Measuring poverty with the Foster, Greer and Thorbecke indexes based on the Gamma distribution

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Abstract

The purpose of this paper is the estimation of the Foster, Greer and Thorbecke family of poverty indexes, P_ε, using the Gamma distribution as a continuous representation of the distribution of incomes. The expressions of the P_ε family of poverty indexes associated with the Gamma probability model and their asymptotic distributions are derived in the text, both for an exogenous and a relative (to the mean) poverty line. Finally, a Monte Carlo experiment is performed to compare three different methods of estimation for grouped data. The results of the experiment showed that, for data grouped in deciles, the non linear least squares performed worse than the other two (minimum χ^2 and scoring estimator) for sample sizes smaller than 1000. For the larger samples studied (1000 and 2000) the three methods performed similarly, with a slight predominance of the minimum χ^2 estimator.

Keywords— Poverty Indexes, Income Distribution, Gamma Distribution.
JEL Classification— I32, C13, C46.

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1 Introduction

Probability models are often used for describing the distribution of incomes and estimating the associated inequality and poverty indexes ([1], [2], [3], [4], [5]). This method is particularly useful when the information is available as grouped data [6].
The purpose of this article is the estimation of the Foster, Greer and Thorbecke family of poverty indexes, P_ε, using the Gamma distribution as a continuous representation of the distribution of incomes. The P_ε family of indexes was developed by Foster *et al.* [7] and includes the Head Count Ratio ($\varepsilon = 1$) and the Per Capita Income Gap Ratio ($\varepsilon = 2$) as particular cases. They exhibit a series of good properties, like additive decomposability and several transfer properties depending on the value of ε (see [8]).
On the other hand, the Gamma probability model provides good fits to the distribution of income ([9], [10], [11]). At the same time, it is a relatively simple model with only two parameters and it is easy to interpret ([12], [13]).
In section two we derive the expressions of the P_ε indexes for the Gamma distribution, and their asymptotic distributions in section three. Finally, the results of a Monte Carlo experiment for comparing the properties of alternative methods of estimation for grouped data are briefly discussed.

2 The Foster, Greer and Thorbecke poverty indexes based in the gamma distribution

In this section, the expressions of the P_ε indexes associated with the Gamma distribution are derived. For a continuous income variable $y > 0$ with density $f(y)$, the family of Foster, Greer and Thorbecke poverty indexes is defined by:

$$P_\varepsilon = \int_0^z \left(\frac{(z-y)}{z}\right)^{\varepsilon-1} f(y)dy \tag{1}$$

where z stands for the poverty line. When the density of income is represented by the Gamma probability model, $Y \sim (\alpha, \beta)$, with density $f(y) = (e^{-(y/\beta)})y^{(\alpha-1)})/(\beta^\alpha\Gamma(\alpha))$, the P_ε indexes are:

$$P_\varepsilon = \frac{1}{z^{\varepsilon-1}} \int_0^z (z-y)^{\varepsilon-1} y^{\alpha-1} \frac{e^{-\frac{y}{\beta}}}{\beta^\alpha\Gamma(\alpha)} dy \tag{2}$$

which can be rewritten:

$$P_\varepsilon = \frac{1}{z^{\varepsilon-1}\beta^\alpha\Gamma(\alpha)} \int_0^z (z-y)^{\varepsilon-1}\, y^{\alpha-1}\, {}_0F_0\left[\begin{matrix} -; & -\dfrac{y}{\beta} \\ -; & \end{matrix}\right] dy \tag{3}$$

Using theorem 5.38 from [14], the integral is solved in:

$$P_\varepsilon = \frac{1}{z^{\varepsilon-1}\beta^\alpha\Gamma(\alpha)} B(\alpha,\varepsilon) z^{\alpha+\varepsilon-1} \, {}_1F_1 \left[\begin{array}{cc} \alpha; & \\ \alpha+\varepsilon; & -\dfrac{z}{\beta} \end{array} \right] \tag{4}$$

And applying the first Kummer formula, P_ε indexes can be expressed as:

$$P_\varepsilon = \frac{B(\alpha,\varepsilon)}{\Gamma(\alpha)} \left(\frac{z}{\beta}\right)^\alpha e^{-\frac{z}{\beta}} {}_1F_1 \left[\begin{array}{cc} \varepsilon; & \\ \alpha+\varepsilon; & \dfrac{z}{\beta} \end{array} \right] \tag{5}$$

In the preceding expressions, ${}_nF_n[\cdot]$, $B(\cdot,\cdot)$ and $\Gamma(\cdot)$ stand for the confluent hypergeometric function, and the Beta and Gamma functions, respectively [14]. From this general form (5), and giving any particular value to ε, the different members of the Foster, Greer and Thorbecke family of poverty indexes are generated.

