Epistemic democracy with correlated voters

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Abstract

We develop a general theory of epistemic democracy in large societies, which subsumes the classical Condorcet Jury Theorem, the Wisdom of Crowds, and other similar results. We show that a suitably chosen voting rule will converge to the correct answer in the large-population limit, even if there is significant correlation amongst voters, as long as the *average* correlation between voters becomes small as the population becomes large. Finally, we show that these hypotheses are consistent with models where voters are correlated via a social network, or through the DeGroot model of deliberation.

Keywords: Condorcet Jury Theorem; Wisdom of Crowds; epistemic social choice; deliberation; social network; DeGroot.
JEL class: D71.

1 Introduction

The epistemic approach to social choice theory originates with Condorcet (1785). Suppose a group of people want to obtain the correct answer to some dichotomous (yes/no) question. The question has an objectively correct answer, and everyone has an opinion, but nobody has perfect information. The group could be, for example, a jury trying to determine the guilt or innocence of a defendant in a criminal trial, or a committee of engineers trying to determine whether a bridge is structurally safe. Condorcet's insight was that such a group could efficiently aggregate their private information by *voting*. The famous Condorcet Jury Theorem (CJT) consists of two statements:

- A decision made by a committee using majority vote will be more reliable than the opinion of any single individual. Furthermore, larger committees are more reliable than smaller committees.
- Majority vote will converges in probability to the correct answer as the committee size becomes arbitrarily large.

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The first statement is sometimes called the *nonasymptotic* part of the CJT, while the second statement is the *asymptotic* part. Although it was originally stated only for dichotomous decisions made by majority vote, the CJT has been generalized to polychotomous decisions made by the plurality rule (Ben-Yashar and Paroush, 2001; List and Goodin, 2001), and even other voting rules such as the Kemeny rule and the Borda rule (Young, 1986, 1988, 1995, 1997). Furthermore, in these contexts, the "nonasymptotic" part of the CJT can be refined: under certain conditions, the output of the voting rule is a *maximum likelihood estimator* of the correct answer (see Pivato (2013b) for a general formulation of these results).

A closely related result is the "Wisdom of Crowds" (WoC) principle of Galton (1907): if a large number of people estimate some numerical quantity, then the average of their estimates will converge, in probability, to the true value. However, the WoC, the CJT, and all of its polychotomous generalizations depend on the assumption that the errors made by different voters are *independent* random variables. This is obviously unrealistic: in reality, the opinions of different voters will be strongly correlated, both because they rely on common sources of information and because they influence one another through deliberation and discussion. The goal of this paper is to extend the asymptotic part of the CJT, WoC, and similar theorems to an environment with *correlated* voters.

It has been understood for a long time that the "independence" assumption in the CJT is problematic. Starting in the 1980s, a series of papers gauged the seriousness of this problem and proposed possible solutions. Nitzan and Paroush (1984) demonstrated the sensitivity of the CJT to the independence assumption, while Shapley and Grofman (1984) showed that, with certain patterns of correlations, a *nonmonotonic* rule could actually be more reliable than majority vote. Owen (1986) argued that, if the voters can be divided into subgroups such that voters within each subgroup are correlated, then an "indirect" majority vote (like an electoral college) could be more reliable than direct majority vote. Meanwhile, Ladha (1992) showed that the asymptotic CJT remained true as long as the "average" correlation between the voters was sufficiently small. (This is a special case of Theorem 5.3 in the present paper.) Berend and Sapir (2007) found general conditions for the nonasymptotic part of the CJT to hold in a committee of correlated voters. Kaniovski (2009, 2010) modeled the joint probability distribution of a population of homogeneous correlated voters using a representation by Bahadur, and studied the nonasymptotic part of the CJT in this context. Building on this work, Kaniovski and Zaigraev (2011) showed that a special case of the Bahadur representation admits a quota voting rule which neutralizes the effect of the correlations. Finally, Peleg and Zamir (2012) gave a number of necessary conditions and sufficient conditions for a population of correlated voters to satisfy the CJT.

One natural source of voter correlation is "contagion" of opinions (e.g. due to deliberation). Berg (1993a,b) and Ladha (1995) supposed that the voters' errors were correlated according to hypergeometric or Pólya-Eggenberger urn processes, which is a simple model of such "contagion". They showed that the asymptotic CJT holds for the former, but does not hold for the latter (although a group is still more reliable than an individual). See Berg (1996) for a summary.

Another possible cause of voter correlation is a common source of information. For example, in a criminal trial, all jurors observe exactly the same evidence. In a committee of engineers, everyone reads the same technical reports and has access to the same data. In other situations, the voters might all be influenced by an "opinion leader". Boland (1989) and Boland et al. (1989) developed a version of the CJT with such an opinion leader. Later, Berg (1994) extended this to a setting with weighted voting rules. Estlund (1994) also considered a model with opinion leaders, but he showed that, under certain conditions, such opinion leaders could actually improve the reliability of majority vote. Meanwhile, Ladha (1993, Proposition 1) proved a version of the CJT when the voter errors are not independent, but are *exchangeable* random variables. By a theorem of de Finetti, this is equivalent to a model where all the voters are independent Bernoulli random variables with a common parameter α, which is itself another random variable; thus, α can be interpreted as representing a common information source. (The hypergeometric distributions studied by Berg (1993a,b) and Ladha (1995) are also examples of exchangeable distributions.) Peleg and Zamir (2012, Theorem 5) also proved a version of the CJT for exchangeable random variables. Dietrich and List (2004) demonstrated that if all voters draw only on a small set of (unreliable) information sources, then the asymptotic part of the CJT fails: even a very large population of voters cannot be any more reliable than the (small) set of information sources on which they all base their opinions. Dietrich and List represented this situation as a Bayesian network; this approach was further developed by Dietrich and Spiekermann (2013a,b), who showed that, in the presence of common causes, the asymptotic reliability of a large committee can be good, but less than perfect.

A third possible cause of correlation is strategic voting. Even if all voters want the group to get the correct answer, they may have incentives to vote strategically (Austen-Smith and Banks, 1996). However, McLennan (1998, Theorem 1) has shown that any profile of voting strategies which maximizes the probability that the group gets the right answer will be a Bayesian Nash equilibrium (BNE). This holds even if the voters' types (i.e. their private information) are correlated. As observed by Peleg and Zamir (2012), this means that we only need to prove the existence of *some* pattern of voting behaviour which satisfies the CJT; it then follows that the CJT will also hold in BNE. Thus, we do not need to explicitly consider strategic behaviour in our analysis.

All of the aforementioned papers deal only with *dichotomous* decision problems and majority rule. In contrast, the asymptotic results of this paper are applicable to a *polychotomous* decisions and a large class of epistemic voting rules, including majority rule, plurality rule, the Kemeny rule, the median rule, the average rule, the Borda rule, and other scoring rules. To obtain this level of generality, we will introduce a single broad class of voting rules which includes all of the aforementioned rules as special cases: the class of *mean partition rules*. Furthermore, we will provide a concrete illustration of the economic relevance of our general results, by connecting them with the theory of social networks and the DeGroot (1974) model of consensus formation.

The remainder of this paper is organized as follows. Section 2 introduces notation and terminology which will be maintained throughout the paper. Section 3 defines the class of mean partition rules and gives several examples, including majority rule, plurality rule, the median rule, and other scoring rules. Section 4 describes a special case of our model, which we call a *populace*: this is a family of probability distributions, describing a society where voters make *independent* random errors. It contains special cases of our main result (Propositions 4.1 and 4.3), which state that, if the populace satisfies certain mild conditions, then an appropriate mean partition rule will select the correct answer with very high probability in a large population.

Section 5 describes a general case of the model, which we call a *culture*: this is a family of probability distributions, describing a society where the errors of the voters are *correlated* random variables. It then states the general version of our main result (Theorem 5.3): if the culture is *sagacious* (meaning that it satisfies certain mild conditions —in particular, the "average correlation" between voters is not too large), then an appropriate mean partition rule will select the correct answer with very high probability in a large population.

The rest of the paper explores applications. Section 6 considers cultures based on social networks, and contains results (Propositions 6.2 and 6.5) stating that, as long as the social network is not too richly connected, the resulting culture will be sagacious, so that Theorem 5.3 applies. Finally, Section 7 considered the effects of deliberation on an already sagacious culture, and contains a result (Proposition 7.1) saying that, as long as no individuals are too "influential" in this deliberation, the culture will remain sagacious after deliberation. All proofs are in the Appendix.

2 Notation and terminology

We now fix some notation which will be maintained throughout the paper. Let $\mathbb{N} := \{1, 2, \ldots\}$ denote the set of natural numbers. Let $\mathbb{R}$ denote the set of real numbers, and let $\mathbb{R}_+$ denote the set of nonnegative real numbers. Let $\mathcal{I}$ denote a finite set of voters, and let $I := |\mathcal{I}|$. (We will typically assume that I is very large; indeed, we will mainly be interested in asymptotic properties as $I \to \infty$.)

We will assume throughout the paper that the set of possible states of the world is a topological space (for example, a subset of some Euclidean space). If $\mathcal{S}$ is a finite set, then we will just use the discrete topology, where every singleton set is open. We will also assume that the possible votes sent by the voters are elements of a vector space equipped with a norm or an inner product. If $\mathbb{V}$ is a vector space, then a ***norm*** on $\mathbb{V}$ is a function $\|\bullet\| : \mathbb{V} \longrightarrow \mathbb{R}_+$ such that, for any $\mathbf{v}, \mathbf{w} \in \mathbb{V}$ and $r \in \mathbb{R}$: (1) $\|r\,\mathbf{v}\| = |r| \cdot \|\mathbf{v}\|$; (2) $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = 0$; and (3) $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$. Such a norm defines a metric d on $\mathbf{V}$ by $d(\mathbf{v}, \mathbf{w}) := \|\mathbf{v} - \mathbf{w}\|$. For example, the ***Euclidean norm*** on $\mathbb{R}^N$ is defined by $\|\mathbf{v}\| := \sqrt{v_1^2 + \cdots + v_N^2}$.

An ***inner product*** on $\mathbb{V}$ is a function $\langle \bullet, \bullet \rangle : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{R}$ such that, for any $\mathbf{v}, \mathbf{w} \in \mathbb{V}$: (1) The functions $\langle \mathbf{v}, \bullet \rangle : \mathbb{V} \longrightarrow \mathbb{R}$ and $\langle \bullet, \mathbf{w} \rangle : \mathbb{V} \longrightarrow \mathbb{R}$ are linear; (2) $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$; and (3) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, and furthermore, $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$. An inner product defines a norm by setting $\|\mathbf{v}\| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$. For example, if $\mathbb{V} = \mathbb{R}$, then we could simply take $\langle r, s \rangle := r \cdot s$ for any $r, s \in \mathbb{R}$. If $\mathbb{V} = \mathbb{R}^N$, then we could use the standard dot product: $\langle \mathbf{v}, \mathbf{w} \rangle = v_1\, w_1 + \cdots + v_N\, w_N$ for any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^N$.

Let ρ be a probability measure on a vector space $\mathbb{V}$. The ***expected value*** of a ρ-random

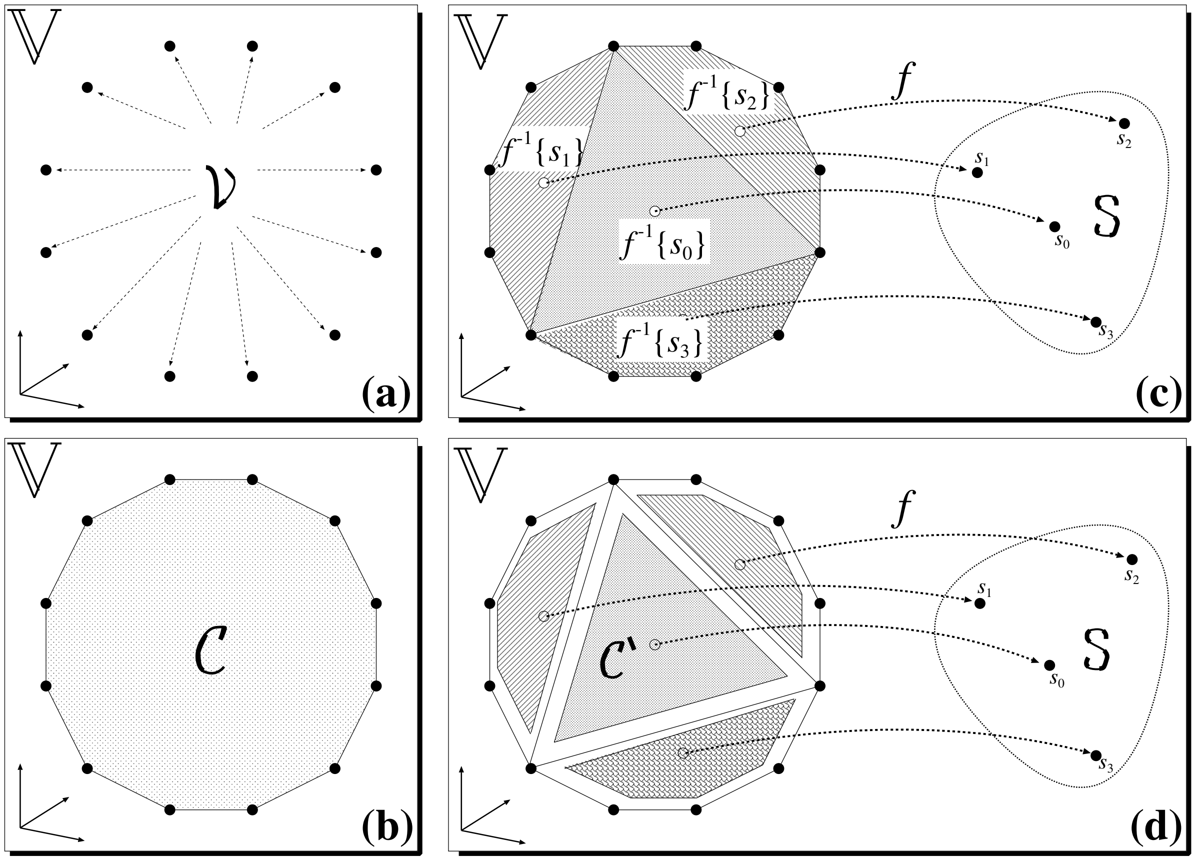

Figure 1: A mean partition rule. **(a)** $\mathcal{V}$ is a subset of the vector space $\mathbb{V}$. **(b)** $\mathcal{C}$ is the convex hull of $\mathcal{V}$. **(c)** $f^{-1}\{s\}$ is a convex subset of $\mathcal{C}$, for each $s \in \mathcal{S}$. **(d)** The continuity set $\mathcal{C}'$.

variable is defined $\mathbb{E}[\rho] := \int_{\mathcal{V}} \mathbf{v}\ \mathrm{d}\rho[\mathbf{v}]$. If $\mathbb{V}$ has a norm $\|.\|$, then the variance is defined $\mathrm{var}[\rho] := \displaystyle\int_{\mathcal{V}} \|\mathbf{v} - \overline{\mathbf{v}}\|^2\ \mathrm{d}\rho[\mathbf{v}]$, where $\overline{\mathbf{v}} := \mathbb{E}[\rho]$.

3 Mean partition rules

Let $\mathcal{I}$ be a set of individuals. Let $\mathcal{S}$ be a topological space of social alternatives. An (anonymous) *mean partition rule* on $\mathcal{S}$ is a voting rule defined by a data structure $F := (\mathbb{V}, \mathcal{V}, f)$ with three properties:

(M1) $\mathbb{V}$ is a normed vector space, and $\mathcal{V} \subseteq \mathbb{V}$ (as shown in Figure 1(a)).

(M2) If $\mathcal{C}$ is the convex hull of $\mathcal{V}$ (as in Figure 1(b)), then $f : \mathcal{C} \longrightarrow \mathcal{S}$ is a surjective function, such that for all $s \in \mathcal{S}$, the preimage set $f^{-1}\{s\}$ is a convex subset of $\mathcal{C}$ (as in Figure 1(c)).

(M3) There is a relatively open subset $\mathcal{C}' \subseteq \mathcal{C}$ (as in Figure 1(d)) such that f is continuous and surjective when restricted to $\mathcal{C}'$.

