Unified quasi-maximum likelihood estimation theory for stable and unstable Markov bilinear processes

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Abstract

A unified quasi-maximum likelihood (QML) estimation theory for stationary and nonstationary simple Markov bilinear (SMBL) models is proposed. Such models may be seen as generalized random coefficient autoregressions (GRCA) in which the innovation and the random coefficient processes are fully correlated. It is shown that the QML estimate (QMLE) for the SMBL model is always asymptotically Gaussian without assuming strict stationarity, meaning that there is no knife edge effect. The asymptotic variance of the QMLE is different in the stationary and nonstationary cases but is consistently estimated using the same estimator. A perhaps surprising result is that in the nonstationary domain, all SMBL parameters are consistently estimated in contrast with unstable GARCH and GRCA models where the QMLE of the conditional variance intercept is inconsistent. As a result, strict stationarity testing for the SMBL is studied. Simulation experiments and a real application to strict stationarity testing for some financial stock returns illustrate the theory in finite samples.

Keywords: Markov bilinear process, random coefficient process, stability, instability, Quasi-maximum likelihood, knife edge effect, strict stationarity testing.

AMS Subject Classification (2000) Primary 62M10; Secondary 62M04.

Proposed running head: Inference for stable and unstable SMBL

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1. Introduction

Over the past few decades, there has been a very abundant literature on conditional mean and volatility (CMV) models because of their ability to describe both level and variability of a broad array of observed time series such as financial stock returns (see e.g. Engle, 1982; Nicholls and Quinn, 1982; Weiss, 1984; Bollerslev, 1986; Taylor, 1986; Tsay, 1987, 2002; Holan et al, 2010; Francq and Zakoïan, 2010). An essential common specification for such models is that their conditional mean and conditional variance are stochastic, generally function of the past of the observed phenomenon, from which they can be evaluated for level and volatility predictions. In particular, when the conditional variance (resp. conditional mean) is non-stochastic the CMV model is simply called purely conditional mean (resp. purely conditional volatility) model. Among the most popular specifications are: the ARMA model with a GARCH innovation (ARMA-GARCH), the ARMA model with a stochastic volatility (ARMA-SV) innovation, the ARMA model with a bilinear innovation (ARMA-BL), the subdiagonal bilinear (BL) model, the conditionally heteroskedastic ARMA (CHARMA) model, the double autoregressions (DAR) (Ling and Li, 2008; Chen et al, 2014) and the random coefficient autoregression (RCA) with a special case in which the random coefficient is finite-valued like the Markov mixture autoregression (MAR) and the threshold autoregression (TAR). In fact, all aforementioned models are subclasses of the general class of weak (or nonlinear) ARMA models (e.g. Amendola and Francq, 2009) which consist of ARMA equations with uncorrelated, but not necessarily independent innovations. When the innovation is independent, the ARMA model is simply called strong (or linear).

While (G)ARCH-type models seem to have dominated the literature on CMV models, a renewed interest has been paid recently to RCA models which were initially considered as purely conditional mean models. The most popular RCA model is an autoregressive equation driven by an independent and identically distributed (iid) innovation where the corresponding autoregressive coefficient is an iid process. Statistical analysis for RCA models usually assumes that the random coefficient and the innovation processes are uncorrelated (e.g. Nicholls and Quinn, 1982; Feigin and Tweedie, 1985; Schick, 1996; Aue et al, 2006;
Berkes et al, 2009; Aue and Horváth, 2011; Aknouche, 2013 etc.). The case of RCA models in which the random coefficient and the innovation are permitted to be correlated (which is called generalized RCA) has seen less interest despite its practical importance as it allows more flexible volatility representation including asymmetry in level and volatility (e.g. Hwang and Basawa, 1998; Zhao and Wang, 2012, 2013; Truquet and Yao, 2012; Aknouche, 2015a).

A special case of generalized RCA models in which the random coefficient and the innovation are fully correlated is the SMBL (1) given by the stochastic equation

\[ y_t = (\phi + \beta \varepsilon_t) y_{t-1} + \varepsilon_t, \quad t \in \mathbb{N}^*, \]

where \( y_0 \) is a given random variable and

\[ \{\varepsilon_t, t \in \mathbb{N}\} \text{ is an independent and identically distributed (iid) process} \quad (A1) \]

with

\[ E(\varepsilon_1) = 0 \text{ and } E(\varepsilon_1^2) = \sigma^2 > 0, \quad (A2) \]

\( \mathbb{N}^* = \mathbb{N} - \{0\} \) being the set of positive integers. The SMBL equation introduced by Tong (1981) is related to many volatility models. Indeed, it can be seen as a double autoregression, a subdiagonal bilinear model or a generalized RCA in which the random coefficient is fully correlated with the innovation. Probabilistic properties of the SMBL model (1.1) such as stationarity, ergodicity, geometric ergodicity and some Markov chain solidarity properties have been extensively studied (e.g. Tong, 1981; Feigin and Tweedie, 1985; Goldie and Maller, 2000; Cline and Pu, 2002; Meyn and Tweedie, 2009) where some singular properties on the stochastic unit root (\( \phi = 1 \)) have been revealed (Cline and Pu, 2002). Some generalizations of the original formulation have been developed and their structures have been studied (e.g. Ferrante et al, 2003; Cline, 2007). However, statistical properties of the SMBL model have received much less interest. Indeed, at the knowledge of the author, it appears that the first work concerning estimation of the SMBL model (1.1) is the one of Aknouche (2013, Section 3.2) who studied asymptotic distribution of the QMLE for a nonstationary SMBL model (1.1) with \( \beta = 1 \). It turns out that the QMLE coincides with the two-stage weighted least squares estimate, 2SWLSE (cf. Aknouche, 2012a, 2012b, 2013, 2014, 2015a).
This Chapter proposes a unified quasi-maximum likelihood (QML) estimation theory for stable and unstable SMBL models (assuming $\beta$ known, say $\beta = 1$), i.e.

$$y_t = (\phi + \varepsilon_t) y_{t-1} + \varepsilon_t, \quad t \in \mathbb{N}^*.$$  \hspace{1cm} (1.2)

Our aim is threefold. i) First, under stability of (1.2) with respect to strict stationarity, we show that the QMLE of $(\phi, \sigma^2)'$ is asymptotically Gaussian when $\phi \neq 1$ and inconsistent in the stochastic unit root case $\phi = 1$. The result is valid regardless of any moment requirement on the observed process $\{y_t, t \in \mathbb{N}\}$. ii) Second, we shall see that when $\phi \neq 1$, the QMLE of $(\phi, \sigma^2)'$ is always $\sqrt{n}$-Gaussian irrespective of the strict stationarity requirement, meaning that there is no knife edge effect (Lumsdaine, 1996; Jensen and Rahbek, 2004) for the SMBL model. The corresponding asymptotic distribution is different in the stationary and nonstationary cases but is consistently estimated using the same estimator. This parallels recent results by Aue and Horváth (2011) for RCA(1) models (see also Hwang and Basawa, 2005) and Francq and Zakoïan (2012, 2013a) for GARCH(1,1) and asymmetric GARCH (1,1) models, respectively. iii) Third, as an application of the proposed unified estimation theory, strict stationarity testing for the SMBL equation is studied. A perhaps surprising result is that all parameters of the SMBL are consistently estimated when $\phi \neq 1$. This is in contrast with RCA(1) and GARCH(1,1) models where the QMLE of the conditional variance intercept is inconsistent in the nonstationary domain (see Aue and Horváth, 2011; Francq and Zakoïan, 2012; Aknouche, 2013, 2015a). Moreover, in the nonstationary stochastic unit root case, the QMLE is still consistent when (1.2) is appropriately started.