In the previous paragraphs, the poverty line z has been considered as an exogenous value; however, the use of poverty lines relative to some location measure of the population, generally the mean or the median ([15], [16], [17]), is very frequent. When the poverty line is expressed as some fraction k of the mean, and if the population is distributed following the Gamma model, then $z = k\mu = k\alpha\beta$. This fact simplifies the expression of the P_ε indexes, which only depend on the parameter α, since the scale factor β vanishes:

$$P_\varepsilon = \frac{B(\alpha,\varepsilon)}{\Gamma(\alpha)} (k\alpha)^\alpha \, e^{-k\alpha} {}_1F_1 \left[\begin{array}{cc} \varepsilon; & \\ \alpha+\varepsilon; & k\alpha \end{array} \right] \tag{6}$$

3 Asymptotic distribution of the P_ε indexes

Now we turn to the asymptotic distribution of the estimators of the P_ε indexes, which are functions of the parameters of the Gamma distribution. The maximum-likelihood estimators of α and β are the solutions of the equation system (see [18]) :

$$\begin{gathered} \hat{\beta} = \frac{\overline{x}}{\hat{\alpha}} \\ Ln(\hat{\alpha}) - \Psi(\hat{\alpha})) = Ln(\overline{x}) - Ln(\tilde{x}) \end{gathered} \tag{7}$$

where $\overline{x}$ and $\tilde{x}$ are the arithmetic and geometric means, respectively, and $\Psi(\cdot)$ is the digamma function.

The asymptotic distribution of these estimators is given by:

$$\sqrt{n}\left(\left(\begin{array}{c} \hat{\alpha} \\ \hat{\beta} \end{array}\right) - \left(\begin{array}{c} \alpha \\ \beta \end{array}\right)\right) \xrightarrow[n\to\infty]{} N(0,\mathbf{\Sigma}), \tag{8}$$

where

$$\mathbf{\Sigma} = \frac{1}{\alpha\Psi'(\alpha)-1} \left(\begin{array}{cc} \alpha & -\beta \\ -\beta & \beta\Psi'(\alpha) \end{array}\right) \tag{9}$$

and $\Psi'(\cdot)$ is the trigamma function.
Therefore, if the estimators of α and β are the maximum likelihood estimators, those of P_ε based on them are also maximum-likelihood estimators, due to the invariance principle for parametric transformations of this class of estimators [19]. Following this principle, the estimator of P_ε has an asymptotic distribution given by:

$$\sqrt{n}\left(\hat{P}_\varepsilon - P_\varepsilon\right) \xrightarrow[n\to\infty]{} N(0, \mathbf{\Sigma}_{P_\varepsilon}), \tag{10}$$

where

$$\mathbf{\Sigma}_{P_\varepsilon} = \left[\frac{dP_\varepsilon(\boldsymbol{\theta})}{d\,\boldsymbol{\theta}}\right]' \mathbf{\Sigma} \left[\frac{dP_\varepsilon(\boldsymbol{\theta})}{d\,\boldsymbol{\theta}}\right] \tag{11}$$

and $\boldsymbol{\theta}$ is the vector of parameters (α, β).
The derivatives of P_ε with respect to the parameters α and β in (11) are

$$\frac{\partial P_\varepsilon}{\partial \alpha} = P_\varepsilon Ln\left(\frac{z}{\beta}\right) - \frac{\Gamma(\varepsilon)e^{-\frac{z}{\beta}}\left(\frac{z}{\beta}\right)^\alpha}{\Gamma(\alpha+\varepsilon)} \sum_{i=0}^{\infty} \frac{(\varepsilon)_i \psi(\alpha+\varepsilon+i)}{(\alpha+\varepsilon)_i} \frac{\left(\frac{z}{\beta}\right)^i}{i!} \tag{12}$$

$$\frac{\partial P_\varepsilon}{\partial \beta} = \frac{\Gamma(\varepsilon)e^{-\frac{z}{\beta}}\left(\frac{z}{\beta}\right)^\alpha}{\Gamma(\alpha+\varepsilon)} \frac{z}{\beta^2} \frac{\alpha}{\alpha+\varepsilon} {}_1F_1\left[\begin{matrix}\varepsilon; \\ \alpha+\varepsilon+1;\end{matrix}\; \frac{z}{\beta}\right] - \frac{\alpha}{\beta} P_\varepsilon \tag{13}$$

