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# Cournot, Bertrand or Chamberlin: Toward a reconciliation\*

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## Abstract

The purpose of this paper is to provide a comparison of three types of competition in a differentiated industry: *Cournot*, *Bertrand*, and *monopolistic competition*. This is accomplished in an economy involving one sector and a population of consumers endowed with separable preferences and a given number of labor units. When firms are free to enter the market, monopolistically competitive firms charge lower prices than oligopolistic firms, while the mass of varieties provided by the market is smaller under the former than the latter. If the economy is sufficiently large, Cournot, Bertrand and Chamberlin solutions converge toward the same market outcome, which may be a competitive or a monopolistically competitive equilibrium, depending on the nature of preferences.

**Keywords:** Cournot competition, Bertrand competition, monopolistic competition, free entry.

**JEL Classification:** D43, D41, F12, L13.

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# 1 Introduction

The theory of imperfect competition is dominated by two approaches that seem to clash with each other. Whereas industrial organization stresses the importance of strategic interactions among firms, the model of imperfect competition used in economic fields such as trade, economic geography and growth is the CES model of monopolistic competition developed by Dixit and Stiglitz (1977). In this model, any form of interaction among firms is absent. In addition, in oligopolistic markets, price (Bertrand) and quantity (Cournot) competition deliver market solutions that typically differ, making it hard to formulate robust predictions. The purpose of this paper is to contribute to this debate by providing a comparison of these three types of competition. This is accomplished in an economy involving one sector and a population of consumers endowed with separable preferences and a finite number of labor units. Although we recognize that additive preferences are restrictive, they are widely used in the literature and suffice to shed new light on old questions. Note also that the budget constraint implies that firms do not behave like monopolists.

According to the “Folk Theorem of Competitive Markets,” perfect competition almost holds when firms are small relative to the size of the market. In the same spirit, there has been a vivid debate in the 1930s between, on the one side, Chamberlin (1933) and, on the other, Robinson (1934) and Kaldor (1935) about the relevance of monopolistic competition as a possible market structure. Robinson and Kaldor maintained against Chamberlin that perfect competition must emerge when the number of firms becomes arbitrarily large relative to market size. No clear answer came out of this debate because these authors lacked the analytical tools to study the convergence issue. Our paper shows that the answer depends on the nature of preferences.

It was not until 1980 that Novshek was able to tackle the convergence issue rigorously for Cournot games in which firms produce a homogeneous good and face U-shaped average costs. In the spirit of methods used in general equilibrium theory, Novshek (1980) chose to make firms small relative to the market by replicating the demand side. When the number of replications is sufficiently large, the equilibrium is nearly competitive. As for Bertrand differentiated oligopoly, Novshek and Chowdhury (2003) showed that the convergence of the Bertrand equilibria toward the perfectly competitive equilibrium may not take place, even under strong assumptions on technologies.

Our main findings are as follows. We first show that a Cournot differentiated oligopoly generates a higher markup than a Bertrand differentiated oligopoly when the number of firms is exogenously given. This is in accordance with the folk wisdom of industrial organization according to which Cournot competition is softer than Bertrand competition. Second, as the number of competitors becomes arbitrarily large, both types of competition deliver the same equilibrium outcome. Whether the limit of Cournot and Bertrand competition is perfectly competitive or monopolistically competitive depends on consumers’ attitude toward product differentiation. Using the concept of relative love for variety, which measures the intensity of the preference for variety, we show that each firm operating in a large economy retains enough market power to enjoy a positive markup when the relative love for variety remains bounded away from zero at arbitrarily low consumption levels. On the contrary, when the relative love for variety vanishes at zero, consumers cease to value

product differentiation. A growing number of firms thus leads to the perfectly competitive outcome. In sum, *the market structure that emerges as the limit of oligopolistic competition depends on the nature of preferences*. Last, when firms are free to enter the market, monopolistically competitive firms are more aggressive than oligopolistic firms in that these firms charge lower prices, while the mass of varieties provided by the market is smaller  $\$\$$ under oligopolistic competition than under monopolistic competition. $\$\$$  If the economy is sufficiently large, Cournot, Bertrand and Chamberlin solutions converge toward the same market outcome, which need not be a competitive equilibrium.

## 2 The model

### 2.1 Firms and consumers

There is one sector supplying a horizontally differentiated good and one production factor - labor - and a mass  $L$  of identical consumers. Each consumer supplies one unit of labor and owns  $1/L$  of firms' profits. The labor market is perfectly competitive and labor is chosen as the numéraire. The differentiated good is made available under the form of a finite number  $n \geq 2$  of varieties. Each variety is produced by a single firm and each firm produces a single variety. To operate every firm needs a fixed requirement  $f \geq 0$  and a marginal requirement  $c > 0$  of labor. Since wage is normalized to 1, the cost of producing  $q_i$  units of variety  $i = 1, \dots, n$  is equal to  $f + cq_i$ .

Consumers share the same additive preferences given by

$$U(\mathbf{x}) = \sum_{i=1}^n u(x_i), \quad (1)$$

where  $u$  is thrice continuously differentiable, strictly increasing and strictly concave over  $\mathbb{R}_+$ . The strict concavity of  $u$  implies that consumers have a *love for variety*: when a consumer is allowed to consume  $X$  units of the differentiated good, she strictly prefers the consumption profile  $x_i = X/n$  to any other profile  $\mathbf{x} = (x_1, \dots, x_n)$  such that  $\sum_i x_i = X$ .

Following Zhelobodko et al. (2012), we define the *relative love for variety* (RLV) as follows:

$$r_u(x) \equiv -\frac{xu''(x)}{u'(x)},$$

which is strictly positive for all  $x > 0$ . Very much like the Arrow-Pratt's relative risk-aversion, the RLV is a local measure of consumers' variety-seeking behavior. A higher value of the RLV means a stronger love for variety. On the contrary,  $r_u(x) = 0$  means that the consumer perceives the varieties as perfect substitutes. Under the CES, we have  $u(x) = x^\rho$  where  $\rho$  is a constant such that  $0 < \rho \leq 1$ , thus implying a constant RLV given by  $1 - \rho$ . Other examples include: (i) the CARA utility  $u(x) = 1 - \exp(-\alpha x)$  where  $\alpha > 0$  is the absolute love for variety (Behrens and Murata, 2007), while the RLV is increasing and given by  $\alpha x$ ; and (ii) the quadratic utility  $u(x) = \alpha x - \beta x^2/2$ , with  $\alpha, \beta > 0$ ; the RLV is increasing and given by  $\beta x/(\alpha - \beta x)$ .

