The Associated Solidarity Game of n-Person Transferable Utility Games: Linking the Solidarity Value to the Shapley Value.

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Abstract: We introduce for any TU-game, a new TU-game referred as its associated solidarity game (ASG). The ASG gives more power to the grand coalition by reducing the payoffs of others coalitions. It comes that, the Shapley value of the ASG is the Solidarity value of the initial game.

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1- Introduction and preliminaries

Let N=$\{1,2,...,n\}$ be a finite set of n players called the grand coalition and let Γ denote the linear space of the n-person transferable utility game V on N. $\Gamma=\{V/V:2^N \to R\}$. For any subset (coalition) in N, $V(S)$ is the payoff (worth) of coalition S when all the players in S collaborate and $V(\varnothing)=0$.

A value $\psi(V)=(\psi_1(V),\psi_2(V),...,\psi_n(V))$ on Γ, is a vector-valued mapping: $\psi:\Gamma \to R^n$, which uniquely determines, for each $V \in \Gamma$, a distribution of wealth available to the players 1,2,…,n through their participation in the game V.

Many values have been defined in the literature and we will focus our study on two of them: The Shapley and the Solidarity Value, in view to establish a relation between those values.

2- Main Results and discussion

Let us recall formula of the Shapley and the Solidarity Value.

Solidarity value

To define the solidarity value, one compute for any non empty coalition S, the quantity:

$$A^V(S)=\frac{1}{k}\sum_{i\in S}[V(S)-V(S-\{i\})] \quad \text{where } k=|S| \text{ is the cardinality of the coalition } S.$$

It is clear that $A^V(S)$ is the average marginal contribution of a member of the coalition S.

The solidarity value is the unique value (A.S.Nowak and T.Radzik, 1994) of the form $\phi(V)=(\phi_1(V),\phi_2(V),...,\phi_n(V))$ where for $i=1, 2, \ldots, n$,

$$\phi_i(V)=\sum_{\substack{S\ni i\\ |S|=k}}\frac{(n-k)!(k-1)!}{n!}A^V(S) \qquad (1)$$

Shapley Value

The Shapley value is the unique value (L.S. Shapley, 1953)

$\varphi(V)=(\varphi_1(V),\varphi_2(V),...,\varphi_n(V))$ where for $i=1, 2, \ldots, n$,

$$\varphi_i(V)=\sum_{\substack{S\ni i\\ |S|=k}}\frac{(n-k)!(k-1)!}{n!}[V(S)-V(S-\{i\})] \qquad (2)$$

Definition 2.1: (*The solidarity game of a TU game*)

Let V be a game in Γ, the associated solidarity game (ASG) of V is the game $\hat{V}\in\Gamma$ defined as:

$$\hat{V}(S)=\begin{cases}\dfrac{V(S)}{k+1} & if\ |S|=k<n\\[2ex] V(N) & if\ |S|=n\end{cases}$$

The game $\hat{V}$, refereed as solidarity game relatively to the game V , is obtained from V by reduction the gap between the payoffs of coalitions of the same cardinality. In fact, given two coalitions S and S' with equal cardinality k, if $V(S)\geq V(S')$ the gap between their payoffs in the game V is $V(S)-V(S')$ while in the game $\hat{V}$ this gap becomes $\dfrac{V(S)-V(S')}{k+1}$, that is k+1 times less. Secondly, the payoff of the grand coalition is unchanged in the game $\hat{V}$ while all others coalitions of cardinality k<n have their payoff divided by k+1 and hence, considerably reduced.

This mechanism reduces the power of the coalitions and gives more importance to the grand coalition (that is the all society) in $\hat{V}$ relatively to V . Since all players belong to the grand coalition, each of them takes advantage of its increasing importance in the new game. In particular, this shows that, the Shapley null-player, if there is one, in the initial game V will change status in the associated solidarity game. Every player, at least, through the grand coalition, participates substantially in the new game. This consideration does not mean that the players will benefit from the passage of a game to its associated solidarity game in the same level. When use the Shapley value rule, some of the players will gain by the change while others will loss; it depends on the structure of the coalition's payoffs in the initial game. But what is sure is that every player will have a substantial importance relatively to the Shapley marginal contribution rule.

