A bargaining-Walras approach for finite economies

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Abstract. We give a notion of bargaining set for finite economies and show its coincidence with the set of Walrasian allocations. Moreover, we also show that justified objections equate with Walrasian objections. Our bargaining-Walras equivalence provides a discrete approach to the characterization of competitive equilibria obtained by Mas-Colell (1989) for continuum economies.

Some further results highlight whether it is possible to restrict the formation of coalitions and still get the bargaining set. Finally, recasting some known characterizations of Walrasian allocations, we state additional interpretations of the bargaining set.

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1 Introduction

The core of an economy is defined as the set of allocations which cannot be blocked by any coalition. Thus, the veto mechanism that defines the core implicitly assumes that individuals are not forward-looking. However, one may ask whether an objection or veto is credible or, on the contrary, not consistent enough so other agents in the economy may react to it and propose an alternative or counter-objection.

The first outcome of this two-step conception of the veto mechanism was the work by Aumann and Maschler (1964), who introduced the concept of bargaining set, containing the core of a cooperative game. This original concept of bargaining set was later adapted to atomless economies by Mas-Colell (1989). The main idea is to inject a sense of credibility and stability to the veto mechanism, hence permitting the implementation of some allocations which otherwise would be formally blocked, although in a non-credible way. Thus, only objections without counter-objections are considered as credible or justified, and consequently, blocking an allocation becomes more difficult.

In the case of pure exchange economies with a finite number of traders the set of Walrasian allocations is a strict subset of the core which is also strictly contained in the bargaining set. Under conditions of generality similar to those required in Aumann's (1964) core-Walras equivalence theorem, Mas-Colell (1989) showed that the bargaining set and the competitive allocations coincide for continuum economies.

Instead of starting from Aumann's core-Walras equivalence, in this paper we build upon the Edgeworth equilibrium notion and its coincidence with the Walrasian allocations as shown by Debreu-Scarf's (1963) core convergence result. An Edgeworth equilibrium for an economy with a finite number of agents is an attainable allocation whose r-fold repetition belongs to the core of the r-fold replica of the original economy, for any positive integer r; it can also be defined as an attainable allocation which cannot be blocked by a coalition with rational rates of participation. Enlarging the set of coalitions in order to allow a participation of the agents with any rate belonging to the real unit interval, Aubin (1979) considered a limit notion of Edgeworth equilibrium. The core resulting from this blocking system à la Aubin equals the set of Walrasian allocations in economies with a finite set of agents.[1]

The Aubin core-Walras equivalence leads us to consider this limit veto in the spirit of Edgeworth to define objections and counter-objections. Thus, we define a concept of bargaining set for finite economies that involves not only more possible objections but also counter-objections. Note that enlarging the number of coalitions in this way may be a double-edged sword. Having more coalitions implies more possibilities to object but, at the same time, produces more ways of counter-objecting. That is, objecting becomes easier but having a justified objection becomes harder. This highlights the fact that the overall effect of enlarging the number of coalitions is not straightforward.

It could appear that this notion is nothing but Mas-Colell's for the particular case of a n-types continuum economy, but it is not. There are actually conceptual differences between both concepts with important implications regarding the nature of justified objections. Any coalition which makes a justified Mas-Colell objection to an allocation must contain all the agents of the type that becomes strictly better off and therefore, if a coalition with a justified objection includes only part of some type of agents then it is not possible for these agents to strictly improve at the objection. These properties make the notion of a Mas-Colell's justified objection very stringent and his bargaining set may become very large. In contrast, our definition of a justified objection does not require total participation of the agents which are involved, and it also allows a member who participates without the whole of her endowments to strictly improve at the objection. (See the remarks after the Definition 3.1 where we deal in detail with the main distinction between both notions).

Our first result states that the set of Walrasian allocations coincides with this Aubin bargaining set, providing a finite approach to the characterization obtained by Mas-Colell (1989) of competitive allocations. The connection of the bargaining-Walras equivalence we obtain with the related literature can be summarized in the following table:

Atomless economies	core-Walras equivalence. (Aumann, 1964)	Mas-Colell's bargaining set-Walras equivalence. (Mas-Colell, 1989)
Finite economies	Aubin core-Walras equivalence. (Aubin, 1979)	(Aubin) bargaining set -Walras equivalence. This paper: Theorem 3.1.

[1]Florenzano (1990) extends this result to production economies without ordered preferences defined in a Hausdorff linear topological space.

The bargaining-Walras equivalence we show allows us to deduce that the bargaining set we have defined is also consistent in the sense of Dutta *et al.* (1989) as happens with the Mas-Colell bargaining set for atomless economies. Furthermore, we also provide a discrete approach to the characterization of justified objections stated by Mas-Colell (1989) by means of a notion of Walrasian objections which reflects the main differences between Mas-Colell's bargaining set and ours. The fact that any Walrasian objection is justified and vice-versa for finite economies, allows us to refine our bargaining-Walras equivalence and its proof in terms of Walrasian objections.