The budget constraint is given by

$$\sum_{i=1}^n p_i x_i = y. \quad (2)$$

A consumer's income  $y$  is equal to her wage plus her share of total profits:

$$y = 1 + \frac{1}{L} \sum_{i=1}^n \Pi_i \geq 1,$$

where the profits earned by firm  $i$  is given by

$$\Pi_i = (p_i - c)q_i - f, \quad (3)$$

$p_i$  being the price of variety  $i$ .

The first-order condition for utility maximization yields

$$u'(x_i) = \lambda p_i,$$

where  $\lambda$  is the Lagrange multiplier defined by

$$\lambda(\mathbf{x}, y) = \frac{\sum_{j=1}^n x_j u'(x_j)}{y} \geq 0. \quad (4)$$

A consumer's inverse demand for variety  $i$  is such that

$$p_i(x_i, \mathbf{x}_{-i}, y) = \frac{u'(x_i)}{\lambda}, \quad (5)$$

where  $\mathbf{x}_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ .

## 2.2 Market equilibrium

The market equilibrium is defined by the following conditions.

(E.1) Each consumer maximizes her utility (1) subject to (2).

(E.2) Each firm  $i$  maximizes its profit (3) with respect to  $q_i$  (Cournot) or  $p_i$  (Bertrand).

(E.3) Product market clears:

$$Lx_i = q_i \quad \text{for } i = 1, \dots, n.$$

(E.4) Labor market clears:

$$nf + c \sum_{i=1}^n q_i = L.$$

The last condition implies that

$$\bar{q} \equiv \frac{1}{c} \left( \frac{L}{n} - f \right) \iff \bar{x} \equiv \frac{1}{c} \left( \frac{1}{n} - \frac{f}{L} \right) \quad (6)$$

are the only candidate symmetric equilibrium output and consumption, which both decrease with  $n$ . As a consequence, Cournot competition and Bertrand competition are equally efficient when the number of firms

is the same, which contradicts Vives (1985). The reason for this difference in results lies in the wide-spread assumption made in standard oligopoly models: the labor (or input) supply is perfectly elastic. In contrast, labor supply is perfectly inelastic in our setup. Between these two extreme cases, there is a continuum of possibilities. To be specific, a labor supply with a positive and finite elasticity implies that things work as if a firm's marginal cost  $q\gamma(q)$  were increasing. In this case, the equilibrium consumption and output are the same under Bertrand and Cournot and given by

$$Lx^* = q^* \equiv \beta^{-1} \left( \frac{L}{n} - f \right),$$

where  $\beta(q) \equiv q\gamma(q)$  is strictly increasing from 0 to  $\infty$ . Note also that  $nf$  is the minimum labor requirement for  $n$  firms to operate. Therefore,  $n$  cannot exceed  $L/f$ , which implies  $\bar{x} \geq 0$ .

### 2.2.1 Cournot

Using (4) and (5), we obtain firm  $i$ 's inverse demand:

$$p_i(\mathbf{x}) = \frac{y^C u'(x_i)}{\sum_{j=1}^n x_j u'(x_j)}, \quad (7)$$

where  $y^C$  is a consumer's income under Cournot competition. Firm  $i$ 's profit function is then given by

$$\Pi_i^C(\mathbf{x}) = [p_i(x_i, \mathbf{x}_{-i}) - c]Lx_i - f = \left[ \frac{y^C u'(x_i)}{\sum_{j=1}^n x_j u'(x_j)} - c \right] Lx_i - f. \quad (8)$$

For any given  $n \geq 2$ , a *Cournot equilibrium* is a vector  $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$  such that each strategy  $x_i^*$  is firm  $i$ 's best reply to the strategies  $\mathbf{x}_{-i}^*$  chosen by the other firms. This equilibrium is symmetric if  $x_i^* = x^C$  for all  $i = 1, \dots, n$ .

### 2.2.2 Bertrand

Assume now that firms compete in prices. Let  $\mathbf{p} = (p_1, \dots, p_n)$  be a price vector. In this case, consumers' demand functions  $x_i(\mathbf{p})$  are obtained by solving the system of equations (7) with  $i = 1, \dots, n$ , where  $y^C$  is replaced with  $y^B$  that is, a consumer's income under Bertrand competition. A consumer's demand for variety  $i$  is then

$$x_i(\mathbf{p}) = \xi(p_i \lambda(p_i, \mathbf{p}_{-i})), \quad (9)$$

where  $\xi(\cdot)$  is the inverse function of  $u'(\cdot)$ . Thus, firm  $i$ 's profits are given by

$$\Pi_i^B(\mathbf{p}) = (p_i - c)q_i(\mathbf{p}) - f = (p_i - c)L\xi(p_i \lambda(p_i, \mathbf{p}_{-i})) - f. \quad (10)$$

It follows from (4) and (9) that  $\lambda$  can be rewritten as a function of  $\mathbf{p}$ . Indeed, the budget constraint

$$\sum_{j=1}^n p_j x_j(\mathbf{p}) = y^B$$

implies that

$$\lambda(\mathbf{p}) = \frac{1}{y^B} \sum_{j=1}^n x_j(\mathbf{p}) u_j(\mathbf{x}(\mathbf{p})),$$

where the income  $y^B$  is given by

$$y^B = 1 + \frac{1}{L} \sum_{j=1}^n \Pi_j^B(\mathbf{p}).$$

A Nash equilibrium  $\mathbf{p}^* = (p_1^*, \dots, p_n^*)$  of this game is called a *Bertrand equilibrium*. This equilibrium is symmetric if  $p_i^* = p^B$  for all  $i$ .

