Nash equilibrium uniqueness in nice games with isotone best replies

Ceparano, Maria Carmela and Quartieri, Federico

Dipartimento di Scienze Economiche e Statistiche, Università degli Studi di Napoli Federico II, Naples, Italy

5 October 2015

Nash equilibrium uniqueness in nice games with isotone best replies

Ceparano, Maria Carmela
email: mariacarmela.ceparano@unina.it ; milena.ceparano@gmail.com

Quartieri, Federico
email: federico.quartieri@unina.it ; quartieri.f@alice.it

**Dipartimento di Scienze Economiche e Statistiche,
Università degli Studi di Napoli Federico II, Naples, Italy**

Abstract We prove the existence of a unique pure-strategy Nash equilibrium in nice games with isotone chain-concave best replies and compact strategy sets. We establish a preliminary fixpoint uniqueness argument, thus showing sufficient assumptions on the best replies of a nice game that guarantee the existence of exactly one Nash equilibrium. Then, by means of a comparative statics analysis, we examine the necessity and sufficiency of the conditions on marginal utility functions for such assumptions to be satisfied; in particular, we find necessary and sufficient conditions for the isotonicity and chain-concavity of best replies. We extend the results on Nash equilibrium uniqueness to nice games with upper unbounded strategy sets and we present "dual" results for games with isotone chain-convex best replies. A final application to Bayesian games is exhibited.

Keywords Nash equilibrium uniqueness; Chain-concave best reply; Nice game; Comparative statics; Strategic complementarity.
JEL classification: C61 · C72

1 Introduction

Nash equilibrium uniqueness has been a point of interest since the inception of non-cooperative game theory. In his Ph.D. dissertation (see [25]), John Forbes Nash posed the following rhetorical question about a possible interpretation of the solution concept that took name after him:

> 'What would be a "rational" prediction of the behavior to be expected of rational playing the game in question?'

He answered that (Nash) equilibrium uniqueness, together with other conditions of epistemic nature, are sufficient to expect that rational agents end up behaving as prescribed by the solution concept he proposed for noncooperative situations of strategic interaction:

> 'By using the principles that a rational prediction should be unique, that the players should be able to deduce and make use of it, and

> that such knowledge on the part of each player of what to expect the others to do should not lead him to act out of conformity with the prediction, one is led to the concept of a solution defined before.'

His reasoning is not a conclusive argument by which one should expect that the Nash' solution concept can be considered the reasonable prediction of players' behavior only in a non-cooperative game with exactly one Nash equilibrium. Indeed, John Nash himself maintained later on in his thesis that in some classes of noncooperative games some subsolutions can shrink the set of reasonable predictions to a singleton; besides, he offered also a mass-action interpretation of his solution concept for which solution multiplicity is not a problem. Nonetheless, the quotation well enlightens about the historical importance of the issue of Nash equilibrium uniqueness in (non-cooperative) game-theoretic thought. The present paper is devoted to analyze such issue.

On Nash equilibrium uniqueness in the class of games under examination

Many games are known to possess a multiplicity of equilibria and one cannot hope to derive general conditions for the existence of a unique Nash equilibrium. Thus, in this work we shall restrict attention to a particular class of games: the class of nice games[1] with isotone best reply functions.

The "isotonicity" of best reply correspondences, in some loose sense, is a very general expression of the strategic complementarity among optimal choices of agents. Games with "isotone" best reply correspondences have received special attention in the economic and game-theoretic literature because of the richness and easy intelligibility of their equilibrium structure and properties. Such a literature, started from [32] and [33], had been popularized in economics by several articles during the 1990s: [21], [35], [23] and [22] just to mention a few. Some of these articles showed interesting properties implied by Nash equilibrium uniqueness in classes of games admitting isotone selections from best replies. For example, in some of such classes Nash equilibrium uniqueness was proved to be: equivalent to dominance solvability (see Theorem 5 and the second Corollary at p. 1266 in [21], Theorem 12 in [23] and Proposition 4 in [1]); sufficient to establish an equivalence between the convergence to Nash equilibrium of an arbitrary sequence of joint strategies and its consistency with adaptive learning processes (see the first Corollary at p. 1270 in [21] and Theorem 14 in [23]); sufficient to infer the existence—and uniqueness—of coalition-proof Nash equilibria (see Theorem A1 and the last Remark at p.127 in [22]). However, these articles do not provide sufficient structural conditions for Nash equilibrium uniqueness.

A new strand of the literature on nice games with isotone best replies played on networks started a still partial investigation about the conditions on utility functions for the existence of a unique Nash equilibrium in that class of games:

[1] I.e., games with a finite set of players whose strategy space is a closed proper real interval with a minimum and whose utility function is strictly pseudoconcave and upper semicontinuous in own strategy. The term *nice game* is introduced in [24] and our definition is similar—but not identical—to the one therein.

[3], [2], [16] and [13] to mention a few. Except for [16],[2] in such papers Nash equilibrium uniqueness is guaranteed by a type of fixpoint argument—introduced by [19] in the economic literature—whose application requires the isotonicity of best reply functions.[3] However, the general structures of the primitives of a game with isotone best replies ensuring the existence of a unique Nash equilibrium are still unclear, despite a natural interest of economic and game theorists in the understanding thereof; in particular, the possible role played by the isotonicity of best replies is unclear. Of course, the literature offers conditions on the primitives of a game for the existence of a unique Nash equilibrium, but not many results seem to crucially depend on the condition of isotonicity of best replies. Restricting attention to nice games with isotone best reply functions, can we add something to known Nash equilibrium uniqueness results?

Our contribution

We examine the conditions on the primitives of a nice game with isotone best replies that ensure Nash equilibrium uniqueness. The investigation makes use of a fixpoint argument—similar but not identical to the one in [19]—which employs a notion of generalized concavity that we name chain-concavity (see Sect. 3 for the definition). A particular version of the argument goes as follows.

> *Let f be a self-map of $[0,1]^n$ with no fixpoints on the boundary of $\mathbb{R}^n_+$ (e.g., each f_i could be positive). Then f has exactly one fixpoint if each component function f_i is isotone and chain-concave.*

We derive four theorems on Nash equilibrium uniqueness in nice games. Such theorems dispense with any differentiability assumption. In case of compact nice games with differentiable utility functions, a corollary of one of our main results—by which the reader might already gain an insight of our findings—can be stated thus (see Sect. 2, 3 and 5 for all definitions).

> *Let Γ be a smooth compact nice game. Suppose each strategy set S_i has minimum 0. Then Γ has exactly one Nash equilibrium if, for each player i, the marginal utility function $\mathcal{M}_i$:*

- *is quasiincreasing in every argument other than the i-th one;*
- *has a chain-convex upper level set at height zero;*
- *is positive at $(0, \ldots, 0)$.*

Our main Nash equilibrium uniqueness results do not rely on the differentiability of utility functions and are formulated in terms of Dini derivatives (here regarded as "generalized marginal utilities"). The prime contribution of these results is not, however, the lack of any differentiability assumption: it will be

[2] Equilibrium uniqueness in [16] follows from Theorem 5.1 in [18].

[3] An alternative argument, still relying on the isotonicity of best replies, is provided in [3].

shown that three classical theorems on Nash equilibrium uniqueness (i.e., [29]'s Theorem 2, [18]' s Theorem 5.1 and [10]'s Theorem 4.1) do not directly imply our results even when utility functions are infinitely many times differentiable.

Our investigation proceeds as follows. First, we interpret the mentioned fixpoint uniqueness argument as a set of sufficient conditions on the best replies of a compact nice game that guarantee Nash equilibrium uniqueness. Then we characterize these conditions in terms of "generalized marginal utilities". This characterization is carried out through the examination of the necessity and sufficiency of the conditions of a Choice Problem for the isotonicity and chain-concavity of its Choice function: this examination is our key contribution. A Choice Problem is—in the terminology of [26] and [20]—a Type A problem of comparative statics where a parametrized (strictly pseudoconcave upper semicontinuous) function is optimized on a fixed choice set (a compact proper real interval) for each given value of the parameter; its Choice function associates with each value of the parameter the optimal solution of the Choice Problem.

Sect. 4 and Appendix B provide new results in terms of the necessity and sufficiency of the conditions for both the concavity/chain-concavity and isotonicity of Choice functions. To the best of our knowledge, the concavity/chain-concavity of optimal solutions has not been systematically studied in the literature, but results that guarantee the concavity or the chain-concavity of Choice functions are useful also for game-theoretic analyses of problems not related to Nash equilibrium uniqueness[4]. To the contrary, the isotonicity of Choice functions has been extensively investigated. However, our results on this issue do not follow from known theorems such as [23]'s Monotonicity Theorem or similar results of the subsequent literature: for example, those in [30], [9], [1] and—though in a more abstract spirit—in [20]. In fact our results on the isotonicity of Choice functions are structurally similar to Theorem 1 in [28] and hold for a class of problems which is properly included in that for which Theorem 1 in [28] guarantee the isotonicity of Choice functions; nevertheless, as shown in Appendix D, the conditions involved in our differential characterization differ from the sufficient conditions on derivatives obtained in Sect. 2.4 in [28].

Structure of the paper

The paper is organized as follows: Sect. 2 presents preliminaries; Sect. 3 exposits novel notions of generalized convexity/concavity; Sect. 4 introduces the definition of a (Normalized) Choice function for a Choice Problem and examines the necessity and sufficiency of the conditions of a Choice Problem for the isotonicity and chain-concavity of the Choice function and the positivity of the Normalized Choice function; Sect. 5 illustrates the main Nash equilibrium uniqueness results of this work and relates them to some known theorems of the literature; Sect. 6 shows an extension of one of our uniqueness results to games of incomplete information. Appendices A–F show a fixpoint argument, examine the concavity of Choice functions and contain other mathematical facts.

[4] E.g., [5] and [6] use this type of results in the analysis of multi-leader multi-follower games.

2 Preliminary notation, definitions and results

2.1 Notation

Let I be a proper real interval and $f : I \to \mathbb{R}$. There are several standard notations for the four Dini derivatives of f. Just to provide a precise reference, our notation is the same of [17]: see (3.1.4–7) at p. 56 therein. Thus the upper (resp. lower) right Dini derivative of f at $x_0 \neq \sup I$ is denoted by $D^+ f(x_0)$ (resp. $D_+ f(x_0)$) and the upper (resp. lower) left Dini derivative of f at $x_0 \neq \inf I$ is denoted by $D^- f(x_0)$ (resp. $D_- f(x_0)$). We recall that $D^+ f(x_0)$, $D_+ f(x_0)$, $D^- f(x_0)$ and $D_- f(x_0)$ are well-defined elements of the set of the extended reals $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

Let $f : A \times B \to \mathbb{R}$, where A and B are nonempty subsets of Euclidean spaces. Let $(a^*, b^*) \in A \times B$. Sometimes we write $f(\cdot, b^*)$ to denote the function $A \to \mathbb{R} : a \mapsto f(a, b^*)$ and $f(a^*, \cdot)$ to denote the function $B \to \mathbb{R} : b \mapsto f(a^*, b)$. Thus, for instance, the expression $f(\cdot, b^*)(a^*)$ is perfectly equivalent to the expression $f(a^*, b^*)$. This notation is standard; however, for clarity, we remark that when ($A \subseteq \mathbb{R}$ and) we write $D^+ f(\cdot, b^*)(a^*)$—or an analogous expression—we mean to indicate the upper right Dini derivative of $f(\cdot, b^*)$ at a^*.

2.2 Generalized monotonicity: standard concepts

For real-valued functions, the following notions of generalized monotonicity are standard and, for instance, can be found at p. 1199 in [12]. In our definitions, we prefer to use the term "increasing" instead of "monotone" to remark the fact that the domains are totally ordered sets.

Definition 1 *A function $f : X \subseteq \mathbb{R} \to \overline{\mathbb{R}}$ is, respectively,* ***increasing****,* ***strictly increasing****,* ***strictly pseudoincreasing****,* ***quasiincreasing*** *iff, respectively,*

- $(\underline{x}, \overline{x}) \in X \times X$ *and* $\underline{x} < \overline{x} \Rightarrow f(\underline{x}) \leq f(\overline{x})$,
- $(\underline{x}, \overline{x}) \in X \times X$ *and* $\underline{x} < \overline{x} \Rightarrow f(\underline{x}) < f(\overline{x})$,
- $(\underline{x}, \overline{x}) \in X \times X$, $\underline{x} < \overline{x}$ *and* $f(\underline{x}) \geq 0 \Rightarrow f(\overline{x}) > 0$,
- $(\underline{x}, \overline{x}) \in X \times X$, $\underline{x} < \overline{x}$ *and* $f(\underline{x}) > 0 \Rightarrow f(\overline{x}) \geq 0$.

To dispel any doubts, the standard notion of a quasiincreasing function employed in this paper is very different from that in [19].

Definition 2 *A function $f : X \subseteq \mathbb{R} \to \overline{\mathbb{R}}$ is, respectively,* ***decreasing****,* ***strictly decreasing****,* ***strictly pseudodecreasing****,* ***quasidecreasing*** *iff $-f$ is, respectively, increasing, strictly increasing, strictly pseudoincreasing, quasiincreasing.*

Henceforth, we assume the usual convention $\pm\infty \times 0 = 0$.