The rest of this Chapter proceeds as follows. In Section 2, stability of the SMBL equation (1.1) with arbitrary $\beta$ is revisited. A necessary and sufficient condition for the SMBL model with $\phi \neq 1$ to admit a unique (asymptotically) strictly stationary solution is provided. Furthermore, various modes of divergence to infinity in the nonstationary case are also presented. Assuming strict stationarity of the model and $\beta = 1$, Section 3 establishes asymptotic normality of QMLE of $(\phi, \sigma^2)'$ when $\phi \neq 1$ and its inconsistency when $\phi = 1$. In Section 4, a consistent estimate for the asymptotic variance of the QMLE in both strict stationarity and non strict stationarity situations is given when $\phi \neq 1$. Then, a unified
asymptotic theory for the QMLE in both stable and unstable situations is provided. Section 5 proposes strict stationarity and non-strict stationarity testing procedures for the SMBL. In particular, consistent interval estimates for the parameters are given without assuming strict stationarity. In addition, a simulation study is conducted to assess the theory in finite samples and application to strict stationarity testing for some financial stock returns is provided. Finally, Section 6 concludes.

2. Stability analysis for the SMBL model

Existence of a nonanticipative strictly stationary solution of (1.1) is now considered. It is clear that studying stationarity of the one-sided equation (1.1) translates immediately into studying stationarity of the two-sided version of (1.1)

\[ y_t = (\phi + \beta \varepsilon_t) y_{t-1} + \varepsilon_t, \quad t \in \mathbb{Z}, \]  

(\mathbb{Z} being the set of integers). This of course implies that \( y_0 \) in (1.1) should have the same distribution as the unique strictly stationary solution of (2.1) when exists. Otherwise, we rather speak about the unique "asymptotically" strictly stationary solution \( \{y_t, t \in \mathbb{N}\} \) in the sense that the limiting distribution of \( y_t \) (as \( t \to \infty \)) exists and is unchanged whatever the distribution of \( y_0 \). For both situations we are then interested in the stability of (1.1) with respect to strict stationarity. Notice that the finite second moment assumption \( A2 \) on the innovation sequence \( \{\varepsilon_t, t \in \mathbb{Z}\} \) is unnecessary for that purpose and is replaced by the weaker condition of finiteness of absolute log-moments:

\[ E(|\log |\varepsilon_1||) < \infty \text{ and } E(|\log |\phi + \beta \varepsilon_1||) < \infty. \]  

(A3)

For model (2.1), assumption \( A1 \) corresponds to

\[ \{\varepsilon_t, t \in \mathbb{Z}\} \text{ is an independent and identically distributed (iid) process.} \]  

(A1')

The following result, by now classical, provides a necessary and sufficient condition for strict stationarity of model (2.1) and hence stability of (1.1) with respect to strict stationarity.
Theorem 2.1 Consider equation (2.1) subject to (A1′) and (A3).
i) (2.1) admits a unique nonanticipative strictly stationary and ergodic solution given by
\[ y_t = \sum_{j=0}^{\infty} \prod_{i=0}^{j-1} (\phi + \beta \varepsilon_{t-i}) \varepsilon_{t-j}, \quad t \in \mathbb{Z}, \tag{2.2} \]
where the latter series converges absolutely almost surely if
\[ \gamma := E(\log|\phi + \beta \varepsilon_1|) < 0. \tag{2.3} \]
ii) Conversely, if (2.1) admits a nonanticipative strictly stationary solution, \( \phi \neq 1 \) and
\[ P(\varepsilon_1 = c) < 1, \tag{2.4} \]
for all \( c \in \mathbb{R} \), then (2.3) holds.
iii) If \( \phi = 1 \) then model (2.1) is not irreducible in the sense of Bougerol and Picard (1992) and the Markov chain \( \{y_t, t \in \mathbb{N}\} \) defined by (1.1), starting from \( y_0 \), is not ergodic. Moreover, under (2.3) and assuming that \( E\left(\log\left|y_0 + \frac{1}{\beta}\right|\right) < \infty \),
\[ y_t \xrightarrow{a.s.} -\frac{1}{\beta}. \tag{2.5} \]

Proof i) The first part of the theorem follows from Brandt (1986).
ii) It is clear that when \( \phi \neq 1 \) and \( \varepsilon_1 \) is nondegenerate (i.e. (2.4) holds), model (2.1) is irreducible in the sense of Bougerol and Picard (1992), so ii) follows from their Theorem 2.5.
iii) If \( \phi = 1 \) then (2.2) reduces to \( y_t = -\frac{1}{\beta} \) for all \( t \in \mathbb{Z} \) (cf. Cline and Pu, 2002, p. 287) which is a strictly stationary solution whatever \( \gamma \in [-\infty, +\infty) \). Considering the one-sided equation (1.1), if \( y_0 = -\frac{1}{\beta} \) a.s. then \( y_1 = (1 + \beta \varepsilon_1) y_0 + \varepsilon_1 = -\frac{1}{\beta} \) a.s., so any subspace of \( \mathbb{R} \) containing \( \left\{-\frac{1}{\beta}\right\} \) is invariant under (2.1). This shows that model (2.1) is not irreducible in the sense of Bougerol and Picard (1992). Moreover, non ergodicity of the Markov chain \( \{y_t, t \in \mathbb{N}\} \) starting from \( y_0 \) has been proved by Cline and Pu (2002, Theorem 2.1). Finally, (2.5) trivially follows when \( y_0 = -\frac{1}{\beta} \) a.s. since as seen above \( y_t = -\frac{1}{\beta} \) a.s. for all \( t \in \mathbb{N} \). If,
however, \( P \left( y_0 \neq -\frac{1}{\beta} \right) < 1 \), then iterating (1.1) with \( \phi = 1 \), we have

\[
y_t + \frac{1}{\beta} = (1 + \beta \varepsilon_t) y_{t-1} + \varepsilon_t + \frac{1}{\beta} = (1 + \beta \varepsilon_t) \left( y_{t-1} + \frac{1}{\beta} \right)
\]

\[
= \ldots = \prod_{k=1}^{t} \left( 1 + \beta \varepsilon_k \right) \left( y_0 + \frac{1}{\beta} \right), \quad t \in \mathbb{N}^*.
\]

From the strong law of large numbers and under (2.3) and \( E \left( \log |y_0 + \frac{1}{\beta}| \right) < \infty \), it follows that

\[
\frac{1}{t} \log \left| y_t + \frac{1}{\beta} \right| = \frac{1}{t} \sum_{k=1}^{t} \log |1 + \beta \varepsilon_k| + \frac{1}{t} \log \left| y_0 + \frac{1}{\beta} \right| \xrightarrow{a.s. \ \ t \to \infty} \gamma < 0.
\]

This shows that \( \log \left| y_t + \frac{1}{\beta} \right| \xrightarrow{a.s. \ \ t \to \infty} -\infty \), so \( y_t + \frac{1}{\beta} \xrightarrow{a.s. \ \ t \to \infty} 0 \) proving (2.5). \( \blacksquare \)

So in all, assuming (A1), (A3), \( \phi \neq 1 \) and (2.4), condition (2.3) is the necessary and sufficient condition for model (2.1) to have a unique (nonanticipative) strictly stationary and ergodic solution. For \( \phi = 1 \) the SMBL model (1.1) is (tied-down line) degenerate in the sense of Goldie and Maller (2000, p. 1199) and Babillot et al (1997, p. 480) since when \( c = -\frac{1}{\beta} \), then \( c = (1 + \beta \varepsilon_t) c + \varepsilon_t \) for all \( t \in \mathbb{N} \). As a consequence, if \( y_0 = -\frac{1}{\beta} \) a.s. then \( y_t = -\frac{1}{\beta} \) a.s. for all \( t \in \mathbb{N} \). However, when \( \gamma < 0 \), even though the Markov chain \( \{y_t, t \in \mathbb{N}\} \) is not ergodic, it has a unique stationary distribution given by \( \delta_{-\frac{1}{\beta}} \) (Cline and Pu, 2002), where \( \delta_x \) denotes the degenerate distribution concentrated at \( x \).