In this model, $\mathcal{V}$ is the set of possible *votes* which could be sent by each individual. Given any set finite $\mathcal{I}$ of individuals, and any profile $\mathbf{V} = (\mathbf{v}_i)_{i\in\mathcal{I}}$ of votes (where $\mathbf{v}_i \in \mathcal{V}$ for all $i \in \mathcal{I}$), the output of the rule is obtained by applying f to the *average* of the vectors $\{\mathbf{v}_i\}_{i\in\mathcal{I}}$. Formally,

$$F(\mathbf{V}) \quad := \quad f\left(\frac{1}{|\mathcal{I}|}\sum_{i\in\mathcal{I}}\mathbf{v}_i\right).$$

A few remarks are in order. First, note that the voting rule F is anonymous by construction (i.e. the outcome is invariant under permutation of the voters). Second, if f is injective (so that $f^{-1}\{s\}$ is a singleton for all $s \in \mathcal{S}$), then the convexity condition (M2) is automatically satisfied, while the continuity condition (M3) simply says that f is a continuous function. At the other extreme, if $\mathcal{S}$ is finite, then (M2) says that f defines an $\mathcal{S}$-labelled partition of $\mathcal{C}$ into convex subsets. Third, given condition (M2), we can (and will) assume without loss of generality that the set $\mathcal{C}'$ in (M3) has been chosen to also satisfy:

(M2′) For any $s \in \mathcal{S}$, the f-preimage of s inside $\mathcal{C}'$ is convex. (See Figure 1(d).)

Finally, note that the norm on $\mathbb{V}$ and the topology on $\mathcal{S}$ are only needed to state condition (M3). Furthermore, the content of (M3) is really determined by the norm *topology*, rather than the norm itself. Thus, if $\mathbb{V} = \mathbb{R}^N$ with the standard topology, then we can assume without loss of generality that (M3) invokes the Euclidean norm. If $\mathbb{V}$ is some other N-dimensional vector space, then we can define a norm via any linear isomorphism from $\mathbb{V}$ to $\mathbb{R}^N$. If $\mathbb{V}$ is infinite-dimensional, then the issue is more subtle, as there are many topologically non-equivalent norms to choose from. Note that (M3) does *not* require f to be continuous everywhere on $\mathcal{C}$. (Indeed, if $\mathcal{S}$ was a discrete set, this would be impossible.) We will refer to $\mathcal{C}'$ as the *continuity set* of F.

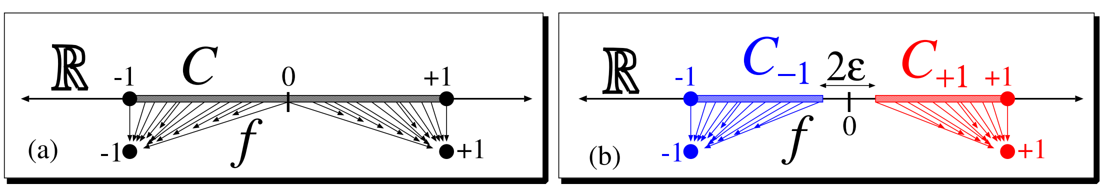

Figure 2: Simple majority vote as a mean partition rule.

Example 3.1. (a) (*Simple majority rule*) Let $\mathcal{S} := \{\pm 1\}$. Let $\mathbb{V}_{\rm maj} := \mathbb{R}$. Let $\mathcal{V}_{\rm maj} := \{\pm 1\}$, so that $\mathcal{C} = [-1, 1]$, as shown in Figure 2(a). Define $f_{\rm maj} : \mathcal{C} \longrightarrow \mathcal{S}$ by setting $f_{\rm maj}(r) := \text{sign}(r)$ for all nonzero $r \in [-1, 1]$, while $f_{\rm maj}(0) := 1$ (an arbitrary tie-breaking rule). Then $F_{\rm maj} = (\mathbb{V}_{\rm maj}, \mathcal{V}_{\rm maj}, f_{\rm maj})$ is the simple majority rule. Now, fix $\epsilon > 0$, and let $\mathcal{C}' := \mathcal{C}_{-1} \sqcup \mathcal{C}_{+1}$, where $\mathcal{C}_{-1} := [-1, -\epsilon)$ and $\mathcal{C}_{+1} := (\epsilon, 1]$, as shown in Figure 2(b). Then (M3) (and (M2′)) are satisfied.

Throughout the remaining examples, let $\mathcal{P}(\mathcal{S})$ be the power set of $\mathcal{S}$, and let $\tau : \mathcal{P}(\mathcal{S}) \longrightarrow \mathcal{S}$ be some function; we will use τ as a "tiebreaker" in the definition of the following rules.

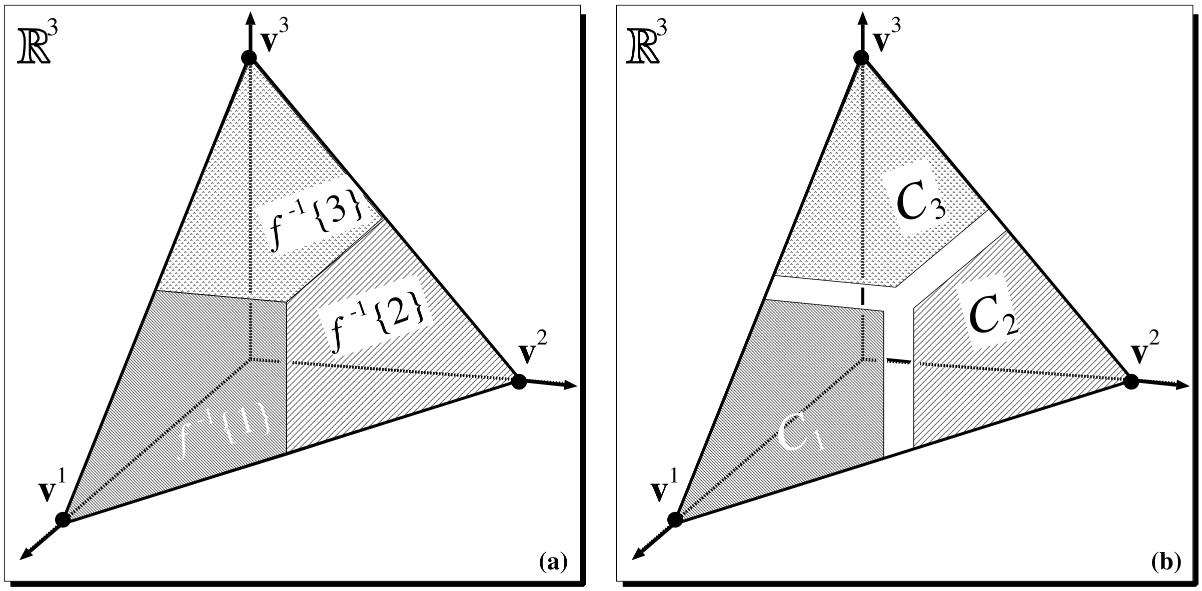

Figure 3: The plurality rule as a mean partition rule.

(b) (*Plurality rule*) Let $N \geq 2$, and let $\mathcal{S} := \{1, 2, \ldots N\}$ (a set of N alternatives). Let $\mathbb{V}_{\text{plu}} := \mathbb{R}^N$. For all $n \in [1 \ldots N]$, let $\mathbf{v}^n := (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 appears in the nth coordinate. Let $\mathcal{V}_{\text{plu}} := \{\mathbf{v}^1, \ldots, \mathbf{v}^N\}$ (a subset of $\mathbb{R}^N$). If $\mathcal{C}$ is the convex hull of $\mathcal{V}$, then $\mathcal{C}$ is the unit simplex in $\mathbb{R}^N$, as shown in Figure 3(a). For any $\mathbf{c} \in \mathcal{C}$, let $\mathcal{S}_{\mathbf{c}} := \{s \in \mathcal{S};\ c_s \geq c_t \text{ for all } t \in \mathcal{S}\}$ be the set of maximal coordinates. Define $f_{\text{plu}} : \mathcal{C} \longrightarrow \mathcal{S}$ by setting $f_{\text{plu}}(\mathbf{c}) := \tau(\mathcal{S}_{\mathbf{c}})$, for all $\mathbf{c} \in \mathcal{C}$. Then $F_{\text{plu}} = (\mathbb{V}_{\text{plu}}, \mathcal{V}_{\text{plu}}, f_{\text{plu}})$ is the plurality rule. Fix $\epsilon > 0$, and for all $s \in \mathcal{S}$, define $\mathcal{C}_s := \{\mathbf{r} \in \mathbb{R}^N;\ r_s > r_t + \epsilon \text{ for all } t \neq s\}$, as shown in Figure 3(b). Let $\mathcal{C}' := \mathcal{C}_1 \sqcup \mathcal{C}_2 \sqcup \cdots \sqcup \mathcal{C}_N$; then (M3) (and (M2′)) are satisfied.

(c) (*The average rule*) Let $N \geq 2$, and let $\mathcal{S}$ be a convex subset of $\mathbb{R}^N$. Let $\mathcal{C} = \mathcal{V} = \mathcal{S}$, and let $f_{\text{ave}} : \mathcal{C} \longrightarrow \mathcal{S}$ be the identity function. This represents the rule where each voter declares an "ideal point" in $\mathcal{S}$, and the outcome is the arithmetic average of these ideal points. Note that (M2) and (M3) are satisfied (with $\mathcal{C}' := \mathcal{S}$), because the identity function is continuous, and the preimage of each point is a singleton.

(d) (*The median rule*) Let $(\mathcal{S}, <)$ be a finite or countable linearly ordered set. Represent $\mathcal{S}$ as a subset of $\mathbb{R}$ in an order-preserving way. Let $\mathbb{V}_{\text{med}} := \mathbb{R}^{\mathcal{S}}$. For all $s \in \mathcal{S}$, define $\mathbf{v}^s := (v^s_t)_{t \in \mathcal{S}} \in \mathbb{V}$, by setting $v^s_t := |s - t|$ for all $t \in \mathcal{S}$. Let $\mathcal{V}_{\text{med}} := \{\mathbf{v}^s\}_{s \in \mathcal{S}}$ (a subset of $\mathbb{V}_{\text{med}}$), and let $\mathcal{C}$ be the convex hull of $\mathcal{V}$. For any $\mathbf{c} \in \mathcal{C}$, let $\mathcal{S}_{\mathbf{c}} := \{s \in \mathcal{S};\ c_s \leq c_t \text{ for all } t \in \mathcal{S}\}$ be the set of minimal coordinates of $\mathbf{c}$ —in effect, these are the element(s) of $\mathcal{S}$ which minimize the *average distance* to the points chosen by the voters. It is easy to see that $\mathcal{S}_{\mathbf{c}}$ is always a closed interval inside $\mathcal{S}$. Define $f_{\text{med}} : \mathcal{C} \longrightarrow \mathcal{S}$ by setting $f_{\text{med}}(\mathbf{c}) := \tau(\mathcal{S}_{\mathbf{c}})$, for all $\mathbf{c} \in \mathcal{C}$. In other words, each voter chooses a point s in $\mathcal{S}$ (represented by $\mathbf{v}^s$), and F_{med} chooses a point in $\mathcal{S}$ which minimizes the *average distance* to the points chosen by the voters (using τ to break ties). As is well-known, this point will be a *median* of the points chosen by the voters.

(e) (*The generalized median rule*) Let $(\mathcal{S}, d)$ be a metric space.[1] Let $\mathbb{V}_{\text{med}} := \mathbb{R}^{\mathcal{S}}$. For all $s \in \mathcal{S}$, define $\mathbf{v}^s := (v^s_t)_{t\in\mathcal{S}} \in \mathbb{V}$, by setting $v^s_t := d(s,t)$ for all $t \in \mathcal{S}$. Let $\mathcal{V}_{\text{med}} := \{\mathbf{v}^s\}_{s\in\mathcal{S}}$ (a subset of $\mathbb{V}_{\text{med}}$), and let $\mathcal{C}$ be the convex hull of $\mathcal{V}_{\text{med}}$. For any $\mathbf{c} \in \mathcal{C}$, let $\mathcal{S}_{\mathbf{c}} := \{s \in \mathcal{S};\ c_s \leq c_t$ for all $t \in \mathcal{S}\}$, as in example (d). Define $f_{\text{med}} : \mathcal{C} \longrightarrow \mathcal{S}$ by setting $f_{\text{med}}(\mathbf{c}) := \tau(\mathcal{S}_{\mathbf{c}})$, for all $\mathbf{c} \in \mathcal{C}$. As in example (d), each voter chooses a point s in $\mathcal{S}$ (represented by $\mathbf{v}^s$), and F_{med} selects a point in $\mathcal{S}$ which minimizes the *average distance* to the points chosen by the voters (using τ to break ties).

(f) (*The Kemeny rule*) Let $\mathcal{A}$ be a finite set of social alternatives. Let $\mathcal{S}$ be the set of all linear orders over $\mathcal{A}$. The *Kendall metric* on $\mathcal{S}$ is defined by declaring $d(s,r)$ to be the number of pairwise comparisons where the orders s and r disagree. In this case, the generalized median rule from example (e) is the *Kemeny rule* for preference aggregation.

(g) (*Any scoring rule*) Let $\mathcal{S}$ be a set of alternatives. Let $\mathbb{V}_{\text{scr}} := \mathbb{R}^{\mathcal{S}}$, and let $\mathcal{V}$ be any subset of $\mathbb{V}$. Intuitively, an element $\mathbf{v} = (v_s)_{s\in\mathcal{S}}$ in $\mathcal{V}$ represents a vote which assigns a "score" of v_s to each alternative in $\mathcal{S}$. Let $\mathcal{C}$ be the convex hull of $\mathcal{V}$. For any $\mathbf{c} \in \mathcal{C}$, let $\mathcal{S}_{\mathbf{c}} := \{s \in \mathcal{S};\ c_s \geq c_t$ for all $t \in \mathcal{S}\}$ be the set of maximal coordinates. Define $f_{\text{scr}} : \mathcal{C} \longrightarrow \mathcal{S}$ by setting $f_{\text{scr}}(\mathbf{c}) := \tau(\mathcal{S}_{\mathbf{c}})$, for all $\mathbf{c} \in \mathcal{C}$. Then $F_{\text{scr}} = (\mathbb{V}_{\text{scr}}, \mathcal{V}, f_{\text{scr}})$ is called a *scoring rule*. All of the examples above are special cases of scoring rules. Other well-known scoring rules include the Borda rule and the approval voting rule.

(h) (*Mean proximity rules*) Let $\mathcal{S}$ be a finite set of alternatives, and for each $s \in \mathcal{S}$, let $\mathbf{r}_s \in \mathbb{R}^N$. Let $\mathcal{V}$ be another finite subset of $\mathbb{R}^N$. Let $\mathcal{C}$ be the convex hull of $\mathcal{V}$. For any $\mathbf{c} \in \mathcal{C}$, let $\mathcal{S}_{\mathbf{c}} := \{s \in \mathcal{S};\ \|\mathbf{r}_s - \mathbf{c}\|$ is minimal$\}$. Define $f_{\text{mpr}} : \mathcal{C} \longrightarrow \mathcal{S}$ by setting $f_{\text{mpr}}(\mathbf{c}) := \tau(\mathcal{S}_{\mathbf{c}})$, for all $\mathbf{c} \in \mathcal{C}$. Then $F_{\text{mpr}} = (\mathbb{V}_{\text{scr}}, \mathcal{V}, f_{\text{mpr}})$ is called a *mean proximity rule*. ◇

When $\mathcal{V}$ and $\mathcal{S}$ are finite, Zwicker (2008, Theorem 4.2.1) has shown that an anonymous voting rule is a scoring rule (as in Example 3.1(g)) if and only if it is a mean proximity rule (as in Example 3.1(h)).[2] So these two classes are equivalent. However, not every mean partition rule is a mean proximity rule, even when $\mathcal{V}$ and $\mathcal{S}$ are finite.[3]

When $\mathcal{V}$ and $\mathcal{S}$ are finite and the voting rule is both anonymous and *neutral* (i.e. it treats all elements of $\mathcal{V}$ the same, in a certain sense), Myerson (1995) has shown that it is a scoring rule if and only if it satisfies two axioms. The first, *Consistency* (also called *Reinforcement*) means, roughly, that given two profiles $(\mathbf{v}_1, \ldots, \mathbf{v}_I)$ and $(\mathbf{w}_1, \ldots, \mathbf{w}_J)$, we have $F(\mathbf{v}_1, \ldots, \mathbf{v}_I, \mathbf{w}_1, \ldots, \mathbf{w}_J) = F(\mathbf{v}_1, \ldots, \mathbf{v}_I) \cap F(\mathbf{w}_1, \ldots, \mathbf{w}_J)$ whenever this intersection is nonempty. The second axiom, *Continuity* (which Myerson called *Overwhelming majority*) says, roughly, that a sufficiently small change to a sufficiently large population profile cannot change the outcome. Pivato (2013a, Theorem 2) proved a generalization of Myerson's result which keeps neutrality but relaxes the *Continuity* condition, by allowing the score

[1] A *metric space* is a set $\mathcal{S}$ together with a function $d : \mathcal{S} \times \mathcal{S} \longrightarrow \mathbb{R}_+$ such that, for any $r, s, t \in \mathcal{S}$: (1) $d(s,t) = d(t,s)$; (2) $d(s,t) = 0$ if and only if $s = t$; and (3) $d(r,t) \leq d(r,s) + d(s,t)$.