Remark 1 *Suppose X is a nonempty Cartesian product of m subsets of $\mathbb{R}$. Let $f : X \to \overline{\mathbb{R}}$, $g : X \to \mathbb{R}_+$ and $h : X \to \mathbb{R}_{++}$. If f is increasing (resp. strictly increasing) in every argument then $f \cdot g$ (resp. $f \cdot h$) is quasiincreasing (resp. strictly pseudoincreasing) in every argument.*

Table 1. Relation diagram for an extended real-valued function f on a real interval

$$\begin{array}{ccc} \text{incr.} & \Rightarrow & \text{quasiincr.} \\ \Uparrow & & \Uparrow \\ \text{str. incr.} & \Rightarrow & \text{str. pseudoincr.} \end{array}$$

Definition 3 *A function $f : X \subseteq \mathbb{R}^m \to \overline{\mathbb{R}}$ is **isotone** (resp. **antitone**) iff*

$$\begin{gathered} (\underline{x}, \overline{x}) \in X \times X \ \textit{and} \ \underline{x}_i \leq \overline{x}_i \ \textit{for all} \ i = 1, \ldots, m \\ \Downarrow \\ f(\underline{x}) \leq f(\overline{x}) \ \ (\textit{resp.} \ f(\overline{x}) \leq f(\underline{x})\,). \end{gathered}$$

Remark 2 *Suppose X is a Cartesian product of m subsets of $\mathbb{R}$ then a function $f : X \subseteq \mathbb{R}^m \to \overline{\mathbb{R}}$ is isotone (resp. antitone) if and only if f is increasing (resp. decreasing) in every argument.*

2.3 Generalized convexity: standard concepts

The standard definitions of a convex set $X \subseteq \mathbb{R}^m$ and of a real-valued (strictly) convex function defined thereon are assumed to be known: just to provide a precise reference, see Definitions 1.2.1 and 1.3.1 in [4]. As usual, we say that a function f is (strictly) concave iff $-f$ is (strictly) convex. We shall now formally recall some standard definitions of generalized convexity.

Definition 4 *Let $X \subseteq \mathbb{R}^m$ be convex. Then $f : X \to \overline{\mathbb{R}}$ is **quasiconcave** iff its upper level sets at finite height are convex. (The upper level set at height $\lambda \in \overline{\mathbb{R}}$ of f is $\{x \in X : f(x) \geq \lambda\}$.)*

Remark 3 *Let $X \subseteq \mathbb{R}^m$ be convex and $g : X \to \mathbb{R}_{++}$. If $f : X \to \overline{\mathbb{R}}$ is quasiconcave then $f \cdot g$ has a convex upper level set at height 0.*

We recall a characterization of a real-valued quasiconcave function (see Theorem 2.2.3 in [4]) and a definition of a strict variant thereof.

Remark 4 *Let $X \subseteq \mathbb{R}^m$ be convex, $f : X \to \mathbb{R}$ is **quasiconcave** if and only if*

$$\lambda \in \left]0,1\right[, \ (\underline{x}, \overline{x}) \in X \times X \ \textit{and} \ \underline{x} \neq \overline{x} \Rightarrow f(\lambda \underline{x} + (1-\lambda)\overline{x}) \geq \min\{f(\underline{x}), f(\overline{x})\}.$$

Definition 5 *Let $X \subseteq \mathbb{R}^m$ be convex, $f : X \to \mathbb{R}$ is **strictly quasiconcave** iff*

$$\lambda \in \left]0,1\right[, \ (\underline{x}, \overline{x}) \in X \times X \ \textit{and} \ \underline{x} \neq \overline{x} \Rightarrow f(\lambda \underline{x} + (1-\lambda)\overline{x}) > \min\{f(\underline{x}), f(\overline{x})\}.$$

Remark 5 *Let $X \subseteq \mathbb{R}^m$ be convex and $f : X \to \mathbb{R}$.*

(i) If f is strictly concave then f is concave;

(ii) If f is concave then f is quasiconcave;

(iii) If f is strictly quasiconcave then f is quasiconcave.

Our definition of strict pseudoconcavity in terms of Dini derivatives is due to [8]: see Definition 9 therein. On the history of the concept see Sect. 1 in [14] and see also Definition 2 in [15] for recent further generalizations.

Definition 6 *Let $X \subseteq \mathbb{R}$ be convex. Then $f : X \to \mathbb{R}$ is **strictly pseudoconcave** iff*

$$(\underline{x}, \overline{x}) \in X \times X,\ \underline{x} < \overline{x} \text{ and } f(\underline{x}) \leq f(\overline{x}) \Rightarrow D^{+} f(\underline{x}) > 0$$

and

$$(\underline{x}, \overline{x}) \in X \times X,\ \underline{x} < \overline{x} \text{ and } f(\underline{x}) \geq f(\overline{x}) \Rightarrow D_{-} f(\overline{x}) < 0;$$

*$f : X \to \mathbb{R}$ is **strictly pseudoconvex** iff $-f$ is strictly pseudoconcave.*

Remark 6 recalls some known facts: part (i) follows from part (ii) of Theorem 14 in [8]; part (ii) follows from the definition of strict pseudoconcavity; part (iii) follows from Corollary 20 in [8].

Remark 6 *Let $X \subseteq \mathbb{R}$ be convex and $f : X \to \mathbb{R}$.*

(i) If f is strictly concave then f is strictly pseudoconcave;

(ii) If f is strictly pseudoconcave then f has at most one maximizer;

(iii) If f strictly pseudoconcave and upper semicontinuous then f is strictly quasiconcave.

Examples of real-valued strictly pseudoconcave functions on $\mathbb{R}$ which are neither quasiconcave nor upper semicontinuous can be constructed by the reader.

Table 2. Relation diagram for an upper semicontinuous real-valued function f on a real interval

$$\begin{array}{ccccc} \text{conc.} & & \Rightarrow & & \text{quasiconc.} \\ \Uparrow & & & & \Uparrow \\ \text{str. conc.} & \Rightarrow & \text{str. pseudoconc.} & \Rightarrow & \text{str. quasiconc.} \end{array}$$

Finally, a characterization of strictly pseudoconcave functions is recalled: for a proof see, e.g., Theorem 1 at p. 1199 in [12] and references therein.

Remark 7 *A real-valued differentiable function f on an open proper real interval is strictly pseudoconcave if and only if Df is strictly pseudodecreasing.*

3 Generalized convexity: chain-convexity

We now introduce some notions of generalized convexity: to the best of our knowledge all definitions and results of this Sect. 3 are new.

Definition 7 *A subset X of $\mathbb{R}^m$ is said to be **chain-convex** iff*

$$\gamma \in [0,1]\,,\ (\underline{x}, \overline{x}) \in X \times X \text{ and } \underline{x}_i \leq \overline{x}_i \text{ for all } i = 1, ..., m \ \Rightarrow\ \gamma \underline{x} + (1-\gamma)\,\overline{x} \in X.$$

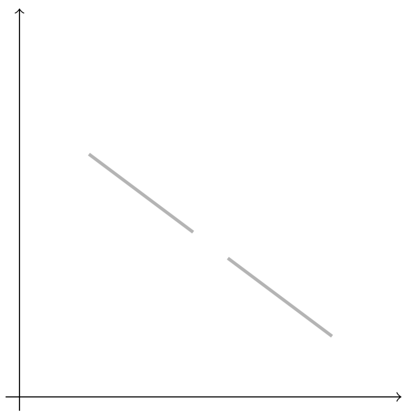

Fig 1. A chain-convex set

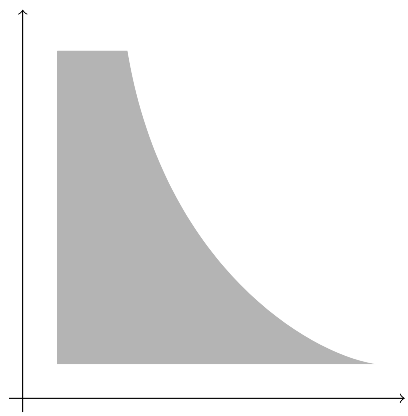

Fig 2. A chain-convex set

Remark 8 *Let $X \subseteq \mathbb{R}^m$. If X is convex then X is chain-convex. When $m = 1$ the converse is true but is generally false when $m > 1$.*

Definition 8 *Let $X \subseteq \mathbb{R}^m$ be chain-convex. A function $f : X \to \mathbb{R}$ is said to be **chain-concave** iff*

$$\gamma \in [0,1]\,,\ (\underline{x}, \overline{x}) \in X \times X \text{ and } \underline{x}_i \leq \overline{x}_i \text{ for all } i = 1, ..., m$$
$$\Downarrow$$
$$\gamma f(\underline{x}) + (1-\gamma)\, f(\overline{x}) \leq f(\gamma \underline{x} + (1-\gamma)\,\overline{x})\,.$$

*A function $f : X \to \mathbb{R}$ is said to be **chain-convex** iff $-f$ is chain-concave.*

Remark 9 *If g and h are chain-concave real-valued functions on a chain-convex subset X of $\mathbb{R}^m$ then so is $g + h$. Also, when $X \subseteq \mathbb{R}^m$ is convex, a concave function $f : X \to \mathbb{R}$ is chain-concave; the converse is true when $m = 1$ but is generally false when $m > 1$ (examples of chain-concave functions with convex domains that are not concave are shown after Remark 11). Clearly, every real-valued function on a disconnected antichain in $\mathbb{R}^2$—e.g., on a set like the one in Fig. 1—is chain-concave but not concave.*

We preliminarily recall a fact used in the proof of Lemma 1.

Remark 10 *Let $I_X \subseteq \mathbb{R}$ be a proper interval and $I_Y \subseteq \mathbb{R}$ be an open superset of I_X. Suppose $g : I_Y \to \mathbb{R}$ is twice continuously differentiable. A necessary and sufficient condition for the concavity of g on I_X—i.e., for the concavity of $g|_{I_X}$—is that $D^2 g(t) \leq 0$ for all $t \in I_X$.*

Definition 9 *An $m \times m$ matrix H is conegative iff $v^T \cdot H \cdot v \leq 0$ for all $v \in \mathbb{R}_+^m$.*

Lemma 1 *Let $Y \subseteq \mathbb{R}^m$ be open and nonempty and $f : Y \to \mathbb{R}$ be twice continuously differentiable. Let X be a chain-convex subset of Y. Then f is chain-concave on X if the Hessian matrix*

$$\mathrm{H}(x) = \left[\frac{\partial^2 f}{\partial x_i \partial x_l}(x) \right]_{i,l}$$

is conegative at all $x \in X$.

Proof. If $X = \emptyset$ we are done. Assume that $X \neq \emptyset$. By contradiction, suppose that $\mathrm{H}(x)$ is conegative at all $x \in X$ and f is not chain-concave. Then there exist $\gamma \in [0,1]$ and $(\underline{z}, \overline{z}) \in X \times X$ such that $\underline{z}_i \leq \overline{z}_i$ for all $i = 1, ..., m$ and that $f(\gamma \underline{z} + (1-\gamma)\overline{z}) < \gamma f(\underline{z}) + (1-\gamma) f(\overline{z})$. Thus $\underline{z} \neq \overline{z}$ and $\gamma \in]0,1[$. Put

$$v = \overline{z} - \underline{z},\ I_X = \{t \in \mathbb{R} : (\underline{z} + tv) \in X\} \text{ and } I_Y = \{t \in \mathbb{R} : (\underline{z} + tv) \in Y\}.$$

Note that $[0,1] \subseteq I_X \subseteq I_Y \subseteq \mathbb{R}$, that I_X is an interval and that I_Y is open. Let

$$\varphi : I_Y \to \mathbb{R} : t \mapsto f(\underline{z} + tv).$$

As $\varphi(0) = f(\underline{x})$, $\varphi(1) = f(\overline{z})$ and $\varphi(1-\gamma) = f(\gamma \underline{z} + (1-\gamma)\overline{z})$, we have that

$$\varphi(1-\gamma) < \gamma \varphi(0) + (1-\gamma)\varphi(1).$$

Thus φ is twice continuously differentiable on I_X but not concave on I_X. Thus $D^2\varphi(t) > 0$ for some $t \in I_X$. But this is impossible as

$$D^2\varphi(t) = v^T \cdot \mathrm{H}(\underline{z} + tv) \cdot v \text{ for all } t \in I_X$$

by the twice continuous differentiability of f and

$$v^T \cdot \mathrm{H}(\underline{z} + tv) \cdot v \leq 0 \text{ for all } t \in I_X$$

by the conegativity of $\mathrm{H}(\underline{z} + tv)$ for all $t \in I_X$. ■

A nonpositive $m \times m$ matrix is conegative, and Corollary 1 readily follows.

Corollary 1 *Under the conditions of Lemma 1, f is chain-concave on X if $\mathrm{H}(x)$ is nonpositive at all $x \in X$ (i.e., if*

$$\frac{\partial^2 f}{\partial x_i \partial x_l}(x) \leq 0$$

for all $i = 1, \ldots, m$, all $l = 1, \ldots, m$ and all $x \in X$).

A characterization of chain-concave functions is provided.

Theorem 1 *Let $X \subseteq \mathbb{R}^m$ be nonempty, open and chain-convex and $f : X \to \mathbb{R}$ be twice continuously differentiable. Then f is chain-concave if and only if the Hessian matrix*

$$\mathrm{H}(x) = \left[\frac{\partial^2 f}{\partial x_i \partial x_l}(x) \right]_{i,l}$$

is conegative at all $x \in X$.