Existence condition of a unique strictly stationary solution to (2.1) with a finite second moment is given by the following result.

**Theorem 2.2** Under (A1’), (A3) and (2.4), equation (2.1) admits a unique nonanticipative strictly stationary solution given by (2.2) with \( E(y_0^2) < \infty \), where the corresponding series converges a.s. and in mean square, if and only if

\[
\phi^2 + \beta^2 \sigma^2 < 1. \quad (2.6)
\]

**Proof** See e.g. Nicholls and Quinn (1982) and Feigin and Tweedie (1985) for the sufficiency part. For the necessity part, assume that \( \{y_t, t \in \mathbb{N}\} \) is a stationary solution to (2.1)
with \( E(y_t^2) < \infty \). Then, from (2.1) we have

\[
y_t^2 = \phi^2 y_{t-1}^2 + \varepsilon_t^2 (1 + \beta y_{t-1})^2 + 2\phi y_{t-1} (1 + \beta y_{t-1}) \varepsilon_t,
\]

so

\[
E(y_t^2) = \phi^2 E(y_{t-1}^2) + \sigma^2 (1 + \beta y_{t-1})^2,
\]

and

\[
(1 - (\phi^2 + \beta^2 \sigma^2)) E(y_t^2) = \sigma^2,
\]

implying that (2.6) should be satisfied. ■

It is clear that (2.6) implies (2.3), so the second-order stationarity domain is strictly included in the strict stationarity one. Therefore, there is non-invariance of the stability domains. When the strict stationarity condition (2.3) is dropped, the two-sided equation (2.1) has no interest, but asymptotic behavior of the solutions of the one-sided equation (1.1) could be studied. The following result (cf. Aknouche, 2013 when \( \beta = 1 \)) gives the limit of \( y_t \) as \( t \to \infty \) under each one of the following instability conditions

\[
\gamma = 0. \quad (2.7a)
\]

\[
\gamma > 0. \quad (2.7b)
\]

**Theorem 2.3** Consider model (1.1) subject to (A1) and (A3).

i) Under \( \phi \neq 1 \) and (2.7a),

\[
|y_t| \xrightarrow{p} t \to \infty \infty. \quad (2.8a)
\]

ii) Under \( \phi \neq 1 \) and (2.7b), there exists \( 0 < \lambda < 1 \) such that

\[
\lambda^t |y_t| \xrightarrow{a.s.} t \to \infty. \quad (2.8b)
\]

iii) Under \( \phi = 1 \), (2.7b) and \( P \left( y_0 \neq -\frac{1}{\beta} \right) = 1 \), there exists \( 0 < \lambda < 1 \) such that

\[
\lambda^t |y_t| \xrightarrow{a.s.} t \to \infty. \quad (2.9)
\]

**Proof** See Aknouche (2013, Lemma 1) when \( \beta = 1 \). ■
Thus, the asymptotic behavior of $y_t$ can be summarized for the two cases $\phi \neq 1$ and $\phi = 1$ as follows:

i) When $\phi \neq 1$:
   - Under stability ($\gamma < 0$) (Vervaat, 1979),
     $$ y_t \xrightarrow{\mathcal{L}} \sum_{j=1}^{\infty} \prod_{i=1}^{j-1} (\phi + \beta \varepsilon_i) \varepsilon_j, $$
   - Under instability ($\gamma = 0$),
     $$ |y_t| \xrightarrow{P} \infty. $$
   - Under strict instability ($\gamma > 0$),
     $$ \lambda^t |y_t| \xrightarrow{a.s.} \infty \quad \text{for some} \ 0 < \lambda < 1. $$

ii) When $\phi = 1$:
   - Under stability ($\gamma < 0$) and $E \left( \log \left| y_0 + \frac{1}{\beta} \right| \right) < \infty$,
     $$ y_t \xrightarrow{a.s.} \frac{-1}{\beta}. $$
   - Under strict instability ($\gamma > 0$) and $P \left( y_0 \neq -\frac{1}{\beta} \right) = 1$,
     $$ \lambda^t |y_t| \xrightarrow{a.s.} \infty, \quad \text{for some} \ 0 < \lambda < 1. $$
   - If $P \left( y_0 = -\frac{1}{\beta} \right) = 1$ then whatever $\gamma \in [-\infty, +\infty)$,
     $$ y_t = -\frac{1}{\beta}, \ a.s. \ \forall t \in \mathbb{N}. $$

iii) The case $\phi = 1$, $\gamma = 0$ and $P \left( y_0 \neq -\frac{1}{\beta} \right) < 1$ remains open.

3. QML estimation for stable SMBL models

In the sequel, we consider model (1.2) (i.e. with $\beta = 1$) started with an arbitrary random variable $y_0$ and subject to (A1), (A2), the fourth moment assumption

$$ E \left( \varepsilon_1^4 \right) < \infty, \quad \text{(A4)} $$
and the non-degeneracy condition
\[ P(\varepsilon_1 = 0) = 0. \] (A5)

The parameter of the model about which we will make inference is denoted by \( \theta = (\phi, \sigma^2)' \).

Notice that the conditional mean and conditional variance of the SMBL process given the past information are respectively given by
\[
E(y_t | \mathcal{F}_{t-1}) = \phi y_{t-1} \quad \text{and} \quad \text{Var}(y_t | \mathcal{F}_{t-1}) = \sigma^2 (1 + y_{t-1})^2,
\]
where \( \mathcal{F}_t \) denotes the \( \sigma \)-algebra generated by \( \{ \varepsilon_s, s \leq t \} \). Observe that the SMBL model is with an endogenous volatility since \( \text{Var}(y_t | \mathcal{F}_{t-1}) \) depends on \( \{ y_t, t \in \mathbb{N} \} \).

Therefore, given a series \( y_1, y_2, \ldots, y_n \) generated from (1.2) the logarithmed (Gaussian) quasi-likelihood function of \( \theta \) conditional on \( y_0 \) is written as follows
\[
\log l = -\frac{1}{2} \sum_{t=1}^{n} \log \left( \sqrt{2\pi} \sigma |1 + y_{t-1}| \right) - \frac{1}{2\sigma^2} \sum_{t=1}^{n} \frac{(y_t - \phi y_{t-1})^2}{(1 + y_{t-1})^2}.
\] (3.1)

Thanks to the form of the log-likelihood in (3.1), the QMLE, \( \hat{\theta}_{QLM} = (\hat{\phi}_{QLM}, \hat{\sigma}^2_{QLM}) \), which is the maximizer of (3.1), is given in a closed form
\[
\hat{\phi}_{QLM} = \left( \sum_{t=1}^{n} \frac{y_t^2}{(1 + y_{t-1})^2} \right)^{-1} \sum_{t=1}^{n} \frac{y_{t-1}y_t}{(1 + y_{t-1})^2}, \quad (3.2)
\]
\[
\hat{\sigma}^2_{QLM} = \frac{1}{n} \sum_{t=1}^{n} \frac{(y_t - \hat{\phi}_{QLM} y_{t-1})^2}{(1 + y_{t-1})^2}. \quad (3.3)
\]

It turns out that the QMLE defined by (3.2)-(3.3) is also the two-stage weighted least squares estimate (2SWLSE) in which the weight is the inverse of the conditional variance (see Aknouche, 2013). Consistency and asymptotic normality of the QMLE given by (3.2)-(3.3) are now established under in particular the stability condition (2.3).