Our result (and also Mas-Colell's) implicitly requires the formation of all coalitions. In other words, the bargaining set concept requires checking the whole set of possible coalitions in order to test whether any group of agents can improve upon an allocation by using their own resources, both in the objection and counter-objection processes. It is usually argued that the costs arising from forming a coalition are not at all negligible; incompatibilities among different agents may appear and a large amount of information and communication might be needed to really get together a coalition. This idea leads us to study the possibility of restricting the formation of coalitions by assuming that not all the parameters, which specify the degree of participation of agents when they become members of a coalition, are admissible. Then, we analyze the consequences that this condition has with regard to the bargaining set solution. We show that both for objections and counter-objections, the participation rates of the agents can be restricted to those arbitrarily small without changing the bargaining set. However, we show that this does not hold if we consider parameters close enough to complete participation. We also prove that the participation rates in the counter-objection system can be restricted to rational numbers, which is the veto power we get when the economy is enlarged via replicas.

Finally, we try to make the best use of our results by recasting in terms of the bargaining set some characterizations of the Walrasian allocations already present throughout the literature. First, we focus on a result by Hervés-Beloso, Moreno-García and Yannelis (2005) that characterizes Walrasian allocations as those that are not blocked by the coalition formed by all the agents in a collection of perturbed economies. Then, we revisit the approach followed by Hervés-

Beloso and Moreno-García (2009), who showed that Walrasian equilibria can be identified by using a non-cooperative two-player game. Both equivalence theorems constitute now additional characterizations of the bargaining set for finite economies.

The rest of the work is structured as follows. In Section 2 we collect notations and preliminaries. In Section 3, a bargaining-Walras equivalence and a characterization of justified objections via Walrasian objections are provided. Section 4 elaborates on the possibility of restricting the coalitions that are allowed to form and still get the bargaining set. In Section 5, specific equivalence theorems for Walrasian equilibrium are presented as further characterizations of the bargaining sets. In order to facilitate the reading of the paper, the proofs of the results are contained in a final Appendix.

2 Preliminaries

Let $\mathcal{E}$ be an exchange economy with a finite number n of agents, who trade a finite number ℓ of commodities. Each consumer i has a preference relation $\succsim_i$ on the set of consumption bundles $\mathbb{R}^\ell_+$, with the properties of continuity, convexity[2] and strict monotonicity. This implies that preferences are represented by utility functions $U_i, i \in N = \{1, \ldots, n\}$. Let $\omega_i \in \mathbb{R}^\ell_{++}$ denote the endowments of consumer i. So the economy is $\mathcal{E} = (\mathbb{R}^\ell_+, \succsim_i, \omega_i, i \in N)$.

An allocation x is a consumption bundle $x_i \in \mathbb{R}^\ell_+$ for each agent $i \in N$. The allocation x is feasible in the economy $\mathcal{E}$ if $\sum_{i=1}^n x_i \leq \sum_{i=1}^n \omega_i$. A price system is an element of the $(\ell - 1)$-dimensional simplex of $\mathbb{R}^\ell_+$. A Walrasian equilibrium for the economy $\mathcal{E}$ is a pair (p, x), where p is a price system and x is a feasible allocation such that, for every agent i, the bundle x_i maximizes the utility function U_i in the budget set $B_i(p) = \{y \in \mathbb{R}^\ell_+ \text{ such that } p \cdot y \leq p \cdot \omega_i\}$. We denote by $W(\mathcal{E})$ the set of Walrasian allocations for the economy $\mathcal{E}$.

A coalition is a non-empty set of consumers. An allocation y is said to be attainable or feasible for the coalition S if $\sum_{i \in S} y_i \leq \sum_{i \in S} \omega_i$. Let $x \in \mathbb{R}^{\ell n}_+$ be a feasible allocation in the economy $\mathcal{E}$. The coalition S blocks x if there exists

[2]The convexity of preferences we require is the following: If a consumption bundle z is strictly preferred to $\hat{z}$ so is the convex combination $\lambda z + (1 - \lambda)\hat{z}$ for any $\lambda \in (0, 1)$. This convexity property is weaker than strict convexity and it holds, for instance, when the utility functions are concave.

an allocation y which is attainable for S, such that $y_i \succsim_i x_i$ for every $i \in S$ and $y_j \succ_j x_j$ for some member j in S. A feasible allocation is efficient if it is not blocked by the grand coalition, formed by all the agents. The core of the economy $\mathcal{E}$, denoted by $C(\mathcal{E})$, is the set of feasible allocations which are not blocked by any coalition of agents.

It is known that, under the hypotheses above, the economy $\mathcal{E}$ has Walrasian equilibrium and that any Walrasian allocation belongs to the core (in particular, it is efficient). Moreover, the blocking power of coalitions in finite economies is not able to eliminate every non-Walrasian allocation. Therefore, in order to characterize the Walrasian equilibria in terms of the core, we have to enlarge the set of coalitions or, alternatively, increase somehow their veto power. This line of arguments has been carried out in different ways. For instance, Aubin (1979) extended the notion of ordinary veto by allowing members to participate with a portion of their endowments when joining a coalition. We refer to this veto system as *Aubin veto* or *veto in the sense of Aubin*. An allocation x is blocked in the sense of Aubin by the coalition S via the allocation y if there exist coefficients $\alpha_i \in (0, 1]$, for each $i \in S$, such that (i) $\sum_{i \in S} \alpha_i y_i \leq \sum_{i \in S} \alpha_i \omega_i$, and (ii) $y_i \succsim_i x_i$, for every $i \in S$ and $y_j \succ_j x_j$ for some $j \in S$. The Aubin core of the economy $\mathcal{E}$, denoted by $C_A(\mathcal{E})$, is the set of all feasible allocations which cannot be blocked in the sense of Aubin. Under the standard assumptions stated above, Aubin (1979) showed that $C_A(\mathcal{E}) = W(\mathcal{E})$.