Proof. The *if part* follows from Lemma 1. We prove the *only if part.* By contradiction, assume that f is chain-concave and $\mathrm{H}(\underline{z})$ is not conegative for some $\underline{z} \in X$. Then there exists $v \in \mathbb{R}^m_+ \setminus \{0\}$ such that $v^T \cdot \mathrm{H}(\underline{z}) \cdot v > 0$. As X is open, there exists $\lambda \in \mathbb{R}_{++}$ and $\overline{z} \in X$ such that $v = \lambda(\overline{z} - \underline{z})$. Thus $(\lambda(\overline{z} - \underline{z}))^T \cdot \mathrm{H}(\underline{z}) \cdot (\lambda(\overline{z} - \underline{z})) > 0$ and hence

$$(\overline{z} - \underline{z})^T \cdot \mathrm{H}(\underline{z}) \cdot (\overline{z} - \underline{z}) > 0. \tag{1}$$

As f is chain-concave, the function

$$\varphi : I \to \mathbb{R} : t \mapsto f(\underline{z} + t(\overline{z} - \underline{z})) \text{ with } I = \{t \in \mathbb{R} : \underline{z} + t(\overline{z} - \underline{z}) \in X\}$$

must be concave. Note that I is open and includes $[0,1]$. Thus $D^2\varphi(0) \leq 0$, in contradiction with (1) and the fact that $D^2\varphi(0) = (\overline{z} - \underline{z})^T \cdot \mathrm{H}(\underline{z}) \cdot (\overline{z} - \underline{z})$. ∎

Remark 11 *In Theorem 1, a necessary and sufficient condition for the conegativity of $\mathrm{H}(x)$ is the semimonotonicity of $-\mathrm{H}(x)$: see Definition 3.9.1 and Proposition 3.9.8 in [7]. Clearly, a sufficient condition for the conegativity of $\mathrm{H}(x)$ is the negative semidefiniteness[5] of $\mathrm{H}(x)$. Also, a sufficient condition for the negative semidefiniteness of $\mathrm{H}(x)$ is that $\mathrm{H}(x)$ is a diagonally dominant matrix with nonpositive diagonal entries: see Definition 2.2.19 and part (c) of Proposition 2.2.20, in [7]. All these sufficient conditions can be used to check the conegativity of $\mathrm{H}(x)$; however, for the examples of this article, the nonpositivity condition mentioned in Corollary 1 remains the "easiest-to-check" condition.*

We clarify further the relation between chain-concavity and (quasi)concavity:

- $f : \mathbb{R}^2 \to \mathbb{R} : (x_1, x_2) \mapsto -x_1 x_2$ is chain-concave but not concave (nor quasiconcave or isotone or antitone);
- $f : [0,1]^2 \to [-1,0] : (x_1, x_2) \mapsto -x_1 x_2$ is antitone and chain-concave but not concave (nor quasiconcave);
- $f : [0,1]^4 \to [0,1] : (x_1, x_2, x_3, x_4) \mapsto x_1 + x_2 - x_1 x_2$ is isotone and chain-concave but not concave (nor quasiconcave);
- $f : \mathbb{R}^2_+ \to \mathbb{R}_+ : (x_1, x_2) \mapsto x_1 + x_2 - e^{-(x_1+1)(x_2+1)} + e^{-1}$ is isotone, Lipschitz continuous and chain-concave but not concave (nor quasiconcave).

[5] I.e., a sufficient condition is that $v^T \cdot \mathrm{H}(x) \cdot v \leq 0$ for all $v \in \mathbb{R}^m$.

Changing the sign of the four functions above, one easily obtains examples of chain-convex functions which are not convex. For instance, the function

- $f : [0,1]^2 \to [0,1] : (x_1, x_2) \mapsto x_1 x_2$ is isotone and chain-convex but not convex.

Alternatively, one can consider variants of the second and fourth examples above to construct nonnegative chain-convex functions which are not convex. For instance, the nonnegative function

- $f : \mathbb{R}^2_+ \to \mathbb{R}_+ : (x_1, x_2) \mapsto x_1 + x_2 + e^{-(x_1+1)(x_2+1)} - e^{-1}$ is chain-convex, isotone and Lipschitz continuous but not convex.

Proposition 1 adapts to chain-concavity/convexity well-known results of convex analysis. Part (i) of Proposition 1 can be considerably generalized and is conveniently stated here for future reference.

Proposition 1 *Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}$ be chain-convex sets. Let $f : Y \to \mathbb{R}$ be isotone and let $g : X \to \mathbb{R}$. Suppose $g[X] \subseteq Y$ and put $h = f \circ g$.*

(i) If g is isotone then h is isotone.

(ii) If g is chain-convex and f is chain-convex then h is chain-convex.

(iii) If g is chain-concave and f is chain-concave then h is chain-concave.

Proof. The proof of (i) is trivial and hence is omitted. The proof of (ii) is as follows. Suppose g is chain-convex. Choose an arbitrary $\gamma \in [0,1]$ and an arbitrary pair $(x, z) \in X \times X$ such that $x_i \leq z_i$ for all $i = 1, \ldots, m$. Then

$$g(y) \leq \gamma g(x) + (1-\gamma) g(z) \text{ with } y = \gamma x + (1-\gamma) z.$$

Thus $f(g(y)) \leq f(\gamma g(x) + (1-\gamma) g(z))$ by the isotonicity of f. Suppose f is chain-convex. Then f is convex as $Y \subseteq \mathbb{R}$. By the convexity of f the last inequality becomes $f(g(y)) \leq \gamma f(g(x)) + (1-\gamma) f(g(z))$. We conclude that $h(y) \leq \gamma h(x) + (1-\gamma) h(z)$ and thus that h is chain-convex by the arbitrariness of γ and (x, z). The proof of (iii) is analogous to the proof of (ii). ■

Definition 10 *A function $f : X \to \overline{\mathbb{R}}$ on a chain-convex subset X of $\mathbb{R}^m$ is* ***chain-quasiconcave*** *iff its upper level sets at finite height are chain-convex.*

The following conclusions can be easily derived by the reader.

Remark 12 *Let $X \subseteq \mathbb{R}^m$ be chain-convex and $f : X \to \overline{\mathbb{R}}$.*

(i) If f is either isotone or antitone then f is chain-quasiconcave;

(ii) If X is convex and f is quasiconcave then f is chain-quasiconcave;

(iii) If f is real-valued and chain-concave then f is chain-quasiconcave;

(iv) If $g : X \to \mathbb{R}_{++}$ and f is chain-quasiconcave then $f \cdot g$ has a chain-convex upper level set at height 0.

4 On three properties of a C-function

We make use of the following definition of a Choice Problem.

Definition 11 *By a **Choice Problem** (**CP** in short) we mean a triple* (A, B, f) *where: (i)* A *is a compact proper real interval; (ii)* B *is a nonempty subset of* $\mathbb{R}^m$ *with* $m \in \mathbb{N}$*; (iii)* f *is a function from* $A \times B$ *into* $\mathbb{R}$ *such that* $f(\cdot, b)$ *is strictly pseudoconcave and upper semicontinuous for all* $b \in B$*.*

Notation ($\mathcal{D}_f$) *With each CP we associate the function*

$$\mathcal{D}_f : \operatorname{int}(A) \times B \to \overline{\mathbb{R}} : (a, b) \mapsto D_- f(\cdot, b)(a).$$

We now define a Choice function for a CP and a normalization thereof. These two functions are used in this Sect. 4 when analyzing the change of optimal choices in a parameter. It is perhaps worth mentioning that, given a CP and $b \in B$, the set $\arg\max f(\cdot, b)$ is nonempty—as A is a compact proper real interval and $f(\cdot, b)$ is upper semicontinuous—and hence contains exactly one element as $f(\cdot, b)$ is strictly pseudoconcave (see Remark 6).

Definition 12 *Given a CP, by the **Choice function** (**C-function** in short) associated to such a CP we mean the function*

$$\beta : B \to A \text{ such that } \{\beta(b)\} = \arg\max f(\cdot, b) \text{ at all } b \in B$$

*and by the **Normalized Choice function** (**NC-function** in short) associated to such a CP we mean the function* $\beta^* : B \to \mathbb{R}_+ : b \mapsto \beta(b) - \min A$.

We now examine the necessity and sufficiency of the conditions for the isotonicity and chain-concavity of a C-function and for the positivity of an NC-function. We refer to Appendix B for an examination of the necessity and sufficiency of the conditions for the concavity of a C-function.

4.1 Isotonicity of a C-function

The following Theorem 2 is the first main result of this Sect. 4. We refer to Appendix D for a comparison with related results of the literature.

Theorem 2 *Consider a CP and the associated function* β*. Suppose* B *is the Cartesian product of* m *subsets of* $\mathbb{R}$*. Then,* β *is isotone if and only if* $\mathcal{D}_f(a, \cdot)$ *is quasiincreasing in every argument*[6] *for all* $a \in \operatorname{int}(A)$*.*

Proof. *If part.* Suppose $\mathcal{D}_f(a, \cdot)$ is quasiincreasing in every argument for all $a \in \operatorname{int}(A)$. Pick $(x, y) \in B \times B$ such that $x \neq y$ and $x_l \leq y_l$ for all $l = 1, ..., m$.

[6] Recall that $\mathcal{D}_f(a, \cdot) : \prod_{i=1}^m B_i \to \overline{\mathbb{R}}$. Thus the quasiincreasingness of $\mathcal{D}_f(a, \cdot)$ in every argument is—somehow incorrectly—the quasiincreasingness of $\mathcal{D}_f\left(a, (x_l)_{l=1}^m\right)$ in every x_l.

It suffices to show that $\beta(x) \leq \beta(y)$. If $\beta(x) = \min A$ then $\beta(x) \leq \beta(y)$. Suppose $\beta(x) > \min A$. By the strict pseudoconcavity of $f(\cdot, x)$,

$$D^+ f(\cdot, x)(a) > 0 \text{ for all } a \in [\min A, \beta(x)[\,.$$

Thus, by part (ii) of Theorem 1.13 in [11],

$$\mathcal{D}_f(a, x) = D_- f(\cdot, x)(a) \geq 0 \text{ for all } a \in]\min A, \beta(x)[$$

and hence, by Lemma C1 in Appendix C,

$$\mathcal{D}_f(a, y) = D_- f(\cdot, y)(a) \geq 0 \text{ for all } a \in]\min A, \beta(x)[$$

because $\mathcal{D}_f(a, \cdot)$ is quasiincreasing in every argument. Hence $\beta(x) \leq \beta(y)$: otherwise $\beta(y) < \beta(x)$ and $D_- f(\cdot, y)(a) \geq 0$ for some $a \in]\beta(y), \beta(x)[$ in contradiction with the strict pseudoconcavity of $f(\cdot, y)$.

Only if part. Assume that β is isotone and, by contradiction, suppose that $\mathcal{D}_f(\overline{a}, \cdot)$ is not quasiincreasing in the j-th argument for some $\overline{a} \in \text{int}(A)$. Then there exist $\overline{a} \in \text{int}(A)$, $x \in B$ and $y \in B$ such that $x_j < y_j$, $x_l = y_l$ for all $l = \{1, ..., m\} \setminus \{j\}$ and

$$D_- f(\cdot, x)(\overline{a}) > 0 > D_- f(\cdot, y)(\overline{a})\,.$$

By part (iii) of Remark 6 and Corollary 2.5.2 in [4], $f(\cdot, x)$ is strictly decreasing on $[\beta(x), \max A]$ and $f(\cdot, y)$ is strictly increasing on $[\min A, \beta(y)]$. Hence

$$a \in A \text{ and } a > \beta(x) \;\; \Rightarrow D_- f(\cdot, x)(a) \leq 0$$

and

$$a \in A \text{ and } \min A < a \leq \beta(y) \;\; \Rightarrow D_- f(\cdot, y)(a) \geq 0.$$

We conclude that $\overline{a} \leq \beta(x)$ and $\overline{a} > \beta(y)$, which implies $\beta(y) < \beta(y)$ in contradiction with the isotonicity of β. ■

4.2 Positivity of an NC-function

Our results on the chain-concavity of the C-function β will be established on the subset of B where β is greater than $\min A$: such a subset coincides with the support of the NC-function β^* (i.e., the set of points where β^* does not vanish). Clearly, the support of β^* is B if and only if β^* is positive. Some simple facts about the necessity and sufficiency of the conditions for B to coincide with the support of β^* are provided by the following Proposition 2.

Proposition 2 *Consider a CP and the associated functions β and β^*. Suppose B has a least element, say ω. Besides assume that β is isotone. The support of β^* is B (or equivalently, β^* is positive) if and only if $D^+ f(\cdot, \omega)(\min A) > 0$.*

Proof. First note that the isotonicity of β is equivalent to the isotonicity of β^*.

If part. Suppose $D^+ f(\cdot, \omega)(\min A) > 0$. Then $\beta^*(\omega) > 0$ and ω is in the support of β^*. The isotonicity of β^* implies that the support of β^* is B.

Only if part. Suppose the support of β^* is B. If $D^+ f(\cdot, \omega)(\min A) \leq 0$. Then $f(\cdot, \omega)(\min A) > f(\cdot, \omega)(x)$ for all $x > \min A$ by the definition of a strictly pseudoconcave function. Hence ω, which is an element of B, would not be in the support of β^*: a contradiction. ■

4.3 Chain-concavity of a C-function

Theorem 3 is the second main result of this Sect. 4.

Theorem 3 *Consider a CP and the associated functions β and β^*. Suppose B is chain-convex. Besides assume that β is isotone and β^* is positive. Then β is chain-concave if and only if $\mathcal{D}_f$ has a chain-convex upper level set at height* 0.

Proof. Without loss of generality, we shall put $\min A = 0$. Thus β equals the NC-function β^*.