\textbf{Theorem 3.1} Let \( \{ y_t, t \in \mathbb{N} \} \) be the unique (asymptotically) strictly stationary solution of model (1.2) which is subject to (A1), (A2), (2.3) and (A5) and let \( \hat{\phi}_{QLM} \) and \( \hat{\sigma}^2_{QLM} \) given by (3.2)-(3.3). Then:

i) When \( \phi \neq 1 \),
\[
\hat{\phi}_{QLM} \xrightarrow{a.s.} \phi. \quad (3.4a)
\]
\[
\hat{\sigma}^2_{QLM} \xrightarrow{a.s.} \sigma^2. \quad (3.4b)
\]
ii) When $\phi = 1$ and $E (\log |y_0 + 1|) < \infty$, $\hat{\theta}_{QML}$ is inconsistent.

**Proof** i) From (3.2) and (1.2) we have

$$
\hat{\phi}_{QML} - \phi = \left( \frac{1}{n} \sum_{t=1}^{n} \frac{y_{t-1}^2}{(1 + y_{t-1})^2} \right)^{-1} \frac{1}{n} \sum_{t=1}^{n} \frac{y_{t-1}}{1 + y_{t-1}} \varepsilon_t.
$$

(3.5)

So (3.4a) follows from the ergodic theorem, (A5) and the fact that $E (\varepsilon_1) = 0$. To show (3.4b), we rewrite (3.3) as follows:

$$
\hat{\sigma}_{QML}^2 = \frac{1}{n} \sum_{t=1}^{n} \left( \frac{y_t - \hat{\phi} y_{t-1} - \left( \hat{\phi}_{QML} - \phi \right) y_{t-1}}{(1 + y_{t-1})^2} \right)^2
$$

$$
= \frac{1}{n} \sum_{t=1}^{n} \left( \frac{y_t - \hat{\phi} y_{t-1}}{(1 + y_{t-1})^2} \right)^2 + \left( \hat{\phi}_{QML} - \phi \right) \frac{y_{t-1}^2}{(1 + y_{t-1})^2} - \frac{2}{n} \sum_{t=1}^{n} \left( \hat{\phi}_{QML} - \phi \right) \frac{y_{t-1} \varepsilon_t}{(1 + y_{t-1})^2}.
$$

(3.6)

Using (3.4a) and the Césaro lemma, the last two terms of the right hand side of (3.6) converge a.s. to zero. Thus, (3.4b) follows from the strong law of large numbers and (A2).

ii) When $y_0 = -1$ a.s., we have seen that $y_t = -1$ a.s. for all $t \in \mathbb{N}$. So $\hat{\theta}_{QML}$ given by (3.2)-(3.3) is undefined and hence inconsistent. If, however, $P(y_0 = -1) < 1$ then under (2.3) and $E (\log |y_0 + 1|) < \infty$, result (2.5) clearly holds, so $\hat{\theta}_{QML}$ is still inconsistent. ■

Now we establish asymptotic normality of $\hat{\theta}_{QML}$ under in particular the stability condition (2.3). For an asymptotically stationary process $\{z_t, t \in \mathbb{N}\}$ denote by $E_\infty (z_t) = \lim_{t \to \infty} E (z_t)$. Let

$$
\Sigma = \begin{pmatrix}
\sigma^2 & E (\varepsilon_1^3) E_\infty \left( \frac{y_1}{1 + y_1} \right) & E_\infty \left( \frac{y_1}{1 + y_1} \right) \\
E (\varepsilon_1^3) E_\infty & \left( \frac{y_1}{1 + y_1} \right) Var (\varepsilon_1^2) & \left( \frac{y_1}{1 + y_1} \right) Var (\varepsilon_1^2)
\end{pmatrix}^{-1}
$$

(3.7)

In order that $\Sigma$ exists, $y_t^2$ should be non-null almost surely as $t \to \infty$. This holds if we assume that $\{\varepsilon_t, t \in \mathbb{N}\}$ is non-degenerate in the sense of (A5). Thus, we have the following result.
Theorem 3.2 Let \( \{y_t, t \in \mathbb{N}\} \) be the unique (asymptotically) strictly stationary solution to equation (1.2) which is subject to (A1), (A2), (A4), (2.3), (A5) and \( \phi \neq 1 \). Then,

\[
\sqrt{n} \left( \hat{\theta}_{QML} - \theta \right) \xrightarrow{L} N(0, \Sigma), \tag{3.8}
\]

where \( \Sigma \) is given by (3.7).

Proof First, we rewrite (3.5) and (3.6) as follows

\[
\sqrt{n} \left( \hat{\phi}_{QML} - \phi \right) = \left( \frac{1}{n} \sum_{t=1}^{n} \frac{y_{t-1}^2}{(1 + y_{t-1})^2} \right)^{-1} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \frac{y_{t-1} \varepsilon_t}{1 + y_{t-1}}. \tag{3.9}
\]

\[
\sqrt{n} \left( \hat{\sigma}_{QML}^2 - \sigma^2 \right) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (\varepsilon_t^2 - \sigma^2) + \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left( \hat{\phi}_{QML} - \phi \right)^2 \frac{y_{t-1}^2}{(1 + y_{t-1})^2}
- 2 \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left( \hat{\phi}_{QML} - \phi \right) \frac{y_{t-1} \varepsilon_t}{(1 + y_{t-1})^2}. \tag{3.10}
\]

Using strong consistency of \( \hat{\phi}_{QML} \) (see (3.4a)) we have (see e.g. Nicholls and Quinn, 1982; Aknouche, 2015a)

\[
\hat{\phi}_{QML} - \phi = n^{-\frac{1}{2}} O_p(1),
\]

so from Césaro lemma and the ergodic theorem (3.10) becomes

\[
\sqrt{n} \left( \hat{\sigma}_{QML}^2 - \sigma^2 \right) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (\varepsilon_t^2 - \sigma^2) + o_p(1). \tag{3.11}
\]

In vector form, (3.9) and (3.11) may be expressed as follows

\[
\sqrt{n} \left( \hat{\theta}_{QML} - \theta \right) = \left( \frac{1}{n} \sum_{t=1}^{n} \frac{y_{t-1}^2}{(1 + y_{t-1})^2} 0 \right)^{-1} \left( \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \frac{y_{t-1} \varepsilon_t}{1 + y_{t-1}} \right) + o_p(1). \tag{3.12}
\]

Using the ergodic theorem we have

\[
\left( \frac{1}{n} \sum_{t=1}^{n} \frac{y_{t-1}^2}{(1 + y_{t-1})^2} 0 \right)^{a.s.} \xrightarrow{n \to \infty} \left( E_{\infty} \left( \frac{y_{t}^2}{(1 + y_{t})^2} \right) 0 \right) = \left( 0 1 \right). \tag{3.13}
\]
On the other hand, the sequence \( \{W_t, t \in \mathbb{N}\} \) defined by \( W_t = \left( \frac{y_{t-1} \varepsilon_t}{1 + y_{t-1}}, \varepsilon_t^2 - \sigma^2 \right) \) is clearly a bounded Martingale difference with respect to \( \{F_t, t \in \mathbb{N}\} \). Moreover, using again the ergodic theorem it follows that

\[
\frac{1}{n} \sum_{t=1}^{n} E(W_t W'_t / F_{t-1}) = \frac{1}{n} \sum_{t=1}^{n} \left( \begin{array}{c}
\frac{\sigma^2 y_{t-1}^2}{(1 + y_t)^2} \\
y_{t-1} E(\varepsilon_1 (\varepsilon_1^2 - \sigma^2))
\end{array} \right)
\begin{array}{c}
\frac{y_{t-1} E(\varepsilon_1 (\varepsilon_1^2 - \sigma^2))}{1 + y_{t-1}}
\end{array}
\begin{array}{c}
\frac{E(\varepsilon_t^2 - \sigma^2)^2}{E(\varepsilon_1^2 - \sigma^2)^2}
\end{array}
\xrightarrow{a.s. n \to \infty}
\left( \begin{array}{c}
\sigma^2 E_{\infty} \left( \frac{y_t^2}{(1 + y_t)^2} \right) \\
E(\varepsilon_1^2) E_{\infty} \left( \frac{y_t}{1 + y_t} \right)
\end{array} \right) := \Omega.
\]

Therefore, the Martingale central limit theorem yields

\[
\frac{1}{\sqrt{n}} \left( \sum_{t=1}^{n} \frac{y_{t-1} \varepsilon_t}{1 + y_{t-1}} \sum_{t=1}^{n} (\varepsilon_t^2 - \sigma^2) \right) \xrightarrow{L_n} N(0, \Omega). \tag{3.14}
\]

So result (3.8) follows while combining (3.12)-(3.14).