As with the core, the Aubin core does not assess the "credibility" of the objections; any attainable allocation which is blocked by a coalition is dismissed. The argument that objections might be met with counter-objections leads to bargaining set notions. Since the original bargaining set notion was introduced by Aumann and Maschler (1964) for cooperative games, several versions have been defined and studied. More specifically, Mas-Colell (1989) defined the bargaining set for atomless economies.[3] The idea of the definition is that this set contains all the feasible allocations of the economy that are not blocked in a credible, justified way. Recently, the original bargaining set was extended by Yang, Liu and Liu (2011) to Aubin bargaining sets for games which they refer to as convex cooperative fuzzy games. Shortly after, Liu and Liu (2012) gave a modification of the previous extension and obtained both existence and equivalence results

[3]Mas-Colell (1989) not only adapted the original concept of bargaining set to atomless economies but also proved, under conditions of generality similar to the Aumann's (1964) core equivalence theorem, that the bargaining set and the set of competitive allocations coincide.

with other cooperative solutions. However, they remarked that finding a proper definition of the Aubin bargaining set is not an easy task.

In the next section, we provide a concept of bargaining set by means of the Aubin veto instead of the usual blocking mechanism. Thus, we extend and adapt the notions of bargaining sets recently provided by Yang, Liu and Liu (2011) and Liu and Liu (2012) for (transferable utility) cooperative games to finite exchange economies. In addition, we will use the fact that, regarding Walrasian equilibria, a finite economy $\mathcal{E}$ with n consumers is equivalent to a continuum economy $\mathcal{E}_c$ with n-types of agents as we specify below.

Consider a continuum economy where the set of agents is represented by the unit real interval $[0,1]$ endowed with the Lebesgue measure μ (as in Aumann, 1964). There are only a finite number of types of consumers. Thus, $I = [0,1] = \bigcup_{i=1}^{m} I_i$, with $\mu(I_i) = n_i/n$ (i.e., $\mu(I_i)$ is a rational number).[4] Every $t \in I_i$ has the same endowments ω_i and preference $\succsim_i$, that is, all the consumers in I_i are of the same type i. Note that we can write $I_i = \bigcup_{j=1}^{n_i} I_{ij}$ with $\mu(I_{ij}) = 1/n$ for every i, j. Consider now a finite economy with n agents and n_i consumers of each type i. Note that a feasible allocation $x = (x_1, \ldots, x_n)$, with $x_i = (x_{ij}, j = 1, \ldots, n_i)$, in the finite economy defines a feasible allocation f_x in the continuum economy which is given by $f_x(t) = x_{ij}$ for every $t \in I_{ij}$. Reciprocally, a feasible allocation f in the continuum economy defines a feasible allocation x^f in the finite economy which is given by $x_{ij}^f = \frac{1}{\mu(I_{ij})} \int_{I_{ij}} f(t) d\mu(t)$. Moreover, x (respectively f) is an equal-treatment allocation if and only if f_x (respectively x^f) also is.

Under continuity and convexity of preferences, if (x, p) is a Walrasian equilibrium in the n-agent economy, then (f_x, p) is a competitive allocation in the n-types continuum economy. Conversely, if (f, p) is a competitive equilibrium in the continuum economy then (x^f, p) is a Walrasian equilibrium in the finite economy. (See García-Cutrín and Hervés-Beloso, 1993).

Consider now the economy $\mathcal{E}$ that we have defined at the beginning of this section. Let $\mathcal{E}_c$ be the associated continuum economy, where the set of agents is $I = [0,1] = \bigcup_{i=1}^{n} I_i$, where $I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right)$ if $i \neq n$; $I_n = \left[\frac{n-1}{n}, 1\right]$; and all the agents in the subinterval I_i are of the same type i. In this particular case, $x = (x_1, \ldots, x_n)$ is a Walrasian allocation in the finite economy $\mathcal{E}$ if and only if the step

[4]Without loss of generality one can take $I_i = [a_i, a_{i+1})$, for every $i \in \{1, \ldots, m-1\}$; with $a_1 = 0$, $a_{i+1} - a_i = n_i/n$ and $I_m = [a_m, 1]$. Equivalently, we can also take $I = [0, n]$ and $I_i = [n_{i-1}, n_{i-1} + n_i)$, for every $i \in \{1, \ldots, m-1\}$; with $n_0 = 0$ and $I_m = [n_{m-1}, n]$.

function f_x (defined by $f_x(t) = x_i$ for every $t \in I_i$) is a competitive allocation in the continuum economy $\mathcal{E}_c$. In short, the initial finite economy $\mathcal{E}$ and the associated continuum economy $\mathcal{E}_c$ are equivalent regarding market equilibrium.

3 A bargaining-Walras equivalence for finite economies

In economies with a continuum of agents that trade a finite number of commodities, the competitive equilibrium is not only characterized by the core (Aumann, 1964), but also by the bargaining set (Mas-Colell, 1989). The Mas-Colell bargaining set is well defined for finite economies and, in this case, it can be larger than the core (see example in Section 6 in Mas-Colell, 1989).