If part. Assume that D_f has a chain-convex upper level set at height 0. Suppose that x and z are elements of B such that $x_i \leq z_i$ for all $i = 1, ..., m$ and put

$$\xi := \beta(x) \text{ and } \zeta := \beta(z).$$

By the isotonicity of the positive function β,

$$0 = \min A < \xi \leq \zeta.$$

Pick $\gamma \in]0,1[$ and put $y := \gamma x + (1-\gamma) z$. We are done if we prove that

$$\overline{\upsilon} := \gamma \xi + (1-\gamma) \zeta \leq \beta(y) =: \upsilon.$$

Case $\min\{\xi, \zeta\} < \max A$. In this case $\xi = \min\{\xi, \zeta\} < \max A$. Suppose, to the contrary, that $\upsilon < \overline{\upsilon}$. Note that

$$D_- f(\cdot, y)(\overline{\upsilon}) < 0 \tag{2}$$

because $f(\cdot, y)$ is a strictly pseudoconcave function maximized at υ, with

$$\min A \leq \upsilon < \overline{\upsilon} < \max A.$$

Since ξ and ζ are respectively maximizers of $f(\cdot, x)$ and of $f(\cdot, z)$,

$$D_- f(\cdot, x)(\xi) \geq 0 \leq D_- f(\cdot, z)(\zeta)$$

and hence

$$\min\{\mathcal{D}_f(\xi, x), \mathcal{D}_f(\zeta, z)\} \geq 0.$$

Thus (ξ, x) and (ζ, z) belong to the upper level set at height 0 of $\mathcal{D}_f$, and hence so does[7] also $(\overline{\upsilon}, y)$ by the chain-convexity of the upper level set at height 0 of $\mathcal{D}_f$. Therefore

$$\mathcal{D}_f(\overline{\upsilon}, y) = D_- f(\cdot, y)(\overline{\upsilon}) \geq 0,$$

[7] Recall—and this is important here—that $x_i \leq z_i$ for all $i = 1, ..., m$ and that $\xi \leq \zeta$.

in contradiction with (2).

Case $\min\{\xi,\zeta\} \geq \max A$. Thus $\xi = \zeta = \max A$. By the strict pseudoconcavity of $f(\cdot,x)$ and $f(\cdot,z)$,

$$D^{+}f(\cdot,x)(a) > 0 < D^{+}f(\cdot,z)(a) \text{ for all } a \in \operatorname{int}(A) \cup \{\min A\}.$$

By part (ii) of Theorem 1.13 in [11], $f(\cdot,x)$ and $f(\cdot,z)$ are increasing on $\operatorname{int}(A)$; consequently,

$$\mathcal{D}_f(a,x) = D_{-}f(\cdot,x)(a) \geq 0 \leq D_{-}f(\cdot,z)(a) = \mathcal{D}_f(a,z) \text{ for all } a \in \operatorname{int}(A)$$

and hence

$$\mathcal{D}_f(a,y) = D_{-}f(\cdot,y)(a) \geq 0 \text{ for all } a \in \operatorname{int}(A)$$

by the chain-convexity of the upper level set at height 0 of $\mathcal{D}_f$. Thus we must have $\upsilon = \beta(y) = \max A = \overline{\upsilon}$: otherwise $\beta(y) \in \operatorname{int}(A) \cup \{\min A\}$ and $D_{-}f(\cdot,y)(a) \geq 0$ for some $a \in]\beta(y), \max A[$ in contradiction with the strict pseudoconcavity of $f(\cdot,y)$.

Only if part. Assume that β is chain-concave. By contradiction, suppose the upper level set at height 0 of $\mathcal{D}_f$ is not chain-convex. Then there exist $(\hat{a},x) \in \operatorname{int}(A) \times B$, $(\check{a},z) \in \operatorname{int}(A) \times B$ and $\gamma \in]0,1[$ such that

$$\hat{a} \leq \check{a} \text{ and } x_l \leq z_l \text{ for all } l = 1,...,m,$$

$$D_{-}f(\cdot,x)(\hat{a}) \geq 0 \leq D_{-}f(\cdot,z)(\check{a}), \tag{3}$$

and

$$D_{-}f(\cdot,\gamma x + (1-\gamma)z)(\gamma\hat{a} + (1-\gamma)\check{a}) < 0. \tag{4}$$

By the strict pseudoconcavity of $f(\cdot,x)$ and $f(\cdot,z)$, (3) implies

$$\beta(x) \geq \hat{a} \text{ and } \beta(z) \geq \check{a}.$$

Thus $\gamma\beta(x) \geq \gamma\hat{a}$ and $(1-\gamma)\beta(z) \geq (1-\gamma)\check{a}$, and hence

$$\gamma\hat{a} + (1-\gamma)\check{a} \leq \gamma\beta(x) + (1-\gamma)\beta(z).$$

By part (iii) of Remark 6 and Corollary 2.5.2 in [4]—reasoning as in the proof of the *only if part* of Theorem 2—we have that

$$\beta(\gamma x + (1-\gamma)z) < \gamma\hat{a} + (1-\gamma)\check{a}$$

since $f(\cdot,\gamma x + (1-\gamma)z)$ is upper semicontinuous and strictly pseudoconcave and (4) holds true. But then

$$\beta(\gamma x + (1-\gamma)z) < \gamma\beta(x) + (1-\gamma)\beta(z),$$

in contradiction with the chain-concavity of β. ∎

5 Uniqueness of Nash equilibria

By a **game** Γ we mean a triple $\left(N, (S_i)_{i\in N}, (u_i)_{i\in N}\right)$ where $N = \{1, ..., n\}$ is the **set of players** (thus we are tacitly assuming also that N is finite and $n > 1$), $S_i \neq \emptyset$ is **player i's strategy set** and $u_i : \prod_{i\in N} S_i \to \mathbb{R}$ is **player i's utility function**. We denote by S the **joint strategy set** $\prod_{i\in N} S_i$ and by S_{-i} the **joint strategy set of i's opponents** $\prod_{l\in N\setminus\{i\}} S_l$. Sometimes, an element of S_{-i} is denoted by s_{-i} and we write (s_i, s_{-i}) instead of s.

Definition 13 *We say that a game Γ is a **nice game** if, for all $i \in N$:*

- *S_i is a proper closed real interval with a minimum;*
- *u_i is upper semicontinuous in the i-th argument;*
- *u_i is strictly pseudoconcave in the i-th argument.*

Definition 14 *A nice game Γ is a **compact nice game** if each S_i is compact. A nice game Γ is an **unbounded nice game** if each S_i is upper unbounded.*

Notation (ω,α) *The least joint strategy $(\min S_i)_{i\in N}$ of a nice game is denoted by ω and the greatest joint strategy $(\max S_i)_{i\in N}$ of a compact nice game by α.*

Notation ($\mathcal{D}_-^{u_i}$,$\mathcal{D}_{u_i}^+$) *Given a nice game Γ and $i \in N$, we denote*

- *player i's* **lower left marginal utility function** *by*

$$\mathcal{D}_-^{u_i} : \mathrm{int}\,(S_i) \times S_{-i} \to \overline{\mathbb{R}} : (s_i, s_{-i}) \mapsto D_- u_i\,(\cdot, s_{-i})\,(s_i)\,,$$

- *player i's* **upper right marginal utility function** *by*

$$\mathcal{D}_{u_i}^+ : \mathrm{int}\,(S_i) \times S_{-i} \to \overline{\mathbb{R}} : (s_i, s_{-i}) \mapsto D^+ u_i\,(\cdot, s_{-i})\,(s_i)\,.$$

Notation ($\mathcal{D}_-^{\bar{u}_i}$,$\mathcal{D}_{\bar{u}_i}^+$) *Given a nice game Γ and $i \in N$, we denote*

- *player i's* **extended lower left marginal utility function** *by*

$$\mathcal{D}_-^{\bar{u}_i} : (S_i \setminus \{\inf S_i\}) \times S_{-i} \to \overline{\mathbb{R}} : (s_i, s_{-i}) \mapsto D_- u_i\,(\cdot, s_{-i})\,(s_i)\,,$$

- *player i's* **extended upper right marginal utility function** *by*

$$\mathcal{D}_{\bar{u}_i}^+ : (S_i \setminus \{\sup S_i\}) \times S_{-i} \to \overline{\mathbb{R}} : (s_i, s_{-i}) \mapsto D^+ u_i\,(\cdot, s_{-i})\,(s_i)\,.$$

The definition of a smooth game used in the Introduction is the following. (Note that, despite our terminology, a player's "smooth" utility function can well be discontinuous in the opponents' strategies.)

Definition 15 *Let Γ be a nice game. We say that Γ is a* ***smooth nice game*** *if $u_i(\cdot, s_{-i})$ has a differentiable extension $v_i(\cdot, s_{-i})$ to some open superset of S_i, for all $s_{-i} \in S_{-i}$ and for all $i \in N$. Given a smooth nice game Γ and chosen a differentiable extension $v_i(\cdot, s_{-i})$ of $u_i(\cdot, s_{-i})$ for each $i \in N$ and for each $s_{-i} \in S_{-i}$, the function*

$$\mathcal{M}_i : S \to \mathbb{R} : s \mapsto \frac{\partial v_i}{\partial s_i}(s)$$

is called player i's ***marginal utility function.***

As usual, a (pure strategy) **Nash equilibrium** is a fixpoint of the set-valued **joint best reply function**

$$b : S \to \prod_{i \in N} 2^{S_i} : s \mapsto (\arg\max u_i(\cdot, s_{-i}))_{i \in N},$$

that is, e is a Nash equilibrium for Γ if and only if $e_i \in b_i(e)$ for all $i \in N$. When **player i's best reply function** b_i is single-valued, such b_i can be understood as a function into S_i: this observation will be often used without further mention in the sequel of Sect. 5.

Remark 13 *In any nice game player i's best reply b_i can be understood as a partial function $b_i : S \nrightarrow S_i$ defined by $\{b_i(s)\} = \arg\max u_i(\cdot, s_{-i})$ whenever $\arg\max u_i(\cdot, s_{-i}) \neq \emptyset$: recall that in any nice game $\arg\max u_i(\cdot, s_{-i})$ is either a singleton or the empty set (see Remark 6). Thus, when b_i is nonempty-valued—like, e.g., in compact nice games—such partial function is indeed a function $b_i : S \to S_i$ defined by $\{b_i(s)\} = \arg\max u_i(\cdot, s_{-i})$.*

5.1 A characterization theorem

Corollaries A1 and A2 in Appendix A state two fixpoint uniqueness results for a self-map of a finite Cartesian product of compact proper real intervals; but as a matter of fact, the two Corollaries provide also sufficient conditions on the joint best reply function of a compact nice game for the existence of exactly one Nash equilibrium. In Sect. 4 we have characterized such conditions in terms of "generalized marginal utilities"; Theorem 4 and its Corollary 2 readily follow from these characterizations.

Theorem 4 *Let Γ be a compact nice game and $i \in N$. The best reply function b_i is (i) isotone, (ii) chain-concave and (iii) greater than ω_i if and only if:*

H1. $\mathcal{D}_{-}^{u_i}$ *is quasiincreasing in the j-th argument, for all $j \in N \setminus \{i\}$;*

H2. $\mathcal{D}_{-}^{u_i}$ *has a chain-convex upper level set at height 0;*

H3. $\mathcal{D}_{\bar{u}_i}^{+}$ *is positive at the least joint strategy ω.*

Proof. Let (A, B, f) be the CP where $A = S_i$, $B = S_{-i}$ and f is defined by $f(s_i, s_{-i}) = u_i(s_i, s_{-i})$. Pick an arbitrary $x \in S_i$ and note that the function $\beta : S_{-i} \to S_i : s_{-i} \mapsto b_i(x, s_{-i})$ is the C-function for (A, B, f). Note also that β is isotone (resp. chain-concave, greater than ω_i) if and only if so is b_i.

If part. Suppose H1–3 hold. As H1 holds, β is isotone by Theorem 2; thus b_i is isotone. As β is isotone and H3 holds, β is greater than ω_i by Proposition 2; thus b_i is greater than ω_i. As β is isotone and greater than ω_i and H2 holds, β is chain-concave by Theorem 3; thus b_i is chain-concave.

Only if part. Suppose b_i is isotone, chain-concave and greater than ω_i; then so is also β. As β is isotone, H1 holds by Theorem 2. As β is isotone and greater than ω_i, H3 holds by Proposition 2. As β is isotone, chain-concave and greater than ω_i then H2 holds by Theorem 3. ■

Corollary 2 is only a “dual” reformulation of Theorem 4.

Corollary 2 *Let Γ be a compact nice game and $i \in N$. The best reply function b_i is (i) isotone, (ii) chain-convex and (iii) smaller than α_i if and only if:*

H1'. $\mathcal{D}^{+}_{u_i}$ is quasiincreasing in the j-th argument, for all $j \in N \setminus \{i\}$;

H2'. $\mathcal{D}^{+}_{u_i}$ has a chain-convex lower level set at height 0;

H3'. $\mathcal{D}^{\bar{u}_i}_{-}$ is negative at the greatest joint strategy α.

Proof. To prove the thesis for $\Gamma = \left(N, (S_l)_{l\in N}, (u_l)_{l\in N}\right)$, it suffices to consider the game $\left(N, (-S_l)_{l\in N}, (u_l \circ (-\operatorname{id}_{-S}))\right)$ and apply Theorem 4. ■

Theorem 5 is worth to be stated separately: its proof follows the same reasoning of that of Theorem 4 and is omitted.

Theorem 5 *Let Γ be a compact nice game and $i \in N$. The best reply function b_i is isotone if and only if H1 is satisfied for i.*

Example 1 shows compact nice games where H1 is satisfied for all players.

