_{k\in\mathcal{I}} \int_d^{\infty} v_k^s(x) N_k(x) \mathrm{d}x \qquad s.t. \quad \int_d^{\infty} N_i(x)\mathrm{d}x = N_i^s \quad \forall i \in \mathcal{I}. \tag{40}$$

(b) *The long-run equilibrium number* $[N_i^{dt*}(x)]$ $(x \in \mathcal{X}^d)$ *of downtown commuters at x coincides with the solution of the following optimization problem:*

$$\max_{[N_i(x)]} \sum_{k\in\mathcal{I}} \int_0^{d} v_k^d(x) N_k(x) \mathrm{d}x \qquad s.t. \quad \int_0^{d} N_i(x)\mathrm{d}x = N_i^d \quad \forall i \in \mathcal{I}. \tag{41}$$

(c) *For any* $x_i \in \mathrm{supp}\,(N_i^{st*})$, $x_j \in \mathrm{supp}\,(N_j^{st*})$, *and* $i, j \in \mathcal{I}$, *the following condition holds at the long-run equilibrium:*

$$\left(\alpha_i - \alpha_j\right)\left(x_i - x_j\right) \geq 0. \tag{42}$$

This condition also holds for any $x_i \in \mathrm{supp}\,(N_i^{dt*})$, $x_j \in \mathrm{supp}\,(N_j^{dt*})$, *and* $i, j \in \mathcal{I}$.

We denote the utilities of commuters i residing in the suburb and downtown under the congestion toll by $v_i^{st*}(\boldsymbol{N}^s)$ and $v_i^{dt*}(\boldsymbol{N}^d)$, respectively, which are derived from (13a), (13b), (13c), and (13d) with the use of (39). Then, under the assumption in (26), $v_i^{st*}(\boldsymbol{N}^s)$ and $v_i^{dt*}(\boldsymbol{N}^d)$ are obtained in the same manner as in (28):

$$v_i^{st*}(\boldsymbol{N}^s) = y_i - c_i^{st*}(\boldsymbol{N}^{st}) - \kappa \ln[r^{\mathrm{A}}] - \alpha_i \tau X_i^{st} - \sum_{k=i}^{I} \alpha_k \tau (X_{k+1}^{st} - X_k^{st}) + \epsilon, \tag{43a}$$

$$v_i^{dt*}(\boldsymbol{N}^d) = y_i - \kappa \ln[r(d)] - \alpha_i \tau X_i^{dt} - \sum_{k=i}^{I} \alpha_k \tau (X_{k+1}^{dt} - X_k^{dt}) + \epsilon, \tag{43b}$$

where X_k^{st} and X_k^{dt} are, respectively, the residential locations for suburban and downtown commuters i closest to the CBD, which are given by (29). Thus, $v_i^{dt*}(\boldsymbol{N}^d) - v_i^{ds*}(\boldsymbol{N}^s)$ is represented as

$$v_i^{dt*}(\boldsymbol{N}^d) - v_i^{st*}(\boldsymbol{N}^s) = c_i^{st*}(\boldsymbol{N}^s) - \kappa \ln\left[\frac{r(d)}{r^{\mathrm{A}}}\right] + \alpha_i \tau \left(X_i^{st} - X_i^{dt}\right) + \sum_{k=i}^{I} \alpha_k \tau \left\{\left(X_{k+1}^{st} - X_k^{st}\right) - \left(X_{k+1}^{dt} - X_k^{dt}\right)\right\}. \tag{44}$$

It follows from this and (33) that

$$\left(v_i^{dt*}(\boldsymbol{N}^d) - v_i^{st*}(\boldsymbol{N}^s)\right) - \left(v_i^{d*}(\boldsymbol{N}^d) - v_i^{s*}(\boldsymbol{N}^s)\right) = c_i^{st*}(\boldsymbol{N}^s) - c_i^{s*}(\boldsymbol{N}^s) \qquad \forall i \in \mathcal{I}. \tag{45}$$

Furthermore, following the same procedure as for Proposition 5, we obtain $(v_i^{dt*}(\boldsymbol{N}^d) - v_i^{st*}(\boldsymbol{N}^s)) - (v_{i-1}^{dt*}(\boldsymbol{N}^d) - v_{i-1}^{st*}(\boldsymbol{N}^s))$ as follows.