To specify the notion of the Mas-Colell bargaining set for the finite economy $\mathcal{E}$, let x be a feasible allocation that is blocked by a coalition S via the allocation y. Thus, the objection (S, y) to x has a counter-objection if there exists a coalition T and an attainable allocation z for T such that $z_i \succ_i y_i$ for every $i \in T \cap S$ and $z_i \succ_i x_i$ for every $i \in T \setminus S$, where $T \setminus S$ is the set of agents which are in T but not in S.

An objection which cannot be counter-objected is said to be justified. Thus, the Mas-Colell bargaining set of an economy contains all the feasible allocations which, if they are objected (or blocked), could also be counter-objected. Let $B_{MC}(\mathcal{E})$ denote the Mas-Colell bargaining set for the economy $\mathcal{E}$ with n consumers.

3.1 A bargaining set notion for finite economies

In this section we provide a definition of bargaining set for finite economies using Aubin's veto mechanism that will allow us to prove that the set of Walrasian allocations and the bargaining set coincide.

An *Aubin objection* to x in the economy $\mathcal{E}$ is a pair (S, y), where S is a coalition that blocks x via y in the sense of Aubin. Note that the coalition S can be also defined by the parameters which specify the participation of its members.

An *Aubin counter-objection* to the objection (S, y) is a pair (T, z), where T is a coalition and z is an allocation defined on T, for which there exist $\lambda_i \in (0, 1]$

for each $i \in T$, such that:

(i) $\sum_{i \in T} \lambda_i z_i \leq \sum_{i \in T} \lambda_i \omega_i$,

(ii) $z_i \succ_i y_i$ for every $i \in T \cap S$ and

(iii) $z_i \succ_i x_i$ for every $i \in T \setminus S$.

Remark. Consider that the parameters defining the participations rates of each member in a blocking coalition S are rational numbers. Then, there are natural numbers $a_i, i \in S$ and $r \geq \max\{a_i, i \in S\}$, such that $\lambda_i = a_i/r$ for every $i \in S$. That is, we can say that the blocking coalition is formed by a_i agents of type i. Therefore, when the participation rates are rational numbers, the veto mechanism in the sense of Aubin is the standard veto system in sequence of replicated economies.

From now on in this section and in the related proofs, every time we are in a finite economy framework and write block, objection, counter-objection, or any other concept related with a veto system, we refer to those notions in the sense of Aubin unless stated otherwise.

Definition 3.1 *A feasible allocation belongs to the (Aubin) bargaining set of the finite economy if it has no justified objection. A justified objection is an objection that has no counter-objection.*

We denote by $B(\mathcal{E})$ the bargaining set of the economy $\mathcal{E}$ as we have defined above. Note that $W(\mathcal{E}) = C_A(\mathcal{E}) \subseteq B(\mathcal{E})$.

Our notion of bargaining set differs from the one by Mas-Colell. To clarify this point, let us highlight the main differences between the sets $B_{MC}(\mathcal{E})$ and $B(\mathcal{E})$. In our definition agents can join a coalition for objection or counter-objection process, with a part of their initial endowments. That is, regarding the bargaining system, agents can cooperate with different participation levels and the attainable bundles depend on these degrees of involvement. Furthermore, whenever an agent i is assigned the commodity bundle y_i within a coalition involved in an objection, if she also joins a coalition for a counter-objection, then she necessarily needs to be assigned a bundle that improves her upon y_i, independently of the rate of participation of agent i in the coalition.[5] This fact embodies one of the main conceptual differences between the Mas-Colell bargaining set and the bargaining set using the veto mechanism in the sense of Aubin.

To be precise, considering the notion of the Mas-Colell bargaining set, if a coalition with a justified objection includes only part of some type of agents then it is not possible for these agents to strictly improve at the objection.[6] This is not the case with our notion of justified objections. In particular, if we have a justified objection (S, y) to the allocation x in a finite economy $\mathcal{E}$, with rates of participation $\lambda_i, i \in S$, then the pair $(\tilde{S}, \tilde{y})$ given by any coalition $\tilde{S}$ in the associated continuum economy $\mathcal{E}_c$, such that the set of members in $\tilde{S}$ of type i (denoted by $\tilde{S}_i$) has measure λ_i, and $\tilde{y}(t) = y_i$ for every $t \in \tilde{S}_i$, is an objection to the step allocation f_x in $\mathcal{E}_c$, although it is not necessarily a justified objection. Basically, this contrast is due to the somehow leadership condition that a type obtains whenever any agent of such a type takes part in an objection, independently of the degree of participation.

3.2 A bargaining-Walras equivalence result

The bargaining set we consider constitutes indeed an adequate way of "enlarging" the economy, allowing us to characterize Walrasian allocations in finite economies as allocations with no justified objections. To this end, we show a preliminary result that we will use in the proof of our bargaining-Walras equivalence for economies with a finite number of consumers.