It is interesting to examine the case where n equal 2. Suppose that N= $\{i, j\}$:

- If $V\{i\}=V\{j\}$, Shapley rule attributes the same value to the two players in the game V as in the game $\hat{V}$.
- If $V\{i\}>V\{j\}$, the Shapley value is:

$$\varphi_i(V)=\frac{V(N)}{2}+\frac{(V\{i\}-V\{j\})}{2} \qquad \varphi_j(V)=\frac{V(N)}{2}+\frac{(V\{j\}-V\{i\})}{2}$$

The associated solidarity game of V is : $\hat{V}\{i\}=\dfrac{V\{i\}}{2}$, $\hat{V}\{j\}=\dfrac{V\{j\}}{2}$ and $\hat{V}\{i, j\}=V\{i, j\}$

And its Shapley value is: $\varphi_i(\hat{V})=\dfrac{V(N)}{2}+\left(\dfrac{V\{i\}-V\{j\}}{4}\right)$ $\varphi_j(\hat{V})=\dfrac{V(N)}{2}+\left(\dfrac{V\{j\}-V\{i\}}{4}\right)$

One can easily observes that, player i who is the leader player in the game V is still leading in the associated solidarity game $\hat{V}$ but the gap between the payoff of the two players has been divided by 2 in $\hat{V}$. So that the Shapley value of $\hat{V}$ is obtained from the Shapley value of V as a transfer of the quantity $\dfrac{(V\{i\}-V\{j\})}{4}$ from the leader player to the second player.

The aim of this paper is to determine in the general case ($n\geq 2$) the relation between the Shapley value of a game and the Shapley value of its associated solidarity game. Before investigate our object, let us look at some examples. For simplicity, we take n=3.

As first example, let consider the unanimity game *Tree Brothers* introduced by Nowak and Radzik in 1994. That is, N=$\{1,2,3\}$, $V\{i\}=0$ $V\{1,2\}=1$ $V\{1,3\}=V\{2,3\}=0$ and $V\{1,2,3\}=1$.

The Shapley value of V is $\left(\frac{1}{2},\frac{1}{2},0\right)$

The solidarity associated game of V is : $\hat{V}\{i\}=0$ $\hat{V}\{1,2\}=\frac{1}{3}$ $\hat{V}\{1,3\}=\hat{V}\{2,3\}=0$.

$\hat{V}\{1,2,3\}=1$. It is interesting to note that $\hat{V}$ is no more a unanimity game. Player 3 who was a Shapley-null player has changed status; he substantially contributes in the game via the grand coalition. The Shapley value of $\hat{V}$ is $\left(\frac{7}{18},\frac{7}{18},\frac{4}{18}\right)$, that is exactly the solidarity value of V found by the authors. In this case, this Shapley value of $\hat{V}$, could be interpreted relatively to the Shapley value of V as a transfer of value from the two brothers 1, 2 to their disabled brother 3. Note that this transfer maintains the rank of the values. But do these properties always valid?

The second example involves the case where a player who has the greatest Shapley value in the initial game V obtains a more great value in the associated solidarity game $\hat{V}$:

N=$\{1,2,3\}$, $V\{1\}=4$ $V\{2\}=V\{3\}=0$ $V\{1,2\}=5$ $V\{1,3\}=5$ $V\{2,3\}=\frac{17}{2}$ and $V\{1,2,3\}=21$.

The Shapley value of the game V is $\varphi(V)=\left(\frac{86}{12},\frac{83}{12},\frac{83}{12}\right)$

And the Shapley value of the game $\hat{V}$ is $\varphi(\hat{V})=\left(\frac{262}{36},\frac{247}{36},\frac{247}{36}\right)$

This shows that, the player 1 who is the richest by the Shapley value rule of V is richer in the Shapley value rule of its associated solidarity game $\hat{V}$.

Another interesting example is the game N=$\{1,2,3\}$, $V\{1\}=2$ $V\{2\}=4$ $V\{3\}=0$
$V\{1,2\}=6$ $V\{1,3\}=5$ $V\{2,3\}=x$.

Where, for $2<x<3$, we have the Shapley value which verifies: $\varphi_2(V)<\varphi_1(V)$ and $\varphi_2(\hat{V})>\varphi_1(\hat{V})$.