Example 1 *Put $X = [0, 1]$ and let Γ be a multiplayer game where, for all $i \in N$,*

$$S_i = X \text{ and } u_i(s) = g(s_{-i}) - d_i(s_i, f(s_{-i}))$$

for some function $g_i : S_{-i} \to \mathbb{R}$, some isotone function $f_i : S_{-i} \to X$ and some premetric[8] $d_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is lower semicontinuous in the first argument and strictly pseudoconvex in the first argument: e.g., letting $\gamma \geq 0$, $\delta \geq 0$ and $\lambda > 0$, we might have that

$$d_i(x, y) = \lambda |x - y| + \gamma R(x - y) + \delta y R(x - y)$$

or that

$$d_i(x, y) = |x - y|^{\lambda} + \gamma H(x - y) + \delta y R(x - y)$$

[8] A premetric $d_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a nonnegative function such that $d_i(x, x) = 0$ for all $x \in \mathbb{R}$.

(where R denotes the so-called ramp function $R : \mathbb{R} \to \mathbb{R}$ defined by $R(z) = \max(0, z)$ and H denotes the so-called Heavyside step function $H : \mathbb{R} \to \mathbb{R}$ defined by $H(z) = 0$ if $z \leq 0$ and by $H(z) = 1$ if $z > 0$).

In the statement of Theorem 5 one can replace H1 with H1': this would be only an equivalent reformulation. Also, by reversing the order of the parameter set of the CP considered in Theorem 2, one readily obtains a necessary and sufficient condition[9] for the antitonicity of best replies in compact nice games: such a result, however, is only another reformulation of Theorem 5 and hence we omit it. The following consequence of Tarski's fixpoint theorem is recalled.

Remark 14 *Let Γ be a compact nice game where H1 is satisfied for all players. Then a greatest Nash equilibrium and a least Nash equilibrium exist.*

5.2 Bounded strategy sets

Theorem 6 readily follows from Theorem 4 and Corollary A1: we omit the proof.

Theorem 6 *Let Γ be a compact nice game where H1, H2 and H3 are satisfied for all $i \in N$. Then Γ has exactly one Nash equilibrium.*

Theorem 7 readily follows from Corollaries 2 and A2: we omit the proof.

Theorem 7 *Let Γ be a compact nice game where H1', H2' and H3' are satisfied for all $i \in N$. Then Γ has exactly one Nash equilibrium.*

Though evident, the following fact is explicitly remarked.

Remark 15 *In Theorem 6 no i-th component of the unique Nash equilibrium equals ω_i (an analogous remark holds for Theorem 8). Similarly, in Theorem 7 no i-th component of the unique Nash equilibrium equals α_i.*

Example 2 *Consider again Example 1 and additionally assume that each function f_i (resp. $1 - f_i$) is chain-concave and positive: a unique Nash equilibrium exists as H1, H2 and H3 (resp. H1', H2' and H3') are satisfied for all $i \in N$.*

5.3 Unbounded strategy sets

Theorems 8–9 extend Theorems 6–7 to the case of upper unbounded strategy sets: their proofs are contained in Appendix E.

Theorem 8 *Let Γ be an unbounded nice game where H1, H2 and H3 are satisfied for all $i \in N$. Suppose there exists s^* in the interior of S such that*

$$\mathcal{D}_-^{u_i}\left(s_i^*, s_{-i}^*\right) < 0, \quad \text{for all } i \in N. \tag{5}$$

Then Γ has exactly one Nash equilibrium.

[9] Which would be the quasidecreasingness of each $\mathcal{D}_-^{u_i}$ in the opponents' strategies.

We remark that Theorem 9 below is in no way a "dual" of Theorem 8.

Theorem 9 *Let Γ be an unbounded nice game where H1' and H2' are satisfied for all $i \in N$. Suppose*

$$\mathcal{D}_{-}^{u_i}(t, \ldots, t) < 0 \text{ for all } i \in N, \text{ for all sufficiently large } t \in \mathbb{R}_{++}. \tag{6}$$

Then Γ has exactly one Nash equilibrium.

Example 3 *Consider again Example 1. Replace the assumption $X = [0, 1]$ with the assumption $X = \mathbb{R}_+$, leaving unaltered all the other conditions. Additionally assume that each f_i is positive and chain-concave (resp. chain-convex). Finally, assume that $f_i(x, \ldots, x) < x$ for all sufficiently large x, for all $i \in N$. Then Theorem 8 (resp. Theorem 9) ensure the existence of a unique Nash equilibrium.*

5.4 Further examples and relation to other results

Theorems 6–9 can be certainly applied to games on networks: conditions H1–3 and H1'–3' are compatible with a utility function u_i that is constant in the strategy s_l of some player $l \neq i$ that does not belong to player i's neighbourhood $N_i \subseteq N \setminus \{i\}$. It should be clear, however, that this compatibility would not have occurred in general if, for instance, in H2 the condition "$\mathcal{D}_{-}^{u_i}$ has a chain-convex upper level set at height 0" had been the much stronger "$\mathcal{D}_{-}^{u_i}$ is strictly concave" or in H1 the condition "$\mathcal{D}_{-}^{u_i}$ is quasiincreasing in the j-th argument, for all $j \in N \setminus \{i\}$" had been the much stronger "$\mathcal{D}_{-}^{u_i}$ is strictly increasing in the j-th argument, for all $j \in N \setminus \{i\}$". Examples 4-5 below are conceived as possible examples of games on networks (note that the "functional form" of u_i in Examples 4-5 is similar to that defined by (2) in [3]); but the structure of the system of neighbours in the network is not important for the application of our Nash equilibrium uniqueness results, and hence we shall not mention it.

Example 4 *Let Γ be a game where each $S_i = [0, \alpha_i]$ (with $\alpha_i \in \mathbb{R}_{++}$) and each u_i is defined by*

$$u_i : s \mapsto f_i(\sigma_i(s_{-i}))\, s_i + \gamma_i s_i - \delta_i s_i^{\mu_i}$$

for some isotone chain-concave function $\sigma_i : S_{-i} \to \mathbb{R}_+$ and some concave[10] *function $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ and with*

$$\gamma_i > 0,\ \delta_i > 0 \text{ and } \mu_i \geq 2.$$

Then Γ satisfies all conditions of Theorem 6: each $u_i(\cdot, s_{-i})$ is strictly concave and continuous; each $u_i(\cdot, \omega_{-i})$ is not decreasing (hence $D^+ u_i(\cdot, \omega_{-i})(\omega_i) > 0$ by the strict concavity of $u_i(\cdot, \omega_{-i})$); each function $\operatorname{int}(S_i) \times S_{-i} \to \mathbb{R} : s \mapsto D_- u_i(\cdot, s_{-i})(s_i)$ is increasing in every argument $j \neq i$ and chain-concave.

[10] Recall that any concave function $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ is necessarily increasing. Also, recall that Proposition 1 guarantees that $f_i \circ \sigma_i : S_{-i} \to \mathbb{R}_+$ is chain-concave and isotone.

Example 5 *Example 4 can be readily generalized. Let Γ be a game where each $S_i = [0, \alpha_i]$ (with $\alpha_i \in \mathbb{R}_{++}$) and—adopting the convention $0^0 = 1$—each u_i is defined by*

$$u_i : s \mapsto f_i\left(\sigma_i\left(s_{-i}\right)\right) s_i^{\lambda_i} + \gamma_i s_i - \delta_i s_i^{\mu_i}$$

for some isotone chain-concave function $\sigma_i : S_{-i} \to \mathbb{R}_+$ and some concave function $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ and with $\lambda_i \in [0, 1]$, $\gamma_i > 0$, $\delta_i > 0$ and $\mu_i \geq 2$. Then Γ satisfies all conditions of Theorem 6: Appendix F proves this claim.[11]

Theorem 6 does not follow from Theorem 2 in [29]: Remark 16 clarifies.

Remark 16 *Theorem 2 in [29] does not guarantee the existence of exactly one Nash equilibrium for some games described by Example 4 (and hence, more generally, satisfying the conditions of Theorem 6). This is evident, for instance, if we consider the game Γ where $N = \{1, 2\}$, $S_1 = S_2 = [0, 1]$,*

$$u_1\left(s_1, s_2\right) = s_1\left(1 + s_2\right) - 2s_1^3 \quad \text{and} \quad u_2\left(s_1, s_2\right) = s_2\left(1 + s_1\right) - 2s_2^3.$$

This is the particular game described in Example 4 where $N = \{1, 2\}$ and for all $i \in N$: $\gamma_i = 1$, $\delta_i = 2$, $\mu_i = 3$, $f_i = \mathrm{id}_{\mathbb{R}_+}$ and $\sigma_i : s_{-i} \mapsto \sum_{l \in N_i} s_l$ with $N_i = N \setminus \{i\}$. Pick a player $i \in N$ of this symmetric game: the function

$$\xi :]0, 1[\to \mathbb{R} : t \mapsto \frac{\partial u_i}{\partial s_i}(t, t)$$

is strictly increasing on $]0, 1/12[$ and we can conclude that Γ does not satisfy the assumptions of Theorem 2 in [29] since those assumptions would imply the decreasingness of ξ (on the entire $]0, 1[$).

Let us now consider an "unbounded" version of Example 4.

Example 6 *Consider again Example 4 and suppose each $\mu_i \neq 2$. Replace the assumption that each S_i is a compact proper interval with $S_i = \mathbb{R}_+$. Now Γ satisfies all conditions of Theorem 8. (Note that $f_i \circ \sigma_i$ is isotone concave and nonnegative on $L_k = \{(x, \ldots, x) : x \geq k\} \subseteq \mathbb{R}^{n-1}$ for all $k > 0$, thus $(f_i \circ \sigma_i)|_{L_k}$ is Lipschitz continuous when $k > 0$ and $\mathcal{D}_-^{u_i}(x, \ldots, x) < 0$ for some large $x > 0$.)*

Remark 17 clarifies that Theorem 8 does not follow from Theorem 5.1 in [18] or from Theorem 4.1 in [10].

Remark 17 *Reconsider the game described in Remark 16, but now put $S_1 = S_2 = \mathbb{R}_+$. Such a modified game Γ is certainly compatible with the conditions of Example 6 (and, more generally, with the conditions of our Theorem 8). Pick a player $i \in N$ of this modified symmetric game Γ: the function*

$$\xi : \mathbb{R}_{++} \to \mathbb{R} : t \mapsto \frac{\partial u_i}{\partial s_i}(t, t)$$

[11] Note that some condition listed at the end of Example 4 need not be satisfied when $\lambda_i \neq 1$.

is strictly increasing on $]0,1/12[$ *and*

$$\frac{\partial^2 u_1}{\partial s_1 \partial s_1}(1/24, 1/24) = -\frac{1}{2} < 1 = \frac{\partial^2 u_1}{\partial s_1 \partial s_2}(1/24, 1/24).$$

Thus Γ *does not satisfy the conditions of Theorem 5.1 in [18] since those conditions would imply the decreasingness of* ξ *(on the entire* $\mathbb{R}_{++}$*); also,* Γ *does not satisfy the conditions of Theorem 4.1 in [10] since those conditions would imply*

$$\left|\frac{\partial^2 u_1}{\partial s_1 \partial s_1}(1/24, 1/24)\right| > \left|\frac{\partial^2 u_1}{\partial s_1 \partial s_2}(1/24, 1/24)\right|.$$

6 Incomplete information

Some of our equilibrium uniqueness results extend to certain incomplete information games. Following the interim formulation of the Bayesian game in Sect. 3 of [34], we show a possible extension to Bayesian games with finite types.

Definition 16 *A* ***Bayesian game*** *is a quintuple*

$$G = \left(M, (Z_l)_{l\in M}, (T_l)_{l\in M}, ((p_l(\cdot|\theta))_{\theta\in T_l})_{l\in M}, (\upsilon_l)_{l\in M}\right)$$

where $M = \{1, ..., m\}$ *is a finite set of elements called players and for all* $l \in M$*:*

- Z_l *is a nonempty set of elements called player* l*'s actions;*
- T_l *is a nonempty finite set of elements called player* l*'s types;*
- $p_l(\cdot|\theta) : T_{-l} \to [0,1]$ *is a probability measure*[12] *on* T_{-l}*, for all* $\theta \in T_l$*;*
- $\upsilon_l : Z_l \times Z_{-l} \times T_l \times T_{-l} \to \mathbb{R}$ *associates a payoff to player* l *with each joint action* (z_l, z_{-l}) *in* $Z_l \times Z_{-l}$ *and each joint type* $(t_l, t_{-l}) \in T_l \times T_{-l}$*.*

To avoid confusion, we clarify that $m > 1$ and that

$$T_{-l} := \prod_{k\in M\setminus\{l\}} T_k \quad \text{and} \quad Z_{-l} := \prod_{k\in M\setminus\{l\}} Z_k.$$

Definition 17 *A* ***Bayesian Nash equilibrium*** *for a Bayesian game* G *is an* m*-tuple* $\sigma = (\sigma_l : T_l \to Z_l)_{l\in M}$ *of functions such that, for all* $l \in M$*,*

$$\sigma_l(\theta) \in \arg\max \sum_{\tau\in T_{-l}} \upsilon_l(\cdot, \sigma_{-l}(\tau), \theta, \tau) \cdot p_l(\tau|\theta) \;\; \textit{for all } \theta \in T_l$$

where $\sigma_{-l}(\tau) = (\sigma_k(\tau_k))_{k\in M\setminus\{l\}}$*.*

[12] Henceforth we shall write $p_l(\tau|\theta)$ instead of $p_l(\cdot|\theta)(\tau)$. Clearly, $\sum_{\tau\in T_{-l}} p_l(\tau|\theta) = 1$. One might interpret $p_l(\tau|\theta)$ as the conditional probability for l that the joint type of l's opponents is τ when l's type is θ. However such an interpretation is not very important here.