Lemma 3.1 *Let x be an allocation in $\mathcal{E}$. If (S, g) is a justified objection (in the sense of Mas-Colell) to f_x in the associated n-types continuum economy $\mathcal{E}_c$, then $(\bar{S}, \bar{g})$ is a justified objection to x in the finite $\mathcal{E}$, where $\bar{S} = \{i \in N \mid \mu(S \bigcap I_i) > 0\}$ and $\bar{g}_i = \frac{1}{\mu(S_i)} \int_{S_i} g(t) d\mu(t)$, for every $i \in \bar{S}$.*

Note that, in particular, we can conclude that if (S, g) is a justified objection (in the sense of Mas-Colell) to f_x in $\mathcal{E}_c$, then so is $(S, \hat{g})$, where $\hat{g}(t) = \bar{g}_i$ for

[5] This remark provides a different way to overcome the weakness (pointed out by Liu and Liu, 2012) of the related fuzzy bargaining set introduced by Yang, Liu and Liu (2011) for (transferable utility) cooperative games.

[6] For more details, see Remark 5 in Mas-Colell (1989). See also the related Lemma 3.4 in Anderson, Trockel and Zhou (1997).

every $t \in S_i = S \cap I_i$ and every $i \in \bar{S}$.[7] We remark that, in the proof of this Lemma, we just use the corresponding notions of justified objections in $\mathcal{E}$ and $\mathcal{E}_c$, respectively, and we do not use the characterization of justified objections that Mas-Colell (1989) showed and which can be applied to the associated n-types continuum economy.

Theorem 3.1 *The bargaining set of the finite economy $\mathcal{E}$ coincides with the set of Walrasian allocations.*

Enlarging the set of coalitions has a double effect. On the one hand, objecting is easier and allows for more justified objections which, in turn, would make the bargaining set smaller. On the other hand, counter-objecting is also easier, which would eliminate more objections, making it more difficult for the equivalence to hold. As we have already pointed out, there is still another effect that comes from the aforementioned fact that if a consumer participates in both an objection and counter-objection, then an improvement is required in the counter-objection with respect the objection for such an agent, independently of the participation rate in the objection. The aggregate effect is therefore not clear, which makes our equivalence result not trivial.

Let us remember that Dutta *et al.* (1989) introduced the concept of consistency regarding the bargaining set, going one step further and trying to assess not only the credibility of the objections but also of the counter-objections involved in the process. They establish a notion of consistent bargaining set meaning that each objection in a "chain" of objections is tested (credible) in precisely the same way as its predecessor. However, the authors recognize that in a context of an exchange economy with a continuum of agents, the equivalence result by Mas-Colell (1989) implies that his bargaining set is consistent. Since we provide an equivalence result , there is also consistency in our bargaining set.

3.3 Justified objections as Walrasian objections

We remark that Theorem 3.1 states that any non Walrasian allocation has a justified objection. We finish this section by characterizing justified objections

[7] We stress that when preferences are not strictly convex we cannot ensure that every justified objection in the n-types continuum economy has the equal-treatment property. However, the Lemma 3.1 ensures that given a justified objection in $\mathcal{E}_c$, there is also an equal-treatment justified objection.

as Walrasian objections. This characterization is a discrete approach to the one stated by Mas-Colell (1989) for continuum economies. The concept of Walrasian objection requires the introduction of a price system p, and is based on a self selection property: members that participate in a coalition in a Walrasian objection against an allocation are those who would rather trade at the price vector p than get the consumption bundle they receive by such an allocation. The following notion of Walrasian objection differs from the one by Mas-Colell (1989) and reflects the differences between $B_{MC}(\mathcal{E})$ and $B(\mathcal{E})$.

Definition 3.2 *Let x be an allocation in the finite economy $\mathcal{E}$. An (Aubin) objection (S, y) to x is said to be Walrasian if there exists a price system p such that (i) $p \cdot v \geq p \cdot \omega_i$ if $v \succsim_i y_i$, $i \in S$ and (ii) $p \cdot v \geq p \cdot \omega_i$ if $v \succsim_i x_i$, $i \notin S$.*

We remark that, under the assumptions of monotonicity and strict positivity of the endowments, we know that $p \gg 0$, and therefore conditions *(i)* and *(ii)* above can be written as follows: $v \succ_i y_i$ implies $p \cdot v > p \cdot \omega_i$, for $i \in S$ and $v \succ_i x_i$ implies $p \cdot v > p \cdot \omega_i$ for $i \notin S$.

Observe that the notion of Walrasian objection in the finite economy $\mathcal{E}$ does not depend explicitly on the rates of participation of the members in the coalition that objects an allocation. To be precise, in order to check whether the objection (S, y) is Walrasian, no importance is attached to the degree of participation of the individuals joining the coalition S that make the allocation y attainable *à la Aubin*; what does become important is the set of consumers who are involved in the objection.

Proposition 3.1 *Let x be a feasible allocation in the finite economy $\mathcal{E}$. Then, any objection to the allocation x is justified if and only if it is a Walrasian objection.*

The fact that any Walrasian objection is a justified objection in finite economies allows us to refine our bargaining-Walras equivalence and its proof in terms of Walrasian objections. To see this, let x be a feasible allocation in $\mathcal{E}$. Note that we can now guarantee that if x is not a Walrasian allocation, then it has a Walrasian objection. Moreover, applying Proposition 3.1, Lemma 3.1 states that if (S, g) is a Walrasian objection (in the sense of Mas-Colell) to f_x in the associated n-types continuum economy $\mathcal{E}_c$, then $(\bar{S}, \bar{g})$ is a Walrasian objection to x in the finite $\mathcal{E}$, where $\bar{S} = \{i \in \{1, \ldots, n\} \mid \mu(S_i) = \mu(S \bigcap I_i) > 0\}$ and $\bar{g}_i = \frac{1}{\mu(S_i)} \int_{S_i} g(t) d\mu(t)$, for every $i \in \bar{S}$.