### 3 Comparing Cournot and Bertrand

One major difficulty in general equilibrium with oligopolistic firms is the income effect. Ever since Gabszewicz and Vial (1972), it is well known that firms operating in an imperfectly competitive environment are able to manipulate individual incomes through the profits they redistribute to consumers. By changing consumers' incomes, firms affect their demand functions, whence their profits. Accounting for such feedback effects typically leads to the nonexistence of an equilibrium because the resulting profit functions are not quasi-concave (Roberts and Sonnenschein, 1977). This negative result probably explains why many economic models involving imperfectly competitive product markets rely on the CES model of monopolistic competition, where the existence of an equilibrium is easy to show because firms are non-strategic. In this paper, we assume that firms recognize that income is endogenous because they operate in a general equilibrium environment. However, firms treat income parametrically, which means that they behave like "income-takers." This approach is in the spirit of Hart (1985) for whom firms may take into account only some effects of their policy on the whole economy.<sup>1</sup> Even though our model does not capture all possible strategic aspects, it is a full-fledged general equilibrium model in which oligopolistic firms account for strategic interactions within their group, as well as for endogenous incomes through the distribution of profits.

A consumer's income

$$y = 1 + \frac{1}{L} \sum_{j=1}^n [(p_j - c)q_j - f] = 1 - nf/L + \sum_{j=1}^n (p_j - c)x_j$$

depends on  $\mathbf{x}$  under Cournot and  $\mathbf{p}$  under Bertrand. Firms are said to be *income-takers* when they are aware that the income is endogenous, but treat  $y$  parametrically:

$$\frac{\partial y}{\partial x_i} = 0 \quad \left( \frac{\partial y}{\partial p_i} = 0 \right) \quad \text{for all } i. \quad (11)$$

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<sup>1</sup>When product markets are imperfectly competitive, it is common to assume that firms do not manipulate wages, even though firms also have market power on the labor market. d'Aspremont et al. (1996) is a noticeable exception.

### 3.1 Cournot

Firm  $i$ 's profits may be expressed as follows:

$$\Pi_i^C = \left[ \frac{y^C u'(x_i)}{\sum_{j=1}^n x_j u'(x_j)} - c \right] Lx_i - f.$$

Using (11), the first-order condition for profit maximization yields

$$\frac{y^C u'(x_i)}{\sum_{j=1}^n x_j u'(x_j)} - c + \frac{u''(x_i) \sum_{j=1}^n x_j u'(x_j) - u'(x_i) [u'(x_i) + x_i u''_i(x_i)]}{\left[ \sum_{j=1}^n x_j u'(x_j) \right]^2} x_i y^C = 0.$$

Given (7), this expression is equivalent to

$$m_i^C \equiv \frac{p_i - c}{p_i} = \frac{x_i}{u'(x_i)} \frac{u'(x_i) [u'(x_i) + x_i u''_i(x_i)] - u''(x_i) \sum_{j=1}^n x_j u'(x_j)}{\sum_{j=1}^n x_j u'(x_j)}.$$

Along the diagonal, the candidate equilibrium markup  $m^C$  is equal to

$$m^C = r_u(x^C) + \frac{x^C u'(x^C) (1 - r_u(x^C))}{n x^C u'(x^C)} = \frac{1}{n} + \frac{n-1}{n} r_u(x^C), \quad (12)$$

where  $x^C = \bar{x}$ . Note that  $r_u(x^C)$  must be smaller than 1 for  $m^C < 1$  to be satisfied. Since  $x^C$  can take on any positive value, for an equilibrium to exist under any collection of the parameter values, it must be that

$$r_u(x) < 1 \quad \text{for all } x \geq 0. \quad (13)$$

It is well known that a firm's profit function is strictly quasi-concave if the second-order condition for profit-maximization is satisfied at any solution to the first-order condition. We show in Appendix A that the second-order condition always holds if

$$r_{u'}(x) = -\frac{x u'''(x)}{u''(x)} < 2. \quad (14)$$

This condition highlights the need to impose restrictions on the third derivative of the utility  $u$  to prove the existence and uniqueness of a Nash equilibrium. The HARA functions satisfy both (13) and (14) conditions, but the CARA does not because  $r_u(x) = r_{u'}(x) = \alpha x$ . However, our results remain valid when the number of firms is sufficiently large for  $\bar{x} \in (0, \alpha^{-1})$  to hold (see (6)).

To sum up, *if (13) and (14) hold, then (12) is the unique symmetric equilibrium markup of the Cournot game.*

### 3.2 Bertrand

Applying the first-order condition to (10) yields the markup

$$m_i^B \equiv \frac{p_i - c}{p_i} = -\frac{\xi(p_i \lambda)}{\xi'(p_i \lambda) p_i \left( \lambda + p_i \frac{\partial \lambda}{\partial p_i} \right)}, \quad (15)$$



which involves  $\partial\lambda/\partial p_i$  because  $\lambda$  depends on  $\mathbf{p}$ . Differentiating both sides of the budget constraint

$$\sum_{j=1}^n p_j \xi(p_j \lambda(\mathbf{p})) = y^B \quad (16)$$

with respect to  $p_i$  when  $y^B$  is treated parametrically, we get

$$\xi(p_i \lambda(\mathbf{p})) + p_i \xi'(p_i \lambda(\mathbf{p})) \lambda + \sum_{j=1}^n p_j^2 \xi'(p_j \lambda(\mathbf{p})) \frac{\partial \lambda}{\partial p_i} = 0,$$

or, equivalently,

$$\frac{\partial \lambda}{\partial p_i} = - \frac{\xi(p_i \lambda(\mathbf{p})) + p_i \lambda(\mathbf{p}) \xi'(p_i \lambda(\mathbf{p}))}{\sum_{j=1}^n p_j^2 \xi'(p_j \lambda(\mathbf{p}))}. \quad (17)$$

Substituting (17) into (15) and symmetrizing leads the candidate equilibrium markup:

$$m^B = - \frac{\xi(\lambda p)}{\xi'(\lambda p) \lambda p \left(1 - \frac{1}{n} + r_u(\xi(\lambda p))\right)} = \frac{n}{n - 1 + r_u(x^B)} r_u(x^B) < 1, \quad (18)$$

where we have used the identity

$$r_u(x) \equiv - \frac{\xi(\lambda p)}{\xi'(\lambda p) \lambda p},$$

along the diagonal. As in the Cournot case,  $x^B$  is equal to  $\bar{x}$ .