Proposition 9. *Under the optimal congestion toll, the difference* $v_i^{dt*}(\boldsymbol{N}^d) - v_i^{st*}(\boldsymbol{N}^s)$ *between utilities from residing in the suburb and downtown has the following properties:*

(a) For all $i \in \mathcal{I}\backslash\{1\}$,

$$\left(v_i^{dt*}(\boldsymbol{N}^d) - v_i^{st*}(\boldsymbol{N}^s)\right) - \left(v_{i-1}^{dt*}(\boldsymbol{N}^d) - v_{i-1}^{st*}(\boldsymbol{N}^s)\right) = \left(v_i^{d*}(\boldsymbol{N}^d) - v_i^{s*}(\boldsymbol{N}^s)\right) - \left(v_{i-1}^{d*}(\boldsymbol{N}^d) - v_{i-1}^{s*}(\boldsymbol{N}^s)\right), \tag{46}$$

if and only if there exists δ such that $c_i^{st*}(\boldsymbol{N}^s) = c_i^{s*}(\boldsymbol{N}^s) + \delta$ for all $i \in \mathcal{I}$.

(b) For all $i \in \mathcal{I}\backslash\{1\}$,

$$\left(v_i^{dt*}(\boldsymbol{N}^d) - v_i^{st*}(\boldsymbol{N}^s)\right) - \left(v_{i-1}^{dt*}(\boldsymbol{N}^d) - v_{i-1}^{st*}(\boldsymbol{N}^s)\right) = c_i^{st*}(\boldsymbol{N}^s) - c_{i-1}^{st*}(\boldsymbol{N}^s) - (\alpha_{i-1} - \alpha_i)\,\tau\left(X_i^{st} - X_i^{dt}\right). \tag{47}$$

Note that, in general, there is no δ such that $c_i^{st*} = c_i^{s*} + \delta\ \forall i \in \mathcal{I}$ when we consider commuter heterogeneity in the value of travel time. Unlike Arnott (1998), therefore, Proposition 9 (a) implies that *imposing the optimal congestion toll does change spatial distribution of commuters in our model.*

Note also that the results presented thus far are obtained under the assumption that toll revenues are not redistributed. Since the optimal congestion toll minimizes short-run social cost of commuting, commuting costs of all commuters can be reduced by appropriately redistributing toll revenues. That is, if policymakers appropriately redistribute toll revenues, the following condition is satisfied for all $i \in \mathcal{I}$:

$$v_i^{dt*}(\boldsymbol{N}^d) - v_i^{st*}(\boldsymbol{N}^s) - \rho_i(\boldsymbol{N}^s) < v_i^{d*}(\boldsymbol{N}^d) - v_i^{s*}(\boldsymbol{N}^s), \tag{48}$$

where $\rho_i(\boldsymbol{N}^s)$ denotes the toll-revenue redistribution for a commuter i. This indicates that appropriate redistribution attracts commuters to the suburb and causes urban sprawl.

Proposition 9 (b) shows that if $c_i^{st*}(\boldsymbol{N}^s) < c_{i-1}^{st*}(\boldsymbol{N}^s)$ for all $i \in \mathcal{I}$, the following condition is satisfied:

$$v_i^{dt*}(\boldsymbol{N}^d) - v_i^{st*}(\boldsymbol{N}^s) < v_{i-1}^{dt*}(\boldsymbol{N}^d) - v_{i-1}^{st*}(\boldsymbol{N}^s) \qquad \forall i \in \mathcal{I}\backslash\{1\}. \tag{49}$$

That is, at the long-run equilibrium with tolling, commuters with a high value of travel time reside downtown, while those with a low value of travel time reside in the suburb.

5 A simple example

In this section, we analyze our model in a simple setting to show concretely the properties of equilibrium and effects of the congestion toll. Specifically, we assume that the following conditions hold.

$$\alpha_{i-1} > \alpha_i, \quad \beta_{i-1} > \beta_i, \quad \gamma_{i-1} > \gamma_i, \quad \frac{\beta_{i-1}}{\alpha_{i-1}} < \frac{\beta_i}{\alpha_i}, \quad \frac{\gamma_{i-1}}{\alpha_{i-1}} < \frac{\gamma_i}{\alpha_i}, \quad y_{i-1} > y_i \qquad \forall i \in \mathcal{I}\backslash\{1\}. \tag{50a}$$

For expositional clarity, we further introduce the assumption common to literature that employs a bottleneck model with commuter heterogeneity (Arnott et al., 1992, 1994; van den Berg and Verhoef, 2011):

$$\frac{\gamma_i}{\beta_i} = \eta \quad \forall i \in \mathcal{I}. \tag{50b}$$

These assumptions mean that commuters with smaller i are rich and those with larger i are poor.