[2] Zwicker's model is slightly different: instead of using a tiebreaker rule, he allows voting rules to be multivalued in the case of a tie.

[3] Mean proximity rules correspond to the special case when the partition of $\mathcal{C}$ into F-preimages is the *Voronoi partition* induced by the set $\{\mathbf{r}_s\}_{s\in\mathcal{S}}$.

vectors to take "infinitesimal" values. The axiom of *Consistency* alone characterizes the class of *balance rules*, which are almost the same as mean partition rules except for more precisely specified behaviour in the case of ties (Pivato, 2013a, Theorem 1). However, there is not yet any axiomatic characterization of the class of mean partition rules itself.

4 Epistemic social choice with independent voters

Let $\mathcal{S}$ be the topological space of the possible states of the world (the true state being unknown). Let $(\mathbb{V}, \mathcal{V}, F)$ be a mean partition rule taking outcomes in $\mathcal{S}$. Let $\mathcal{I}$ be a finite set of individuals, and let $I := |\mathcal{I}|$. We suppose that each individual's vote is a random variable, which is dependent on the true state of nature. The idea is that each individual obtains some information about the state of nature (possibly incomplete and/or incorrect), combines it with her own pre-existing beliefs, and formulates a belief about the state of nature, which she expresses using her vote. Our goal is to use the pattern of these votes to estimate the true state of nature.

Formally, for each individual $i \in \mathcal{I}$, we posit a ***behaviour model*** $\rho^i : \mathcal{S} \longrightarrow \Delta(\mathcal{V})$; if the true state is $s \in \mathcal{S}$, then the probability distribution of individual i's vote will be $\rho^i(s)$. For any $\mathbf{v} \in \mathcal{V}$, we will write $\rho^i(\mathbf{v}|s)$ for the value of $\rho^i(s)$ evaluated at $\mathbf{v}$ —i.e. the probability that individual i votes for $\mathbf{v}$, given that the true state is s.

Different voters may have different behaviour models (due to differing competency, different prior beliefs, or access to different information sources). Furthermore, it is not realistic to suppose that we have precise knowledge of the behaviour model of every voter (or even of *any* voter); in general, we only know some broad qualitative properties of their behaviour models. Thus, we will suppose that there is some set $\mathcal{P}$ of possible behaviour models (i.e. functions from $\mathcal{S}$ into $\Delta(\mathcal{V})$), and all we know is that $\rho^i \in \mathcal{P}$ for all $i \in \mathcal{I}$. We will refer to $\mathcal{P}$ as a ***populace***. We will say that $\mathcal{P}$ is ***sagacious*** for a mean partition rule $F = (\mathbb{V}, \mathcal{V}, f)$ if it satisfies two conditions:

Identification. For any $\rho \in \mathcal{P}$ and any $s \in \mathcal{S}$, the *expected* value of a $\rho(s)$-random variable lies in the f-preimage of s inside the continuity set $\mathcal{C}'$. In other words, if $\mathbb{E}[\rho(s)]$ denotes the mean value of the distribution $\rho(s)$, then $\mathbb{E}[\rho(s)] \in \mathcal{C}'$ and $f(\mathbb{E}[\rho(s)]) = s$.

Minimal reliability. There is some $M \geq 0$ such that $\mathrm{var}[\rho(s)] \leq M$ for all $\rho \in \mathcal{P}$ and $s \in \mathcal{S}$.

The *Identification* condition says that, while an individual's *actual* vote may be incorrect, the *expected value* of her vote is a good indicator of the true state of nature —at least once it has been "interpreted" using the function f. The variance of an individual's vote distribution is a measure of her reliability: if the variance is large, then this voter has a high probability of picking the wrong answer. The *Minimal Reliability* condition says that all voters meet at least some minimum standard of reliability. Note that, if the set $\mathcal{V}$ is bounded (in particular, if $\mathcal{V}$ is finite), then *Minimal Reliability* is automatically satisfied (because there will be some M such that $\mathrm{var}(\rho) \leq M$ for any $\rho \in \Delta(\mathcal{V})$).

Our first result says that, if $\mathcal{S}$ is finite and a large number of voters are drawn from a sagacious populace, and their votes are independent random variables, then the output of the voting rule will be the true state of nature, with very high probability.

Proposition 4.1 *Let F be a mean partition rule ranging over a finite set $\mathcal{S}$, and let $\mathcal{P}$ be a populace which is sagacious for F. For all $i \in \mathbb{N}$, let $\rho_i \in \mathcal{P}$. Fix $s \in \mathcal{S}$, and suppose $\{\mathbf{v}_i\}_{i=1}^{\infty}$ are all independent random variables, where, for all $i \in \mathbb{N}$, $\mathbf{v}_i$ is drawn from distribution $\rho_i(s)$. Then $\lim_{I\to\infty} \text{Prob}\left[F\left(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_I\right) = s\right] \ = \ 1$.*

Example 4.2. (a) (*Condorcet Jury Theorem*) Let $\mathcal{S} = \mathcal{V} := \{\pm 1\}$ and let F_{maj} be as in Example 3.1(a). Let $\mathcal{P}$ be the set of all behaviour models $\rho : \{\pm 1\} \longrightarrow \Delta\{\pm 1\}$ such that $\rho(s|s) > \frac{1}{2} + \epsilon$ (and thus, $\rho(-s|s) < \frac{1}{2} - \epsilon$) for both $s \in \{\pm 1\}$. Let $\mathcal{C}_{-1} := [-1, -\epsilon)$ and $\mathcal{C}_1 := (\epsilon, 1]$. Then $\mathbb{E}[\rho(s)] \in \mathcal{C}_s$ for any $\rho \in \mathcal{P}$ and $s \in \{\pm 1\}$. Thus, $F_{\text{maj}}\left(\mathbb{E}[\rho(s)]\right) = s$, so *Identification* is satisfied. Furthermore, $\text{var}(\rho) < 4$ for any $\rho \in \Delta\{\pm 1\}$, so *Minimal Reliability* is always satisfied. Thus, Proposition 4.1 yields an extension of the Condorcet Jury Theorem to heterogenous voters, originally stated by Paroush (1998): If the voter's opinions about some dichotomous choice are independent random variables, and each voter satisfies some minimal level of competency (i.e. her probability of identifying the correct answer is ϵ-better than a coin flip), then the outcome of a simple majority vote will converge in probability to the correct answer as the voting population becomes large.

(b) (*Plurality CJT*) Let $N \geq 2$, and let $\mathcal{S} := \{1, 2, \ldots N\}$. Define $(\mathbb{V}, \mathcal{V}, F_{\text{plu}})$ as in Example 3.1(b). Let $\mathcal{P}$ be the set of all behaviour models $\rho : \mathcal{S} \longrightarrow \Delta(\mathcal{V})$ such that $\rho(\mathbf{v}^s|s) > \rho(\mathbf{v}^t|s) + \epsilon$, for all $s, t \in \mathcal{S}$ with $s \neq t$. For all $s \in \mathcal{S}$, define $\mathcal{C}_s$ as in Example 3.1(b). Then $\mathbb{E}[\rho(s)] = (\rho^i(1|s), \rho^i(2|s), \ldots, \rho^i(N|s)) \in \mathcal{C}_s$ for all $\rho \in \mathcal{P}$ and $s \in \mathcal{S}$; thus, *Identification* is satisfied. Furthermore, $\text{var}(\rho) < N$ for any $\rho \in \Delta(\mathcal{V})$, so *Minimal Reliability* is always satisfied. Thus, Proposition 4.1 yields a "polychotomous" extension of the CJT, originally stated by Goodin and List (2001; Proposition 2): if each voter satisfies some minimal level of competency (i.e. is ϵ-better than a random guess), then the outcome of the *plurality rule* will converge in probability to the correct answer as the voting population becomes large. ♢

In fact, Proposition 4.1 is a special case of the next result, which also applies when $\mathcal{S}$ is infinite. This result says that, if a large number of voters are drawn from a sagacious populace, and their votes are independent random variables, then the output of the voting rule will be *very close* to the true state of nature, with very high probability.

Proposition 4.3 *Let F be a mean partition rule ranging over an arbitrary set $\mathcal{S}$, and let $\mathcal{P}$ be a populace which is sagacious for F. For all $i \in \mathbb{N}$, let $\rho_i \in \mathcal{P}$. Fix $s \in \mathcal{S}$, and suppose $\{\mathbf{v}_i\}_{i=1}^{\infty}$ are all independent random variables, where, for all $i \in \mathbb{N}$, $\mathbf{v}_i$ is drawn from distribution $\rho_i(s)$. Then for any open subset $\mathcal{U} \subset \mathcal{S}$ containing s, we have* $\lim_{I\to\infty} \text{Prob}\left[F\left(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_I\right) \in \mathcal{U}\right] \ = \ 1.$

Example 4.4. (*The Wisdom of Crowds*) Let $N \geq 1$, let $\mathcal{V} = \mathcal{S}$ be some convex subset of $\mathbb{R}^N$, and let F_{ave} be the average rule, as in Example 3.1(c).

Fix $M > 0$, and let $\mathcal{P}$ be the set of all behaviour models $\rho : \mathcal{S} \longrightarrow \Delta(\mathcal{V})$ such that, for all $\mathbf{s} \in \mathcal{S}$, $\mathbb{E}[\rho(\mathbf{s})] = \mathbf{s}$ and $\text{var}[\rho(s)] \leq M$. Then *Identification* and *Minimal Reliability* are satisfied. Thus, Proposition 4.3 yields the Wisdom of Crowds principle for the estimation of some real-valued (or, more generally, vector-valued) quantity: if each voter estimates the quantity, and their estimates are independent, unbiased, and have finite variance, then the average of their estimates will converge in probability to the correct answer. ◊

Examples 4.2 and 4.4 are well-known results from epistemic social choice theory. However, the next example is new.

Example 4.5. (*Log-likelihood scoring rules*) Let $\mathcal{S}$ be a finite set. Let $p : \mathcal{S} \longrightarrow \Delta(\mathcal{S})$ be a function (called an ***error model***). For any $s, t \in \mathcal{S}$, we interpret $p(t|s)$ be the probability that a voter will *believe* that the true state is t, when it is actually s. Let $\mathbb{V} := \mathbb{R}^{\mathcal{S}}$, and for all $r \in \mathcal{S}$, define $\mathbf{v}^r := (v^r_s)_{s\in\mathcal{S}} \in \mathbb{V}$ by setting $v^r_s := \log[p(r|s)]$, for all $s \in \mathcal{S}$. Let $\mathcal{V} := \{\mathbf{v}^r\}_{r\in\mathcal{S}}$, let $\mathcal{C}$ be the convex hull of $\mathcal{V}$, and let $f^p_{\log} := f_{\text{scr}} : \mathcal{C} \longrightarrow \mathcal{S}$ be the scoring rule defined in Example 3.1(g). We will refer to this as a ***log-likelihood*** scoring rule.

Assume the votes of the different voters are independent random variables (conditional on the true state of nature). Any error model p' induces a behaviour model ρ' by setting $\rho'(\mathbf{v}^r|s) := p'(r|s)$ for all $r, s \in \mathcal{S}$. For any $\delta > 0$, let $\mathcal{P}_{p,\delta}$ be the populace consisting of all behaviour models ρ' induced by an error model p' such that $|p'(t|s) - p(t|s)| < \delta$ for all $t, s \in \mathcal{S}$. If $p(t|s) > 0$ for all $t, s \in \mathcal{S}$, then the populace $\mathcal{P}_{p,\delta}$ satisfies *Minimal Reliability* (see Proposition A.2(a) in the Appendix). Now fix $\epsilon > 0$, and for all $s \in \mathcal{S}$, define $\mathcal{C}^\epsilon_s := \{\mathbf{c} \in \mathcal{C};\ \ c_s > c_t + \epsilon \text{ for all } t \neq s\}$. If $\mathcal{C}'_\epsilon := \bigcup_{s\in\mathcal{S}} \mathcal{C}^\epsilon_s$, then $f^p_{\log}$ satisfies (M3) when restricted to $\mathcal{C}'$. If ϵ and δ are small enough, then $\mathcal{P}_{p,\delta}$ satisfies *Identification* with respect to $f^p_{\log}$ and $\mathcal{C}'_\epsilon$ (see Proposition A.2(b) in the Appendix).

Thus, Proposition 4.1 yields an extension of the Condorcet Jury Theorem to *any* log-likelihood scoring rule. If a sufficiently large number of independent random voters are drawn from the populace $\mathcal{P}_{p,\delta}$, then the log-likelihood scoring rule $F^p_{\log}$ will select the true state of nature, with probability arbitrarily close to 1. For example, if $\mathcal{S}$ is the space of preference orders on some set of alternatives, then this conclusion holds for the Kemeny rule, given the error model proposed by Young (1986, 1988, 1995, 1997). ◊

If the error model p in Example 4.5 has "sufficient symmetry", then the outcome of the rule $F^p_{\log}$ will be the maximum likelihood estimator (MLE) of the true state. Conversely, *any* scoring rule can be interpreted as a log-likelihood scoring rule for some error model, and in many cases, these are in fact maximum likelihood estimators (Pivato, 2013b, Theorem 2.2(b)). For example, the Kemeny rule (Example 3.1(f)) is the MLE for a natural error model on the space of preference orders (Young, 1986, 1988, 1995, 1997). More generally, on any metric space $(\mathcal{S}, d)$ which is "sufficiently symmetric", the generalized median rule (Example 3.1(e)) is the MLE for any *exponential* error model, where $p(s|t) = C\ \exp[-\alpha\, d(s,t)]$, for some constants $\alpha, C > 0$ (Pivato, 2013b, Corollary 3.2). Example 4.5 is a complementary result: not only is $F^p_{\log}$ an MLE, but it is in fact a *consistent estimator* of the true state, in the large-population limit.

5 Correlated Voters

The problem with the model in Section 4 is the assumption that the errors of the voters are stochastically independent. We will now extend this model to allow for correlated voters.

Culture. If the voters are correlated, then we can no longer consider their vote distributions separately. Instead, we must consider the *joint* distribution of all the voters. Given a set $\mathcal{I}$ of individuals and a set $\mathcal{V}$ of votes, a *profile* is an element $\mathbf{V} = (\mathbf{v}_i)_{i\in\mathcal{I}}$ of $\mathcal{V}^{\mathcal{I}}$, which assigns a vote $\mathbf{v}_i$ to each individual i in $\mathcal{I}$. A *collective behaviour model* is a function $\rho : \mathcal{S}\longrightarrow\Delta(\mathcal{V}^{\mathcal{I}})$, which determines a probability distribution $\rho(s)$ over the set of possible profiles, for each possible state $s \in \mathcal{S}$. We cannot assume that we have detailed knowledge of the collective behaviour model of a society. We will only suppose that it arises from some family of collective behaviour models with certain qualitative properties. For this reason, we define a *culture* on $\mathcal{V}$ to be a sequence $\mathfrak{R} = (\mathcal{R}_I)_{I=1}^{\infty}$ where, for all $I \in \mathbb{N}$, $\mathcal{R}_I$ is a set of collective behaviour models on $\mathcal{V}$, for a population of size I. Note that a culture is not intended as a description of a *single* society facing a single epistemic problem. It describes an infinite family of *possible* societies, of all possible sizes, facing a family of possible decision problems.