Definition 18 *Let* $\Gamma = \left(N, (S_i)_{i\in N}, (u_i)_{i\in N}\right)$ *be a (complete information) game. We say that* Γ *satisfies* ***property P*** *if, for all* $i \in N$*:*

- S_i *is a compact proper real interval with minimum* ω_i*;*
- *each function* $u_i(\cdot, s_{-i})$ *is (i) strictly concave and (ii) continuous;*
- *the function* $u_i(\cdot, \omega_{-i})$ *is not decreasing (where* $\omega_{-i} = (\omega_l)_{l\in N\setminus\{i\}}$*);*
- *the function* $\operatorname{int}(S_i) \times S_{-i} \to \mathbb{R} : s \mapsto D_- u_i(\cdot, s_{-i})(s_i)$ *is (i) increasing in every argument* $j \neq i$ *and (ii) chain-concave.*

If a game satisfies property P then $D^+ u_i(\cdot, \omega_{-i})(\omega_i) > 0$ (as $u_i(\cdot, \omega_{-i})$ is strictly concave and not decreasing) and

$$D_- u_i(\cdot, s_{-i})(s_i) = D^- u_i(\cdot, s_{-i})(s_i)$$

whenever $s_i \in \operatorname{int} S_i$. Noted this, one can readily verify that the use of the "Selten trick" allows to infer Corollary 3 from Theorem 6: other corollaries can be inferred from Theorems 7-9 and are left to the reader. Clearly, the use of such a "trick" is allowed by our definition of a Bayesian game which is restricted to the particular case of a finite set of players with finite sets of types.

Corollary 3 *Let* G *be a Bayesian game where, for each joint type* $t \in \prod_{i\in M} T_i$*, the (complete information) game*

$$\Gamma^{(t)} = \left(M, (Z_i)_{i\in M}, (\upsilon_i(\cdot, \cdot, t_i, t_{-i}))_{i\in M}\right)$$

satisfies property P. Then G *has exactly one Bayesian Nash equilibrium.*

Using Corollary 3 one can easily specify classes of Bayesian games with exactly one Bayesian Nash equilibrium like, for instance, in Example 7 below.

Example 7 *Let* G *be a Bayesian game where, for all* $t \in \prod_{i\in M} T_i$*, the (complete information) game* $\left(M, (Z_i)_{i\in M}, (\upsilon_i(\cdot, \cdot, t_i, t_{-i}))_{i\in M}\right)$ *is specified like in Example 4. Then* G *has exactly one Bayesian Nash equilibrium by Corollary 3.*

Acknowledgments We heartily thank Prof. Jacqueline Morgan for proposing the problem of concavity of best replies. We also thank the audiences of GMA2015 and SING11-GTM2015 for comments. The first and second author gratefully acknowledge financial support from, respectively, POR Campania FSE 2007-2013/ POR Campania FSE 2014-2020 and Programma STAR Napoli Call 2013 89 "Equilibrium with ambiguity" (financially supported by UniNA and Compagnia di San Paolo).

Appendix A: Fixpoint uniqueness

Theorem A1 *Let I be a finite index set, $\{F_i\}_{i\in I}$ be a family of compact proper real intervals and f be a self-map of $F = \prod_{i\in I} F_i$. Suppose that each component function f_i of f is isotone and chain-concave and that f has no fixpoints in*

$$F^{\llcorner} := \{t \in F : \min\{t_i - \min F_i : i \in I\} = 0\}.$$

Then f has exactly one fixpoint.

Proof. Each $(F_i, \leq)$ is a complete lattice, where $\leq$ denotes the usual partial order relation[13] on $\mathbb{R}$ induced on F_i. Denote by $\preceq$ the usual product partial order relation on F. Also $(F, \preceq)$ is a complete lattice. By Tarski's fixpoint theorem there exist a least fixpoint for f, say y, and a greatest fixpoint for f, say z. We are done if we prove that $y = z$. By contradiction, suppose $y \neq z$. Note that

$$\min F_i < y_i \leq z_i \text{ for all } i \in I,$$

where the first inequality holds because f has no fixpoints in $F^{\llcorner}$ and the second because z is the greatest fixpoint for f. Let

$$y_{\downarrow} := \{t \in F : t \preceq y\} \text{ and } y_{\downarrow\downarrow} := y_{\downarrow} \setminus \{y\},$$

and let $\operatorname{aff}(\{y, z\})$ denote the affine hull of $\{y, z\}$. The finiteness of I guarantees that $\operatorname{aff}(\{y, z\}) \cap y_{\downarrow\downarrow} \neq \emptyset$.[14] Pick

$$x \in (\operatorname{aff}(\{y, z\}) \cap y_{\downarrow\downarrow})$$

and let $\gamma \in]0, 1[$ be such that

$$y = \gamma x + (1 - \gamma) z.$$

By Tarski's fixpoint theorem (see the last equality in the statement of Theorem 1 in [31]), $f(t) \npreceq t$ for all $t \in y_{\downarrow\downarrow}$. Then

$$x_l < f_l(x) \text{ for some } l \in I.$$

Since $f_l(y) - y_l = f_l(z) - z_l = 0 < f_l(x) - x_l$, we have

$$f_l(y) - y_l < \gamma (f_l(x) - x_l) + (1 - \gamma)(f_l(z) - z_l);$$

hence, since $y_l = \gamma x_l + (1 - \gamma) z_l$, we have

$$f_l(y) < \gamma f_l(x) + (1 - \gamma) f_l(z).$$

But the last strict inequality contradicts the chain-concavity of f_l. ■

For clarity, when we shall write that "f_i is greater than ω_i" and that "f_i is smaller than α_i" in the statements of Corollaries A1–2 we shall respectively mean that "$f_i(x) > \omega_i$ for all $x \in F$" and that "$f_i(x) < \alpha_i$ for all $x \in F$".

[13] The lack of an index for $\leq$ (i.e., the fact that we write $\leq$ instead of the more correct $\leq_i$) should not be a source of confusion.

[14] Indeed, one might reason as follows: put $I^+ = \{i \in I : z_i - y_i > 0\}$—where $I^+ \neq \emptyset$—and $\lambda = \min\left\{\frac{y_i - \min F_i}{z_i - y_i} : i \in I^+\right\}$ and note that $\lambda > 0$ and $y - \frac{\lambda}{2}(z - y) \in (\operatorname{aff}(\{y, z\}) \cap y_{\downarrow\downarrow})$.

Corollary A1 *Let I be a finite nonempty index set and $\{F_i\}_{i\in I}$ be a family of compact proper real intervals. Let f be a self-map of $F = \prod_{i\in I} F_i$ and denote by ω the least element of F. Then f has exactly one fixpoint if each component function f_i is (i) isotone,(ii) chain-concave and (iii) greater than ω_i.*

Corollary A1, and a fortiori Theorem A1, cannot be directly inferred from Theorem 3.1 in [19] for at least two reasons: in Theorem 3.1 in [19] the domain is unbounded and $f - \mathrm{id}$ is "strictly R-concave" while in Corollary A1 the domain is bounded and $f - \mathrm{id}$ need not be "strictly R-concave" (e.g., the self-map of $[-2, 1] \times [-1, 1]$ defined by $f : (x_1, x_2) \mapsto (x_2, x_1/2 + 1/4)$ satisfies all conditions[15] of Corollary A1 but no extension of f to $\mathbb{R}^{|I|}$ can be "strictly R-concave" in the precise sense of Definition 2.1 in [19] because $f(1/2, 1/2) - (1/2, 1/2) = 0$ and $f_1(\lambda, \lambda) - \lambda = 0$ for all $\lambda \in [0, 1/2]$).

The following Corollary A2 is nothing but the "dual" of Corollary A1: its proof in fact consists of the reversion of the order of F.

Corollary A2 *Let I be a finite nonempty index set and $\{F_i\}_{i\in I}$ be a family of compact proper real intervals. Let f be a self-map of $F = \prod_{i\in I} F_i$ and denote by α the greatest element of F. Then f has exactly one fixpoint if each component function f_i is (i) isotone, (ii) chain-convex and (iii) smaller than α_i.*

Appendix B: Concavity of a C-function

We prove a variant of Theorem 3 about the concavity of a C-function β on the support of the NC-function β^*. The variant is established without preliminary assumptions on the isotonicity of β and the positivity of β^*. Sufficient conditions for the concavity of β can be easily derived by applying Proposition B1.

Theorem B1 *Consider a CP and the associated functions β and β^*. Suppose B is convex. Then β^* has convex support and β is concave thereon if and only if $\mathcal{D}_f$ has a convex upper level set at height 0.*

Proof. Without loss of generality, we shall put $\min A = 0$. Thus $\beta = \beta^*$.

If part. Suppose the upper level set at height 0 of $\mathcal{D}_f$ is convex. Choose x and z in B such that

$$\xi := \beta(x) > 0 < \beta(z) =: \zeta.$$

(Therefore $\min\{\xi, \zeta\} > 0 = \min A$.) Pick $\gamma \in]0, 1[$ and put $y := \gamma x + (1 - \gamma) z$. We are done if we prove that

$$\overline{\upsilon} := \gamma\xi + (1 - \gamma)\zeta \leq \beta(y) =: \upsilon.$$

[15] Alternatively, one might also consider the self-map f of $F = [-2, 1] \times [-1, 2]$ defined by

$$f : (x_1, x_2) \mapsto (\min\{x_2, 1\}, x_1/2 + 1/4),$$

noting that the first component of $f - \mathrm{id}$ (i.e., $F \to \mathbb{R} : (x_1, x_2) \mapsto \min\{x_2, 1\} - x_1$) is even constant in the second argument on the subset $[-2, 1] \times [1, 2]$ of its domain F.

Case $\min\{\xi,\zeta\} < \max A$. Suppose, to the contrary, that $\upsilon < \overline{\upsilon}$. Note that

$$D_{-}f(\cdot,y)(\overline{\upsilon}) < 0 \tag{7}$$

because $f(\cdot,y)$ is a strictly pseudoconcave function maximized at υ, with

$$\min A \leq \upsilon < \overline{\upsilon} < \max A.$$

Since ξ and ζ are respectively maximizers of $f(\cdot,x)$ and of $f(\cdot,z)$,

$$D_{-}f(\cdot,x)(\xi) \geq 0 \leq D_{-}f(\cdot,z)(\zeta)$$

and hence

$$0 \leq \min\{\mathcal{D}_f(\xi,x), \mathcal{D}_f(\zeta,z)\}.$$

Therefore (ξ,x) and (ζ,z) belong to the upper level set at height 0 of $\mathcal{D}_f$, and then so does also $(\overline{\upsilon},y)$ by the convexity of the upper level set at height 0 of $\mathcal{D}_f$. Thus

$$\mathcal{D}_f(\overline{\upsilon},y) = D_{-}f(\cdot,y)(\overline{\upsilon}) \geq 0$$

in contradiction with (7).

Case $\min\{\xi,\zeta\} \geq \max A$. In this case $\xi = \zeta = \max A$. By the strict pseudoconcavity of $f(\cdot,x)$ and $f(\cdot,z)$,

$$D^{+}f(\cdot,x)(a) > 0 < D^{+}f(\cdot,z)(a) \quad \text{for all } a \in \{\min A\} \cup \operatorname{int}(A).$$

By part (ii) of Theorem 1.13 in [11], $f(\cdot,x)$ and $f(\cdot,z)$ are increasing on $\operatorname{int}(A)$; consequently,

$$\mathcal{D}_f(a,x) = D_{-}f(\cdot,x)(a) \geq 0 \leq D_{-}f(\cdot,z)(a) = \mathcal{D}_f(a,z) \quad \text{for all } a \in \operatorname{int}(A)$$

and hence

$$\mathcal{D}_f(a,y) = D_{-}f(\cdot,y)(a) \geq 0 \quad \text{for all } a \in \operatorname{int}(A)$$

by the convexity of the upper level set at height 0 of $\mathcal{D}_f$. Thus $\upsilon = \beta(y) = \max A = \overline{\upsilon}$: otherwise $\beta(y) \in \{\min A\} \cup \operatorname{int}(A)$ and $D_{-}f(\cdot,y)(a) \geq 0$ for some $a \in \left]\beta(y), \max A\right[$ in contradiction with the strict pseudoconcavity of $f(\cdot,y)$.

Only if part. Suppose β has convex support and is concave thereon. By contradiction, suppose the upper level set at height 0 of $\mathcal{D}_f$ is not convex. Then there exist $(\hat{a},x) \in \operatorname{int}(A) \times B$, $(\check{a},z) \in \operatorname{int}(A) \times B$ and $\gamma \in \left]0,1\right[$ such that

$$D_{-}f(\cdot,x)(\hat{a}) \geq 0 \leq D_{-}f(\cdot,z)(\check{a}), \tag{8}$$

and

$$D_{-}f(\cdot,\gamma x + (1-\gamma)z)(\gamma\hat{a} + (1-\gamma)\check{a}) < 0. \tag{9}$$

By the strict pseudoconcavity of $f(\cdot,x)$ and $f(\cdot,z)$, (8) implies

$$\beta(x) \geq \hat{a} > \min A = 0 \text{ and } \beta(z) \geq \check{a} > \min A = 0.$$

Thus $\gamma\beta(x) \geq \gamma\hat{a}$ and $(1-\gamma)\beta(z) \geq (1-\gamma)\check{a}$, and hence

$$\gamma\hat{a} + (1-\gamma)\check{a} \leq \gamma\beta(x) + (1-\gamma)\beta(z).$$

Note that x and z belong to the support of β—which is convex—and hence so does also $\gamma x + (1-\gamma) z$. By part (iii) of Remark 6 and Corollary 2.5.2 in [4]—reasoning as in the proof of the *only if part* of Theorem 2—we have that

$$\beta(\gamma x + (1-\gamma) z) < \gamma\hat{a} + (1-\gamma)\check{a}$$

since $f(\cdot, \gamma x + (1-\gamma) z)$ is upper semicontinuous and strictly pseudoconcave and (9) holds true. But then

$$\beta(\gamma x + (1-\gamma) z) < \gamma\beta(x) + (1-\gamma)\beta(z),$$

in contradiction with the concavity of β on its support. ■

We show conditions for the support of β^* to coincide with B. Clearly, if the support of β^* coincides with B and β is concave thereon, then β is concave.