4. Unified \(QML\) estimation theory for stable and unstable \(SMBL\) models

Having established asymptotics for the \(QMLE\) in the stable case, we now use asymptotic results by Aknouche (2013, Section 3.2) for the \(QMLE\) in the unstable \(SMBL\) case, giving unified theory for the \(QMLE\) irrespective of stability issues.

**Theorem 4.1** Let \( \{y_t, t \in \mathbb{N}\} \) be a solution to equation (1.2) which is subject to (A1), (A2), (A4) and (A5).

i) If \( \phi \neq 1 \),

\[
\hat{\theta}_{QML} \xrightarrow{a.s. n \to \infty} \theta \quad \text{if} \quad E(\log |\phi + \varepsilon_1|) \neq 0. \quad \tag{4.1a}
\]

\[
\hat{\theta}_{QML} \xrightarrow{p n \to \infty} \theta \quad \text{if} \quad E(\log |\phi + \varepsilon_1|) = 0. \quad \tag{4.1b}
\]

ii) In addition,

\[
\sqrt{n} \left( \hat{\theta}_{QML} - \theta \right) \xrightarrow{L_n} N(0, \Delta), \quad \tag{4.1c}
\]

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where
\[
\Delta = \begin{cases}
\Sigma & \text{if } E (\log |\phi + \varepsilon_1|) < 0, \\
\begin{pmatrix}
\sigma^2 & E (\varepsilon_1^3) \\
E (\varepsilon_1^3) & Var (\varepsilon_1^3)
\end{pmatrix} & \text{if } E (\log |\phi + \varepsilon_1|) \geq 0,
\end{cases}
\] (4.2)
and \( \Sigma \) is given by (3.7).

iii) If, however, \( \phi = 1 \), \( E (\log |\phi + \varepsilon_1|) \geq 0 \) and \( P (y = -1) = 0 \) then (4.1c) still holds.

**Proof**
i) (4.1a) follows from (2.9) and (3.5) when \( E (\log |\phi + \varepsilon_1|) > 0 \) (see Aknouche, 2013), and from (3.4) when \( E (\log |\phi + \varepsilon_1|) < 0 \). Result (4.1b) easily follows from (2.8a) and (3.5) (see Aknouche, 2013).

ii) See Aknouche (2013, Theorem 4, (i)) for the proof of (4.2) in the case where (2.3) is not satisfied. If, however, (2.3) holds then (4.2) reduces to (3.8) which has been already proved.

iii) See Aknouche (2013, Theorem 4, (ii)) for the proof. ■

Assuming \( \phi \neq 1 \), we now propose for the asymptotic variance \( \Delta \) given by (4.2), an estimate that is consistent in the strict stationary and nonstationary cases. Set
\[
\tilde{\varepsilon}_t = \frac{y_t - \hat{\phi}_{QML} y_{t-1}}{1 + y_{t-1}}, \quad (4.3a)
\]
\[
\hat{\mu}_r = \frac{1}{n} \sum_{t=1}^{n} \tilde{\varepsilon}_t^r, \quad (4.3b)
\]
for some \( r \in \{1, ..., 4\} \). Clearly, \( \hat{\mu}_2 \) reduces to \( \hat{\sigma}_{QML}^2 \).

**Theorem 4.2** i) Under (A1), (A2), (A5) and \( \phi \neq 1 \),
\[
\tilde{\varepsilon}_t - \varepsilon_t \xrightarrow{a.s.}_{t \to \infty} 0 \quad \text{if } E (\log |\phi + \varepsilon_1|) \neq 0. \quad (4.4a)
\]
\[
\tilde{\varepsilon}_t - \varepsilon_t \xrightarrow{p}_{t \to \infty} 0 \quad \text{if } E (\log |\phi + \varepsilon_1|) = 0. \quad (4.4b)
\]

ii) If, in addition, \( E (\varepsilon_1^4) < \infty \), then
\[
\hat{\mu}_r \xrightarrow{a.s.}_{n \to \infty} E (\varepsilon_1^r) \quad \text{if } E (\log |\phi + \varepsilon_1|) \neq 0. \quad (4.5a)
\]
\[
\hat{\mu}_r \xrightarrow{p}_{n \to \infty} E (\varepsilon_1^r) \quad \text{if } E (\log |\phi + \varepsilon_1|) = 0. \quad (4.5b)
\]
Proof i) From (4.3a) and (1.2) we have

\[ \hat{\varepsilon}_t - \varepsilon_t = \left( \phi - \hat{\phi}_{QML} \right) \frac{y_{t-1}}{1 + y_{t-1}}. \]  

(4.6)

Hence, (4.4a) follows from (4.1a) and the a.s. boundedness of \( \frac{y_{t-1}}{1 + y_{t-1}} \). Result (4.5b) follows from (4.6), (4.1b) and the boundedness in probability of \( \frac{y_{t-1}}{1 + y_{t-1}} \).

ii) (4.6) and elementary algebras yield

\[
\hat{\mu}_r = \frac{1}{n} \sum_{t=1}^{n} (\varepsilon_t + (\hat{\varepsilon}_t - \varepsilon_t))^r
\]

\[ = \frac{1}{n} \sum_{t=1}^{n} \varepsilon_t^r + \frac{1}{n} \sum_{t=1}^{n} \sum_{i=0}^{r-1} \binom{r}{i} \varepsilon_t^i (\hat{\varepsilon}_t - \varepsilon_t)^{r-i}
\]

\[ = \frac{1}{n} \sum_{t=1}^{n} \varepsilon_t^r + \frac{1}{n} \sum_{t=1}^{n} \sum_{i=0}^{r-1} \binom{r}{i} \varepsilon_t^i \left( \left( \phi - \hat{\phi}_{QML} \right) \frac{y_{t-1}}{1 + y_{t-1}} \right)^{r-i}. \]  

(4.7)

From (4.1a) and the Césaro lemma, (4.7) becomes

\[ \hat{\mu}_r = \frac{1}{n} \sum_{t=1}^{n} \varepsilon_t^r + o_{a.s.}(1), \]

so (4.5a) follows from the ergodic theorem. If, however, \( E(\log |\phi + \varepsilon_1|) = 0 \), then we can use (4.1b) to easily show that the last term in the right hand side of (4.7) is \( o_p(1) \). So (4.5b) is established from the ergodic theorem.