Let x be a feasible allocation in $\mathcal{E}$ and (S, y) an objection to x, being α_i the participation of each $i \in S$. Denote by $\mathcal{E}_S(\alpha)$ the continuum economy formed only by consumers of types in S and such that the measure of the set of agents of type i is α_i. From Proposition 3.1, we can deduce that when $S = N$, the objection (S, y) is justified if and only if y is a competitive allocation in the restricted continuum economy $\mathcal{E}_N(\alpha)$. However, note that in general an objection given by a coalition S and a competitive allocation of $\mathcal{E}_S(\alpha)$ is not necessarily a justified (or Walrasian) objection. Being a Walrasian objection is much more demanding. We also remark that the fact that (S, y) is a justified objection to x and $y_i \succ_i x_i$ does not imply $\alpha_i = 1$. This is in contrast to Mas-Colell's notion for which if a coalition with a justified objection includes only part of some type of agents then it is not possible for these agents to strictly improve with the objection.

In short, we stress that, since justified and Walrasian objections coincide, one can conclude that such a characterization points out that the concept of Walrasian objection in the finite framework is also more than a technical tool to refine the bargaining-Walras equivalence.

4 Restricting coalition formation

Both Mas Colell's (1989) result and our bargaining-Walras equivalence implicitly require the formation of all coalitions in the objection and counter-objecting mechanism. That is, checking whether a given allocation belongs to the bargaining set seems to require contemplating the whole set of possible coalitions in order to test whether any group of agents, by using their own resources, can improve upon an allocation either in the objection or counter-objection process. This will be a complicated task, even when the economy is small, provided that agents can participate in a coalition with a part of their endowments. Indeed, the Aubin veto system in a finite economy is equivalent to the blocking scheme in the associated continuum economy, with a finite number of types, conducted by equal-treatment allocations.

We also remark that the formation of coalitions may imply some theoretical difficulties. It is usually argued that the costs, which arise from forming a coalition, are not at all negligible. Incompatibilities among different agents may appear and a large amount of information and communication might be needed to really form a coalition. Thus, sometimes, it will not suffice to merely say that several agents constitute a coalition since it may result in high formation costs, commitments and constraints, which make it difficult to assume that the veto mechanism underlying cooperative solutions, like the core or the bargaining set, works freely and spontaneously.

In this section, the difficulty in arguing that coalition formation is costless leads us to consider a restricted veto mechanism in the procedure leading to the bargaining set. Thus, we assume that not all the parameters, which specify the degree of participation of agents when they become members of a coalition, are admissible. Next we will study the consequences that this assumption has with regard to the bargaining set solution.

To this end, we consider that a coalition S is defined by the rates of participation of its members, which is given by a vector $\lambda_S = (\lambda_i, i \in S) \in (0,1]^{|S|}$, where $|S|$ denotes the cardinality of S.

Consider that for each coalition S the participation rates are restricted to a subset $\Lambda_S \subset [0,1]^{|S|}$. Let us denote by $B_\Lambda(\mathcal{E})$ (respectively $B^\Lambda(\mathcal{E})$) the bargaining set where a coalition S can object (respectively counter-object) only with participation rates in Λ_S. When the set of coalitions is restricted in the objection (respectively counter-objection) process, it becomes harder to block an allocation (respectively to counter-object an objection) and thus we have $B^\Lambda(\mathcal{E}) \subseteq B(\mathcal{E}) \subseteq B_\Lambda(\mathcal{E})$. In addition, if $\Lambda, \widehat{\Lambda}$ are such that $\Lambda_S \subseteq \widehat{\Lambda}_S$ for every coalition S, then $B^\Lambda(\mathcal{E}) \subseteq B^{\widehat{\Lambda}}(\mathcal{E})$ but $B_{\widehat{\Lambda}}(\mathcal{E}) \subseteq B_\Lambda(\mathcal{E})$. Therefore, restricting the set of coalitions which are able to object enlarges the bargaining set, whereas restricting the coalition formation in the counter-objection mechanism diminishes the bargaining set. This is so because when not all the coalitions can take part in the bargaining mechanism, on the one hand, blocking is harder but on the other hand, it is easier for an admissible objection to become credible or justified.

In the case of continuum economies, following Schmeidler (1972), we can interpret the measure of a coalition as the amount of (or cost of) information and communication needed in order to form such a coalition. Consequently, it may be meaningful to consider those coalitions whose size converges to zero; that is, the coalitions with small formation cost. We apply this argument to economies with a finite number of agents where the veto system in the sense of Aubin is considered. To this effect, given $\delta \in (0,1]$, let δ-$B(\mathcal{E})$ denote the bargaining set of the economy $\mathcal{E}$ where the participation rate of any agent in any coalition, both in the objection and counter-objection procedure, is restricted to be less or equal than δ.