The considered examples reveal in particular that, except the case where n=2, there is not an order relation between the Shapley value of a player in a game V and his Shapley value in the associated solidarity game $\hat{V}$ of V. This leads again to question on the justification to refer to the game $\hat{V}$ as solidarity one regard to the initial game V. In other words, one can ask itself in which sense the game $\hat{V}$ may be considered as a solidarity game relatively to the game V ?

An other answer of this question, in addition to the fact that the importance of the grand coalition (the whole society) is reinforce in the associated solidarity game relatively to the initial game, is given in the following theorem and constitutes the state of our main result:

Theorem 2.2: *For a given game* $V \in \Gamma$ *on* $N=\{1,2,...,n\}$*, the Shapley value of its associated solidarity game* $\hat{v}$ *coincides with the solidarity value of the initial game* V *; that this: For any player* $i \in N$*,* $\varphi_i(\hat{V}) = \phi_i(V)$.

3- Proof

To prove our result, we need to rewrite in an extensive way, the classical formula of the Shapley value (see equation (2)) and the Solidarity value given in equation (1). This is the formulation of the following lemma.

Lemma 3.1: *For a given game* $V \in \Gamma$*, The Shapley value and the solidarity value can be expressed as:*

1) Shapley value: $$\varphi_i(V) = \frac{V(N)}{n} + \frac{1}{n!}\sum_{k=1}^{n-1}\left((n-k)!(k-1)!\sum_{\substack{S \ni i \\ |S|=k}} V(S) - (n-k-1)!k!\sum_{\substack{i \notin S \\ |S|=k}} V(S) \right) \quad (3)$$

2) Solidarity value: $$\phi_i(V) = \frac{V(N)}{n} + \frac{1}{n!}\sum_{k=1}^{n-1}\left(\frac{(n-k)!(k-1)!}{k+1}\sum_{\substack{S \ni i \\ |S|=k}} V(S) - \frac{(n-k-1)!k!}{k+1}\sum_{\substack{i \notin S \\ |S|=k}} V(S) \right) \quad (4)$$

Proof 3.2:

1) Shapley value

It is well kwon that $$\varphi_i(V) = \sum_{k=1}^{n}\sum_{\substack{S \ni i \\ |S|=k}} \frac{(n-k)!(k-1)!}{n!}\left(V(S) - V(S-\{i\})\right)$$

$$= \sum_{k=1}^{n}\sum_{\substack{S \ni i \\ |S|=k}} \frac{(n-k)!(k-1)!}{n!}V(S) - \sum_{k=1}^{n}\sum_{\substack{S \ni i \\ |S|=k}} \frac{(n-k)!(k-1)!}{n!}V(S-\{i\})$$

Given a coalition S with a fixed cardinality equal k $(k = 1,2,...,n)$, we have two possibilities:

- either $S \ni i$ and then the coefficient of $V(S)$ is (n-k)!(k-1)!/n!
- or $i \notin S$ and then S may be obtain as $S = S' - \{i\}$ with $|S'| = k+1$ $(k < n)$, hence the coefficient of $V(S)$ is $-(n-(k+1))!(k+1-1)!/n! = -(n-k-1)!k!/n!$.

This imply,

$$\varphi_i(V) = \frac{1}{n!}\sum_{k=1}^{n}\left((n-k)!(k-1)!\sum_{\substack{S \ni i \\ |S|=k}} V(S) - (n-k-1)!k!\sum_{\substack{i \notin S \\ |S|=k}} V(S) \right)$$

$$= \frac{V(N)}{n} + \frac{1}{n!}\sum_{k=1}^{n-1}\left((n-k)!(k-1)!\sum_{\substack{S \ni i \\ |S|=k}} V(S) - (n-k-1)!k!\sum_{\substack{i \notin S \\ |S|=k}} V(S) \right)$$

And thus, (3) holds.▪

2) Solidarity value.