Using Theorem 4.2, a consistent estimate for the asymptotic covariance matrix \( \Delta \) is now given. Define \( \hat{\Delta} \) by

\[
\hat{\Delta}_{11} = \hat{\sigma}_{QML}^2 \left( \frac{1}{n} \sum_{t=1}^{n} \frac{y_t^2}{(1 + y_t)^2} \right)^{-1}. \]  

(4.8a)

\[
\hat{\Delta}_{12} = \hat{\Delta}_{21} = \hat{\mu}_3 \frac{1}{n} \sum_{t=1}^{n} \frac{y_t}{1 + y_t} \left( \frac{1}{n} \sum_{t=1}^{n} \frac{y_t^2}{(1 + y_t)^2} \right)^{-1}. \]  

(4.8b)

\[
\hat{\Delta}_{22} = \frac{1}{n} \sum_{t=1}^{n} (\hat{\varepsilon}_t^2 - \hat{\mu}_2)^2. \]  

(4.8c)

Then, we state the main result of this Section.
Corollary 4.1 Under \((A1), (A2), (A4), \phi \neq 1 \text{ and } (A5)\),

\[ \hat{\Delta} \xrightarrow{a.s.} \Delta \quad \text{if } E(\log |\phi + \varepsilon|) \neq 0. \] \hspace{1cm} (4.9a)

\[ \hat{\Delta} \xrightarrow{p} \Delta \quad \text{if } E(\log |\phi + \varepsilon|) = 0. \] \hspace{1cm} (4.9b)

In addition,

\[ \sqrt{n} \hat{\Delta}^{-1} \left( \hat{\theta}_{QMLE} - \theta \right) \xrightarrow{n \to \infty} N(0, I), \] \hspace{1cm} (4.10)

where \(I\) denotes the identity matrix of dimension 2.

Proof i) (4.9) follows from (4.8), (4.4), (4.5) and the ergodic theorem.

ii) (4.10) is a consequence of (4.1c) and (4.9). 

In practice, result (4.10) is useful in getting confidence interval estimates and significance tests for the SMBL parameters (see Section 5). It is the analog of results by Aue and Horváth (2011) for RCA models and Francq and Zakoïan (2012, 2013a) for GARCH and asymmetric GARCH models (see also Aknouche, 2012a, 2012b, 2014, 2015a; Aknouche and Al-Eid, 2012; Aknouche et al, 2011).

5. Strict stationarity testing and illustrations

5.1. Strict stationarity testing

For CMV models with endogenous volatility, EnCMV (e.g. GARCH, RCA, DAR, SMBL), second-order stationarity and unit root testing seem to have a little interest compared to CMV models with exogenous volatility (e.g. strong ARMA, ARMA-GARCH) because outside the second-order stationarity domain, the observed process may still remain strictly stationary. This is in contrast with CMV models (e.g. strong ARMA, ARMA-GARCH) with exogenous volatility in which both regions of strict and second-order stationarities (with respect to the conditional mean parameter) coincide. An important consequence is that the asymptotic distribution of the QMLE for such endogenous volatility models is invariant inside or outside the second-order stationary domain and only depends on strict stationarity (see e.g. Francq and Zakoïan 2012, 2013a; Aue and Horváth, 2011; Aknouche, 2013
and the references therein). Thus, for SMBL modeling, strict stationarity and non-strict stationarity testing are appealing.

For the strict stationarity testing problems

\[ H_0 : \gamma < 0 \quad \text{against} \quad H_1 : \gamma \geq 0, \quad (5.1) \]

and

\[ H_0 : \gamma \geq 0 \quad \text{against} \quad H_1 : \gamma < 0, \quad (5.2) \]

(\( \gamma = E \log |\phi + \varepsilon_1| \)) consider the estimate \( \hat{\gamma}_n \) of \( \gamma \) given by

\[
\hat{\gamma}_n = \frac{1}{n} \sum_{t=1}^{n} \log \left| \phi_{QML} + \hat{\varepsilon}_t \right|, \]

where \( \hat{\varepsilon}_t \) is obtained from (4.3a). If we set

\[
\gamma_n (\varphi) = \frac{1}{n} \sum_{t=1}^{n} \log \left| \varphi + \frac{y_t - \varphi y_{t-1}}{1 + y_{t-1}} \right|, \]

for some \( \varphi \), then clearly \( \hat{\gamma}_n = \gamma_n (\phi_{QML}) \).

Let

\[
e_t = \log |\phi + \varepsilon_t| - E \log |\phi + \varepsilon_t|, \quad t \in \mathbb{N} \]

\[
\sigma^2_e = E \left( e_t^2 \right), \]

and assume that

\[
E \left( (\log |\phi + \varepsilon_1|)^2 \right) < \infty. \quad (A6) \]

Therefore, the following result provides the asymptotic distribution of \( \hat{\gamma}_n \) under \( \gamma \in [-\infty, +\infty) \).

**Theorem 5.1** Consider model (1.2) subject to A1, A3, A4, A5, A6 and \( \phi \neq 1 \). Then,

\[
\sqrt{n} (\hat{\gamma}_n - \gamma) \xrightarrow{n \to \infty} N \left( 0, \sigma^2_\gamma \right), \quad (5.3a) \]

where

\[
\sigma^2_\gamma = \begin{cases} 
\sigma^2_e + \sigma^2 \left( E_{\infty} \left( \frac{y_t^2}{(1 + y_t)^2} \right) \right)^{-1} \left( E_{\infty} \left( \frac{1}{\phi + y_t} \right) \right)^2 & \text{if } \gamma < 0, \\
\sigma^2_e & \text{if } \gamma \geq 0.
\end{cases} \quad (5.3b)
\]

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Proof The Taylor formula gives

\[ \hat{\gamma}_n = \gamma_n \left( \phi_{QML} \right) \]

\[ = \gamma_n (\phi) + \left( \phi_{QML} - \phi \right) \frac{\partial \gamma_n (\phi)}{\partial \phi} + o_p \left( n^{-\frac{1}{2}} \right) \]

\[ = \gamma_n (\phi) + \frac{1}{n} \left( \phi_{QML} - \phi \right) \sum_{t=1}^{n} \frac{1}{\phi + y_t} + o_p \left( n^{-\frac{1}{2}} \right). \]

So

\[ \sqrt{n} (\hat{\gamma}_n - \gamma) = \sqrt{n} (\gamma_n (\phi) - \gamma) + \sqrt{n} \left( \gamma_n \left( \phi_{QML} \right) - \gamma_n (\phi) \right) \]

\[ = \sqrt{n} (\gamma_n (\phi) - \gamma) + \sqrt{n} \left( \phi_{QML} - \phi \right) \frac{1}{n} \sum_{t=1}^{n} \frac{1}{\phi + y_t} + o_p (1). \quad (5.4) \]

If \( \gamma < 0 \) the ergodic theorem yields

\[ \frac{1}{n} \sum_{t=1}^{n} \frac{1}{\phi + y_t} \xrightarrow{a.s.} E \left( \frac{1}{\phi + y_t} \right). \quad (5.5) \]

If, however, \( \gamma \geq 0 \) then from (2.8) we have

\[ \frac{1}{n} \sum_{t=1}^{n} \frac{1}{\phi + y_t} \xrightarrow{p} 0. \quad (5.6) \]

Thus (5.3) follows from (5.4), (5.5), (5.6) and (4.1c).

Like the \textit{GARCH} model (cf. Francq and Zakoïan, 2012, Theorem 3.1), the asymptotic variance of \( \hat{\gamma}_n \) is larger in the strict stationarity domain than in the non strict stationarity one.

To make inference about \( \hat{\gamma}_n \) we need to estimate its asymptotic variance \( \sigma^2_\gamma \). Let

\[ \hat{\sigma}^2_\gamma = \hat{\sigma}^2_e + \hat{\sigma}^2_{QML} \left( \frac{1}{n} \sum_{t=1}^{n} \frac{y^2_t}{(1 + y_t)^2} \right)^{-1} \left( \frac{1}{n} \sum_{t=1}^{n} \frac{1}{\phi_{QML} + y_t} \right)^2, \]

where

\[ \hat{\sigma}^2_e = \frac{1}{n} \sum_{t=1}^{n} \left( \log \left| \phi_{QML} + \hat{\varepsilon}_t \right| - \hat{\gamma}_n \right)^2. \]

The following result establishes consistency of \( \hat{\sigma}^2_\gamma \).
Corollary 5.1 Under the same assumptions of Theorem 5.1 we have

\[
\frac{\sigma_\gamma^2}{\sigma_e^2} \xrightarrow{a.s.} \sigma_\gamma^2 \quad \text{if } E(\log|\phi + \varepsilon_1|) \neq 0. \]

\[
\frac{\sigma_\gamma^2}{\sigma_e^2} \xrightarrow{p} \sigma_\gamma^2 \quad \text{if } E(\log|\phi + \varepsilon_1|) = 0. \]

An important consequence of Theorem 5.1 and Corollary 5.1 is that we can get a consistent interval estimate for \(\gamma\).