The next result is related to the remark on the core of atomless economies stated by Schmeilder (1972), showing that in order to obtain the core of a continuum economy, it is enough to consider the blocking power of arbitrarily small coalitions.

Lemma 4.1 *All the δ-bargaining sets are equal and coincide with the bargaining set in the finite economy $\mathcal{E}$. That is, δ-$B(\mathcal{E}) = B(\mathcal{E})$, for every $\delta \in (0,1]$.*

The above result is in contrast to the work by Schjødt and Sloth (1994) who showed that, in continuum economies, when one restricts the coalitions participating in objections and counter-objections to those whose size is arbitrarily small, then the Mas-Colell bargaining set becomes strictly larger than the original one.[8] In other words, in atomless economies and contrary to the core, the formation of only arbitrarily small coalitions in the bargaining process does not allow the characterization of the competitive allocations. This is due to the fact that limiting the size of coalitions in continuum economies prevents obtaining justified objections. This is not the case in economies with a finite number of agents when one restricts the participation rates of members forming a coalition to those arbitrarily small. Thus, the previous lemma marks a further contrast between Mas-Collel's bargaining set for continuum economies and our finite approach.

Symmetrically to Schmeidler's (1972) and Grodal's (1972)[9] core characterizations for atomless economies, Vind (1972) showed that in order to block any non-competitive allocation it is enough to consider the veto power of arbitrarily large coalitions. This result allows us to show that in order to obtain the Aubin

[8]Moreover, Hervés-Estévez and Moreno-García (2015) show that, in order to obtain the Mas-Colell bargaining set in atomless economies, it is not possible to restrict coalitions in the objection process, independently of the kind of restriction we consider. In particular, it is shown the impossibility of restricting to arbitrarily small or large coalitions to obtain a Mas-Colell justified objection.

[9]Grodal extended Schmeidler's result by showing that, given $\delta \in (0,1)$, the blocking coalitions can be restricted to those with measure less than δ that are also union of at most $\ell + 1$ subcoalitions with diameter less than δ.

core the formation of only one coalition is sufficient, namely, the big coalition, which is formed by all the agents in the economy; moreover, for every consumer the endowment participation rate can be chosen to be arbitrarily close to one, i.e., the parameters defining the degree of joining in the big coalition can be restricted to those close to the total participation (see Hervés-Beloso and Moreno-García, 2001 and Hervés-Beloso, Moreno-García and Yannelis, 2005). The next example shows that this restriction on coalition formation cannot be adapted to the bargaining set solution we address.

Example 1. Let $\mathcal{E}$ be an economy with two consumers who trade two commodities, a and b. Both agents have the same preference relation represented by the utility function $U(a,b) = ab$, and both are initially endowed with one unit of each commodity. Let us consider the feasible allocation x which assigns the bundle $x_1 = (2,2)$ to the individual 1 and the bundle $x_2 = (0,0)$ to individual 2. The allocation x does not belong to the bargaining set (it does not belong to the core and it is not a Walrasian allocation). In fact, x is blocked in the sense of Aubin by $S = \{2\}$ with any participation rate $\lambda \in (0,1]$. Moreover, every objection $(\{2\},(1,1))$, with any $\lambda \in (0,1]$, has no counter-objection *à la Aubin* and, therefore, is justified.

Note that there exists y such that the coalition $\{1,2\}$ objects x in the sense of Aubin via $y = (y_1, y_2)$, with strictly positive weights. That is, there exists $(\lambda_1, \lambda_2) \in (0,1]^2$ such that $\lambda_1 y_1 + \lambda_2 y_2 \leq (\lambda_1 + \lambda_2)(1,1)$. In addition, $U(y_1) \geq 4$ and $U(y_2) \geq 0$, with at least one strict inequality. This implies that $U(y_2) < U(\omega_2) = 1$. Therefore, any objection where the participation parameters are restricted to be strictly positive for every consumer is counter-objected by individual 2.

We conclude that in contrast to the Aubin core, we cannot restrict the coalition formation to the grand coalition with parameters close enough to the total participation. Next we state a similar example showing that we cannot state such a restriction in the counter-objecting mechanism either.

Example 2. Let $\mathcal{E}$ be an economy with three consumers who trade two commodities, a and b. All the agents have the same preference relation represented by the utility function $U(a,b) = ab$, and are initially endowed with one unit of each commodity. Let us consider the feasible allocation x which assigns the bundle $x_1 = (3,3)$ to individual 1 and the bundle $x_2 = x_3 = (0,0)$ to individuals 2 and 3. The allocation x is blocked in the sense of Aubin by $S = \{2\}$ with any participation rate $\lambda \in (0,1]$. Note also that $(\{3\},(1,1))$ is a counter-objection to the objection $(\{2\},(1,1))$. However, there is no counter-objection to $(\{2\},(1,1))$ if all the participation rates are required to be, for instance, larger than 1/2.[10] To see this, assume that $\{1,2,3\}$ counter-objects, with weights $\lambda_i, i = 1,2,3$. Given the preference relations, we can conclude that $3\lambda_1 + \lambda_2 < \lambda_1 + \lambda_2 + \lambda_3$. We obtain a contradiction with the fact that $\lambda_1, \lambda_3 \in (1/2, 1]$.