Correlation. We will need to quantify the correlation between voters arising from a culture. Let $I \in \mathbb{N}$ and let $\mathcal{I} := [1 \ldots I]$. An element $\mathbf{V} = (\mathbf{v}_i)_{i\in\mathcal{I}}$ of $\mathcal{V}^{\mathcal{I}}$ will be interpreted as a profile of I voters. Fix a collective behaviour model $\rho : \mathcal{S}\longrightarrow\Delta(\mathcal{V}^{\mathcal{I}})$, and some state $s \in \mathcal{S}$. For all $i \in \mathcal{I}$, let

$$\widehat{\mathbf{v}}_i \quad := \quad \int_{\mathcal{V}} \mathbf{v}_i \ \mathrm{d}\rho[\mathbf{V}|s]$$

be the expected value of individual i's vote, given the state s.

Let $\langle \bullet, \bullet \rangle$ be an inner product structure on $\mathbb{V}$. Fix $s \in \mathcal{S}$, and let $\mathbf{V} = (\mathbf{v}_i)_{i\in\mathcal{I}}$ be a $\rho(s)$-random profile. For any $i \in \mathcal{I}$, the random vector $(\mathbf{v}_i - \widehat{\mathbf{v}}_i)$ measures the amount by which individual i's vote deviates from its expected value (if the voters satisfy *Identification*, then we can think of this as the "error" in i's vote). The inner product $\langle \mathbf{v}_i - \widehat{\mathbf{v}}_i,\ \mathbf{v}_j - \widehat{\mathbf{v}}_j \rangle$ measures the extent to which the errors of voters i and j are "aligned" with respect to the geometry of $\mathbb{V}$. The *covariance* of voters i and j is the expected value of this inner product:

$$\mathrm{cov}(\mathbf{v}_i, \mathbf{v}_j) \quad := \quad \mathbb{E}[\langle \mathbf{v}_i - \widehat{\mathbf{v}}_i,\ \mathbf{v}_j - \widehat{\mathbf{v}}_j \rangle].$$

This measures the amount, *on average*, by which we can expect the errors of i and j to align in same direction in $\mathbb{V}$. Note that $\mathrm{var}[\mathbf{v}_i] = \mathrm{cov}(\mathbf{v}_i, \mathbf{v}_i)$. We then define the *covariance matrix* of $\rho(s)$ to be the $I \times I$ matrix $\mathrm{cov}[\rho(s)] := [b_{i,j}]_{i,j=1}^{I}$, where, for all $i, j \in [1 \ldots I]$, $b_{i,j} := \mathrm{cov}(\mathbf{v}_i, \mathbf{v}_j)$.

It is important to note that $b_{i,j}$ measures the correlation of *errors*, not the correlation of *votes*. For example, if i and j were both perfectly reliable (so that, with probability 1, we have $\mathbf{v}_i = \mathbf{v}_j = \mathbf{v}$ for some $\mathbf{v} \in \mathcal{V}$ such that $F(\mathbf{v}) = s$), then their votes would be perfectly correlated, but we would have $b_{i,j} = 0$, since the error term is zero. Likewise, if

$b_{i,j} = -1$, this means that the *errors* of i and j are always exact negations of one another —it does not mean their *votes* are always exact negations of one another.

Since we do not know the true collective behaviour model of society, and we don't know the true state of nature, we also do not know the true covariance matrix of the voters. We can only assume that it comes from some family satisfying certain broad qualitative properties. For this reason, we define a ***correlation structure*** to be a sequence $\mathfrak{B} = (\mathcal{B}_I)_{I=1}^{\infty}$, where, for all $I \in \mathbb{N}$, $\mathcal{B}_I$ is a collection of $I \times I$ symmetric, positive definite matrices. The elements of $\mathcal{B}_I$ are the *possible* covariance matrices that we could see in a society of size I. We say that $\mathfrak{B}$ is the correlation structure of the culture $\mathfrak{R}$ if, for every $I \in \mathbb{N}$, $\mathcal{B}_I$ is the set of all covariance matrices $\text{cov}[\rho(s)]$, for any $\rho \in \mathcal{R}_I$ and $s \in \mathcal{S}$.

For any collective behaviour model $\rho \in \mathcal{R}_I$, and any state $s \in \mathcal{S}$, the covariance matrix $\mathbf{B} = \text{cov}[\rho(s)]$ combines two sorts of information: the diagonal entries encode the "reliability" of individual voters, whereas the off-diagonal entries encode the correlations *between* voters. To be precise, for any $i \in [1 \ldots I]$, the diagonal entry $b_{i,i}$ is the variance of individual i's vote in a $\rho(s)$-random profile; this is inversely proportional to i's "reliability". For any distinct $i, j \in [1 \ldots I]$, the off-diagonal entry $b_{i,j}$ is the covariance between the error of individual i's vote and the error of individual j's vote, in a $\rho(s)$-random profile. (Note that $b_{i,j}$ could be negative, reflecting *anti*correlation between the errors of i and j.) For this reason, we will associate two distinct numerical values with each covariance matrix $\mathbf{B} \in \mathcal{B}_I$. We define

$$\sigma(\mathbf{B}) \;:=\; \frac{1}{I}\sum_{i=1}^{I} b_{i,i}, \qquad \text{and} \qquad \kappa(\mathbf{B}) \;:=\; \frac{1}{I(I-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{I} b_{i,j}. \tag{1}$$

In other words, $\sigma(\mathbf{B})$ is the average of the diagonal entries (in effect: the average variance of the voters' errors), while $\kappa(\mathbf{B})$ is the average of the off-diagonal entries (in effect: the average covariance *between* the voters's errors).

Let $\mathfrak{R} = (\mathcal{R}_I)_{I=1}^{\infty}$ be a culture, with correlation structure $(\mathcal{B}_I)_{I=1}^{\infty}$. Let $F = (\mathbb{V}, \mathcal{V}, f)$ be a mean partition rule with continuity set $\mathcal{C}'$. We will say that $\mathfrak{R}$ is ***sagacious*** with respect to F if it satisfies the following three properties.

Identification. For any $I \in \mathbb{N}$, any $\rho \in \mathcal{R}_I$, and any $s \in \mathcal{S}$, if $(\mathbf{v}_i)_{i \in \mathcal{I}}$ is a $\rho(s)$-random profile, then for all $i \in [1 \ldots I]$, the *expected* value of $\mathbf{v}_i$ is in the f-preimage of s inside $\mathcal{C}'$ —i.e. $\mathbb{E}_{\rho(s)}[\mathbf{v}_i] \in f^{-1}\{s\} \cap \mathcal{C}'$.

Asymptotic Reliability. For any $I \in \mathbb{N}$, let $\sigma(I) := \sup_{\mathbf{B} \in \mathcal{B}_I} \sigma(\mathbf{B})$. Then $\lim_{I \to \infty} \dfrac{\sigma(I)}{I} = 0$.

Asymptotically Weak Average Correlation. For any $I \in \mathbb{N}$, let $\kappa(I) := \sup\{\kappa(\mathbf{B});\ \mathbf{B} \in \mathcal{B}_I\}$. Then $\lim_{I \to \infty} \kappa(I) = 0$.

Here, the key condition is *Asymptotically weak average correlation.* This says that voters' errors can be correlated, but as the society grows large, the *average* correlation between the errors of different voters must become small. *Identification* has exactly the same interpretation as in Section 4. The condition of *Asymptotic Reliability* is a very weak form of the *Minimal reliability* condition from Section 4. To see, this, first note that *Minimal reliability* could be weakened to the following condition:

Average Reliability. There is some constant $M > 0$ such that, for any $I \in \mathbb{N}$, and any $\mathbf{B} \in \mathcal{B}_I$, $\sigma(\mathbf{B}) < M$.

This condition allows some individuals to be *very* unreliable, as long as the *average* reliability is good.[4] Clearly, *Minimal reliability* implies *Average Reliability*. But *Asymptotic Reliability* is even weaker than *Average Reliability*: it says that even the *average* reliability can decay as the population gets larger, as long as it does not decay too quickly. (To be precise: the average variance of the voter's errors can grow with population size, but its growth rate must be sublinear.) Most of our examples satisfy the stronger conditions of *Minimal reliability* or *Average Reliability*. But *Asymptotic Reliability* is all that is required for our main result.

Example 5.1. Let $\mathcal{P}$ be a sagacious populace, as defined in Section 4. Given any behaviour models $\rho_1, \ldots, \rho_I \in \mathcal{P}$, and any $s \in \mathcal{S}$, let $\rho_1 \otimes \cdots \otimes \rho_I(s)$ be the product probability measure on $\mathcal{V}^I$ —that is, the distribution of a random profile where $\mathbf{v}_1, \ldots, \mathbf{v}_I$ are are independent random variables, with $\mathbf{v}_i$ distributed according to $\rho_i(s)$ for all $i \in [1 \ldots I]$. This yields a collective behaviour model $\rho_1 \otimes \cdots \otimes \rho_I : \mathcal{S} \longrightarrow \Delta(\mathcal{V}^I)$.

For all $I \in \mathbb{N}$, define $\mathcal{R}_I := \{\rho_1 \otimes \cdots \otimes \rho_I; \;\; \rho_1, \ldots, \rho_I \in \mathcal{P}\}$, and then let $\mathfrak{R} := (\mathcal{R}_I)_{I=1}^{\infty}$. Then $\mathfrak{R}$ is a sagacious culture. ♢

The next result says that, if $\mathcal{S}$ is finite, and a random profile of votes is drawn from a sagacious culture, and the population is sufficiently large, then with very high probability, the outcome of the voting rule will be the true state of nature.

Proposition 5.2 *Let F be a mean partition rule ranging over a finite set $\mathcal{S}$, and let $(\mathcal{R}_I)_{I=1}^{\infty}$ be a sagacious culture for F. For all $I \in \mathbb{N}$, let $\rho_I \in \mathcal{R}_I$. Then for any $s \in \mathcal{S}$,*

$$\text{Prob}\left(F(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_I) = s \;\middle|\; (\mathbf{v}_i)_{i=1}^I \textit{ is a } \rho_I\textit{-random profile}\right) \xrightarrow[I \to \infty]{} 1.$$

In fact, Proposition 5.2 is a special case of our main result, which also applies when $\mathcal{S}$ is infinite. It says that, if a random profile of votes is drawn from a sagacious culture, and the population is sufficiently large, then with very high probability, the outcome of the voting rule will be *very close* to the true state of nature.

Theorem 5.3 *Let F be a mean partition rule ranging over a set $\mathcal{S}$, and let $(\mathcal{R}_I)_{I=1}^{\infty}$ be a sagacious culture for F. For all $I \in \mathbb{N}$, let $\rho_I \in \mathcal{R}_I$. Then for any $s \in \mathcal{S}$, and any open set $\mathcal{U} \subset \mathcal{S}$ containing s,*

$$\text{Prob}\left(F(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_I) \in \mathcal{U} \;\middle|\; (\mathbf{v}_i)_{i=1}^I \textit{ is a } \rho_I\textit{-random profile}\right) \xrightarrow[I \to \infty]{} 1.$$

[4]For a version of the CJT assuming *Average reliability*, see Theorem II of Grofman et al. (1989).

In the case of dichotomous choice (i.e. the classical Condorcet Jury Theorem), this result is very similar to a result proved by Ladha (1992)[5]. Theorem 5.3 extends this result to a much larger family of epistemic social choice rules, and also weakens *Minimal reliability* to *Asymptotic Reliability*. Using Theorem 5.3, it is straightforward to extend Examples 4.2, 4.4, and 4.5 to a setting with correlated voters; we leave the details to the reader.

What other combinations of voting rules and cultures are sagacious? First, we will see how to eliminate any explicit mention of the voting rule from this question. Let $\mathbb{V}$ be a vector space, and let $\mathfrak{R} = (\mathcal{R}_I)_{I=1}^{\infty}$ be a culture on $\mathbb{V}$. We will say that $\mathfrak{R}$ is ***identifiable*** if, for any $s \in \mathcal{S}$, there is a closed, convex subset $\mathcal{C}'_s \subset \mathbb{V}$ such that, for any $I \in \mathbb{N}$, any $\rho \in \mathcal{R}_I$, and any $s \in \mathcal{S}$, if $(\mathbf{v}_i)_{i \in \mathcal{I}}$ is a $\rho(s)$-random profile, then for all $i \in [1 \ldots I]$, the *expected* value of $\mathbf{v}_i$ is in $\mathcal{C}'_s$. This is a minimal condition for any possibility of epistemic social choice. We will say that a correlation structure $\mathfrak{B} = (\mathcal{B}_I)_{I=1}^{\infty}$ is ***sagacious*** if it satisfies *Asymptotic Reliability* and *Asymptotically Weak Average Correlation.*

Proposition 5.4 *If $\mathfrak{R}$ is an identifiable culture, with correlation structure $\mathfrak{B}$, and $\mathfrak{B}$ is sagacious, then there exists a mean partition rule F on $\mathbb{V}$ such that $\mathfrak{R}$ is sagacious with respect to F.*

So, what sort of correlation structures are sagacious? We now turn to this question.

6 Social networks

A ***graph*** is a set $\mathcal{I}$ equipped with a symmetric, reflexive binary relation $\sim$. When $\mathcal{I}$ is a set of voters, we can interpret a graph as a *social network*: if $i \sim j$, we interpret this to mean that voters i and j are somehow "socially connected" (e.g. friends, family, neighbours, colleagues, classmates, etc.). The main result of this section says that, if this social network satisfies certain mild geometric conditions, and the correlation between voters is a decreasing function of their distance in the network, then the resulting culture will be sagacious.

We cannot assume that we have *exact* knowledge of the social network topology; we can only assume that belongs to some family of graphs satisfying broad qualitative properties. For this reason, we define a ***social web*** to be a sequence $\mathfrak{N} = (\mathcal{N}_I)_{I=1}^{\infty}$, where, for all $I \in \mathbb{N}$, $\mathcal{N}_I$ is a set of possible graphs of size I. Thus, our hypotheses will be formulated in terms of the asymptotic properties of the graphs in $\mathcal{N}_I$, as $I \to \infty$. But before we can formulate these hypotheses, we need some basic concepts from graph theory.

Sublinear average degree growth. For any $i \in \mathcal{I}$, the ***degree*** of i is the number of links i has in the graph $(\mathcal{I}, \sim)$. Formally, $\deg(i, \sim) := \#\{j \in \mathcal{I};\ i \sim j\}$. If $|\mathcal{I}| = I$, then the ***average degree*** of the graph $(\mathcal{I}, \sim)$ is defined:

$$\mathrm{avedeg}(\mathcal{I}, \sim) \quad := \quad \frac{1}{I} \sum_{i \in \mathcal{I}} \deg(i, \sim).$$

[5] See Ladha (1992, Corollary, p.628) and Ladha (1995, Proposition 1).

This is the average number of social links of a voter in the social network described by $(\mathcal{I}, \sim)$. We then define $\overline{\text{avedeg}}(\mathcal{N}_I) := \sup_{(\mathcal{I},\sim)\in\mathcal{N}_I} \text{avedeg}(\mathcal{I}, \sim)$. We will say that a social web $(\mathcal{N}_I)_{I=1}^{\infty}$ exhibits ***sublinear average degree growth*** if

$$\lim_{I\to\infty} \frac{1}{I} \overline{\text{avedeg}}(\mathcal{N}_I) \quad = \quad 0. \tag{2}$$

For instance, if $\overline{\text{avedeg}}(\mathcal{N}_I)$ remains bounded as $I\to\infty$, then the limit (2) is obviously satisfied. However, the limit (2) even allows $\overline{\text{avedeg}}(\mathcal{N}_I)$ to grow as $I\to\infty$, as long as it grows more slowly than a linear function.