Finally, the estimator of $\mathbf{\Sigma}_{P_\varepsilon}$ can be obtained substituting the parameters with their maximum-likelihood estimations in (11), (12) and (13).
As mentioned above, the particular case in which the poverty line is set at a fraction of the mean ($z = k\mu$) simplifies the expressions of the indexes, and this also occurs with their asymptotic distribution, which is normal with its variance equal to:

$$Var\left(\sqrt{n}\hat{P}_\varepsilon\right) = \left(\frac{dP_\varepsilon}{d\,\alpha}\right)^2 Var\left(\sqrt{n}\hat{\alpha}\right) \tag{14}$$

where the derivative of P_ε with respect to α is given by:

$$\begin{aligned}\frac{\partial P_\varepsilon}{\partial \alpha} &= P_\varepsilon\left(Ln(k\alpha) + 1 - k\right) \\ &+ \frac{\Gamma(\varepsilon)e^{-k\alpha}(k\alpha)^\alpha}{\Gamma(\alpha+\varepsilon)} \left\{\frac{\varepsilon k}{\varepsilon+\alpha} {}_1F_1\left[\begin{matrix}\varepsilon+1; \\ \alpha+\varepsilon+1;\end{matrix}\; k\alpha\right]\right. \\ &\left. - \sum_{i=0}^{\infty} \frac{(\varepsilon)_i \psi(\alpha+\varepsilon+i)}{(\alpha+\varepsilon)_i} \frac{(k\alpha)^i}{i!}\right\}\end{aligned} \tag{15}$$

4 Estimation methods

When income data is grouped, there are several methods for estimating the parameters of the Gamma distribution (and the P_ε indexes based on them), that are asymptotically equivalent to the maximum likelihood methods. In the following paragraphs we describe the most popular ones (the minimum χ^2, the scoring and the non linear least squares).
If the income distribution data is grouped in k classes of frequency n_i, and the probability generated by the population model $f(y, \boldsymbol{\theta})$ for the interval i is $p_i(\boldsymbol{\theta}) = \int_i f(y, \boldsymbol{\theta}) dy$; the minimum χ^2 estimator of $\boldsymbol{\theta}$, $\boldsymbol{\theta}_{MCS}$, is that which minimizes the goodness of fit statistic

$$\sum_{i=1}^{k} \frac{(n_i - np_i(\boldsymbol{\theta}))^2}{np_i(\boldsymbol{\theta})} \tag{16}$$

The non linear least squares estimator is the vector of the parameters $\boldsymbol{\theta}$, $\boldsymbol{\theta}_{NLLS}$, which minimizes the sum of squares:

$$\sum_{i=1}^{k} \left(\frac{n_i}{n} - p_i(\boldsymbol{\theta}) \right)^2 \tag{17}$$

and the score estimator is the vector, $\boldsymbol{\theta}_S$, that maximizes the likelihood function:

$$n! \prod_{i=1}^{k} \frac{(p_i(\boldsymbol{\theta}))^{n_i}}{n_i!} \tag{18}$$

These three estimators of $\boldsymbol{\theta}$ are asymptotically equivalent to the maximum likelihood estimators -see [19] or [20]. The estimators of P_1, P_2, P_3 and P_4 are obtained by substituting α and β in (5) with any of the three estimators of $\boldsymbol{\theta} = (\alpha, \beta)$.
In order to compare the main properties of the three methods of estimating P_ε we performed a Monte Carlo experiment. This consists of estimating the indexes P_1, P_2, P_3 and P_4 with the three described methods for 12 sets of 1000 artificial samples. The samples are drawn from Gamma populations of $\alpha = 1, 2, 3$ (we keep $\beta = 1$ without any loss of generality) and have sizes of 100, 500, 1000 and 2000. For each combination of n and α, a set of 1000 artificial samples has been generated. The data of every sample has been grouped into deciles (this is the most common presentation in income data surveys) for the estimation of the indexes.
Tables 1 to 4 show the mean squared errors of the P_1, P_2, P_3 and P_4 estimations for the three methods. The main differences between them appear in the smaller samples, of size 100 and 500, in which the non linear least squares give the higher mean squared errors. For these samples the scoring estimators seem to perform better than the minimum χ^2 in the estimation of P_1 and P_2, while the contrary occurs in the estimation of P_3 and P_4.
On the other hand, for the larger samples considered, sizes 1000 and 2000, the three methods give very similar values in their mean squared errors, especially

in the cases of $n = 2000$ and of P_3 and P_4. However, the lower values are given by the minimum χ^2 estimator in almost all cases.