We show in Appendix B that the second-order condition is satisfied if (13) and (14) hold. Under these circumstances, (18) is the unique symmetric markup of the Bertrand game.

Using (6) and comparing (12) and (18), we have the following proposition.

**Proposition 1.** *Assume that a symmetric equilibrium exists under Cournot and Bertrand competition when the number of firms is equal to  $n < L/f$ . If firms are income-takers, the equilibrium markups are given by*

$$m^C(n) = \frac{1}{n} + \frac{n-1}{n} r_u \left( \frac{1}{cn} - \frac{f}{cL} \right) \quad m^B(n) = \frac{n}{n-1 + r_u \left( \frac{1}{cn} - \frac{f}{cL} \right)} r_u \left( \frac{1}{cn} - \frac{f}{cL} \right),$$

while

$$m^C(n) > m^B(n).$$

The following remarks are in order. First, when the number of firms is given and the same, *Cournot competition always generates a higher markup than Bertrand competition*. This reflects the folk wisdom according to which Cournot competition is “softer” than Bertrand competition (Vives, 1985). However, the mechanism leading to this result differs from that used in standard industrial organization models where the marginal utility of income is constant and the same under the two competition regimes, while Bertrand firms supply more output than Cournot firms. In contrast, the marginal utility of income is variable here. Since the consumption and output of each variety is the same under both regimes while the residual demand curve is steeper under Bertrand than under Cournot, firms charge lower prices and earn smaller profits in the former case than in the latter, which implies  $y^C > y^B$ . By (4), the marginal utility of income is thus lower under Cournot, making competition softer under this regime. Second, when  $f/L \rightarrow 0$ , which may be interpreted as  $f \rightarrow 0$  or  $L \rightarrow \infty$ ,

the number  $n$  of competitors can become arbitrarily large. In this event, both types of oligopolistic competition deliver similar market outcomes, as the two markups are approximately equal to  $r_u(0)$ . Whether the limit of Cournot and Bertrand competition is perfectly competitive or monopolistic competitive thus depends on the value of  $r_u(0) \geq 0$ , which is finite as implied by (13).

When  $r_u(0) = 0$ , an infinitely large number of firms always leads to the perfectly competitive outcome, as maintained by Robinson (1934) and Kaldor (1935). The intuition is easy to grasp. When the love for variety vanishes at 0, consumers no longer value product differentiation and treat varieties as perfect substitutes. In this case, it is hardly a shock that perfect competition prevails. On the contrary, when  $r_u(0) > 0$ , a very large number of firms whose size is small relative to the market size is consistent with the idea that firms retain enough market power for their markup to be bounded away from zero. Intuitively, the love for variety is now strong enough to overcome the decrease in consumption. Since consumers still perceive varieties as being differentiated, firms retain some monopoly power, and thus price above marginal cost. This agrees with Chamberlin (1933). In short, *the nature of the limit of oligopolistic competition depends on preferences.*

To illustrate, consider the HARA utility

$$u(x) = (a + x)^\rho - a^\rho,$$

where  $a$  is a non-negative constant while  $0 < \rho < 1$ . We have

$$r_u(x) = (1 - \rho) \frac{x}{a + x} \in (0, 1).$$

Therefore, as long as  $a > 0$ , monopolistic competition is not the limit of a large group of firms. In contrast, when  $a = 0$ , we have  $r_u(0) = 1 - \rho > 0$ . Therefore, the CES model of monopolistic competition is the limit of a large group of firms, but the other HARA models of monopolistic competition, as well as CARA and quadratic, are not. Another example of preferences that lead to a monopolistically competitive limit is obtained when the subutility is given by *the convex combination of two* CES functions:

$$u(x) = \theta x^{\rho_1} + (1 - \theta)x^{\rho_2},$$

where  $0 < \theta < 1$  and  $0 < \rho_1 \leq \rho_2 < 1$ . When  $\rho_1 = \rho_2$ , we fall back on the CES case. Otherwise, the elasticity of substitution is variable while the RLV is

$$r_u(x) = \frac{\theta \rho_1 (1 - \rho_1) + (1 - \theta) \rho_2 (1 - \rho_2) x^{\rho_2 - \rho_1}}{\theta \rho_1 + (1 - \theta) \rho_2 x^{\rho_2 - \rho_1}},$$

and thus  $r_u(0) = 1 - \rho_1 > 0$ .

## 4 Free entry

### 4.1 Oligopolistic competition

In this section, we assume that fixed costs act as an entry barrier ( $f > 0$ ). In equilibrium, profits must be non-negative for firms to operate. The budget constraint can be rewritten as follows:

$$y = 1 - \frac{nf}{L} + \frac{1}{L} \sum_{j=1}^n \frac{p_j - c}{p_j} p_j q_j,$$

which, after symmetrization, yields

$$y = 1 - \frac{nf}{L} + \frac{1}{L} m \cdot np \cdot q = 1 - \frac{nf}{L} + m \cdot y \iff y = \frac{1 - nf/L}{1 - m}.$$

Ignoring the integer problem, profits are zero or, equivalently,  $y = 1$  if and only if the equilibrium number of firms under free entry is

$$n_F^k = \frac{L}{f} m_F^k < \frac{L}{f}, \quad (19)$$

for  $k = C, B$ , while the subscript  $F$  stands for free entry. Therefore, the equilibrium number of firms increases with the market size and the degree of firms' market power, which is measured by the Lerner index, and decreases with the level of fixed cost. Note also that

$$\bar{x}_F^k = \frac{f(1 - m_F^k)}{cLm_F^k} > 0, \quad (20)$$

provided that  $m_F^k$  satisfies  $0 < m_F^k < 1$ . This implies that the equilibrium markups under free-entry must solve the following equations:

$$m_F^C = \frac{f}{Lm_F^C} + \left(1 - \frac{f}{Lm_F^C}\right) r_u \left[ \frac{f}{cLm_F^C} (1 - m_F^C) \right], \quad (21)$$

$$m_F^B = \frac{f}{L} + \left(1 - \frac{f}{L}\right) r_u \left[ \frac{f}{cLm_F^B} (1 - m_F^B) \right]. \quad (22)$$

Under the CES, the right-hand side (22) is a constant  $K$  while the right-hand side of (21) is a decreasing function of  $m_F^C$ , which exceeds  $K$  over  $[0, 1]$ . Therefore, it must be that  $m_F^B < m_F^C$ . It then follows from (19) and (20) that  $n_F^C > n_F^B$  and  $q_F^C < q_F^B$ . The next proposition is our main result. First, we determine sufficient conditions on preferences and market size for a free-entry equilibrium to exist and to be unique. Second, we show that the above inequalities hold for any utility  $u$ .