Example 6.1. (Asymptotic degree distributions) Let $(\mathcal{I}, \sim)$ be a graph. For all $n \in \mathbb{N}$, let

$$\mu_{(\mathcal{I},\sim)}(n) \quad := \quad \frac{1}{I} \#\{i \in \mathcal{I} \; ; \; \deg(i, \sim) = n\}.$$

This defines a probability distribution $\mu_{(\mathcal{I},\sim)} \in \Delta(\mathbb{N})$, called the the ***degree distribution*** of $(\mathcal{I}, \sim)$. If $\mu \in \Delta(\mathbb{N})$ is another probability distribution, then we define the distance between μ and $\mu_{(\mathcal{I},\sim)}$ by

$$d(\mu, \mu_{(\mathcal{I},\sim)}) \quad := \quad \sum_{n=1}^{\infty} n \cdot \left| \mu_{(\mathcal{I},\sim)}(n) - \mu(n) \right|.$$

We will say that a social web $\mathfrak{N}$ has ***asymptotic degree distribution*** μ if

$$\lim_{I\to\infty} \sup_{(\mathcal{I},\sim)\in\mathcal{N}_I} d(\mu, \mu_{(\mathcal{I},\sim)}) \quad = \quad 0.$$

Let $\text{avedeg}(\mu) := \sum_{n=1}^{\infty} \mu(n)\, n$. If this value is finite, and $\mathfrak{N}$ has asymptotic degree distribution μ, then it is easy to check that $\overline{\text{avedeg}}(\mathcal{N}_I)$ will converge to $\text{avedeg}(\mu)$ as $I\to\infty$; thus, $\mathfrak{N}$ will have sublinear average degree growth.

For example, many social networks seem to exhibit a "power law" degree distribution of the form $\mu(n) \approx K/n^{\alpha}$, for all $n \in \mathbb{N}$, where $\alpha > 1$, and where $K > 0$ is a normalization constant (Barabási and Albert, 1999; Albert et al., 1999). This is a well-defined probability distribution on $\mathbb{N}$, as long as $\alpha > 1$. (Typically, $2 < \alpha < 3$.) Networks with power law distributions contain a surprisingly large number of "superconnected" or "hub" individuals, whose degrees are much larger than that of the typical person. Thus, in such networks, some individuals can be correlated with a very large number of other individuals. However, $\text{avedeg}(\mu)$ is still finite, as long as $\alpha > 2$. Thus, if a social web has a power law asymptotic degree distribution with $\alpha > 2$, then it will have sublinear average degree growth. ◇

Not all social webs have sublinear average degree growth. For example, if $\alpha < 2$ in Example 6.1, then $\overline{\text{avedeg}}(\mathcal{N}_I)$ will grow at a superlinear rate as $I\to\infty$. For another example, suppose $\mathcal{N}_I$ is generated by sampling the Erdös-Renyi "random graph" model, where there is a constant probability p that any two randomly chosen agents are linked. Then $\overline{\text{avedeg}}(\mathcal{N}_I) \approx p\, I$, which grows linearly as $I\to\infty$. However, these are not considered realistic models for social networks in most situations.

Nearest-neighbour correlation structures. Let $\mathfrak{B} = (\mathcal{B}_I)_{I=1}^{\infty}$ be a correlation structure, and let $\mathfrak{N} = (\mathcal{N}_I)_{I=1}^{\infty}$ be a social web. We will say that $\mathfrak{B}$ is a ***nearest-neighbour correlation structure*** for $\mathfrak{N}$ if:

- For any $I \in \mathbb{N}$ and $\mathbf{B} \in \mathcal{B}_I$, there is some graph $(\mathcal{I}, \sim)$ in $\mathcal{N}_I$ and some identification of $\mathcal{I}$ with $[1 \dots I]$ such that, for all $i, j \in [1 \dots I]$, we have $b_{i,j} \neq 0$ only if $i \sim j$,
- There is some constant $M > 0$ such that, for any $I \in \mathbb{N}$ and $\mathbf{B} \in \mathcal{B}_I$, we have $|b_{i,j}| \leq M$ for all $i, j \in [1 \dots I]$.

We now come to the first result of this section.

Proposition 6.2 *If a social web* $\mathfrak{N}$ *has sublinear average degree growth, then any nearest-neighbour correlation structure for* $\mathfrak{N}$ *is sagacious.*

In fact, Proposition 6.2 is only a special case of the main result of this section. But before we can state this result, we must introduce some more terminology.

Generalized degrees. Let $(\mathcal{I}, \sim)$ be a connected graph. A ***path*** in $(\mathcal{I}, \sim)$ is a sequence of vertices $i_0, i_1, \dots, i_L \in \mathcal{I}$ such that $i_0 \sim i_1 \sim \cdots \sim i_L$; we say this path has ***length*** L, and that it ***connects*** i_0 to i_L. For any $i, j \in \mathcal{I}$, let $d_\sim(i,j)$ be the length of the shortest path connecting i to j in $(\mathcal{I}, \sim)$. For completeness, we also define $d_\sim(i,i) := 0$ for all $i \in \mathcal{I}$. Observe that $d_\sim$ is a metric on $\mathcal{I}$. (It is called the ***geodesic metric*** of the graph.) For any $r \in \mathbb{N}$ and $i \in \mathcal{I}$, we define the r-***degree*** of i as $\deg^r(i, \sim) := \#\{j \in \mathcal{I};\ d_\sim(i,j) = r\}$. Thus, $\deg^1(i, \sim)$ is just the degree of i, as defined above. Now let $\gamma : \mathbb{N} \longrightarrow [0, \infty]$ be a function (typically, increasing). For any $i \in \mathcal{I}$, we define the γ-***degree*** of i by

$$\deg^\gamma(i, \sim) \quad := \quad \sup_{r \in \mathbb{N}} \frac{\deg^r(i, \sim)}{\gamma(r)}. \tag{3}$$

We then define

$$\operatorname{avedeg}^\gamma(\mathcal{I}, \sim) \quad := \quad \frac{1}{I} \sum_{i \in \mathcal{I}} \deg^\gamma(i, \sim), \tag{4}$$

$$\text{and} \qquad \overline{\operatorname{avedeg}}(\mathcal{N}_I) \quad := \quad \sup_{(\mathcal{I}, \sim) \in \mathcal{N}_I} \operatorname{avedeg}(\mathcal{I}, \sim). \tag{5}$$

We will say that a social web $(\mathcal{N}_I)_{I=1}^{\infty}$ exhibits ***sublinear average*** γ-***degree growth*** if

$$\lim_{I \to \infty} \frac{1}{I} \overline{\operatorname{avedeg}}^\gamma(\mathcal{N}_I) \quad = \quad 0. \tag{6}$$

For instance, suppose we define $\gamma_1 : \mathbb{N} \longrightarrow \{1, \infty\}$ by

$$\gamma_1(r) \quad := \quad \begin{cases} 1 & \text{if} \quad r = 1; \\ \infty & \text{if} \quad r \geq 2. \end{cases} \tag{7}$$

Then clearly, $\deg^{\gamma_1}(i, \sim) = \deg(i, \sim)$ for all $i \in \mathcal{I}$ and all $(\mathcal{I}, \sim) \in \mathcal{N}_I$. Thus, formula (6) is equivalent to formula (2); thus, a social web will have sublinear average γ_1-degree growth if and only if it has sublinear average degree growth.

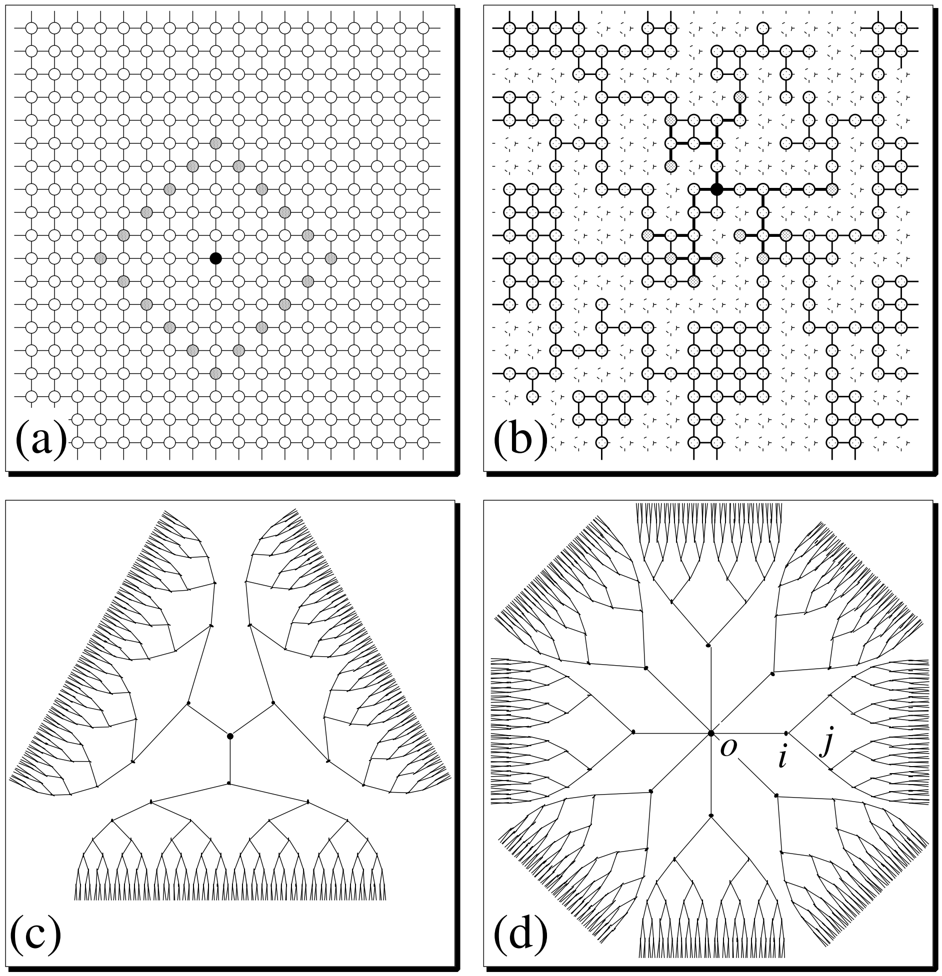

Figure 4: **(a)** A two-dimensional grid has growth bounded by $\gamma(r) = 4\,r$. For example, if i is the black node, then $\deg^5(i,\sim) = 20$ (the number of grey nodes). **(b)** If $(\mathcal{J},\sim)$ is a subgraph of a two-dimensional grid, then its growth is also bounded by $\gamma(r) = 4\,r$. In this case, if i is the black node, then $\deg^5(i,\sim) = 9$. **(c)** If $(\mathcal{J},\sim)$ is a tree where all nodes have 3 edges, then its growth is bounded by $\gamma(r) = 3\cdot(2^{r-1})$. **(d)** If $(\mathcal{J},\sim)$ is an eight binary trees around a hub, then its growth is bounded by $\gamma(r) = 8\cdot(2^{r-1})$.

Example 6.3. (Social networks from infinite graphs) Let $\mathcal{J}$ be an infinite set of vertices, and let $\sim$ be a graph structure on $\mathcal{J}$; this is called an ***infinite graph***. If $\gamma : \mathbb{N}\longrightarrow[0,\infty]$ is some function, then $(\mathcal{J},\sim)$ has γ-***bounded growth*** if we have $\deg^r(j,\sim) \leq \gamma(r)$, for all $j \in \mathcal{I}$ and all $r \in \mathbb{N}$. In other words, $\deg^\gamma(j) \leq 1$ for all $j \in \mathcal{J}$.

For example, if $(\mathcal{J},\sim)$ is the two-dimensional grid shown in Figure 4(a), then $\deg^r(i) = 4\,r$ for all $r \in \mathbb{N}$; thus, $(\mathcal{J},\sim)$ has growth bounded by the function $\gamma(r) := 4\,r$. More generally, if $(\mathcal{J},\sim)$ is an infinite subgraph of a two-dimensonal grid, like the one shown in Figure 4(b), then its growth bounded by the function $\gamma(r) := 4\,r$. Likewise, if $(\mathcal{J},\sim)$ was an infinite subgraph of a D-dimensional grid, then it would have growth bounded by a polynomial function $\gamma(r) := K\,r^{D-1}$ (for some constant $K > 0$). As these examples show, a graph with polynomially bounded growth of degree $D-1$ has a "D-dimensional" geometry.

In contrast, suppose $(\mathcal{J},\sim)$ is an infinite tree where every node has degree 3, as shown in Figure 4(c). Then $(\mathcal{J},\sim)$ has growth bounded by $\gamma(r) = 3 \cdot (2^{r-1})$. More generally, if $M \in \mathbb{N}$, and $(\mathcal{J},\sim)$ is any graph where every vertex has degree $(M+1)$ or less, then $(\mathcal{J},\sim)$ has growth bounded by the exponential function $\gamma(r) := M^r$.

For all $I \in \mathbb{N}$, let $\mathcal{N}_I$ be a collection of connected subgraphs of $(\mathcal{J},\sim)$ with exactly I vertices; then the sequence $\mathfrak{N} = (\mathcal{N}_I)_{I=1}^{\infty}$ is a social web, which we will say is ***subordinate*** to $(\mathcal{J},\sim)$. Heuristically, the vertices in the graph $(\mathcal{J},\sim)$ represent the set of all "potential" people who could exist, and the links in $(\mathcal{J},\sim)$ are all "potential" social connections between them. Thus, any *actual* social network will be some finite subgraph of $(\mathcal{J},\sim)$; these are the graphs which appear in $\mathfrak{N}$. If $(\mathcal{J},\sim)$ has growth bounded by the function γ, then it is easy to see that $\overline{\text{avedeg}}^{\gamma}(\mathcal{N}_I) \leq 1$ for all $I \in \mathbb{N}$; thus, the asymptotic condition (6) is trivially satisfied, so that $\mathfrak{N}$ has sublinear average γ-degree growth. $\diamondsuit$

Correlation decay. Let $(\mathcal{I},\sim)$ be a graph, and let $\mathbf{B} \in \mathbb{R}^{I\times I}$ be an $I \times I$ matrix (e.g. a covariance matrix). Let $\beta : \mathbb{N} \longrightarrow \mathbb{R}_+$ be a function (typically, decreasing). We will say that the matrix $\mathbf{B}$ exhibits β-***decay*** relative to $(\mathcal{I},\sim)$ if (after bijectively identifying $\mathcal{I}$ with $[1 \ldots I]$ in some way), we have $b_{i,j} \leq \beta[d_{\sim}(i,j)]$ for all $i,j \in \mathcal{I}$. In particular, $\mathbf{B}$ exhibits ***exponential decay*** if there are some constants $\lambda \in (0,1)$ and $K \geq 0$ such that $b_{i,j} \leq K \cdot \lambda^{d_{\sim}(i,j)}$ for all $i,j \in \mathcal{I}$. Exponential correlation decay is a typical phenomenon in the spatially distributed stochastic processes studied in statistical physics, such as Ising models of ferromagnetism (Penrose and Lebowitz, 1974; Procacci and Scoppola, 2001; Bach and Møller, 2003). The opinions of the voters in a social network can be seen as such a spatially distributed stochastic process.

We will say that a correlation structure $\mathfrak{B} = (\mathcal{B}_I)_{I=1}^{\infty}$ exhibits β-***correlation decay*** relative to social web $\mathfrak{N} = (\mathcal{N}_I)_{I=1}^{\infty}$ if, for every $I \in \mathbb{N}$, and every matrix $\mathbf{B} \in \mathcal{B}_I$, there is some graph $(\mathcal{I},\sim)$ in $\mathcal{N}_I$ such that $\mathbf{B}$ exhibits β-decay relative to $(\mathcal{I},\sim)$. For example, let $M > 0$, and define $\beta(1) := M$ while $\beta(r) := 0$ for all $r \geq 2$. Then $\mathfrak{B}$ exhibits β-correlation decay relative to $\mathfrak{N}$ if and only if $\mathfrak{B}$ is a nearest-neighbour correlation structure for $\mathfrak{N}$.