5.1 Theoretical analysis

5.1.1 Short-run equilibrium

We first examine the properties of the short-run equilibrium. It follows from assumption (50) and Proposition 2 that early-arriving suburban commuters with smaller i travel earlier at the no-toll short-run equilibrium. Thus, the difference between time-based commuting costs of suburban commuters i and $i-1$ is as follows:

$$\frac{c_i^{s*}(\boldsymbol{N}^s)}{\alpha_i} - \frac{c_{i-1}^{s*}(\boldsymbol{N}^s)}{\alpha_{i-1}} = \frac{\eta}{1+\eta}\left(\frac{\beta_i}{\alpha_i} - \frac{\beta_{i-1}}{\alpha_{i-1}}\right)\frac{\sum_{k=i}^{I} N_k^s}{\mu} > 0. \tag{51}$$

From assumption (50) and Proposition 7, early-arriving suburban commuters with larger i travel earlier at the short-run equilibrium with tolling. Thus, $c_i^{st*}(\boldsymbol{N}^s) - c_{i-1}^{st*}(\boldsymbol{N}^s)$ is given by

$$c_i^{st*}(\boldsymbol{N}^s) - c_{i-1}^{st*}(\boldsymbol{N}^s) = \frac{\eta}{1+\eta}(\beta_i - \beta_{i-1})\frac{\sum_{k=1}^{i-1} N_k^s}{\mu} < 0. \tag{52}$$

(51) and (52) give us $c_i^{s*}(\boldsymbol{N}^s)$ and $c_i^{st*}(\boldsymbol{N}^s)$ as follows:

$$c_i^{s*}(\boldsymbol{N}^s) = \frac{\eta}{1+\eta}\left\{\beta_i\frac{\sum_{k=i+1}^{I} N_k^s}{\mu} + \alpha_i\sum_{k=1}^{i}\frac{\beta_k}{\alpha_k}\frac{N_k^s}{\mu}\right\}, \tag{53a}$$

$$c_i^{st*}(\boldsymbol{N}^s) = \frac{\eta}{1+\eta}\left\{\beta_i\frac{\sum_{k=1}^{i-1} N_k^s}{\mu} + \sum_{k=i}^{I}\beta_k\frac{N_k^s}{\mu}\right\}. \tag{53b}$$

By using (53a), we obtain the difference between the no-toll short-run equilibrium commuting costs of commuters i and $i-1$.

$$c_i^{s*}(\boldsymbol{N}^s) - c_{i-1}^{s*}(\boldsymbol{N}^s) = \frac{\eta}{1+\eta}\left\{(\beta_i - \beta_{i-1})\frac{\sum_i^I N_k^s}{\mu} + (\alpha_i - \alpha_{i-1})\sum_{k=1}^{i-1}\frac{\beta_k}{\alpha_k}\frac{N_k^s}{\mu}\right\} < 0. \tag{54}$$

Results obtained above show that for all $i \in \mathcal{I}\backslash\{1\}$,

$$c_i^{s*}(\boldsymbol{N}^s) - c_{i-1}^{s*}(\boldsymbol{N}^s) < 0, \tag{55a}$$

$$c_i^{st*}(\boldsymbol{N}^s) - c_{i-1}^{st*}(\boldsymbol{N}^s) < 0. \tag{55b}$$

That is, at the short-run equilibrium with and without congestion tolling, commuting costs of rich commuters exceed those of poor commuters.

$c_i^{st*}(\boldsymbol{N}^s) - c_i^{s*}(\boldsymbol{N}^s)$ is obtained from (53) as follows:

$$c_i^{st*}(\boldsymbol{N}^s) - c_i^{s*}(\boldsymbol{N}^s) = \frac{\eta}{1+\eta}\left\{\alpha_i\sum_{k=1}^{i-1}\left(\frac{\beta_i}{\alpha_i} - \frac{\beta_k}{\alpha_k}\right)\frac{N_k^s}{\mu} - \sum_{k=i}^{I}(\beta_i - \beta_k)\frac{N_k^s}{\mu}\right\}. \tag{56}$$

From assumption (50), the first term on the right hand side of (56) is positive and the second is negative. Thus, we have

$$c_i^{st*}(\boldsymbol{N}^s) - c_i^{s*}(\boldsymbol{N}^s) > c_{i-1}^{st*}(\boldsymbol{N}^s) - c_{i-1}^{s*}(\boldsymbol{N}^s) \qquad \forall i \in \mathcal{I}\backslash\{1\}. \tag{57}$$

Furthermore, there exists $i^* \in \mathcal{I}$ such that $c_i^{st*}(\boldsymbol{N}^s) - c_i^{s*}(\boldsymbol{N}^s) \leq 0$ for all $i \leq i^*$ and $c_i^{st*}(\boldsymbol{N}^s) - c_i^{s*}(\boldsymbol{N}^s) > 0$ for all $i > i^*$. Therefore, introducing the congestion toll reduces the commuting cost of rich commuters $i \leq i^*$ and increases the commuting cost of poor commuters $i > i^*$.