Table 1: Estimation of P_1: Sample mean squared error (10^{-8})

Minimum χ^2

α	n			
	100	500	1000	2000
1	81646.52	15907.36	7481.35	3815.32
2	85740.38	15963.52	8124.90	4217.63
3	85753.92	15169.27	7692.36	4074.30

Non Linear Least Squares

α	n			
	100	500	1000	2000
1	87359.40	15989.45	7649.73	3791.20
2	93150.27	16082.94	8390.50	4308.69
3	95746.20	15743.94	7882.57	4195.82

Scoring Estimator

α	n			
	100	500	1000	2000
1	79533.58	15782.04	7500.02	3798.71
2	83701.36	15840.97	8172.44	4237.04
3	84784.11	15186.12	7716.55	4105.86

Table 2: Estimation of P_2: Sample mean squared error (10^{-8})

Minimum χ^2

α	n			
	100	500	1000	2000
1	79409.14	15952.90	7574.27	3859.14
2	39082.85	7430.13	3859.66	2003.20
3	22379.40	4031.50	2049.10	1093.13

Non Linear Least Squares

α	n			
	100	500	1000	2000
1	90192.11	16232.11	7790.01	3845.72
2	46129.86	7640.91	4025.60	2057.05
3	27471.56	4287.49	2123.16	1131.72

Scoring Estimator

α	n			
	100	500	1000	2000
1	79202.80	15892.77	5871.94	3846.04
2	39524.28	7424.35	3895.80	2015.98
3	23006.87	4070.02	2063.36	1103.59

Table 3: Estimation of P_3: Sample mean squared error (10^{-8})

Minimum χ^2

α	n			
	100	500	1000	2000
1	65109.63	13232.19	6312.43	3213.30
2	20600.73	3912.86	2055.44	1065.85
3	8345.63	1485.70	754.51	403.97

Non Linear Least Squares

α	n			
	100	500	1000	2000
1	76493.84	13557.98	6513.75	3207.37
2	25568.61	4075.06	2156.92	1097.97
3	10858.53	1605.18	787.47	419.67

Scoring Estimator

α	n			
	100	500	1000	2000
1	65809.23	13213.54	6348.12	3204.16
2	21278.50	3926.59	2079.14	1073.82
3	8785.01	1508.10	761.65	408.31

Table 4: Estimation of P_4: Sample mean squared error (10^{-8})

Minimum χ^2

α	n			
	100	500	1000	2000
1	53448.9	10926.01	5229.10	2659.56
2	12157.55	2290.67	1212.99	628.21
3	3783.18	658.44	333.71	179.07

Non Linear Least Squares

α	n			
	100	500	1000	2000
1	64381.74	11253.33	5409.14	2657.90
2	15668.82	2408.90	1278.63	648.70
3	5137.48	720.19	350.28	186.53

Scoring Estimator

α	n			
	100	500	1000	2000
1	54553.73	10929.90	5263.12	2653.09
2	12756.56	2306.29	1228.79	633.42
3	4051.39	671.22	337.53	181.16

5 Summary and conclusions

When the distribution of income is modelled by a Gamma distribution, the Foster, Greer and Thorbecke poverty indexes and their asymptotic standard deviations can be obtained from the parameters α and β and the poverty line z by means of the formulae derived in the text. If the poverty line is a fraction k of the mean (a relative poverty line), then the Foster, Greer and Thorbecke indexes depend only on α and k.

The parameters of the model can be estimated from the complete sample information or from grouped data. For the latter, a Monte Carlo experiment showed that, within the three methods studied for data grouped in deciles, the non linear least squares performed worse than the other two (minimum χ^2 and scoring estimator) for sample sizes smaller than 1000. For the larger samples studied (1000 and 2000) the three methods performed similarly, with a slight predominance of the minimum χ^2 estimator.

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