Subordinate correlation structures. We will say that a correlation structure $\mathfrak{B}$ is ***subordinate*** to a social web $\mathfrak{N}$ if there exist functions $\beta : \mathbb{N} \longrightarrow \mathbb{R}^+$ and $\gamma : \mathbb{N} \longrightarrow [0,\infty]$ such that $\mathfrak{N}$ has sublinear average γ-degree growth, $\mathfrak{B}$ exhibits β-correlation decay relative to $\mathfrak{N}$, and also

$$\sum_{r=0}^{\infty} \gamma(r)\,\beta(r) \quad < \quad \infty. \tag{8}$$

(Here, we adopt the convention that $\infty \cdot 0 = 0$.) Note that, the faster $\gamma(r)$ grows as $r \to \infty$, the faster β must decay to zero in order for inequality (8) to be satisfied.

Example 6.4. (a) Let $M, D \in \mathbb{N}$ and suppose that $\mathfrak{N}$ is subordinate to an infinite, D-dimensional grid or an M-ary tree, as described in Example 6.3. Let $\gamma(r) := M^r$ for all $r \in \mathbb{N}$; then $\mathfrak{N}$ has sublinear average γ-degree growth. Let $\lambda < 1/M$, let $\beta(r) := \lambda^r$ for all

$r \in \mathbb{N}$; and suppose that every matrix in $\mathfrak{B}$ exhibits β-exponential correlation decay with respect to some graph in $\mathfrak{N}$. Let $c := M\,\lambda$; then $0 < c < 1$, and

$$\sum_{r=0}^{\infty} \gamma(r)\,\beta(r) \quad = \quad \sum_{r=0}^{\infty} M^r\,\lambda^r \quad = \quad \sum_{r=0}^{\infty} c^r \quad = \quad \frac{1}{1-c} \quad < \quad \infty.$$

Thus, inequality (8) is satisfied, so $\mathfrak{B}$ is subordinate to $\mathfrak{N}$.

(b) Suppose $\mathfrak{N}$ has sublinear average degree growth, and $\mathfrak{B}$ is a nearest-neighbour correlation structure for some social web $\mathfrak{N}$. As we have seen, this means there is some constant $M > 0$ such that $\beta(r) := M$ if $r = 1$ and $\beta(r) := 0$ for all $r > 0$, and $\mathfrak{B}$ exhibits β-correlation decay relative to $\mathfrak{N}$. Now define $\gamma_1 : \mathbb{N} \longrightarrow \{1, \infty\}$ by formula (7). Then inequality (8) is automatically satisfied. By comparing formulae (2) and (6), we see that $\mathfrak{N}$ has sublinear average γ_1-degree growth. Thus, $\mathfrak{B}$ is subordinate to $\mathfrak{N}$. $\diamondsuit$

We now come to the main result of this section.

Proposition 6.5 *Let $\mathfrak{N}$ be a social web. Then any correlation structure which is subordinate to $\mathfrak{N}$ is sagacious.*

For example, Proposition 6.2 follows by applying Proposition 6.5 to Example 6.4(b).

7 Deliberation

We will now show that the sagacity of a culture is preserved under a simple model of deliberation. We will adapt a well-known model of deliberation proposed by DeGroot (1974).[6] Formally, for all distinct $i, j \in \mathcal{I}$, let $d_{i,j} \geq 0$ be the "influence" of voter j on voter i. This could be determined by the level of respect or trust which i has for j. Note that influence is not symmetric: we may have $d_{i,j} \neq d_{j,i}$. The diagonal entry $d_{i,i}$, in effect, measures i's confidence in her own opinions. Let $\mathbf{D} := [d_{ij}]_{i,j \in \mathcal{I}}$. We will assume that $\mathbf{D}$ is a stochastic matrix —that is, $\sum_{j \in \mathcal{I}} d_{i,j} = 1$, for all $i \in \mathcal{I}$. We will refer to $\mathbf{D}$ as an *influence matrix*. We cannot assume exact knowledge of the pattern of social influences in the society. Thus, instead of fixing a single influence matrix $\mathbf{D}$, we will consider an entire family of such influence matrices. Formally, we define a *deliberative institution* to be a sequence $\mathfrak{D} = (\mathcal{D}_I)_{I=1}^{\infty}$, where for all $I \in \mathbb{N}$, $\mathcal{D}_I$ is a family of $I \times I$ stochastic matrices.

A deliberative institution is not a culture. It is a *transformation*, which can be applied to a culture to obtain another culture, as we now explain. For the rest of this section, suppose that $\mathcal{V}$ is a *convex* subset of a vector space $\mathbb{V}$. Let $\mathbf{V} = (\mathbf{v}_i)_{i=1}^{I}$ be an I-voter profile in $\mathcal{V}^I$. Given an $I \times I$ stochastic matrix $\mathbf{D}$ (e.g. an element of $\mathcal{D}_I$), we define $\mathbf{D} \cdot \mathbf{V}$ to be the profile $\mathbf{V}' = (\mathbf{v}'_i)_{i=1}^{I}$, where, for all $i \in \mathcal{I}$,

$$\mathbf{v}'_i \quad := \quad \sum_{j=1}^{I} d_{i,j}\,\mathbf{v}_j.$$

[6] For an interesting recent application of the DeGroot model, see Golub and Jackson (2010).

For all $i \in \mathcal{I}$, $\mathbf{v}_i$ represents the opinion of voter i *before* deliberation, while $\mathbf{v}'_i$ represents her opinion *after* deliberation —it is a weighted average of her own opinion and those of her peers, with the weights reflecting their degree of "influence" over her.

Now, let $\rho : \mathcal{S} \longrightarrow \Delta(\mathcal{V}^{\mathcal{I}})$ be a collective behaviour model, fix $s \in \mathcal{S}$, and suppose $\mathbf{V} = (\mathbf{v}_i)_{i=1}^I$ is a $\rho(s)$-random profile. Then $\mathbf{D} \cdot \mathbf{V}$ is another random profile. We denote the probability distribution of $\mathbf{D} \cdot \mathbf{V}$ by $\mathbf{D} \odot \rho(s)$. If we do this for all $s \in \mathcal{S}$, then we obtain a collective behaviour model $\mathbf{D} \odot \rho : \mathcal{S} \longrightarrow \Delta(\mathcal{V}^I)$.

Now, let $\mathfrak{R} = (\mathcal{R}_I)_{I=1}^{\infty}$ be a culture, and let $\mathfrak{D} = (\mathcal{D}_I)_{I=1}^{\infty}$ be a deliberative institution. For all $I \in \mathbb{N}$, we define

$$\mathcal{D}_I \odot \mathcal{R}_I \quad := \quad \{\mathbf{D} \odot \rho \, ; \, \mathbf{D} \in \mathcal{D}_I \text{ and } \rho \in \mathcal{R}_I\}.$$

This is a collection of collective behaviour models on a population of I voters. Heuristically, it has the following interpretation:

- $\mathcal{R}_I$ is the set of possible collective behaviour models which can exist *before* deliberation.
- $\mathcal{D}_I$ is the set of the possible deliberations which can occur.
- $\mathcal{D}_I \odot \mathcal{R}_I$ is the set of the possible collective behaviour models which can exist *after* deliberation.

We then define the culture $\mathfrak{D} \odot \mathfrak{R} := (\mathcal{R}'_I)_{I=1}^{\infty}$, where, for each $I \in \mathbb{N}$, $\mathcal{R}'_I := \mathcal{D}_I \odot \mathcal{R}_I$. We interpret this as the culture which arises when voters drawn from the culture $\mathfrak{R}$ deliberate according to $\mathfrak{D}$.

The main result of this section says that sagacity survives such deliberation. To explain this result, we will need a bit more notation. For any $j \in \mathcal{I}$, we define $\overline{d}_j := \sum_{i \in \mathcal{I}} d_{i,j}$. This measures the "total influence" of voter j on other voters. A deliberative institution $\mathfrak{D}$ is *local* if there exists a constant $D > 0$ (which we will call the *modulus* of $\mathfrak{D}$) such that, for all $I \in \mathbb{N}$ and all $\mathbf{D} \in \mathcal{D}_I$ we have $\overline{d}_j \leq D$ for all $j \in \mathcal{I}$. In other words, the total influence of each voter in any society is bounded; she can have a significant influence over at most a small number of individuals (although she might also have a very small influence over a much larger number of individuals). In particular, there are no "demagogues" who can strongly influence a large number of people.

Proposition 7.1 *Let $F = (\mathbb{V}, \mathcal{V}, f)$ be a mean partition voting rule, where $\mathcal{V}$ is a convex subset of $\mathbb{V}$. If $\mathfrak{D}$ is a local deliberative institution, and the culture $\mathfrak{R}$ is sagacious for F, then the culture $\mathfrak{D} \odot \mathfrak{R}$ is also sagacious for F.*

To illustrate the scope of this result, we will now construct some examples of local deliberative institutions. Given two deliberative institutions $\mathfrak{D}$ and $\mathfrak{E}$, we define $\mathfrak{D} \cdot \mathfrak{E} := (\mathcal{C}_I)_{I=1}^{\infty}$, where for all $I \in \mathbb{N}$, $\mathcal{C}_I := \{\mathbf{D}\,\mathbf{E};\ \mathbf{D} \in \mathcal{D}_I \text{ and } \mathbf{E} \in \mathcal{E}_I\}$. Informally, $\mathfrak{D} \cdot \mathfrak{E}$ represents a deliberative institution where the voters first deliberate according to an influence matrix drawn from $\mathfrak{E}$, and then deliberate further using a matrix drawn from $\mathfrak{D}$.

Given any $q \in [0,1]$, we define $q\,\mathfrak{D} + (1-q)\,\mathfrak{E} := (\mathcal{C}_I)_{I=1}^{\infty}$, where for all $I \in \mathbb{N}$, $\mathcal{C}_I := \{q\,\mathbf{D} + (1-q)\mathbf{E};\ \mathbf{D} \in \mathcal{D}_I \text{ and } \mathbf{E} \in \mathcal{E}_I\}$. Informally, this represents a deliberative institution where the influence of one voter on another is a weighted average of two forms of influence; one described by $\mathfrak{D}$ and the other by $\mathfrak{E}$. (For example, $\mathfrak{D}$ might describe influences arising from personal affection, while $\mathfrak{E}$ describes influences arising from professional respect and admiration.)

Proposition 7.2 *Let $\mathfrak{D}$ and $\mathfrak{E}$ be two local deliberative institutions. Then $\mathfrak{D} \cdot \mathfrak{E}$ is also local, and $q\,\mathfrak{D} + (1-q)\,\mathfrak{E}$ is local for any $q \in [0,1]$.*

For any deliberative institution $\mathfrak{D}$ and any $n \in \mathbb{N}$, we define $\mathfrak{D}^n := (\mathcal{D}_I^n)_{I=1}^{\infty}$, where for all $I \in \mathbb{N}$, $\mathcal{D}_I^n := \{\mathbf{D}_1 \cdots \mathbf{D}_n;\ \mathbf{D}_1, \ldots \mathbf{D}_n \in \mathcal{D}_I\}$. Informally, $\mathfrak{D}^n$ represents a deliberative institution where the voters deliberate n times, using n influence matrices drawn from $\mathfrak{D}$. Let $\mathfrak{D}^0 := \{\mathbf{I}\}$, where $\mathbf{I}$ is the identity matrix (this represents *no* deliberation). Finally, given any sequence $\mathbf{q} = (q_n)_{n=0}^{\infty}$ in $[0,1]$ with $\sum_{n=0}^{\infty} q_n = 1$, we can define the institution $\sum_{n=0}^{\infty} q_n \mathfrak{D}^n$ in the obvious way; informally, this is an institution where voters have deliberated a very large number of times, and the total influence of one voter on another is a weighted average of more direct, short-term effects (corresponding to small values of n) and more indirect, longer-terms effects (corresponding to larger values of n).

Corollary 7.3 *If $\mathfrak{D}$ is a local deliberative institution with modulus D, then $\sum_{n=0}^{\infty} q_n \mathfrak{D}^n$ is local as long as $\sum_{n=0}^{\infty} q_n\, D^n < \infty$.*

As a simple example, suppose $\mathcal{D}_I$ contains only one matrix, $\mathbf{D}$, and furthermore, suppose that most of the entries in $\mathbf{D}$ are zero. For any $i, j \in \mathcal{I}$, write "$j \rightsquigarrow i$" if $d_{i,j} > 0$. Informally, this means "j has some direct influence on i". The relation $\rightsquigarrow$ defines a directed graph, which we might call the "influence network". Now let $\mathbf{D}^n = [d_{i,j}^{(n)}]$; Thus, $d_{i,j}^{(n)} > 0$ if and only if there is at least one directed path of length n from j to i in the influence network; in this case, $d_{i,j}^{(n)}$ measures the total *indirect* influence which j has on i via such chains of intermediaries. Finally, if $\sum_{n=1}^{\infty} q_n \mathbf{D}^n = [e_{i,j}]_{i,j \in \mathcal{I}}$, then $e_{i,j}$ measures the total influence which j has on i over all possible chains of all possible lengths (weighted by the vector $\mathbf{q}$).

An interesting special case is when $\rightsquigarrow$ is an *acyclic digraph* on $\mathcal{I}$ (that is: a binary relation which is irreflexive, antisymmetric, and whose transitive closure contains no cycles). In this case, the society has a hierarchical structure: there are "opinion leaders" (who are further upstream with respect to $\rightsquigarrow$) and "followers" (who are downstream from the opinion leaders). Informally, "opinion leaders" correspond to pundits, politicians, public intellectuals, and religious authorities, who can influence a large audience of "followers". The deliberative institution will be local as long as the opinion leaders do not have too strong an influence on their followers.

Conclusion

The optimistic predictions of this paper are consistent with most of the previous literature on epistemic social choice theory, but seemingly inconsistent with the empirical evidence.

In reality, modern mass democracies do not seem to be very epistemically competent. What is the reason for this discrepancy? Are some of our hypotheses incorrect?

Perhaps the hypotheses of *Identification* and *Asymptotic reliability* impute an unrealistically high level of epistemic competence to the average voter. There is now abundant empirical evidence that human beings are subject to systematic cognitive biases, particularly in tasks which involve logical or probabilistic reasoning (Kahneman, 2011). They also overestimate small but spectacular risks (e.g. terrorism), while neglecting threats which are less visible but far more pervasive and hazardous (e.g. antibiotic resistant bacteria). They gravitate towards simple solutions, based on simplistic moral narratives. A more sophisticated theory of epistemic democracy should account for such cognitive biases.

Ironically, the purported epistemic competency of large groups may be self-refuting. By combining the strategic analysis of Austen-Smith and Banks (1996) with the "rational ignorance" of Downs (1957), a voter might decide that there is no reason for her to become informed at all, because the group is going to get the right answer anyways. If enough voters behave this way, then the epistemic competency of the group may be undermined. To counteract such "epistemic free-riding", perhaps we must offer each voter an individual incentive to get the right answer. It is notable that Galton's (1907) original inspiration was a betting pool, not a referendum.

We might also question our assumption that the set $\mathcal{S}$ of social alternatives can be identified one-for-one with the possible states of the world. In reality, the alternatives in $\mathcal{S}$ are generated by some murky and epistemically dubious political process, and it is possible that *none* of these alternatives correctly describes the actual state of the world. Suppose $\mathcal{S} = \{f, t_1, t_2, t_3, t_4\}$, where f is a completely false theory, while the theories t_1, t_2, t_3, t_4 are each somewhat flawed but "mostly true". Then even in a society of highly competent voters, where 75% select one of the "mostly true" theories, the false theory f might win a plurality vote through vote-splitting, contradicting the predictions of Example 4.2(b). And this assumes that $\mathcal{S}$ consists of clear descriptions of possible worlds at all; in some cases, the statements in $\mathcal{S}$ may be ambiguous or even meaningless.

Finally, it is possible that modern mass democracies actually exhibit a much higher degree of voter correlation than we allowed in our models. The hypothesis of *Asymptotically Weak Average Correlation* is consistent with a world where most correlations arise from "local" interactions —e.g. through links in a social network, or via person-to-person deliberation. It is even consistent with an Internet-saturated world, where voters are influenced by bloggers and other social media celebrities whose audiences follow a power law distribution (Example 6.1). However, these models assume that the process which generates the social network topology is entirely independent of the process which generates the voters' opinions. In practice, these two processes are highly interdependent, because people preferentially affiliate with other people who share their opinions. This can lead to the formation of "echo chambers", within which deliberation actually *reduces* epistemic competency, by reinforcing voters' ideological biases and cultivating manichean extremism (Sunstein, 2009). A properly functioning epistemic democracy needs mechanisms to prevent the formation of such echo chambers.