At the no-toll long-run equilibrium $\boldsymbol{N}^{s*}$, we can show from (56) that $i^* \geq \min\{\mathrm{supp}\,(\boldsymbol{N}^{s*})\}$. This conclusion implies that the congestion toll reduces the short-run equilibrium commuting cost of commuters $i \in \mathcal{I}^r \equiv \{i \in \mathcal{I} \mid i \leq \min\{\mathrm{supp}\,(\boldsymbol{N}^{s*})\}$ who are downtown commuters or are suburban commuters having the highest value of travel time at the no-toll long-run equilibrium. In addition, if there exist multiple types of suburban commuters at the no-toll long-run equilibrium, the commuting costs $c_{i^s}^{st*}(\boldsymbol{N}^{s*})$ and $c_{i^s}^{s*}(\boldsymbol{N}^{s*})$ of suburban commuters $i^s = \min\{\mathrm{supp}\,(\boldsymbol{N}^{s*})\}$ satisfy

$$c_{i^s}^{st*}(\boldsymbol{N}^{s*}) < c_{i^s}^{s*}(\boldsymbol{N}^{s*}). \tag{58}$$

That is, congestion tolling reduces their commuting cost. This and (57) show that

$$c_i^{st*}(\boldsymbol{N}^{s*}) < c_i^{s*}(\boldsymbol{N}^{s*}) \qquad \forall i \in \mathcal{I}^r \tag{59}$$

if there exist multiple types of suburban commuters at the no-toll long-run equilibrium.

Results obtained above are summarized as follows:

Proposition 10. *Suppose conditions in assumption* (50) *hold. Then, the short-run equilibrium commuting cost has the following properties:*

(a) For any $\boldsymbol{N}^s$ and $i \in \mathcal{I}\backslash\{1\}$, $c_i^{s*}(\boldsymbol{N}^s) < c_{i-1}^{s*}(\boldsymbol{N}^s)$ and $c_i^{st*}(\boldsymbol{N}^s) < c_{i-1}^{st*}(\boldsymbol{N}^s)$.

(b) There exists $i^* \in \mathcal{I}$ such that $c_i^{st*}(\boldsymbol{N}^s) \leq c_i^{s*}(\boldsymbol{N}^s)$ for all $i \leq i^*$.

(c) Consider the no-toll long-run equilibrium $\boldsymbol{N}^{s*}$. Then, $c_i^{st*}(\boldsymbol{N}^{s*}) \leq c_i^{s*}(\boldsymbol{N}^{s*})$ for all $i \in \mathcal{I}^r$. A strict inequality holds if there exist multiple types of suburban commuters.

5.1.2 Long-run equilibrium

We next show properties of spatial distribution of commuters at the long-run equilibrium. Using Propositions 5, 9 (b), and 10 (a), we have

$$\begin{cases} v_i^{d*}(\boldsymbol{N}^d) - v_i^{s*}(\boldsymbol{N}^s) < v_{i-1}^{d*}(\boldsymbol{N}^d) - v_{i-1}^{s*}(\boldsymbol{N}^s) \\ v_i^{dt*}(\boldsymbol{N}^d) - v_i^{st*}(\boldsymbol{N}^s) < v_{i-1}^{dt*}(\boldsymbol{N}^d) - v_{i-1}^{st*}(\boldsymbol{N}^s) \end{cases} \qquad \forall i \in \mathcal{I}\backslash\{1\}. \tag{60}$$

This indicates that commuters with a high value of travel time reside downtown, while those with a low value of travel time reside in the suburb at the long-run equilibrium with and without congestion tolling. Together with Propositions 4 and 8 (c), this shows that commuters with a higher value of travel time reside closer to the CBD. This result is consistent with empirical observation in cities with heavy traffic congestion (McCann, 2013).

The effects of congestion tolling on spatial distribution of commuters can be examined using the following relation at the no-toll long-run equilibrium $\boldsymbol{N}^{s*}$ and $\boldsymbol{N}^{d*}$, which is obtained from

(33) and (44):

$$v_i^{dt*}(\boldsymbol{N}^{d*}) - v_i^{st*}(\boldsymbol{N}^{s*}) = c_i^{st*}(\boldsymbol{N}^{s*}) - c_i^{s*}(\boldsymbol{N}^{s*}) + v_i^{d*}(\boldsymbol{N}^{d*}) - v_i^{s*}(\boldsymbol{N}^{s*}) \qquad \forall i \in \mathcal{I}. \tag{61}$$