According to (1), $\phi_i(V)=\sum_{k=1}^{n}\sum_{\substack{S\ni i\\ |S|=k}}\frac{(n-k)!(k-1)!}{n!}A^V(S)$

$$=\sum_{k=1}^{n}\sum_{\substack{S\ni i\\ |S|=k}}\frac{(n-k)!(k-1)!}{n!}\sum_{j\in S}\frac{(V(S)-V(S-\{j\}))}{|S|}$$

$$=\sum_{k=1}^{n}\left(\sum_{\substack{S\ni i\\ |S|=k}}\left[\frac{(n-k)!(k-1)!}{n!}V(S)-\frac{(n-k)!(k-1)!}{n!}\sum_{j\in S}\frac{V(S-\{j\})}{k}\right]\right)$$

A similar reasoning as above lead us to evaluate the coefficient of $V(S)$ for a given coalition S with a fixed cardinality equal k $(k=1,2,...,n)$.

Suppose $|S|=k$,

If $i\notin S$, then $S=S'-\{j\}$ with $|S'|=k+1$ and the coefficient of $V(S)$ is :

$$-(n-(k+1))!(k-1+1)!/(k+1)n!=-(n-k-1)!k!/(k+1)n!$$

If $i\in S$, we will directly have $V(S)$ with coefficient $(n-k)!(k-1)!/n!$ in one hand, and we will also obtain S as $S=S'-\{j\}$ with $|S'|=k+1$ and $J\notin S$ (there is n-k such possibilities) which imply that , in second hand, we will find $V(S)$ in the summation

$-\frac{(n-k-1)!k!}{n!}\sum_{j\in S'}\frac{V(S'-\{j\})}{k+1}$ and then obtain the second coefficient of $V(S)$:

$-\frac{(n-k-1)!k!}{n!}\frac{(n-k)}{k+1}$.

Summing the two coefficients lead to:

$(n-k)!(k-1)!/n!-\frac{(n-k-1)!k!}{n!}\frac{(n-k)}{k+1}=\frac{(n-k)!(k-1)!}{(k+1)n!}$.

Finally, we can write:

$$\phi_i(V)=\frac{1}{n!}\sum_{k=1}^{n}\left(\frac{(n-k)!(k-1)!}{k+1}\sum_{\substack{S\ni i\\ |S|=k}}V(S)-\frac{(n-k-1)!k!}{k+1}\sum_{\substack{i\notin S\\ |S|=k}}V(S)\right)$$

$$=\frac{V(N)}{n}+\frac{1}{n!}\sum_{k=1}^{n-1}\left(\frac{(n-k)!(k-1)!}{k+1}\sum_{\substack{S\ni i\\ |S|=k}}V(S)-\frac{(n-k-1)!k!}{k+1}\sum_{\substack{i\notin S\\ |S|=k}}V(S)\right)$$

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Proof of the theorem 3.3: According to equation (4),

$$\varphi_i(\hat{V})=\frac{\hat{V}(N)}{n}+\frac{1}{n!}\sum_{k=1}^{n-1}\left((n-k)!(k-1)!\sum_{\substack{S\ni i\\ |S|=k}}\hat{V}(S)-(n-k-1)!k!\sum_{\substack{i\notin S\\ |S|=k}}\hat{V}(S)\right)$$

inserting the definition of $\hat{V}$,

$$\varphi_i(\hat{V}) = \frac{V(N)}{n} + \frac{1}{n!}\sum_{k=1}^{n-1}\left(\frac{(n-k)!(k-1)!}{k+1}\sum_{\substack{S\ni i\\|S|=k}} V(S) - \frac{(n-k-1)!k!}{k+1}\sum_{\substack{i\notin S\\|S|=k}} V(S)\right)$$

Taking into account equation (4), we directly have,

$$\varphi_i(\hat{V}) = \phi_i(V) \qquad \blacksquare$$

Clearly, $\frac{V(N)}{n}$ is the average wealth of the players in N and the lemma reveals that Shapley value and the Solidarity value give a similar treatment to $V(N)$. The formulas given in equations (3) and (4) could be useful for a direct computation of the Shapley and the Solidarity value, especially when n is not great.

4- Conclusion

The solidarity game $\hat{V}$ associated to an n-person transferable utility game V is obtained as a transformation of V so that the new game $\hat{V}$ reflects a kind of solidarity between players. This kind of solidarity could be apprehended as the solidarity in sense of Nowak and Radzik (1994). We have shown that, in this sense, the solidarity could be explained by either considering the Solidarity value in place of Shapley value or considering the associated solidarity game instead of the initial game. This in particular reveals that, a Solidarity value could be obtained as a Shapley value.

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