Furthermore, the growing concentration of media ownership in modern societies means that most voters get most of their information about the world from a very small number of genuinely independent sources. If we take the epistemic view of democracy seriously, then one possible policy implication is that governments should be much more aggressive in preventing the burgeoning oligopolization of radio, television and print media.

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A Appendix

The following result will be used in our analysis of Example 4.5 below. The proof is well-known, but it is short, so we include it for completeness.

Lemma A.1 *Fix* $\mathbf{p} \in \Delta(\mathcal{S})$. *Define* $F_{\mathbf{p}} : \Delta(\mathcal{S}) \longrightarrow \mathbb{R}$ *by* $F_{\mathbf{p}}(\mathbf{q}) := \sum_{s \in \mathcal{S}} p_s \log[q_s]$.[7] *Then* $\mathrm{argmax}(F) = \mathbf{p}$.

Proof: We use the method of Lagrange multipliers. Let $\mathbf{1} \in \mathbb{R}^{\mathcal{S}}$ be the constant 1 vector Note that $\Delta(\mathcal{S}) := \{\mathbf{r} \in \mathbb{R}^{\mathcal{S}}_{+}; \quad \mathbf{1} \bullet \mathbf{r} = 1\}$. Thus, if an interior maximum $\mathbf{q}^*$ exists, it must satisfy the first-order condition that $\nabla F_{\mathbf{p}}(\mathbf{q}^*) = c\,\mathbf{1}$ for some constant $c \in \mathbb{R}$.

Now, for all $s \in \mathcal{S}$, we have $\partial_s F(\mathbf{q}) = p_s/q_s$. Thus, $\nabla F_{\mathbf{p}}(\mathbf{q}) = c\,\mathbf{1}$ if and only if $p_s = c\,q_s$ for all $s \in \mathcal{S}$. Since $\mathbf{p}$ and $\mathbf{q}$ are both probability vectors, this can happen only if $c = 1$ and $\mathbf{p} = \mathbf{q}$. Thus, the unique critical point of $F_{\mathbf{p}}$ is at $\mathbf{p}$ itself.

Finally, observe that $F_{\mathbf{p}}$ is concave (indeed, $\partial_t \partial_s F_{\mathbf{p}} = 0$ if $s \neq t$, whereas $\partial_s^2 F_{\mathbf{p}}(\mathbf{q}) = -p_s/q_s^2 < 0$, so the Hessian is a negative diagonal matrix, hence negative-definite ever-where). Thus, this critical point is a maximum. □

The next result deals with the unproved assertions in Example 4.5.

Proposition A.2 *Let* $\mathcal{S}$ *be finite, let* $p : \mathcal{S} \longrightarrow \Delta(\mathcal{S})$ *be any error model, let* $\delta > 0$, *and define* $F^p_{\log}$ *and* $\mathcal{P}_{p,\delta}$ *as in Example 4.5.*

- **(a)** *If* $p(t|s) > 0$ *for all* $t, s \in \mathcal{S}$, *and* $\delta < \min\{p(t|s); \quad s, t \in \mathcal{S}\}$, *then* $\mathcal{P}_{p,\delta}$ *satisfies* Minimal Reliability.

- **(b)** *Let* $\epsilon > 0$, *and define* $\mathcal{C}'_\epsilon$ *as in Example 4.5. If* ϵ *and* δ *are small enough, then* $\mathcal{P}_{p,\delta}$ *satisfies* Identification *with respect to* $F^p_{\log}$ *and* $\mathcal{C}'_\epsilon$.

[7] The function $-F_{\mathbf{p}}(\mathbf{q})$ is sometimes called the *cross-entropy* of $\mathbf{p}$ and $\mathbf{q}$.

Proof: (a) Let $M := \min\{p(t|s);\ t, s \in \mathcal{S}\}$; then $M > 0$, because $\mathcal{S}$ is finite, and $p(t|s) > 0$ for all $t, s \in \mathcal{S}$. Let $L := |\log(M)|$. Then $L < \infty$, and we have $|v_s^t| \leq L$ for all $s, t \in \mathcal{S}$. Thus, $\|\mathbf{v}^t\|^2 \leq L^2\,|S|$ for all $t \in \mathcal{S}$. Thus, $\mathrm{var}[\rho(t)] \leq L^2\,|S|$ for all $t \in \mathcal{S}$. Thus, *Minimal reliability* is satisfied.

(b) For all $s \in \mathcal{S}$, recall that $\mathcal{C}_s^\epsilon := \{\mathbf{c} \in \mathcal{C};\quad c_s \geq c_t + \epsilon \text{ for all } t \neq s\}$. Suppose p is the error model of a voter. Then for any $s, t \in \mathcal{S}$, we have $\rho(\mathbf{v}^t|s) = p(t|s)$. Thus, $\mathbb{E}[\rho(s)] = \sum_{t\in\mathcal{S}} p(t|s)\mathbf{v}^t = (w_r(s))_{r\in\mathcal{S}}$, where, for all $r \in \mathcal{S}$, $w_r(s) = \sum_{t\in\mathcal{S}} p(t|s) v_r^t = \sum_{t\in\mathcal{S}} p(t|s)\,\log[p(t|r)]$.

We will first construct some $\epsilon_0 > 0$ such that $\mathbb{E}[\rho(s)] \in \mathcal{C}_s^{\epsilon_0}$. To do this, we must show that

$$\sum_{t\in\mathcal{S}} p(t|s)\,\log[p(t|s)] \quad \geq \quad \epsilon_0 + \sum_{t\in\mathcal{S}} p(t|s)\,\log[p(t|r)], \quad \text{for all } r \neq s. \tag{9}$$

Now, for any $s \in \mathcal{S}$, define $\mathbf{p}^s$ by setting $p_t^s := p(t|s)$ for all $t \in \mathcal{S}$. Then Lemma A.1 says that $F_{\mathbf{p}^s}(\mathbf{p}^s) > F_{\mathbf{p}^s}(\mathbf{q})$ for all $\mathbf{q} \in \Delta(\mathcal{S}) \setminus \{\mathbf{p}^s\}$. In particular, this implies that $F_{\mathbf{p}^s}(\mathbf{p}^s) > F_{\mathbf{p}^s}(\mathbf{p}^t)$ for all $t \neq s$. Let $\epsilon_0 := F_{\mathbf{p}^s}(\mathbf{p}^s) - \max\{F_{\mathbf{p}^s}(\mathbf{p}^t);\ t \in \mathcal{S} \setminus \{s\}\}$. Then $\epsilon_0 > 0$, because $\mathcal{S}$ is finite. We have $F_{\mathbf{p}^s}(\mathbf{p}^s) \geq F_{\mathbf{p}^s}(\mathbf{p}^t) + \epsilon_0$ for all $t \neq s$. This implies condition (9).

Now, let $0 < \epsilon < \epsilon_0$. If $\delta > 0$ is small enough, then by continuity we will have

$$\sum_{t\in\mathcal{S}} p'(t|s)\,\log[p'(t|s)] \quad \geq \quad \epsilon + \sum_{t\in\mathcal{S}} p'(t|s)\,\log[p'(t|r)], \quad \text{for all } r \neq s. \tag{10}$$

for all error models p' such that $|p'(t|s) - p(t|s)| < \delta$ for all $s, t \in \mathcal{S}$. Thus implies that $\mathcal{P}_{p,\delta}$ satisfies *Identification* with respect to $F_{\log}^p$ and $\mathcal{C}'_\epsilon$. □

Proposition 5.2 is just a special case of Theorem 5.3 when $\mathcal{S}$ is a finite set with the discrete topology. Proposition 4.1 follows by applying Proposition 5.2 to Example 5.1. Likewise, Proposition 4.3 follows by applying Theorem 5.3 to Example 5.1.

Proof of Theorem 5.3. Let $F = (\mathbb{V}, \mathcal{V}, f)$ be a mean partition rule. Let $\mathcal{C}$ be the convex hull of $\mathcal{V}$, let $\mathcal{C}'$ be the closed subset of $\mathcal{C}$ posited by (M3), and let f_0 be the restriction of f to $\mathcal{C}'$. Fix $s \in \mathcal{S}$. Let $I \in \mathbb{N}$, let $\rho \in \mathcal{R}_I$, and let $\mathbf{V} = (\mathbf{v}_i)_{i\in\mathcal{I}}$ be a $\rho(s)$-random profile of votes (where $|\mathcal{I}| = I$). Let $\overline{\mathbf{v}} := \dfrac{1}{I}\displaystyle\sum_{i\in\mathcal{I}} \mathbf{v}_i$ be the average of these votes.

Claim 1: *Let* $\widehat{\mathbf{v}} := \mathbb{E}[\overline{\mathbf{v}}]$. *Then* $\widehat{\mathbf{v}} \in f_0^{-1}\{s\}$.

Proof: $\mathbb{E}(\overline{\mathbf{v}}) = \mathbb{E}\left(\frac{1}{I}\sum_{i\in\mathcal{I}} \mathbf{v}_i\right) = \frac{1}{I}\sum_{i\in\mathcal{I}} \mathbb{E}(\mathbf{v}_i)$. By *Identification*, we have $\mathbb{E}(\mathbf{v}_i) \in f_0^{-1}\{s\}$ for all i. But $f_0^{-1}\{s\}$ is convex by (M2′). The claim follows. ◇ Claim 1

Claim 2: $\mathrm{var}(\overline{\mathbf{v}}) \leq \dfrac{1}{I}\,\sigma(I) + \dfrac{I-1}{I}\,\kappa(I).$

Proof: For any $i \in \mathcal{I}$, let $\widehat{\mathbf{v}}_i := \mathbb{E}[\mathbf{v}_i]$. Then as we saw in the proof of Claim 1, $\widehat{\mathbf{v}} = \frac{1}{I}\sum_{i\in\mathcal{I}} \widehat{\mathbf{v}}_i$. Thus,

$$\overline{\mathbf{v}} - \widehat{\mathbf{v}} \quad = \quad \frac{1}{I}\sum_{i\in\mathcal{I}} \mathbf{v}_i - \frac{1}{I}\sum_{i\in\mathcal{I}} \widehat{\mathbf{v}}_i \quad = \quad \frac{1}{I}\sum_{i\in\mathcal{I}} (\mathbf{v}_i - \widehat{\mathbf{v}}_i) \quad = \quad \frac{1}{I}\sum_{i\in\mathcal{I}} \mathbf{e}_i,$$

where, for all $i \in \mathcal{I}$, we define $\mathbf{e}_i := \mathbf{v}_i - \widehat{\mathbf{v}}_i$. Thus,

$$\|\overline{\mathbf{v}} - \widehat{\mathbf{v}}\|^2 \quad = \quad \left\langle \frac{1}{I}\sum_{i\in\mathcal{I}} \mathbf{e}_i, \frac{1}{I}\sum_{j\in\mathcal{I}} \mathbf{e}_j \right\rangle \quad = \quad \frac{1}{I^2}\sum_{i,j\in\mathcal{I}} \langle \mathbf{e}_i, \mathbf{e}_j \rangle.$$

Thus, if $\mathbf{B}$ is the covariance matrix of $\rho(s)$, then

$$\begin{aligned} \mathrm{var}(\overline{\mathbf{v}}) \quad &= \quad \mathbb{E}\left[\|\overline{\mathbf{v}} - \widehat{\mathbf{v}}\|^2\right] \quad = \quad \frac{1}{I^2}\sum_{i,j\in\mathcal{I}} \mathbb{E}\left[\langle \mathbf{e}_i, \mathbf{e}_j \rangle\right] \\ &= \quad \frac{1}{I^2}\sum_{i,j\in\mathcal{I}} \mathrm{cov}(\mathbf{v}_i, \mathbf{v}_j) \quad = \quad \frac{1}{I^2}\sum_{i\in\mathcal{I}} \mathrm{var}(\mathbf{v}_i) + \frac{1}{I^2}\sum_{\substack{i,j\in\mathcal{I} \\ i\neq j}} \mathrm{cov}(\mathbf{v}_i, \mathbf{v}_j) \\ &\underset{(*)}{=} \quad \frac{1}{I}\sigma(\mathbf{B}) + \frac{I-1}{I}\kappa(\mathbf{B}) \quad \underset{(\dagger)}{\leq} \quad \frac{1}{I}\sigma(I) + \frac{I-1}{I}\kappa(I), \end{aligned}$$

as claimed. Here, $(*)$ is by the defining equations (1), and $(\dagger)$ is by definition of $\sigma(I)$ and $\kappa(I)$. $\diamond$ Claim 2

Now, let $\mathcal{U} \subset \mathcal{S}$ be any open set containing s. We want to show that $\lim_{I\to\infty} \mathrm{Prob}[f(\overline{\mathbf{v}}) \in \mathcal{U}] = 1$. Claim 1 says that $\widehat{\mathbf{v}} \in f_0^{-1}\{s\} \subseteq \mathcal{C}'$. Since $\mathcal{C}'$ is a relatively open subset of $\mathcal{C}$, there is some $\delta_1 > 0$ such that, for any $\mathbf{c} \in \mathcal{C}$, if $\|\mathbf{c} - \widehat{\mathbf{v}}\| < \delta_1$, then $\mathbf{c} \in \mathcal{C}'$ also. But f_0 is continuous on $\mathcal{C}'$, and $f_0(\widehat{\mathbf{v}}) = s \in \mathcal{U}$, so there is some $\delta_2 > 0$ such that, for any $\mathbf{c}' \in \mathcal{C}'$, if $\|\mathbf{c}' - \widehat{\mathbf{v}}\| < \delta_2$, then $f(\mathbf{c}') = f_0(\mathbf{c}') \in \mathcal{U}$ also. Let $\delta := \min\{\delta_1, \delta_2\}$. Then $\delta > 0$, and, combining the two previous statements, we see that for any $\mathbf{c} \in \mathcal{C}$, if $\|\mathbf{c} - \widehat{\mathbf{v}}\| < \delta$, then $f(\mathbf{c}') \in \mathcal{U}$. In particular, this holds if $\mathbf{c} = \overline{\mathbf{v}}$. Thus,

$$\begin{aligned} \mathrm{Prob}\left[f(\overline{\mathbf{v}}) \notin \mathcal{U}\right] \quad &\leq \quad \mathrm{Prob}\left[\|\widehat{\mathbf{v}} - \overline{\mathbf{v}}\| > \delta\right] \quad \underset{(*)}{\leq} \quad \frac{\mathrm{var}(\overline{\mathbf{v}})}{\delta^2} \\ &\underset{(\dagger)}{\leq} \quad \frac{1}{\delta^2}\left(\frac{1}{I}\,\sigma(I) + \frac{I-1}{I}\,\kappa(I)\right) \quad \xrightarrow[I\to\infty]{(\diamond)} \quad 0, \end{aligned}$$

as desired. Here $(*)$ is by the normed vector space version of Chebyshev's inequality, and $(\dagger)$ is by Claim 2. Finally, $(\diamond)$ is because *Asymptotically minimal reliability* says that $\frac{1}{I}\,\sigma(I) \xrightarrow[I\to\infty]{} 0$, while *Asymptotically weak average correlation* says that $\kappa(I) \xrightarrow[I\to\infty]{} 0$. $\square$

Proof of Proposition 5.4. Let $\mathbb{V}$ be a normed vector space, and suppose that $\mathfrak{R}$ is a culture on $\mathbb{V}$. Observe that the convex sets $\mathcal{C}'_s$ and $\mathcal{C}'_r$ must be disjoint, for any distinct r and s. By repeatedly applying the Separating Hyperplane Theorem, we can define a partition of $\mathbb{V}$ into convex (not necessarily closed) disjoint sets $\{\mathcal{C}_s\}_{s\in\mathcal{S}}$ such that $\mathcal{C}'_s \subseteq \mathcal{C}_s$ for all $s \in \mathcal{S}$. If we define the function $F : \mathbb{V} \longrightarrow \mathcal{S}$ by setting $F^{-1}\{s\} = \mathcal{C}_s$ for all $s \in \mathcal{S}$, then F is a mean partition rule, which satisfies the axioms of *Continuity* and *Identification* with respect to $\mathfrak{R}$. □