Proposition B1 *Consider a CP and the associated function β^*. The support of β^* is B if and only if $D^+ f(\cdot, b)(\min A) > 0$ for all $b \in B$.*
Proof. *If part.* An immediate consequence of the definition of $D^+ f(\cdot, b)$.

Only if part. Suppose the support of β^* is B. If $D^+ f(\cdot, \underline{b})(\min A) \leq 0$ for some $\underline{b} \in B$ then $f(\cdot, \underline{b})(\min A) > f(\cdot, \underline{b})(x)$ for all $x > \min A$ by the the definition of a strictly pseudoconcave function. Hence $\underline{b} \in B$ would not be in the support of β^*: a contradiction. ■

Appendix C: An equivalence lemma

Definition 19 *A function $f : X \subseteq \mathbb{R} \to \overline{\mathbb{R}}$ is $\uparrow$-pseudoincreasing iff*

$$(\underline{x}, \overline{x}) \in X \times X,\ \underline{x} < \overline{x} \text{ and } f(\underline{x}) \geq 0 \Rightarrow f(\overline{x}) \geq 0.$$

If $f : X \subseteq \mathbb{R} \to \overline{\mathbb{R}}$ is $\uparrow$-pseudoincreasing then f is quasiincreasing; however, the converse is generally false. We now establish a particular equivalence result.

Lemma C1 *Let $A \subseteq \mathbb{R}$ be a proper interval, $B \subseteq \mathbb{R}^m$ be the Cartesian product of m nonempty subsets of $\mathbb{R}$ and $f : A \times B \to \mathbb{R}$. Suppose $f(\cdot, b)$ is strictly pseudoconcave and upper semicontinuous for all $b \in B$. Let $L = \{1, ..., m\}$ and*

$$\mathcal{D}_f : \operatorname{int}(A) \times B \to \overline{\mathbb{R}} : (a, b) \mapsto D_- f(\cdot, b)(a).$$

Then assertions A1, A2 and A3 are equivalent.

A1. *$\mathcal{D}_f(a, \cdot)$ is quasiincreasing in every argument for all $a \in \operatorname{int}(A)$.*

A2. *$\mathcal{D}_f(a, \cdot)$ is $\uparrow$-pseudoincreasing in every argument for all $a \in \operatorname{int}(A)$.*

A3. *The following implication is true:*

$$a \in \operatorname{int}(A), \ (\underline{x}, \overline{x}) \in B \times B, \ \underline{x}_l \leq \overline{x}_l \text{ for all } l \in L \text{ and } \mathcal{D}_f(a, \cdot)(\underline{x}) \geq 0$$
$$\Downarrow \tag{10}$$
$$\mathcal{D}_f(a, \cdot)(\overline{x}) \geq 0.$$

Proof. Proof of A1 $\Rightarrow$ A2. Suppose that $\mathcal{D}_f(a, \cdot)$ is quasiincreasing in every argument for all $a \in \operatorname{int}(A)$. Then, equivalently, $\mathcal{D}_f(a, \cdot)$ is quasiincreasing in the l-th argument for all $l \in L$, for all $a \in \operatorname{int}(A)$. Now, by contradiction, suppose there exists $i \in L$ and $\check{a} \in \operatorname{int}(A)$ such that $\mathcal{D}_f(\check{a}, \cdot)$ is not $\uparrow$-pseudoincreasing in the i-th argument. Then there exists a pair $(x^*, x^{**}) \in B \times B$ such that

$$x_i^* \leq x_i^{**}, \, x_l^* = x_l^{**} \text{ for all } l \in L \backslash \{i\} \text{ and } \mathcal{D}_f(\check{a}, \cdot)(x^*) \geq 0 > \mathcal{D}_f(\check{a}, \cdot)(x^{**}).$$

By the quasiincreasingness of $\mathcal{D}_f(\check{a}, \cdot)$ in the i-th argument, we must have that

$$\mathcal{D}_f(\check{a}, \cdot)(x^*) = 0. \tag{11}$$

As $\mathcal{D}_f(\check{a}, \cdot)(x^{**}) < 0$, there exists $\underline{a} \in A$ such that $\underline{a} < \check{a}$ and $f(\cdot, x^{**})(\underline{a}) > f(\cdot, x^{**})(\check{a})$. Thus there exists $\hat{a} \in [\underline{a}, \check{a}[$ that maximizes the upper semicontinuous (and strictly pseudoconcave) function $f(\cdot, x^{**})|_{[\underline{a}, \check{a}]}$; hence

$$\mathcal{D}_f(\cdot, x^{**})(a) < 0 \text{ for all } a \in]\hat{a}, \check{a}[\tag{12}$$

by the strict pseudoconcavity of $f(\cdot, x^{**})|_{[\underline{a}, \check{a}]}$. As (11) is true, the strict pseudoconcavity of $f(\cdot, x^*)$ implies that $f(\cdot, x^*)(\hat{a}) < f(\cdot, x^*)(\check{a})$; thus

$$\mathcal{D}_f(\check{a}, \cdot)(x^*) = 0 < \frac{f(\cdot, x^*)(\check{a}) - f(\cdot, x^*)(\hat{a})}{\check{a} - \hat{a}} \tag{13}$$

and by part (ii) of Theorem 1.8 in [11]—Remark 18 clarifies why part (ii) of Theorem 1.8 in [11] can apply—there exists $a^\circ \in]\hat{a}, \check{a}[$ such that

$$\mathcal{D}_f(\cdot, x^*)(a^\circ) \geq \frac{f(\cdot, x^*)(\check{a}) - f(\cdot, x^*)(\hat{a})}{\check{a} - \hat{a}} > 0. \tag{14}$$

Thus, by (12) and (14), $\mathcal{D}_f(\cdot, x^*)(a^\circ) > 0 > \mathcal{D}_f(\cdot, x^{**})(a^\circ)$ with $a^\circ \in]\hat{a}, \check{a}[$. Equivalently—just changing the notation—we have that

$$\mathcal{D}_f(a^\circ, \cdot)(x^*) > 0 > \mathcal{D}_f(a^\circ, \cdot)(x^{**}) \text{ with } a^\circ \in]\hat{a}, \check{a}[\subseteq \operatorname{int}(A)$$

and hence that $\mathcal{D}_f(a^\circ, \cdot)$ is not quasiincreasing in the i-th argument: a contradiction with the assumption that $\mathcal{D}_f(a, \cdot)$ is quasiincreasing in the l-th argument for all $l \in L$, for all $a \in \operatorname{int}(A)$.

Proof of A2 $\Rightarrow$ A1. Suppose $\mathcal{D}_f(a, \cdot)$ is $\uparrow$-pseudoincreasing in every argument for all $a \in \operatorname{int}(A)$. Fix an arbitrary $i \in L$ and an arbitrary $a \in \operatorname{int}(A)$. Clearly, $\mathcal{D}_f(a, \cdot)$ is $\uparrow$-pseudoincreasing in the i-th argument. Then, as we have in fact already noted just before Lemma C1, $\mathcal{D}_f(a,\cdot)$ is also quasiincreasing in the i-th argument. As i is arbitrary in L and a is arbitrary in $\operatorname{int}(A)$, $\mathcal{D}_f(a,\cdot)$ is quasiincreasing in every argument for all $a \in \operatorname{int}(A)$.

Proof of A2 $\Rightarrow$ A3. Suppose $\mathcal{D}_f(a,\cdot)$ is $\uparrow$-pseudoincreasing in every argument for all $a \in \operatorname{int}(A)$. Then, equivalently, $\mathcal{D}_f(a,\cdot)$ is $\uparrow$-pseudoincreasing in the l-th argument for all $a \in \operatorname{int}(A)$, for all $l \in L$. Now fix $a \in \operatorname{int}(A)$ and $(\underline{x}, \overline{x}) \in B \times B$ and suppose $\underline{x}_l \leq \overline{x}_l$ for all $l \in L$ and $\mathcal{D}_f(a,\cdot)(\underline{x}) \geq 0$. Then there exists a "taxicab" sequence $\left(x^0, \ldots, x^m\right)$ in B such that $x^0 = \underline{x}$ and

$$x^l = x^{l-1} + (\overline{x}_l - \underline{x}_l) \cdot 1_{(l)} \text{ for all } l = 1, \ldots, m$$

where $1_{(l)} \in \mathbb{R}^m$ denotes the unit vector with the l-th component equal to one. Since $\mathcal{D}_f(a,\cdot)$ is $\uparrow$-pseudoincreasing in the l-th argument for all $l \in L$, we have

$$\mathcal{D}_f(a,\cdot)\left(x^l\right) \geq 0 \text{ and } l \in \{0, \ldots, m-1\} \Rightarrow \mathcal{D}_f(a,\cdot)\left(x^{l+1}\right) \geq 0.$$

As $x^0 = \underline{x}$, $x^m = \overline{x}$ and $\mathcal{D}_f(a,\cdot)(\underline{x}) \geq 0$, we infer that $\mathcal{D}_f(a,\cdot)(\overline{x}) \geq 0$. Thus implication (10) is true.

Proof of A3 $\Rightarrow$ A2. Suppose implication (10) is true. Fix an arbitrary $i \in L$ and an arbitrary $a \in \operatorname{int}(A)$. Then, by (10), $\mathcal{D}_f(a,\cdot)$ is $\uparrow$-pseudoincreasing in the i-th argument. As i is arbitrary in L and a is arbitrary in $\operatorname{int}(A)$, $D_f(a,\cdot)$ is $\uparrow$-pseudoincreasing in every argument for all $a \in \operatorname{int}(A)$. ■

Remark 18 *Note that (13) implies that*

$$f(\cdot, x^*)(t) - f(\cdot, x^*)(\hat{a}) > \frac{f(\cdot, x^*)(\check{a}) - f(\cdot, x^*)(\hat{a})}{\check{a} - \hat{a}}(t - \hat{a}) \text{ for some } t \in \left]\hat{a}, \check{a}\right[.$$

To check this fact, one might reason as follows. Were the previous inequality true for no $t \in \left]\hat{a}, \check{a}\right[$, one would have that

$$f(t, x^*) - f(\hat{a}, x^*) \leq \frac{f(\check{a}, x^*) - f(\hat{a}, x^*)}{\check{a} - \hat{a}}(t - \hat{a}) \text{ for all } t \in \left]\hat{a}, \check{a}\right[;$$

thus one would have also that

$$f(t, x^*) - f(\check{a}, x^*) \leq \frac{f(\check{a}, x^*) - f(\hat{a}, x^*)}{\check{a} - \hat{a}}(t - \hat{a}) - f(\check{a}, x^*) + f(\hat{a}, x^*)$$

for all $t \in \left]\hat{a}, \check{a}\right[$ and that

$$f(t, x^*) - f(\check{a}, x^*) \leq \frac{f(\check{a}, x^*) - f(\hat{a}, x^*)}{\check{a} - \hat{a}}(t - \check{a}) < 0$$

for all $t \in \left]\hat{a}, \check{a}\right[$; but the previous inequality would imply that

$$\frac{f(t, x^*) - f(\check{a}, x^*)}{t - \check{a}} \geq \frac{f(\check{a}, x^*) - f(\hat{a}, x^*)}{\check{a} - \hat{a}} > 0$$

for all $t \in \left]\hat{a}, \check{a}\right[$ and hence $\mathcal{D}_f(\check{a}, x^*) > 0$, which contradicts (13).

Appendix D: Relation to other isotonicity theorems

Theorem 4 in [23] provides sufficient—but not necessary—conditions for a CP to possess an isotone C-function β. Example 8 clarifies.

Example 8 *Consider the CP where* $A = [0, 10]$, $B = \{1, 2\}$, $f(\cdot, 1)(a) = 5 - |a - 5|$ *and* $f(\cdot, 2)(a) = 30 - 5|a - 6| - 3a$. *Theorem 2 guarantees the isotonicity of* β *(note, in particular, that*

$$\mathcal{D}_f(a, 1) = \begin{cases} 1 & \text{if } a \leq 5 \\ -1 & \text{if } a > 5 \end{cases} \quad \text{and} \quad \mathcal{D}_f(a, 2) = \begin{cases} 2 & \text{if } a \leq 6 \\ -8 & \text{if } a > 6 \end{cases}$$

and hence that $\mathcal{D}_f(a, \cdot)$ *is quasiincreasing in every argument for all* $a \in A$*). However* f *does not satisfy the single crossing property in* $(a; b)$ *and Theorem 4 in [23] does not guarantee the isotonicity of* β *(note, in particular that we have* $f(8, 1) - f(0, 1) > 0 > f(8, 2) - f(0, 2)$*).*

Analogous examples can show that the *if part* of our Theorem 2 does not follow from any Proposition or Theorem in [20] where at least one of the four conditions (7a), (7b), (7c), (7d) is involved.

From Theorem 1 in [28]—see also Theorem A in [27] and the discussion before it—one easily infers necessary and sufficient conditions for a CP to have an isotone C-function β. At the beginning of Sect. 2.4 in [28], the authors pointed out that such conditions need not be easily checked and with Proposition 2 in [28] they provided other simple sufficient conditions on the derivatives of the function involved in their maximization problem. The following Example 9 shows that Theorem 2 does not follow from Proposition 2 in [28].