**Proposition 2.** *Assume that (13) and (14) hold. If  $f > 0$ , then there is a value  $L_0 > 0$  such that, for every  $L \geq L_0$ , there exists a unique symmetric free-entry Cournot equilibrium and a unique symmetric free-entry Bertrand equilibrium. The equilibrium markups, outputs and numbers of firms satisfy*

$$m_F^C > m_F^B \quad q_F^C < q_F^B \quad n_F^C > n_F^B$$

and

$$\lim_{L \rightarrow \infty} m_F^C(L) = \lim_{L \rightarrow \infty} m_F^B(L) = r_u(0).$$

**Proof.** We show that under each competition regime the first-order condition has a unique solution, while we prove in Appendix A that firms' profit functions are strictly quasi-concave. Therefore, the solution to the first-order condition is the unique symmetric Nash equilibrium.

(i) **Existence.** Setting  $\varphi \equiv f/L$ , we have the following functions:

$$F^C(m, \varphi) \equiv \frac{\varphi}{m} + \left(1 - \frac{\varphi}{m}\right) r_u \left[ \frac{\varphi(1-m)}{cm} \right] - m,$$

$$F^B(m, \varphi) \equiv \varphi + (1 - \varphi) r_u \left[ \frac{\varphi(1-m)}{cm} \right] - m,$$

which are defined for all  $\varphi \in (0, 1)$  and  $m \in (0, 1)$ . Using (21), (22) and (27), the equilibrium markup  $m^k$  solves the equations  $F^k(m, \varphi) = 0$  for  $k = C, B$ .

Note that  $F^k(1, \varphi) < 0$  for all admissible values of  $\varphi$  and  $k = C, B$ . Since  $r_u(1/2c) > 0$ ,  $\bar{\varphi} \in (0, 1/2)$  exists such that

$$G(\varphi) \equiv r_u \left[ \frac{1-2\varphi}{2c} \right] - 2\varphi > 0$$

holds for all  $\varphi \in (0, \bar{\varphi})$ . Then,  $F^C(2\varphi, \varphi) > F^B(2\varphi, \varphi) > G(\varphi) > 0$  implies that for any  $\varphi \in (0, \bar{\varphi})$  and  $k = C, B$ , the equation  $F^k(m, \varphi) = 0$  has at least one solution  $m^k(\varphi) \in (2\varphi, 1)$ . Thus, an equilibrium markup  $m^k(\varphi)$  exists if  $\varphi$  is sufficiently small ( $\varphi < \bar{\varphi}$ ).

(ii) **Uniqueness.** This is done by showing that the derivative of  $F^k(m, \varphi)$  with respect to  $m$  is always negative at any solution to  $F^k(m, \varphi) = 0$ .

Note that the equilibrium individual consumption is such that

$$x^k(\varphi) = \frac{\varphi}{cm^k(\varphi)}(1 - m^k(\varphi)).$$

Therefore, we have

$$\frac{\partial F^C}{\partial m}(m^C(\varphi), \varphi) = - \left[ \frac{\varphi}{(m^C)^2} (1 - r_u(x^C)) + \left(1 - \frac{\varphi}{m^C}\right) \frac{r'_u(x^C) x^C}{m^C(1 - m^C)} + 1 \right]. \quad (23)$$

Differentiating  $r_u(x)$  and rearranging terms yields

$$r'_u(x)x = (1 + r_u(x) - r_{u'}(x))r_u(x)$$

for all  $x > 0$ . Substituting  $F^C(m^C, \varphi) = 0$  and this expression into this (23), we obtain

$$\frac{\partial F^C}{\partial m}(m^C, \varphi) = -\frac{1}{m^C} \left[ 2\frac{\varphi}{m^C} (1 - r_u(x^C)) + \frac{r_u(x^C) (2 - r_{u'}(x^C))}{1 - r_u(x^C)} \right] < 0.$$

Repeating the same arguments *mutatis mutandis* for Bertrand competition, we get

$$\frac{\partial F^B}{\partial m}(m^B, \varphi) = -\frac{1}{m^B} \left[ \frac{r_u(x^B) (2 - r_{u'}(x^B))}{1 - r_u(x^B)} + \varphi (1 - r_u(x^B)) \right] < 0.$$

To sum up, when (13) and (14) hold, for all  $L \geq L_0 \equiv f/\bar{\varphi} > 0$  there exists a unique symmetric free-entry equilibrium under Cournot and Bertrand given by  $(m^k(L), n^k(L))$  for  $k = C, B$ .

(iii) **Ranking.** Note that for any given  $m$  and  $\varphi$  values of functions are ranked as follows:

$$F^C(m, \varphi) > F^B(m, \varphi),$$

which, together with

$$\frac{\partial F^k}{\partial m}(m^k, \varphi) < 0,$$

implies that for any given  $\varphi$  the symmetric free-entry equilibrium markups are ranked as follows:

$$m_F^C(\varphi) > m_F^B(\varphi).$$

Since

$$q_F^k = \frac{f(1 - m_F^k)}{cm_F^k} \quad \text{and} \quad n_F^k = \frac{L}{f}m_F^k,$$

we obtain the ranking for outputs and numbers of firms.

(iv) **Limit markups.** We show that

$$\lim_{\varphi \rightarrow 0} m^C(\varphi) = \lim_{\varphi \rightarrow 0} m^B(\varphi) = r_u(0)$$

hold. Observe, first, that

$$\lim_{\varphi \rightarrow 0} \frac{\varphi}{m^C(\varphi)} = \lim_{\varphi \rightarrow 0} \frac{\varphi}{m^B(\varphi)} = 0, \tag{24}$$

so that

$$\lim_{L \rightarrow \infty} n_F^C(L) = \lim_{L \rightarrow \infty} n_F^B(L) = \infty \tag{25}$$

because  $\varphi \equiv f/L$  and  $n^k = m^k L/f$ .