Proof of Proposition 6.5. Let $\beta : \mathbb{N} \longrightarrow \mathbb{R}^+$ and $\gamma : \mathbb{N} \longrightarrow [0, \infty]$ be functions satisfying the inequality (8), such that $\mathfrak{B}$ exhibits β-correlation decay relative to $\mathfrak{N}$, and $\mathfrak{N}$ has sublinear average γ-degree growth. Let $I \in \mathbb{N}$, let $(\mathcal{I}, \sim)$ be a graph in $\mathcal{N}_I$, and let $\mathbf{B}_I$ be a correlation matrix that exhibits β-decay for $(\mathcal{I}, \sim)$. Let $M := \beta(0)$; then *Minimal Reliability* is automatically satisfied, because $|b_{i,i}| \leq \beta(0)$ for all $i \in \mathcal{I}$. It remains to prove *Asymptotically weak average correlation.* Let $C := \sum_{n=1}^{\infty} \gamma(n)\, \beta(n)$; then C is finite by inequality (8). We have:

$$
\begin{aligned}
\kappa(\mathbf{B}) \quad &= \quad \frac{1}{I(I-1)} \sum_{\substack{i,j\in\mathcal{I} \\ i\neq j}}^{I} b_{i,j} \quad = \quad \frac{1}{I(I-1)} \sum_{i\in\mathcal{I}} \sum_{r=1}^{\infty} \sum_{\substack{j\in\mathcal{I} \\ d(i,j)=r}} b_{i,j} \\
&\underset{\text{(a)}}{\leq} \quad \frac{1}{I(I-1)} \sum_{i\in\mathcal{I}} \sum_{r=1}^{\infty} \sum_{\substack{j\in\mathcal{I} \\ d(i,j)=r}} \beta(r) \quad = \quad \frac{1}{I(I-1)} \sum_{i\in\mathcal{I}} \sum_{r=1}^{\infty} \beta(r)\ \deg^r(i,\sim) \\
&\underset{\text{(b)}}{\leq} \quad \frac{1}{I(I-1)} \sum_{i\in\mathcal{I}} \sum_{r=1}^{\infty} \beta(r)\,\gamma(r)\ \deg^\gamma(i,\sim) \\
&= \quad \frac{1}{I(I-1)} \sum_{i\in\mathcal{I}} \deg^\gamma(i,\sim) \left(\sum_{r=1}^{\infty} \beta(r)\,\gamma(r) \right) \quad \underset{\text{(c)}}{=} \quad \frac{1}{I(I-1)} \sum_{i\in\mathcal{I}} \deg^\gamma(i,\sim) \cdot C \\
&\underset{\text{(d)}}{=} \quad \frac{C}{(I-1)}\ \text{avedeg}^\gamma(\mathcal{I},\sim) \quad \underset{\text{(e)}}{\leq} \quad \frac{C}{(I-1)}\ \overline{\text{avedeg}}^\gamma(\mathcal{N}_I).
\end{aligned}
$$

Here, inequality (a) is because $\mathbf{B}_I$ exhibits β-decay for $(\mathcal{I}, \sim)$, while inequality (b) is by defining formula (3). Equality (c) is by definition of C, and equality (d) is by defining formula (4). Inequality (e) is by defining formula (5).

This inequality holds for all matrices $\mathbf{B} \in \mathcal{B}_I$. It follows that

$$
\kappa(I) \quad \leq \quad \frac{C}{(I-1)}\ \overline{\text{avedeg}}^\gamma(\mathcal{N}_I) \quad \xrightarrow[I\to\infty]{} 0,
$$

as desired, where the last step is by the limit equation (6). □

Proof of Proposition 7.1. We must verify the three conditions for $\mathfrak{R}'$ to be sagacious. First, we will show that $\mathfrak{R}'$ satisfies *Identification.* Suppose the true state of nature is s. Fix $I \in \mathbb{N}$, and let $\mathcal{I} := [1 \ldots I]$. Let $\mathcal{R}'_I = \mathcal{D}_I \odot \mathcal{R}_I$, let $\rho' \in \mathcal{R}'_I$, and let $\mathbf{V}' = (\mathbf{v}'_i)_{i=1}^I$ be a $\rho'(s)$-random profile. Then there exists a collective behaviour model $\rho \in \mathcal{R}_I$, and an influence matrix $\mathbf{D} \in \mathcal{D}_I$ such that $\rho' = \mathbf{D} \odot \rho$. Suppose $\mathbf{D} = [d_{i,j}]_{i,j \in \mathcal{I}}$. Thus, for all $i \in \mathcal{I}$, we have

$$\mathbf{v}'_i \quad := \quad \sum_{j=1}^{I} d_{i,j}\, \mathbf{v}_j,$$

where $\mathbf{V} = (\mathbf{v}_i)_{i=1}^I$ is a $\rho(s)$-random profile. Now, $\mathcal{R}_I$ is sagacious, so it satisfies *Identification*; thus, for all $k \in \mathcal{I}$, we have $\mathbb{E}[\mathbf{v}_k] \in F^{-1}\{s\} \cap \mathcal{C}'$. Thus, for all $i \in \mathcal{I}$, we have

$$\mathbb{E}[\mathbf{v}'_i] \;=\; \mathbb{E}\left[\sum_{k \in \mathcal{I}} d_{i,k}\, \mathbf{v}_k\right] \;=\; \sum_{k \in \mathcal{I}} d_{i,k}\, \mathbb{E}[\mathbf{v}_k] \in F^{-1}\{s\} \cap \mathcal{C}',$$

because $F^{-1}\{s\} \cap \mathcal{C}'$ is convex (by (M2′)), and $\sum_{k \in \mathcal{I}} d_{i,k} = 1$ (because $\mathbf{D}$ is a stochastic matrix). Thus, *Identification* is satisfied.

It remains to show that $\mathfrak{R}'$ satisfies *Asymptotically weak average correlation* and *Asymptotic reliability.* Since the culture $\mathfrak{R}$ is sagacious, it already satisfies these properties. For all $I \in \mathbb{N}$, let $\sigma(I)$ and $\kappa(I)$ be as defined in the statements of these conditions. Let $s \in \mathcal{S}$, $\mathbf{D} \in \mathcal{D}_I$, $\rho \in \mathcal{R}_I$, $\rho' = \mathbf{D} \odot \rho$, $\mathbf{V}'$, $\mathbf{V}$, etc. be as defined in the proof of *Identification* above. Let $\mathbf{B}' = [b'_{i,j}]_{i,j=1}^I$ be the correlation matrix of $\rho'(s)$. That is: $b'_{i,j} := \mathrm{cov}(\mathbf{v}'_i, \mathbf{v}'_j)$, for all $i, j \in \mathcal{I}$. Let D be the modulus of $\mathfrak{D}$ (this is finite because $\mathfrak{D}$ is local).

Claim 1: $\displaystyle \frac{1}{I^2} \sum_{i,j \in \mathcal{I}} b'_{i,j} \;\leq\; D^2 \left(\frac{1}{I}\sigma(I) + \frac{I-1}{I}\kappa(I) \right).$

Proof: Let $\mathbf{B} = [b_{i,j}]_{i,j=1}^I$ be the covariance matrix of $\rho(s)$. Then for all $i, j \in \mathcal{I}$, we have

$$\begin{aligned} b'_{i,j} &= \mathrm{cov}(\mathbf{v}'_i, \mathbf{v}'_j) \;=\; \mathrm{cov}\left(\sum_{k \in \mathcal{I}} d_{i,k}\, \mathbf{v}_k, \sum_{\ell \in \mathcal{I}} d_{j,\ell}\, \mathbf{v}_\ell \right) \\ &= \sum_{k \in \mathcal{I}} \sum_{\ell \in \mathcal{I}} d_{i,k}\, d_{j,\ell}\, \mathrm{cov}(\mathbf{v}_k, \mathbf{v}_\ell) \;=\; \sum_{k,\ell \in \mathcal{I}} d_{i,k}\, d_{j,\ell}\, b_{k,\ell}, \end{aligned} \qquad (11)$$

where the last step is because $\mathrm{cov}(\mathbf{v}_k, \mathbf{v}_k) = b_{k,\ell}$. For all $k \in \mathcal{I}$, let $\overline{d}_k := \sum_{i\in\mathcal{I}} d_{i,k}$. Then

$$\begin{aligned}
\frac{1}{I^2}\sum_{i,j\in\mathcal{I}} b'_{i,j} \underset{\text{(a)}}{=} \frac{1}{I^2}\sum_{i,j,k,\ell\in\mathcal{I}} d_{i,k}\, d_{j,\ell}\, b_{k,\ell} &= \frac{1}{I^2}\sum_{k,\ell\in\mathcal{I}}\left(\sum_{i\in\mathcal{I}} d_{i,k}\right)\left(\sum_{j\in\mathcal{I}} d_{j,\ell}\right) b_{k,\ell} \\
&= \frac{1}{I^2}\sum_{k,\ell\in\mathcal{I}} \overline{d}_k\,\overline{d}_\ell\, b_{k,\ell} \underset{\text{(b)}}{\leq} \frac{1}{I^2}\sum_{k,\ell\in\mathcal{I}} D^2\, b_{k,\ell} \\
&= D^2\left(\frac{1}{I^2}\sum_{k\in\mathcal{I}} b_{k,\ell} + \frac{1}{I^2}\sum_{\substack{k,\ell\in\mathcal{I}\\ k\neq\ell}} b_{k,\ell}\right) = D^2\left(\frac{1}{I}\sigma(\mathbf{B}) + \frac{I-1}{I}\kappa(\mathbf{B})\right) \\
&\underset{\text{(c)}}{\leq} D^2\left(\frac{1}{I}\sigma(I) + \frac{I-1}{I}\kappa(I)\right).
\end{aligned}$$

as claimed. Here, (a) is by equation (11), while (b) is by definition of "modulus". (c) is by the definitions of $\sigma(I)$ and $\kappa(I)$. $\diamond$ `Claim 1`

Now, let $\mathfrak{B} = (\mathcal{B}_I)_{I=1}^{\infty}$ be the correlation structure for the culture $\mathfrak{R}'$. Then for any $I \in \mathbb{N}$ and $\mathbf{B}' \in \mathcal{B}_I$, we can find some $\rho' \in \mathcal{R}'_I$ and $s \in \mathcal{S}$ such that $\mathbf{B}' = \mathrm{cov}[\rho'(s)]$, and thus, Claim 1 applies to $\mathbf{B}'$. However,

$$\frac{1}{I^2}\sum_{i,j\in\mathcal{I}} b'_{i,j} = \frac{I-1}{I}\kappa(\mathbf{B}') + \frac{1}{I}\sigma(\mathbf{B}').$$

Thus, Claim 1 implies that

$$\frac{I-1}{I}\kappa(I) + \frac{1}{I}\sigma(I) \leq D^2\left(\frac{1}{I}\sigma(I) + \frac{I-1}{I}\kappa(I)\right) \xrightarrow[I\to\infty]{} 0,$$

where the last step because $\mathfrak{R}$ satisfies *Asymptotically weak average correlation* and *Asymptotic Reliability*. Thus, the culture $\mathfrak{R}'$ is sagacious. $\square$

Proof of Proposition 7.2. Let $\mathcal{I} := [1 \ldots I]$. For any $I \times I$ matrix $\mathbf{D}$, let $\|\mathbf{D}\| := \max_{j\in\mathcal{I}}\left(\sum_{i\in\mathcal{I}} d_{i,j}\right)$. Thus, a deliberative institution $\mathfrak{C} = (\mathcal{C}_I)_{I=1}^{\infty}$ is local if there is some constant $C > 0$ such that $\|\mathbf{C}\| \leq C$ for all $\mathbf{C} \in \mathcal{C}_I$ and all $I \in \mathbb{N}$. In particular, if $\mathfrak{D}$ and $\mathfrak{E}$ are local, then there are constants D and E such that $\|\mathbf{D}\| \leq D$ and and $\|\mathbf{E}\| \leq E$ for all $\mathbf{D} \in \mathcal{D}_I$, all $\mathbf{E} \in \mathcal{E}_I$, and all $I \in \mathbb{N}$.

Claim 1: *For any $I \times I$ matrices $\mathbf{D}$ and $\mathbf{E}$, we have $\|\mathbf{D}\cdot\mathbf{E}\| \leq \|\mathbf{D}\|\cdot\|\mathbf{E}\|$.*

Proof: Let $\mathbf{C} = \mathbf{D} \cdot \mathbf{E}$. Thus, for all $i, k \in \mathcal{I}$, $c_{i,k} = \sum_{j \in \mathcal{I}} d_{i,j}\, e_{j,k}$. Thus, for all $k \in \mathcal{I}$, we have

$$\begin{aligned} \sum_{i \in \mathcal{I}} c_{i,k} &= \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} d_{i,j}\, e_{j,k} &=& \sum_{j \in \mathcal{I}} \left(\sum_{i \in \mathcal{I}} d_{i,j} \right) e_{j,k} \\ &\leq \sum_{j \in \mathcal{I}} \|\mathbf{D}\| e_{j,k} &=& \|\mathbf{D}\| \sum_{j \in \mathcal{I}} e_{j,k} \quad \leq \quad \|\mathbf{D}\| \cdot \|\mathbf{E}\|. \end{aligned}$$

Thus, $\|\mathbf{D} \cdot \mathbf{E}\| \leq \|\mathbf{D}\| \cdot \|\mathbf{E}\|$, as claimed. ◇ Claim 1

Let $\mathcal{C}_I := \{\mathbf{D} \cdot \mathbf{E};\ \ \mathbf{D} \in \mathcal{D}_I \text{ and } \mathbf{E} \in \mathcal{E}_I\}$. It is well-known that the product of two stochastic matrices is a stochastic matrix. (The proof is very similar to Claim 1.) Thus, every element of $\mathcal{C}_I$ is a stochastic matrix. Meanwhile, it follows from Claim 1 that $\|\mathbf{C}\| \leq D\, E$ for all $\mathbf{C} \in \mathcal{C}_I$ and all $I \in \mathbb{N}$. Thus, $\mathfrak{D} \cdot \mathfrak{E}$ is also local.

Now let $q, q' \in [0, 1]$ such that $q + q' = 1$.

Claim 2: *For any $I \times I$ matrices $\mathbf{D}$ and $\mathbf{E}$, we have $\|q\mathbf{D} + q'\mathbf{E}\| \leq q\|\mathbf{D}\| + q'\|\mathbf{E}\|$.*

Proof: Let $\mathbf{C} = q\mathbf{D} + q'\mathbf{E}$. Thus, for all $i, j \in \mathcal{I}$, $c_{i,j} = q\, d_{i,j} + q'\, e_{i,j}$. Thus, for all $j \in \mathcal{I}$, we have

$$\sum_{i \in \mathcal{I}} c_{i,j} \quad = \quad \sum_{i \in \mathcal{I}} (q\, d_{i,j} + q'\, e_{i,j}) \quad = \quad q \sum_{i \in \mathcal{I}} d_{i,j} + q' \sum_{i \in \mathcal{I}} e_{i,j} \quad \leq \quad q\, \|\mathbf{D}\| + q'\|\mathbf{E}\|.$$

Thus, $\|q\mathbf{D} + q'\mathbf{E}\| \leq q\|\mathbf{D}\| + q'\|\mathbf{E}\|$, as claimed. ◇ Claim 2

Let $\mathcal{C}_I := \{q\, \mathbf{D} + q'\, \mathbf{E};\ \ \mathbf{D} \in \mathcal{D}_I \text{ and } \mathbf{E} \in \mathcal{E}_I\}$. It is well-known that the convex combination of two stochastic matrices is a stochastic matrix. (The proof is very similar to Claim 2.) Thus, every element of $\mathcal{C}_I$ is a stochastic matrix. Meanwhile, it follows from Claim 2 that $\|\mathbf{C}\| \leq q\, D + q'\, E$ for all $\mathbf{C} \in \mathcal{C}_I$ and all $I \in \mathbb{N}$. Thus, $q\mathfrak{D} + q'\mathfrak{E}$ is also local. □

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