From this and Proposition 10 (c), we have

$$v_i^{dt*}(\boldsymbol{N}^{d*}) - v_i^{st*}(\boldsymbol{N}^{s*}) \leq v_i^{d*}(\boldsymbol{N}^{d*}) - v_i^{s*}(\boldsymbol{N}^{s*}) \qquad \forall i \in \mathcal{I}^r, \tag{62}$$

We can verify that for any $i \in \operatorname{supp}(\boldsymbol{N}^{s*})$, $v_i^{d*}(\boldsymbol{N}^{d*}) \leq v_i^{s*}(\boldsymbol{N}^{s*})$. Therefore, we obtain from (60) and (62) that

$$v_i^{dt*}(\boldsymbol{N}^{d*}) \leq v_i^{st*}(\boldsymbol{N}^{s*}) \qquad \forall i \in \operatorname{supp}(\boldsymbol{N}^{s*}). \tag{63}$$

This condition shows that imposing a congestion toll must not create incentives for suburban commuters to relocate downtown. Furthermore, (61) indicates that if commuters $i^s = \max\{\mathcal{I}^r\}$ reside both in suburb and downtown (i.e., $v_{i^s}^{d*}(\boldsymbol{N}^{d*}) = v_{i^s}^{s*}(\boldsymbol{N}^{s*})$) and there exist multiple types of suburban commuters at the no-toll long-run equilibrium (i.e., $c_{i^s}^{st*}(\boldsymbol{N}^{s*}) < c_{i^s}^{s*}(\boldsymbol{N}^{s*})$), at least some downtown commuters i^s relocate to suburb.

By letting $\boldsymbol{N}^{st*} = [N_i^{st*}]$ be the long-run equilibrium number of suburban commuters under the optimal congestion toll, the results presented above can be represented as

$$N_i^{st*} \geq N_i^{s*} \qquad \forall i \in \mathcal{I}. \tag{64}$$

This representation implies that imposing a congestion toll generally increases the suburban population. Furthermore, since X^{B} is given by (30), the population increase in the suburb leads to urban sprawl.

This finding is opposite to the standard results of traditional location models, which consider static flow congestion (Kanemoto, 1980; Wheaton, 1998; Anas et al., 1998). It also differs from the results obtained by Arnott (1998), who considers homogeneous commuters. This demonstrates that *strategic interactions among heterogeneous commuters cause urban sprawl resulting from the imposition of the optimal congestion toll.*

We focus on changes in utility of commuters $i \in \mathcal{I}^p \equiv \mathcal{I} \backslash \mathcal{I}^r$ from introducing the congestion toll. It follows from Proposition 10 and conditions (56) and (64) that all commuters $i \in \mathcal{I}^p$ reside in the suburb at the long-run equilibrium with and without tolling. Therefore, (28) and (43) yield

$$v_i^{st*}(\boldsymbol{N}^{st*}) - v_i^{s*}(\boldsymbol{N}^{s*}) = -\left(c_i^{st*}(\boldsymbol{N}^{st*}) - c_i^{s*}(\boldsymbol{N}^{s*})\right) \qquad \forall i \in \mathcal{I}^p. \tag{65}$$

Furthermore, (53b) and (64) give

$$c_i^{st*}(\boldsymbol{N}^{st*}) \geq c_i^{st*}(\boldsymbol{N}^{s*}) \qquad \forall i \in \mathcal{I}^p. \tag{66}$$

Substituting (66) into (65), we have

$$v_i^{st*}(\boldsymbol{N}^{st*}) - v_i^{s*}(\boldsymbol{N}^{s*}) \leq -\left(c_i^{st*}(\boldsymbol{N}^{s*}) - c_i^{s*}(\boldsymbol{N}^{s*})\right) \leq 0 \qquad \forall i \in \mathcal{I}^p. \tag{67}$$

This indicates that the utility of all commuters $i \in \mathcal{I}^p$ declines after imposing the congestion toll. In addition, (64) shows that congestion tolling reduces the number of downtown commuters, thereby increasing the utility of all commuters residing downtown at the equilibrium with tolling. This finding implies that *rich commuters gain, while poor commuters lose from imposing the optimal congestion toll.* Therefore, when the optimal congestion toll is imposed, it is important to simultaneously introduce some schemes to redistribute toll revenues appropriately.

Expanding bottleneck capacity μ reduces $c_i^{s*}(\boldsymbol{N}^s)$ and $c_i^{st*}(\boldsymbol{N}^s)$, and thus, prompts commuters to relocate to the suburb, and traffic traversing the bottleneck increases. Therefore, expanding the bottleneck capacity does not necessarily reduce total queuing time. Although this result does not always arise in our model, we can show that such a situation exists, as discussed in Section 5.2.