Corollary 5.2 Under the same assumptions of Theorem 5.1, a confidence interval for \(\gamma\) at the asymptotic nominal level \(\alpha \in (0, 1)\) is

\[
\left[\hat{\gamma}_n - \frac{\tilde{\sigma}_e}{\sqrt{n}} \Phi^{-1}\left(1 - \frac{\alpha}{2}\right), \hat{\gamma}_n - \frac{\tilde{\sigma}_e}{\sqrt{n}} \Phi^{-1}\left(1 + \frac{\alpha}{2}\right)\right],
\]

where \(\Phi\) denotes the standard normal (\(N(0, 1)\)) cumulative distribution.

Let \(T_n = \frac{\sqrt{n} \hat{\gamma}_n}{\tilde{\sigma}_e}\) be the test statistic for the problems (5.1) and (5.2). Thanks to the form of \(\sigma_\gamma^2\) in Theorem 5.1, we have taken \(T_n\) to be a function of \(\tilde{\sigma}_e\) not of \(\tilde{\sigma}_\gamma\), allowing to simplify the procedure. The same has been considered earlier by Francq and Zakoïan (2012, 2013a) in the context of GARCH and asymmetric power GARCH models. The following result gives the asymptotic critical regions for the testing problems (5.1) and (5.2).

Corollary 5.3 Under the same assumptions of Theorem 5.1:

i) The asymptotic level of the test \(STS\) defined for the problem (5.1) by the critical region

\[
C^{STS} = \{T_n > \Phi^{-1}(1 - \alpha)\},
\]

is bounded by \(\alpha\) and is equal to \(\alpha\) under \(\gamma = 0\). Moreover, the test \(STS\) is consistent for all \(\gamma > 0\).

ii) The asymptotic level of the test \(NSS\) defined for the problem (5.2) by the critical region

\[
C^{NSS} = \{T_n < \Phi^{-1}(\alpha)\},
\]

is bounded by \(\alpha\) and is equal to \(\alpha\) under \(\gamma = 0\). Moreover, the test \(NSS\) is consistent for all \(\gamma < 0\).

The proofs of Corollary 5.1-5.3 are based on arguments already used in the proofs of Theorem 4.2 and Theorem 5.1 and hence they are omitted.
It is worth noting that as in the \textit{GARCH} (1, 1) case (see Francq and Zakoïan, 2012), the test statistic \( T_n = \sqrt{n} \frac{\hat{\gamma}_n - \gamma}{\hat{\sigma}_e} + \sqrt{n} \frac{\gamma}{\hat{\sigma}_e} \) is such that

\[
T_n \overset{a.s.}{\to} \begin{cases} 
-\infty & \text{if } \gamma < 0, \\
+\infty & \text{if } \gamma > 0.
\end{cases} \tag{5.7}
\]

5.2. Finite sample properties of the proposed inference procedures

This subsection studies the behavior of the \textit{QMLE} and the strict stationarity tests \textit{STS} and \textit{NSS} in finite sample through some simulation experiments and real stock return series.

5.2.1. Finite sample properties of the \textit{QMLE}

The \textit{QMLE} has been run on 1000 simulated series generated from Gaussian \textit{SMBL} models with sample sizes 100 and 1000. Three set of parameters have been considered. The first one corresponds to \((\phi, \sigma^2) = (0.5, 0.7)\) for which the model is strictly stationary \((\gamma = -0.6451 < 0, \text{StS})\) with finite variance \((\phi^2 + \sigma^2 = 0.95 < 1, 2nS)\). For the second one, \((\phi, \sigma^2) = (0.8, 0.7)\), the model is strictly \((\gamma = -0.6451 < 0)\) but not second-order stationary \((N2S)\), having an infinite variance \((\phi^2 + \sigma^2 = 1.34 > 1)\). For the third one, \((\phi, \sigma^2) = (2, 1)\), the model is neither strictly stationary \((\gamma = 0.5203 > 0, \text{NSS})\) nor second-order stationary \((\phi^2 + \sigma^2 = 5 > 1, N2S)\). For all instances, we have obtained bias and standard deviations \((\text{Std})\) for the \textit{QMLE}
over the 1000 replications (cf. Table 5.1).

\[
\begin{array}{ccc}
\gamma = -0.6451 \text{ (STS)} & \gamma = -0.4183 \text{ (STS)} & \gamma = 0.5203 \text{ (NSS)} \\
\phi^2 + \sigma^2 = 0.95 \text{ (2nS)} & \phi^2 + \sigma^2 = 1.34 \text{ (N2S)} & \phi^2 + \sigma^2 = 5 \text{ (N2S)} \\
\phi = 0.5 & \sigma^2 = 0.7 & \phi = 0.8 & \sigma^2 = 0.7 \\
\phi = 2 & \sigma^2 = 1
\end{array}
\]

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Table 5.1 Bias and Std of the QMLE for the Gaussian SMBL under second-order stationarity (2nS), strict stationarity (STS) with infinite variance (N2S) and non-strict stationnarity (NSS).

It may be observed from Table 5.1 that the QMLE results are totally consistent with asymptotic theory. Indeed, for all instances, the QMLE has very small bias and Std irrespective of the stationarity conditions. Moreover, in the unstable case the QMLE of all parameters is consistent contrary to the unstable GARCH (Francq and Zakoïan, 2012) and the unstable RCA (Aue and Horváth, 2011) where the QMLE of the conditional variance intercept is inconsistent.

5.2.2. Finite sample properties of the tests

We have applied the tests STS and NSS on 1000 replications of Gaussian SMBL series with sample sizes 100, 500 and 3000. Various sets of parameters, inside ($\gamma < 0$), (approximately) on the boundary ($\gamma \simeq 0$) and outside the strict stationarity domain ($\gamma > 0$) have been taken (cf. Table 5.2 and Table 5.3). For all instances, we have obtained relative frequency of rejection of the tests STS (cf. Table 5.2) and NSS (cf. Table 5.3) at the nominal level.
\( \alpha = 5\% \).

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<td>(0.8, 2.87)</td>
<td>-0.0005</td>
<td>7.1</td>
<td>6.6</td>
<td>4.8</td>
<td>4.8</td>
</tr>
<tr>
<td>(0.8, 2.88)</td>
<td>0.0008</td>
<td>7.5</td>
<td>6.8</td>
<td>4.8</td>
<td>4.8</td>
</tr>
<tr>
<td>(1.1, 3)</td>
<td>0.1029</td>
<td>27.1</td>
<td>68.2</td>
<td>99.8</td>
<td>99.8</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>0.4508</td>
<td>99.3</td>
<td>100.0</td>
<td>100.0</td>
<td>100.0</td>
</tr>
</tbody>
</table>

Table 5.2 Percentage of rejection of the strict stationarity test \(STS, H_0 : \gamma < 0\), at the nominal level \(\alpha = 5\%\) for the Gaussian \(SMBL\) model.

It may be observed from Table 5.2 that the relative frequency of rejection of the test \(STS\):

i) tends to be close to 0\% as \(\gamma\) decreases negatively \((\gamma < 0)\),

ii) tends to be close to 100\% as \(\gamma\) increases positively \((\gamma > 0)\) and,

iii) is close to the nominal level \(\alpha = 5\%\) around \(\gamma = 0\).