Example 9 *Consider the CP where* $A = [-4, 4]$, $B = \{1, 2\}$ *and*

$$f(\cdot, b)(a) = -|a|^b.$$

Theorem 2 applies and guarantees that β *is increasing. For this CP there does not exist any positive increasing function* $\alpha : A \to \mathbb{R}_{++}$ *such that*

$$Df(\cdot, 2)(a) \geq \alpha(a) \cdot Df(\cdot, 1)(a) \text{ for almost all } a \in A,$$

otherwise we would have

$$\frac{Df(\cdot, 2)(x)}{Df(\cdot, 1)(x)} \geq \alpha(x) \geq \alpha(-3) > 0 \text{ for almost all } x \in]-3, 0[$$

(and $\frac{Df(\cdot,2)(x)}{Df(\cdot,1)(x)} \frac{1}{\alpha(-3)} \geq \frac{\alpha(x)}{\alpha(-3)} \geq 1$ *for a.a.* $x \in]-3, 0[$*) in contradiction with*

$$\limsup_{x \uparrow 0} \frac{Df(\cdot, 2)(x)}{Df(\cdot, 1)(x)} \frac{1}{\alpha(-3)} = \lim_{x \uparrow 0} \frac{-2x}{\alpha(-3)} = 0.$$

In fact, an immediate restatement of Theorem 2 can be used to check whether an IDO relation—in the sense of [28]—exists in some simple classes of IDO families where their Proposition 2 cannot be used.

Appendix E: Proofs of Theorems 8 and 9

Proof of Theorem 8. We split the proof into two parts: existence and uniqueness. In the first part we construct a new game Γ^* which has a common Nash equilibrium with $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$. In the second part we prove the existence of at most one Nash equilibrium for Γ.

Equilibrium existence. As usual, denote by b the joint best reply for Γ, but consider it as a partial function from S into S. Put

$$S_i^* = [\omega_i, s_i^*] \text{ for all } i \in N \text{ and } S^* = \prod_{i \in N} S_i^*.$$

As Γ is a nice game, (5) ensures that each b_i is nonempty-valued at s^*; in particular, we must have that $b_i(s^*) \in [\omega_i, s_i^*]$ for all $i \in N$. We can extend the previous conclusion to the entire S^* asserting that each $b_i|_{S^*}$ is a function into S_i^*: to verify this last fact it suffices to note that (5), condition H1 and Lemma C1 in Appendix C imply[16]

$$\mathcal{D}_-^{u_i}(z) < 0 \text{ for all } z \in S \text{ such that } z_i = s_i^* \text{ and } z_l \leq s_l^* \text{ for all } l \in N \setminus \{i\}$$

and to repeat the previous reasoning for s^* at any such z (considering also that $b_i(z) = b_i(s_i, z_{-i})$ for all $s_i \in S_i$ and for any such z). Hence $b|_{S^*}$ can be understood as a self-map on S^*; this fact in turn implies that $b|_{S^*}$ coincides with the joint best reply, call it b^*, of the game $\Gamma^* = (N, (S_i^*)_{i \in N}, (u_i^*)_{i \in N})$ where

$$u_i^* = u_i|_{S^*} \text{ for all } i \in N.$$

As $b|_{S^*} = b^*$, the fixpoints of b and b^* coincide on S^*; thus each Nash equilibrium for Γ^* is also a Nash equilibrium for Γ. It is easily seen that Γ^* satisfies all conditions of Theorem 6 and hence Γ^* has a (unique) Nash equilibrium.

Equilibrium uniqueness. Suppose there exist two distinct Nash equilibria $e^\bullet$ and e° for Γ. Let $\overline{\Gamma} = (N, (\overline{S}_i)_{i \in N}, (\overline{u}_i)_{i \in N})$ be the game where, for all $i \in N$,

$$\overline{S}_i = [\omega_i, \max\{e_i^\bullet, e_i^\circ\} + 1]$$

and $\overline{u}_i = u_i|_{\overline{S}}$ with $\overline{S} = \prod_{l \in N} \overline{S}_l$. As $\{e^\bullet, e^\circ\} \subseteq \overline{S} \subseteq S$, $e^\bullet$ and e° are distinct Nash equilibria also for $\overline{\Gamma}$. But then we have a contradiction, since $\overline{\Gamma}$ satisfies all conditions of Theorem 6 and hence it has exactly one Nash equilibrium. ■

Proof of Theorem 9. *Equilibrium existence.* By (6), there exists a point s^* in the topological interior of S such that

$$\mathcal{D}_-^{u_i}(s^*) < 0 \text{ for all } i \in N. \tag{15}$$

As already pointed out—see again the discussion after Example 1—conditions H1 and H1' are equivalent (if needed, reason as in the proof of Corollary 2). Now the proof of equilibrium existence is exactly the same proof of that of Theorem 8 above: just replace "(5)" with "(15)" and "Theorem 6" with "Theorem 7".

[16] To prove the implication identify A with S_i, B with S_{-i}, f with u_i and $\mathcal{D}_f$ with $\mathcal{D}_-^{u_i}$.

Equilibrium uniqueness. Suppose there exist two distinct Nash equilibria for Γ, say $e^{\bullet}$ and e°. Put

$$\underline{t} = \max\left\{|e_1^{\bullet}|,\ldots,|e_n^{\bullet}|,|e_1^{\circ}|,\ldots,|e_n^{\circ}|\right\}.$$

Choose $t > \underline{t}$ such that $\mathcal{D}_{-}^{u_i}(t,\ldots,t) < 0$ for all $i \in N$ (such a point t can be found by assumption) and put

$$\overline{\alpha} = (t,\ldots,t) \in \mathbb{R}_{++}^n.$$

Thus we have

$$\mathcal{D}_{-}^{u_i}(\overline{\alpha}_i,\overline{\alpha}_{-i}) < 0 \text{ for all } i \in N.$$

Let $\overline{\Gamma} = (N,(\overline{S}_i)_{i\in N},(\overline{u}_i)_{i\in N})$ be the game where, for all $i \in N$, $\overline{S}_i = [\omega_i,\overline{\alpha}_i]$ and $\overline{u}_i = u_i|_{\overline{S}}$ with $\overline{S} = \prod_{l\in N}\overline{S}_l$. As $\{e^{\bullet},e^{\circ}\} \subseteq \overline{S} \subseteq S$, $e^{\bullet}$ and e° are distinct Nash equilibria also for $\overline{\Gamma}$. But $\overline{\Gamma}$ satisfies all conditions of Theorem 7 and hence it cannot have two distinct Nash equilibria. ■

Appendix F: Proof of a claim in Example 5

We prove that the games in Example 5 satisfy all conditions of Theorem 6 when $\lambda_i \in [0,1]$. In fact, the case $\lambda_i = 1$ has been already discussed in Example 4. The case $\lambda_i = 0$ is evident. Let us consider the case $\lambda_i \in]0,1[$. Note that each $u_i(\cdot,s_{-i})$ is continuous and strictly concave, $\mathcal{D}_{\bar{u}_i}^{+}(\omega) = +\infty$ and

$$\mathcal{D}_{-}^{u_i}(s) = s_i^{\lambda_i-1}(\lambda_i f_i(\sigma_i(s_{-i})) + \gamma_i s_i^{1-\lambda_i} - \delta_i\mu_i s_i^{\mu_i-\lambda_i}).$$

Thus each $\mathcal{D}_{\bar{u}_i}^{+}$ is positive at the least joint strategy ω and—by virtue of Remarks 1 and 12—it can be inferred that each $\mathcal{D}_{-}^{u_i}$ is quasiincreasing in the j-th argument for all $j \in N\setminus\{i\}$ and that each $\mathcal{D}_{-}^{u_i}$ has a chain-convex upper level set at height 0 because $\mathcal{D}_{-}^{u_i}(s)$ is the product of the positive function

$$\operatorname{int}(S_i) \times S_{-i} \to \mathbb{R} : s \mapsto s_i^{\lambda_i-1}$$

and the chain-concave function

$$\operatorname{int}(S_i) \times S_{-i} \to \lambda_i f_i(\sigma_i(s_{-i})) + \gamma_i s_i^{1-\lambda_i} - \delta_i\mu_i s_i^{\mu_i-\lambda_i}$$

that is increasing in the j-th argument for all $j \in N\setminus\{i\}$.

References

[1] Agliardi, E.: A generalization of supermodularity. Economics Letters 68, 251–254 (2000)

[2] Baetz, O.: Social activity and network formation. Theoretical Economics 10, 315–340 (2015)

[3] Belhaj, M., Bramoullé, Y., Deroïan, F.: Network games under strategic complementarities. Games and Economic Behavior 88, 310–319 (2014)

[4] Cambini, A., Martein, L.: Generalized Convexity and Optimization: Theory and Applications. Springer-Verlag, Berlin Heidelberg (2009)

[5] Ceparano, M.C.: Nash equilibrium selection in multi–leader multi–follower games with vertical separation. PhD thesis, University of Naples "Federico II". (2015)

[6] Ceparano, M.C., Morgan, J.: Equilibria for multi-leader multi-follower games with vertical information: existence results. Working paper - https://ideas.repec.org/p/sef/csefwp/417.html (2015). Accessed October 2015

[7] Cottle, R.W., Pang, J.-S., Stone, R.E.: The linear complementarity problem. SIAM, Philadelphia (2009)

[8] Diewert, W.E.: Alternative characterizations of six kinds of quasiconcavity in the nondifferentiable case with applications to nonsmooth programming, In: Schaible, S., Ziemba, W.T. (eds.) Generalized Concavity in Optimization and Economics, pp. 51–93. Academic Press, New York (1981)

[9] Edlin, A.S., Shannon, C.: Strict monotonicity in comparative statics. Journal of Economic Theory 81, 201–219 (1998)

[10] Gabay, D., Moulin, H.: On the Uniqueness and Stability of Nash Equilibria in Noncooperative Games. In: Bensoussan, A., Kleindorfer, P.R., Tapiero, C.S. (eds.) Applied Stochastic Control in Econometrics and Management Science, pp. 271–293. North-Holland, Amsterdam (1980)

[11] Giorgi, G., Komlósi, S.: Dini derivatives in optimization - Part I. Rivista di matematica per le scienze economiche e sociali. 15, 3–30 (1992)

[12] Hadjisavvas, N., Schaible, S.: Generalized monotone single valued maps. In: Floudas, C.A., Pardalos, P.M. (eds.) Encyclopedia of Optimization, second edition, pp. 1197–1202. Springer, New York (2009)

[13] Hiller, T.: Peer effects in endogenous networks. Working paper - http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2478975 (2014). Accessed 11 August 2014

[14] Ivanov, V.I.: A note on strong pseudoconvexity. Central European Journal of Mathematics 6, 576–580 (2008)

[15] Ivanov, V.I.: Characterizations of pseudoconvex functions and semistrictly quasiconvex ones. Journal of Global Optimization 57, 677–693 (2013)

[16] Lagerås, A., Seim, D.: Strategic complementarities, network games and endogenous network formation. International Journal of Game Theory (*forthcoming*) (2015)

[17] Kannan, R., Krueger, C.K.: Advanced Analysis on the Real Line. Springer Verlag, New York (1996)

[18] Karamardian, S.: The nonlinear complementarity problem with applications, part 2. Journal of Optimization Theory and Applications 4, 167–181 (1969)

[19] Kennan, J.: Uniqueness of positive fixed points for increasing concave functions on $\mathbb{R}^n$: an elementary result. Review of Economic Dynamics 4, 893–899 (2001)

[20] Kukushkin, N.S.: Monotone comparative statics: changes in preferences versus changes in the feasible set. Economic Theory 52, 1039–1060 (2013)

[21] Milgrom, P., Roberts, J.: Rationalizability, learning and equilibrium in games with strategic complementarities. Econometrica 58, 1255–1277 (1990)

[22] Milgrom, P., Roberts, J.: Coalition-proofness and correlation with arbitrary communication possibilities. Games and Economic Behavior 17, 113–128 (1996)

[23] Milgrom, P., Shannon, C.: Monotone comparative statics. Econometrica 62, 157–180 (1994)

[24] Moulin, H.: Dominance solvability and Cournot stability. Mathematical Social Sciences 7, 83–102 (1984)

[25] Nash, J.F.: Non-cooperative games. Ph.D. Dissertation. Princeton University (1950)

[26] Quah, J.K.-H.: The comparative statics of constrained optimization problems. Econometrica 75, 401–431 (2007)

[27] Quah, J.K.-H., Shirai, K.: A revealed preference theory of monotone choice and strategic complementarity. Working paper - http://www.kier.kyoto-u.ac.jp/DP/DP914.pdf. Accessed February 2015

[28] Quah, J.K.-H., Strulovici, B.: Comparative statics, informativeness, and the interval dominance order. Econometrica 77, 1949–1992 (2009)

[29] Rosen, J.B.: Existence and uniqueness of equilibrium points for concave N-person games. Econometrica 33, 520–534 (1965)

[30] Shannon, C.: Weak and strong monotone comparative statics. Economic Theory 5, 209–227 (1995)

[31] Tarski, A.: A lattice-theoretical fixpoint theorem and its applications. Pacific Journal of Mathematics 5, 285–309 (1955)

[32] Topkis, D.M.: Minimizing a submodular function on a lattice. Operations Research 26, 305–321 (1978)

[33] Topkis, D.M.: Equilibrium points in nonzero-sum n-person submodular games. SIAM Journal on Control and Optimization 17, 773–787 (1979)

[34] Van Zandt, T.: Interim Bayesian Nash equilibrium on universal type spaces for supermodular games. Journal of Economic Theory 145, 249–263 (2010)

[35] Vives, X.: Nash equilibrium with strategic complementarities. Journal of Mathematical Economics 19, 305–321 (1990)