The following proposition summarizes the results obtained.

Proposition 11. *Suppose conditions in assumption* (50) *hold. Then, the long-run equilibrium has the following properties:*

(a) Commuters with a higher value of travel time reside closer to the CBD.

(b) $N_i^{st*} \geq N_i^{s*}$ *for all* $i \in \mathcal{I}$*. A strict inequality holds at least for* $i^s = \max\{I^r\}$ *if there exist multiple types of suburban commuters at the no-toll equilibrium* $\boldsymbol{N}^{s*}$ *and* $N_{i^s}^{d*} > 0$.

(c) $v_i^{st*}(\boldsymbol{N}^{st*}) \leq v_i^{s*}(\boldsymbol{N}^{s*})$ *for all* $i \in \mathcal{I}^p$.

(d) Expanding bottleneck capacity may increase total queuing time.

5.2 Numerical analysis

Finally, we numerically analyze our model and show effects of the optimal congestion toll. In this analysis, we use the following parameter values:

$$I = 4, \quad d = 10 \text{ (km)}, \quad \tau = 2 \text{ (min/km)}, \quad [N_i] = [1000, 1500, 2000, 2500], \tag{68a}$$

$$[y_i] = [300, 200, 150, 100], \quad \kappa = 10, \quad r^{\mathrm{A}} = 200. \tag{68b}$$

The values of α_i, β_i, and η are set to be consistent with the empirical result (Small, 1982) and (50):

$$[\alpha_i] = [0.3, 0.2, 0.15, 0.1], \quad [\beta_i] = [0.15, 0.11, 0.09, 0.07], \quad \eta = 4. \tag{68c}$$

We conduct a comparative statics with respect to bottleneck capacity μ. The no-toll equilibrium number of commuters $i \in \mathcal{I}$ is described in Figure 3. This figure shows that downtown commuters relocate to the suburb in order of decreasing i with increases in the bottleneck capacity. They do so because increasing μ reduces the commuting cost $c_i^{s*}(\boldsymbol{N}^s)$ of all commuters, creating incentives for downtown commuters to relocate to the suburb. This is consistent with the results in Section 5.1.

Figure 4 illustrates the relation between bottleneck capacity μ and total queuing time Q, which is given by

$$Q = \frac{\eta}{1+\eta} \frac{1}{2\mu} \sum_{i \in \mathcal{I}} \left\{ \frac{\beta_i}{\alpha_i} N_i^s \left(2 \sum_{k=i}^{I} N_k^s - N_i^s \right) \right\}. \tag{69}$$

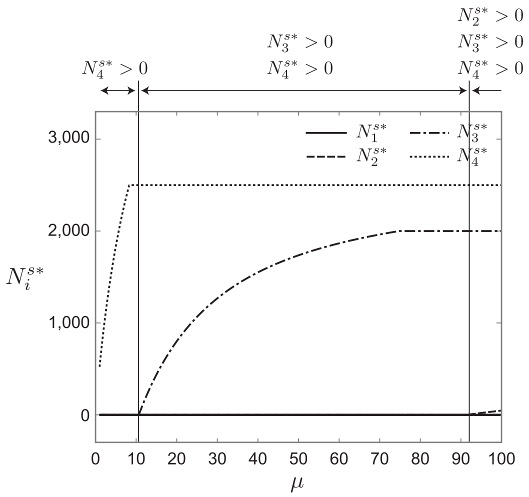

Figure 3: Number N_i^{s*} of suburban commuters i at no-toll equilibrium

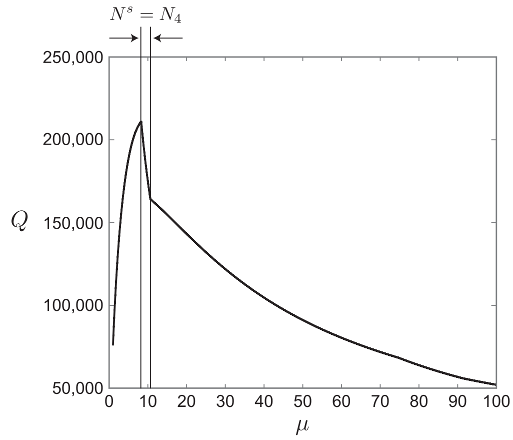

Figure 4: Total queuing time at no-toll equilibrium

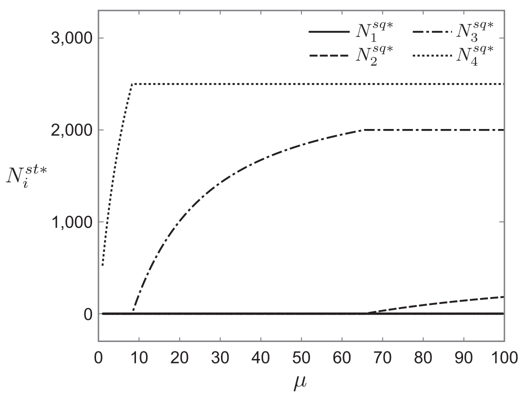

Figure 5: Number N_i^{st*} of suburban commuters i under the optimal congestion toll