These conclusions tend to be true as \(n\) increases confirming consistency of the \(STS\).

<table>
<thead>
<tr>
<th>((\phi, \sigma^2))</th>
<th>(\gamma)</th>
<th>(n)</th>
<th>100</th>
<th>500</th>
<th>3000</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.5, 0.7)</td>
<td>-0.4625</td>
<td>100.0</td>
<td>97.0</td>
<td>97.0</td>
<td>97.0</td>
</tr>
<tr>
<td>(0.9, 0.7)</td>
<td>-0.3312</td>
<td>100.0</td>
<td>100.0</td>
<td>100.0</td>
<td>100.0</td>
</tr>
<tr>
<td>(0.8, 2)</td>
<td>-0.1368</td>
<td>33.8</td>
<td>33.8</td>
<td>33.8</td>
<td>33.8</td>
</tr>
<tr>
<td>(0.8, 2.87)</td>
<td>-0.0005</td>
<td>3.9</td>
<td>4.2</td>
<td>4.2</td>
<td>4.2</td>
</tr>
<tr>
<td>(0.8, 2.88)</td>
<td>0.0008</td>
<td>4.5</td>
<td>3.2</td>
<td>3.2</td>
<td>3.2</td>
</tr>
<tr>
<td>(1.1, 3)</td>
<td>0.1029</td>
<td>0.4</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>0.4508</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table 5.3 Percentage of rejection of the non strict stationarity test \(NTS, H_0 : \gamma \geq 0\), at the nominal level \(\alpha = 5\%\) for the Gaussian \(SMBL\) model.
From Table 5.3 the same conclusion may be done as above: the relative frequency of rejection of the non-strict stationarity test \( NSS \):

i) tends to be close to 100% as \( \gamma \) decreases negatively (\( \gamma < 0 \)),

ii) is close to 0% whenever \( \gamma \) increases positively (\( \gamma > 0 \)) and

iii) is close to the nominal level \( \alpha = 5\% \) when \( \gamma \simeq 0 \) and \( n \) increases.

### 5.2.3. Application: strict stationarity testing for some financial stock returns

We have applied the proposed strict stationarity tests to daily returns of three stock market indices and two oil prices. We have considered the \( SP500 \) from 01/02/1997 to 06/06/2000, the \( CAC40 \) from 06/11/2010 to 06/10/2013, the \( KV \) Pharmaceutical (\( NYSE: KV-A \)) from 09/18/ 2008 to 02/07/2011, the \( BRENT \) oil price from 01/02/2008 to 03/14/2013 and the \( WTI \) oil price from 01/11/2010 to 03/14/2013 (see also Aknouche and Touche, 2015). The \( KV-A \) series has been taken from Francq and Zakoïan (2012). For the \( WTI \) oil price series, missing data have been removed. Table 5.4 displays the strict stationarity test statistic \( T_n \) computed on each return series. In view of the asymptotic property of \( T_n \) in (5.7), the strict stationarity hypothesis of the \( SMBL \) model cannot be rejected at any reasonable level for the return series of \( SP500, CAC40, BRENT \) and \( WTI \). In contrast, a strict stationary \( SMBL \) is not plausible for the \( KV-A \) return series. The same conclusion with a \( GARCH(1,1) \) model has been made by Francq and Zakoïan (2012) for the \( KV-A \) return series.

<table>
<thead>
<tr>
<th></th>
<th>( SP500 )</th>
<th>( CAC40 )</th>
<th>( BRENT )</th>
<th>( WTI )</th>
<th>( KV-A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_n )</td>
<td>-150.8579</td>
<td>-137.4617</td>
<td>-164.4189</td>
<td>-127.8241</td>
<td>0.7933</td>
</tr>
</tbody>
</table>

Table 5.4 The test statistic \( T_n \) of the strict stationarity tests \( STS \) and \( NSS \) for returns of \( SP500, CAC40, BRENT, WTI \) and \( KV-A \).

### 6. Conclusion

In this Chapter statistical properties of the \( SMBL \) model (a random coefficient autoregression in which the random coefficient coincides with the innovation) have been explored
irrespective of its probabilistic structure. In addition to its parsimony and simplicity, the SMBL model allows describing the level and volatility contrary to the pur GARCH process which only models volatility. Testing purely conditional variance effect may then be done while considering the null hypothesis \( H_0: \phi = 0 \) against the alternative \( H_1: \phi \neq 0 \). The test may be obtained irrespective of the stationarity assumption from the distribution of \( \hat{\phi}_{QML} \) given by Corollary 4.1. An interesting statistical property of the SMBL model is that its QMLE has a closed form and surprisingly is consistent for all parameters in the unstable case. This is in contrast with standard RCA and GARCH models where the conditional variance intercept cannot be consistently estimated in the unstable domain (cf. Aue and Horváth, 2011; Aknouche, 2013; Francq and Zakoïan, 2012). Notice that the proposed unified QML theory for the SMBL model was based on the fourth moment assumption \( \text{A4} \) on the innovation, which may be too restrictive when modeling heavy tailed stock returns. So adapting such a theory to some robust methods which do not require \( \text{A4} \), such as the least absolute deviation estimate (LADE) and the generalized QMLE (GQMLE), would be of interest (see e.g. Peng and Yao 2003; Berkes and Horváth, 2004; Francq and Zakoïan, 2013b, Fan et al, 2014 for the GARCH model and Zhu and Ling, 2013 for the DAR model).

7. Appendix: Glossary

\[ \begin{align*}
\overset{\text{a.s.}}{\rightarrow} & \quad \text{Almost sure convergence as } n \to \infty. \\
\overset{\mathcal{L}}{\rightarrow} & \quad \text{Convergence in distribution (law) as } n \to \infty. \\
\overset{\text{P}}{\rightarrow} & \quad \text{Convergence in probability as } n \to \infty. \\
o_p(1) & \quad \text{A term converging in probability to zero as } n \to \infty. \\
o_{a.s.}(1) & \quad \text{A term converging almost surely to zero as } n \to \infty. \\
O_p(1) & \quad \text{A term bounded in probability as } n \to \infty. \\
\mathbb{N} & \quad \text{Set of nonnegative integer numbers.} \\
\mathbb{N}^* & \quad \text{Set of positive integer numbers.} \\
\mathbb{Z} & \quad \text{Set of integer numbers.}
\end{align*} \]
\( \mathbb{R} \) Set of real numbers.

2\(nS \) Second-order stationary, second-order stationarity.

2\(S(WLSE) \) Two-Stage (Weighted Least Squares Estimate).

ARCH Autoregressive Conditionally Heteroskedastic.

ARMA Autoregressive Moving Average.

ARMA-BL ARMA with BiLinear innovation.

ARMA-GARCH ARMA with GARCH innovation.

ARMA-SV ARMA with Stochastic Volatility innovation.

a.s. almost surely.

BL BiLinear.

CHARMA Conditionally Heteroskedastic ARMA.

CMV Conditional Mean and Volatility.

DAR Double AutoRegression.

GARCH Generalized ARCH.

GRCA Generalized RCA.

GQMLE Generalized QMLE.

iid independent and identically distributed.

LADE Least Absolute Deviation Estimate.

MAR Mixture Autoregression.

N2S Non Second-order Stationary, Non Second-order Stationarity.

NSS Non Strict Stationary, Non Strict Stationarity.

QML(E) Quasi Maximum Likelihood (Estimate).

RCA Random Coefficient Autoregression, Random Coefficient Autoregressive.

SMBL Simple Markov BiLinear.

Std Standard deviation.

STS Strict Stationary, Strict Stationarity.

SV Stochastic Volatility.

TAR Treshold autoregression.
References