To show (24), we consider an arbitrary sequence  $\varphi_n \rightarrow 0$ . Since  $m^k(\varphi) > 2\varphi$ , the sequence  $\varphi_n/m^k(\varphi_n)$  belongs to the compact set  $[0, 1/2]$ . Therefore, there exists a subsequence  $n_j \rightarrow \infty$  such that  $\varphi_{n_j}/m^k(\varphi_{n_j})$  is convergent. Let  $\delta$  be the limit of this subsequence. If  $\delta > 0$ , it must be that  $m^k(\varphi_{n_j}) \rightarrow 0$ . Since

$$m^C(\varphi_{n_j}) = \frac{\varphi_{n_j}}{m^C(\varphi_{n_j})} + \left(1 - \frac{\varphi_{n_j}}{m^C(\varphi_{n_j})}\right) r_u \left[ \frac{\varphi_{n_j}}{m^C(\varphi_{n_j})} (1 - m^C(\varphi_{n_j})) \right],$$

$$m^B(\varphi_{n_j}) = \varphi_{n_j} + (1 - \varphi_{n_j}) r_u \left[ \frac{\varphi_{n_j}}{m^B(\varphi_{n_j})} (1 - m^B(\varphi_{n_j})) \right],$$

taking the limit implies, correspondingly, that  $\delta + (1 - \delta)r_u(\delta) = 0$  and/or  $r_u(\delta) = 0$ . This contradicts the inequality  $r_u(x) > 0$  for all  $x > 0$ . As a consequence, it must be that  $\delta = 0$ , which implies (24) due to arbitrariness of sequence  $\varphi_n$ .

Given that  $F^k(m^k(\varphi), \varphi) = 0$ , taking the limit of  $F^k(m^k(\varphi), \varphi)$  for  $\varphi \rightarrow 0$  shows that the limits of  $m^C(\varphi)$  and  $m^B(\varphi)$  are equal to  $r_u(0)$ . Q.E.D.

Thus, regardless of the type of competition, the limit outcome is determined by the very same condition on  $r_u(0)$ , no matter how market size is measured, that is, an exogenously high number of firms or a large population of consumers. Indeed, as shown by (25), the number of firms grows unboundedly with  $L/f$ . Proposition 2 also highlights the existence of a trade-off between per variety consumption and product diversity. To be precise, when free entry prevails, Cournot competition leads to a larger number of varieties, but to a lower consumption level per variety, than Bertrand competition. Therefore, the comparison between  $V^C = n_F^C \cdot u(x_F^C)$  and  $V^B = n_F^B \cdot u(x_F^B)$  is a priori ambiguous. We return to this question in subsection 4.4. In the CES case ( $u(x) = x^\rho$ ), the argument goes as follows. Note that the equilibrium number of firms and the per variety consumption may be expressed as functions of the markup:

$$n_F^k = \frac{L}{f} m_F^k \quad x_F^k = \frac{1}{c n_F^k} - \frac{f}{cL} = \frac{f}{cL} \cdot \frac{1 - m_F^k}{m_F^k},$$

which implies

$$V^C < V^B \iff (m_F^C)^{1-\rho} (1 - m_F^C)^\rho < (m_F^B)^{1-\rho} (1 - m_F^B)^\rho.$$

Since  $m^{1-\rho}(1-m)^\rho$  increases for all  $0 < m < 1 - \rho$  and decreases for all  $1 - \rho < m < 1$ , while  $m_F^C > m_F^B$  by Proposition 2, it is sufficient to show that  $m_F^B > 1 - \rho$ . Using  $r_u(x) = 1 - \rho$  and (22), we obtain  $m_F^B = 1 - \rho + \rho f/L > 1 - \rho$ . As a consequence, Bertrand is more efficient than Cournot ( $V^C < V^B$ ), as in Vives (1985). However, this result need not hold for other utility functions. To show it, assume that  $u(x) = \sqrt{x} + 2x$  and set  $f = 1$ ,  $c = 0.1$  and  $L = 100$ . Then, using Wolfram Mathematica, we get  $n_F^C = 18.3367 > n_F^B = 12.3127$  and  $x^C = 0.4454 < x^B = 0.7122$ , and thus  $V^C = 28.5696 > V^B = 27.9282$ , which runs against the conventional wisdom that holds that Bertrand is more efficient than Cournot.

## 4.2 Monopolistic competition

Whereas the set of firms is finite in oligopolistic competition, in monopolistic competition the set of firms/varieties is given by a continuum of *mass*  $M$ , which is endogenous and pinned down by zero-profit condition (we assume that  $f > 0$ ). The utility function (1) is replaced by the functional

$$U(\mathbf{X}) = \int_0^M u(x_i) di$$

where  $\mathbf{X}$  is a consumption profile defined on  $[0, M]$ , while the budget constraint is

$$\int_0^M p_i x_i di = y.$$

The inverse demand for variety  $i$  is then given by

$$p_i(x_i, \lambda) = \frac{u'(x_i)}{\lambda},$$

so that firm  $i$ 's profits are defined as follows:

$$\pi_i^{MC}(q_i, \lambda) = \left[ \frac{u'(q_i/L)}{\lambda} - c \right] q_i - f. \quad (26)$$

Each firm being negligible to the market, it accurately treats the Lagrange multiplier  $\lambda$  as a parameter in (26). However, firms are aware that  $\lambda$  is endogenous. As a consequence, to determine whether it enters the market a firm must guess what the equilibrium value of the Lagrange multiplier is. Observe the difference between (7) and (26): the former depends on the output vector  $\mathbf{q}$ , whereas the latter depends on  $q_i$  and  $\lambda$ . The Lagrange multiplier has the nature of a market statistic that binds together the markets of all varieties, very much as the budget constraint (2) does under Cournot and Bertrand.