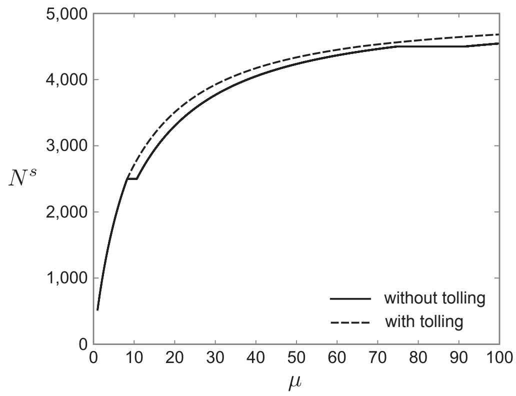

Figure 6: Total number N^s of commuters located in the suburb

From this figure, we see that expanding bottleneck capacity can increase total queuing time Q. More specifically, when capacity μ is quite small, increasing μ increases the population of suburban commuters and exacerbates queuing congestion. As μ keeps rising, total queuing time Q is greatly reduced since the total number $N^s = \sum_{i \in \mathcal{I}} N_i^s$ of suburban commuters remains unchanged. Further increases in μ prompt suburban population increases, but gradually improve traffic congestion.

The effects of the optimal congestion toll are shown in Figures 5–9. Figure 5 represents the long-run equilibrium number N_i^{st*} of suburban commuters i under the optimal congestion toll. Although this result is qualitatively the same as that at the no-toll equilibrium (Figure 3), congestion tolling changes the total number N^s of suburban commuters, as illustrated in Figure 6. Note that when μ is small, imposition of the congestion toll does not alter N^s. This occurs because for small μ, only commuters 4 reside in the suburb (i.e., commuters traversing the bottleneck are homogeneous). Thus, congestion tolling does not affect commuting costs of suburban commuters, as shown in Arnott (1998). Furthermore, a suburban population increase due to congestion tolling leads to expansion of the urban boundary X^{B}, as illustrated in Figure 7. That is, imposing the optimal congestion toll causes urban sprawl.

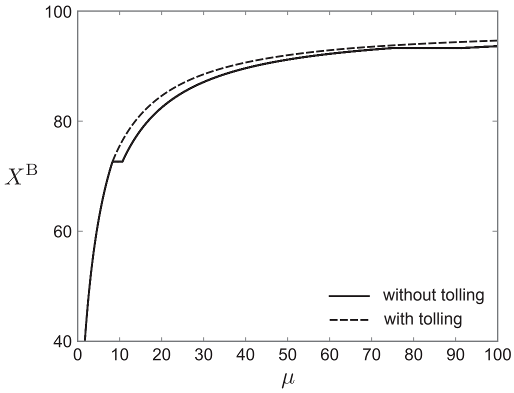

Figure 7: Urban boundary X^{B}

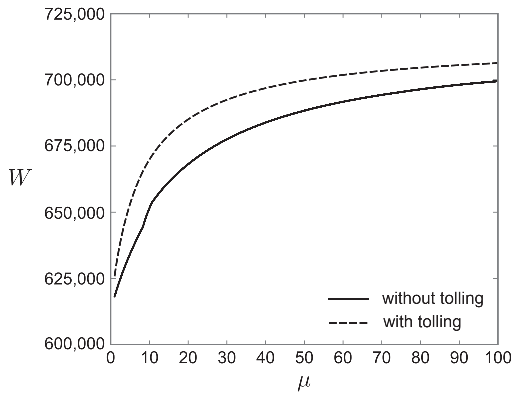

Figure 8: Social welfare W

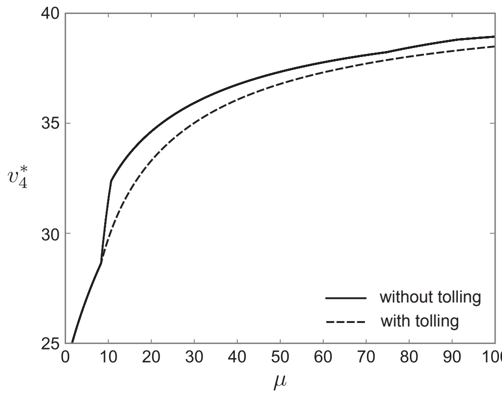

Figure 9: Utility v_4^* of commuters 4

We investigate the effects of congestion tolling on social welfare W defined as the sum of commuters' total utility and toll revenues P:

$$W = \sum_{i \in \mathcal{I}} \int v_i(x) N_i(x) \mathrm{d}x + P, \tag{70}$$

$$P = \frac{\eta}{1+\eta} \frac{1}{2\mu} \sum_{i \in \mathcal{I}} \left\{ \beta_i N_i^s \left(2 \sum_{k=1}^{i} N_k^s - N_i^s \right) \right\}. \tag{71}$$

Figure 8 shows the relations between W and μ with and without congestion tolling. This figure illustrates that expanding the bottleneck capacity increases social welfare. Figure 9 indicates that congestion tolling reduces utility of commuters 4 (i.e., commuters with the lowest value of time). That is, poor commuters lose from congestion tolling. These results are consistent with Proposition 11.

6 Conclusions

This study has developed a model in which heterogeneous commuters choose their departure time from home and residential locations in a monocentric city with a single bottleneck. By using properties of the complementarity problem, we systematically examined spatial distribution of commuters and effects of time-varying congestion tolling. Results indicate that commuters sort themselves temporally and spatially on the basis of their value of time and that imposing an optimal congestion toll shifts population to the suburb and causes urban sprawl. This finding differs fundamentally from results obtained by Arnott (1998), who considers homogeneous commuters. It suggests that strategic interactions among heterogeneous commuters can change the effects of congestion tolling significantly. Furthermore, we clarified that imposing a congestion toll without redistributing toll revenues leads to an undesirable distribution of benefits among commuters. That is, rich commuters gain, while poor commuters lose from tolling. These results suggest that considering commuter heterogeneity is important when we examine the effectiveness of transportation policies intended to alleviate peak-period congestion.

This study considered a city with a single bottleneck. We need to examine the robustness of our result by analyzing a model with multiple bottlenecks.[9] In addition, it would be valuable for future research to investigate effects of policies other than congestion tolling, such as step tolls (Arnott et al., 1990a; Laih, 1994, 2004; Lindsey et al., 2012) and transportation demand management measures for alleviating traffic congestion (Mun and Yonekawa, 2006; Takayama, 2015).

A Proof of Lemma 1

We can show that for any $x^a, x^b \in \mathrm{supp}\,(N^*)$, there is no $x^c \in (x^a, x^b)$ such that $N^*(x^c) = 0$, because the indirect utilities of suburban and downtown commuters i are given by (12). Thus, we obtain (21).

Differentiating the indirect utilities $v_i^s(x)$ and $v_i^d(x)$ with respect to location x, we have

$$\frac{\mathrm{d}v_i^s(x)}{\mathrm{d}x} = \begin{cases} -\alpha_i\tau - \dfrac{1}{N^*(x)}\dfrac{\mathrm{d}N^*(x)}{\mathrm{d}x} & \text{if} \quad \kappa N^*(x) \geq r^{\mathrm{A}}, \\ -\alpha_i\tau & \text{if} \quad \kappa N^*(x) \leq r^{\mathrm{A}}, \end{cases} \tag{72a}$$

$$\frac{\mathrm{d}v_i^d(x)}{\mathrm{d}x} = \begin{cases} -\alpha_i\tau - \dfrac{1}{N^*(x)}\dfrac{\mathrm{d}N^*(x)}{\mathrm{d}x} & \text{if} \quad \kappa N^*(x) \geq r^{\mathrm{A}}, \\ -\alpha_i\tau & \text{if} \quad \kappa N^*(x) \leq r^{\mathrm{A}}. \end{cases} \tag{72b}$$

Therefore, the long-run equilibrium number $N^*(x)$ of commuters residing at x satisfies

$$\kappa N^*(x) \geq r^{\mathrm{A}} \quad \forall x \in \mathrm{supp}\,(N^*). \tag{73}$$

Furthermore, it follows from the long-run equilibrium conditions (13a) and (13c) that $N^*(x)$ also satisfies

$$\begin{cases} \kappa N^*(x) > r^{\mathrm{A}} \quad \forall x \in \mathrm{supp}\,(N^*)\backslash\{X^{\mathrm{B}}\}, \\ \kappa N^*(X^{\mathrm{B}}) = r^{\mathrm{A}}. \end{cases} \tag{74}$$

This completes the proof.

[9] Kuwahara (1990) and Akamatsu et al. (2015) have shown the properties of a bottleneck model with multiple bottlenecks.

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