Since  $u'(x)$  is strictly decreasing, the demand function for variety  $i$  is given by

$$x_i(p_i, \lambda) = \xi(\lambda p_i),$$

where  $\xi$  is the inverse function of  $u'$ . Thus,

$$\pi_i^{MC}(p_i, \lambda) = (p_i - c)L\xi(\lambda p_i) - f.$$

Because each firm treats  $\lambda$  as a given, it behaves like a monopolist facing its own demand and, therefore, choosing price or quantity as a strategy yields the same market outcome. For a given the mass  $M$  of firms, an equilibrium is a function  $\bar{q}_i$  defined over  $[0, M]$  such that (almost) no firm  $i \in [0, M]$  finds it profitable to deviate unilaterally from  $\bar{q}_i$  while anticipating accurately the equilibrium value of  $\lambda$ .

A *monopolistic competitive equilibrium* is defined by the conditions (E.1)-(E.4) plus the zero-profit condition:  $\Pi_i^{MC}(p_i, \lambda) = 0$  for (almost) all  $i$ . Since firms face the same Lagrange multiplier, the solution to the profit-maximizing condition is the same across firms, i.e.  $q_i = \hat{q}$  for (almost) all  $i \in [0, M]$ . In other words, if a monopolistic competitive equilibrium exists, it must be symmetric. As for the equilibrium value of  $\lambda$ , it is given by

$$\hat{\lambda} = M \frac{\hat{q}}{L} u' \left( \frac{\hat{q}}{L} \right),$$

which implies that  $\hat{q}$  depends on  $M$ . A *MC-equilibrium* may thus be defined by a pair  $(q^*, M^*)$  such that (almost) every firm  $i \in [0, M^*]$  maximizes its profits at  $q_i = q^*$ , while the mass  $M^*$  of firms is such that these firms earn zero profits:  $\pi^{MC}(q^*, M^*) = 0$ .

The equilibrium markup under monopolistic competition is the solution to the implicit equation (Zhelobodko et al., 2012):

$$m^{MC} = r_u \left[ \frac{f}{cLm^{MC}} (1 - m^{MC}) \right], \quad (27)$$

which tends to  $r_u(0)$  when  $L/f$  becomes arbitrarily large, as do the markups under Cournot and Bertrand. In this case, the markup  $m^{MC}$  is positive if and only if  $r_u(0) > 0$ , and thus operating profits are positive as predicted by Proposition 2.

### 4.3 Comparing Cournot, Bertrand and Chamberlin

Comparing directly Cournot and Bertrand with Chamberlin is a hard task because oligopolistic competition involves a finite number of firms, whereas monopolistic competition relies on a continuum of firms. Rigorous

techniques have been developed to study when the former models “converge” toward the latter, but such an analysis cannot be developed within the format of this paper (Hildenbrand, 1974). Therefore, in what follows, we propose two heuristic, but intuitive, approaches that both lead to similar results.

In the first one, the number of firms is finite regardless of the competition regime. In this case, what distinguishes monopolistic competition from oligopolistic competition is that each firm treats the marginal utility of income as a parameter in the former, whereas a firm manipulates this magnitude in the latter. To make the comparison possible with a finite number  $n$  of firms, we assume here that each firm behaves *as if* it were a monopolistically competitive firm, that is,  $\partial\lambda/\partial x_i = 0$  or  $\partial\lambda/\partial p_i = 0$ . Under these circumstances, the above analysis holds true provided that  $M$  is replaced with  $n$ . Since  $r_u(x) < 1$ , comparing (22) and (27) shows that  $m_F^B$  is larger than  $m_F^{MC}$ . Furthermore, taking the limit of (21)-(22) and (27), using arguments similar to those developed in part (iv) of the proof of Proposition 2, and summarizing, we obtain the following proposition.

**Proposition 3.** *Assume that symmetric free-entry Cournot, Bertrand and monopolistic competitive equilibria exist. Then, the corresponding markups are such that*

$$m_F^C > m_F^B > m_F^{MC}. \quad (28)$$

Furthermore,  $\lim m_F^C(L/f) = \lim m_F^B(L/f) = \lim m_F^{MC}(L/f) = r_u(0)$  when  $L/f \rightarrow \infty$ .

Hence, *monopolistic competition is tougher than Cournot and Bertrand competition*, where the former is less aggressive than the latter. Furthermore, for a given market size, monopolistic competition is a better approximation of Bertrand than of Cournot.

In the second approach, we assume that an oligopolistic firm is a cartel formed by a mass of negligible firms, which produce each a single firm-specific variety and act at the unison by choosing the output or price that maximizes joint profits. For Proposition 2 to be applied to such cartels, each one must involve a *unit* mass of negligible firms associated with the interval  $[i - 1, i]$ . In other words, the  $n$  cartels provide a total mass  $n$  of varieties. In this context, consumers’ preferences must be reformulated as follows:

$$U(\mathbf{X}) = \sum_{i=1}^n \int_{i-1}^i u(x_{ij})dj,$$

so that varieties produced by firms belonging to the same cartel, or to different cartels, enter preferences symmetrically. By the mean-value theorem,  $k \in [i - 1, i]$  exists such that

$$x_i = x_{ik} \quad u(x_i) = \int_{i-1}^i u(x_{ij})dj,$$

and thus  $u(x_i) = u(x_{ij})$  for all  $j \in [i - 1, i]$  at any outcome symmetric over  $[i - 1, i]$ . The profits earned by a cartel are then given by (8) under Cournot and by (10) under Bertrand. Since the cartels behave strategically, the free entry equilibrium is described by Proposition 2.

Using (19) and (28) yields  $n_F^C > n_F^B > M^*$ . Since each cartel supplies a unit mass of varieties, the range of available varieties provided by the market is the widest under monopolistic competition and the narrowest



under Cournot competition. This is consistent with the above markup ranking: *a soft (tough) competition regime leads to more (fewer) varieties under free entry.*

## 5 Concluding remarks

Additive preferences are widely used in theoretical and empirical applications of monopolistic competition. This is why we have chosen to compare the market outcomes under three different competitive regimes when consumers are endowed with such preferences. It is our belief, however, that most of our results hold true in the case of well-behaved symmetric preferences. Unlike most models of industrial organization which assume the existence of an outside good, we have used a limited labor constraint. This has allowed us to highlight the role of the marginal utility of income in firms' behavior. Another distinctive feature of our approach is that firms recognize that consumers' incomes are endogenous through the distribution of profits. The assumption of income-taking firms seems to be a reasonable alternative to the polar cases in which incomes are taken as exogenous, as in partial equilibrium analyses, or incomes are strategically manipulated by firms, which leads to intractable general equilibrium models. In brief, even though our setup is restrictive, it is sufficient to show that whether monopolistic competition can be the limit of oligopolistic competition depends on the nature of preferences.

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## Appendix

### A. Second-order conditions under Cournot

Differentiating (8) twice and evaluating this expression at symmetric solution  $q_i = \bar{q}$ , we obtain

$$\left. \frac{\partial^2 \Pi_i^C}{\partial q_i^2} \right|_{q_i=\bar{q}} = -\frac{(n-1)y^C}{n^2(\bar{q})^2} \left[ r_u(\bar{q})(2 - r_{u'}(\bar{q})) + \frac{2}{n}(1 - r_u(\bar{q}))^2 \right] < 0.$$

### B. Second-order conditions under Bertrand

It is readily verified that the following two identities hold:

$$r_u(x) = -\frac{\xi(\lambda p)}{\xi'(\lambda p)\lambda p} \quad r_{u'}(x) = \frac{\xi(\lambda p)\xi''(\lambda p)}{(\xi'(\lambda p))^2}. \quad (\text{A.1})$$

Differentiating twice (10) with respect to  $p_i$  yields

$$\frac{\partial^2 \Pi_i^B}{\partial p_i^2} = 2\xi'(p_i\lambda) \left( \lambda + p_i \frac{\partial \lambda}{\partial p_i} \right) + (p_i - c)\xi''(p_i\lambda) \left( \lambda + p_i \frac{\partial \lambda}{\partial p_i} \right)^2 + (p_i - c)\xi'(p_i\lambda) \left( 2\frac{\partial \lambda}{\partial p_i} + p_i \frac{\partial^2 \lambda}{\partial p_i^2} \right). \quad (\text{A.2})$$

Differentiating (16) with respect to  $p_i$ , we obtain

$$\xi(p_i\lambda) + p_i\xi'(p_i\lambda)\lambda + \sum_{j=1}^n p_j^2\xi'(p_j\lambda)\frac{\partial \lambda}{\partial p_i} = 0,$$

which implies

$$\frac{\partial \lambda}{\partial p_i} = -\frac{\xi(p_i\lambda) + p_i\xi'(p_i\lambda)\lambda}{\sum_{j=1}^n p_j^2\xi'(p_j\lambda)}.$$

Substituting this into the right-hand side of (A.2), symmetrizing and using (A.1), we get that second-order condition for any symmetric outcome:

$$2\xi' \frac{\lambda}{n} (n-1+r_u(\bar{x})) + \frac{p-c}{p} p \xi'' \frac{\lambda^2}{n^2} (n-1+r_u(\bar{x}))^2 + \frac{p-c}{p} p \xi' \left[ -2\frac{\lambda}{n} (1-r_u(\bar{x})) + p \frac{\partial^2 \lambda}{\partial p^2} \right] < 0.$$

Substituting the first-order condition

$$m = \frac{p-c}{p} = \frac{nr_u(\bar{x})}{n-1+r_u(\bar{x})}$$

shows that (A.2) evaluated at any solution to the first-order solution is such that

$$\frac{2\xi' \lambda}{n} \left[ (n-1+r_u(\bar{x})) - \frac{nr_u(\bar{x})(1-r_u(\bar{x}))}{n-1+r_u(\bar{x})} \right] + \lambda p \frac{\xi'' \lambda}{n} r_u(\bar{x}) (n-1+r_u(\bar{x})) + \frac{r_u(\bar{x}) n p^2 \xi' \frac{\partial^2 \lambda}{\partial p^2}}{n-1+r_u(\bar{x})} < 0.$$

Differentiating (16) twice with respect to  $p_i$ , symmetrizing and rearranging terms leads to the following expression:

$$np^2 \xi' \frac{\partial^2 \lambda}{\partial p^2} = -\frac{2\xi' \lambda}{n} (n-2(1-r_u(\bar{x}))) - p \frac{\xi'' \lambda^2}{n^2} \left[ (n-1+r_u(\bar{x}))^2 + (1-r_u(\bar{x}))^2 \right],$$

which allows one to rewrite (A.2) as follows:

$$2\xi' \left[ (n-1+r_u(\bar{x}))^2 - nr_u(\bar{x})(1-r_u(\bar{x})) - r_u(\bar{x})(n-2(1-r_u(\bar{x}))) \right] - r_u(\bar{x}) \lambda p \xi'' \left[ (n-1+r_u(\bar{x}))^2 - \frac{1}{n} \left( (n-1+r_u(\bar{x}))^2 + (1-r_u(\bar{x}))^2 \right) \right] < 0.$$

Using (A.1), this inequality is equivalent to

$$2 \left( A - \frac{r_{u'}(\bar{x})}{2} B \right) \xi' < 0,$$

where

$$A \equiv (n-1+r_u(\bar{x}))^2 - nr_u(\bar{x})(1-r_u(\bar{x})) - r_u(\bar{x}) [n-2(1-r_u(\bar{x}))]$$

$$B \equiv (n-1+r_u(\bar{x}))^2 - \frac{1}{n} \left[ (n-1+r_u(\bar{x}))^2 + (1-r_u(\bar{x}))^2 \right].$$

Since  $\xi' = 1/u'' < 0$ , it is sufficient to show that the term between parentheses is positive. Note that (13) implies

$$B = \frac{n-1}{n} (n-1+r_u(\bar{x}))^2 - \frac{1}{n} (1-r_u(\bar{x}))^2 > 0$$

for all  $n \geq 2$ . It then follows from (14) that

$$A - \frac{r_{u'}(\bar{x})}{2} B > A - B = \frac{(1-r_u(\bar{x}))^2}{n} [(n-1)^2 + 1] > 0,$$

which yields the desired inequality. Q.E.D.