To finish this section, we consider a quite different restriction for the participation rates of the agents in coalitions. As the following lemma states, it turns out that the bargaining set is entirely characterized when the participation rates of agents in coalitions involved in counter-objections are rational numbers.

Lemma 4.2 *Let $B^Q(\mathcal{E})$ denote the bargaining set of the economy $\mathcal{E}$ where only rational numbers are allowed as participation rates in the counter-objection process. Then, $B^Q(\mathcal{E}) = B(\mathcal{E})$.*

The restriction in the previous lemma is equivalent to the veto mechanism in the sequence of replicated economies with equal treatment allocations. Then, we conclude that an Aubin objection (S, y) to x is justified if and only if the allocation (feasible or not) which assigns y_i to agents of type $i \in S$ and x_i to agents of type $i \in N \setminus S$ is not objected in any replicated economy.

We remark that, taking into account the observations on restricting coalition formation in the previous section, Lemma 4.2 can be obtained as an immediate consequence of our bargaining-Walras equivalence. However, in the Appendix we provide a proof which does not use the equality $W(\mathcal{E}) = B(\mathcal{E})$.

5 Final additional characterizations

Given our equivalence results, any characterization of Walrasian equilibrium for finite economies turns immediately into an additional characterization of the bargaining set. In this section we pick up two different ways of identifying Walrasian allocations and recast them in terms of bargaining sets as corollaries.

First, let us consider a feasible allocation $x = (x_1, \ldots, x_n)$ in the economy $\mathcal{E}$. Following Hervés-Beloso, Moreno-García and Yannelis (2005a, 2005b), we

[10] The same remains true if the parameters are required to be larger than any number in $(1/2, 1)$.

define a family of economies denoted by $\mathcal{E}(a,x)$, $a = (a_1, \ldots, a_n) \in [0,1]^n$, which coincide with $\mathcal{E}$ except for the endowments that, for each agent $i \in N$, are defined by $\omega_i(a,x) = a_i x_i + (1-a_i)\omega_i$. An allocation (feasible or not) is said to be dominated in the economy $\mathcal{E}$ if it is blocked by the grand coalition N.

In the aforementioned works it was proved that, under the assumptions we have considered, an allocation x is Walrasian in the economy $\mathcal{E}$ if and only if it is not dominated in any perturbed economy $\mathcal{E}(a,x)$. This characterization allows us to write the next corollary as an immediate consequence of the bargaining-Walras equivalence we have obtained in Theorem 3.1.

Corollary 5.1 *An allocation x belongs to the bargaining set of $\mathcal{E}$ (equivalently, to the leader bargaining set of every replicated economy $r\mathcal{E}$) if and only if it is not dominated in any economy $\mathcal{E}(a,x)$.*

An alternative way of stating the above result is: *The allocation x has a justified objection (equivalently, a Walrasian objection) in the economy $\mathcal{E}$ if and only if x is blocked by the grand coalition in some economy $\mathcal{E}(a,x)$.*

The essence of the second characterization of Walrasian equilibrium that we recast for bargaining sets differs substantially from the previous ones. It follows a non-cooperative game theoretical approach and provides insights into the mechanism through which the bargaining process is conducted.

Given the finite economy $\mathcal{E} = (\mathbb{R}^\ell_+, \succsim_i, \omega_i, i \in N)$, let us define an associated game $\mathcal{G}$ as follows. There are two players. The strategy sets for the players are given by:

$$S_1 = \{\ x = (x_1, ..., x_n) \in \mathbb{R}^{\ell n}_+ \ \text{ such that } \ x_i \neq 0 \text{ and } \textstyle\sum_{i=1}^n x_i \leq \sum_{i=1}^n \omega_i\}.$$

$$S_2 = \{(a,y) \in [\alpha,1]^n \times \mathbb{R}^{\ell n}_+ \ \text{ such that } \ \textstyle\sum_{i=1}^n a_i y_i \leq \sum_{i=1}^n a_i \omega_i\},$$

where α is a real number such that $0 < \alpha < 1$.

Given a strategy profile $s = (x,(a,y)) \in S_1 \times S_2$, the payoff functions Π_1 and Π_2, for player 1 and 2, respectively, are defined as $\Pi_1(x,(a,y)) = \min_i\{U_i(x_i) - U_i(y_i)\}$ and $\Pi_2(x,(a,y)) = \min_i\{a_i\ (U_i(y_i) - U_i(x_i))\}$.

Note that if $\Pi_2(x,(a,y)) > 0$, then the allocation x is blocked via y by the big coalition being a_i the participation rate of each consumer i. Actually, player 2 gets a positive payoff if and only if the big coalition objects in the sense of Aubin the allocation proposed by player 1.

As an immediate consequence of our bargaining-Walras equivalence and Theorem 4.1 in Hervés-Beloso and Moreno-García (2009), we obtain the following corollary.

Corollary 5.2 *x belongs to the bargaining set of the economy $\mathcal{E}$, if and only if $(x, (\mathbf{b}, x))$ with $\mathbf{b}_i = b$, for every $i = 1, \ldots, n$, (for instance $(x, (\mathbf{1}, x))$) is a Nash equilibrium for the game $\mathcal{G}$.*

To finish, we remark that the spirit of the bargaining set notions we have considered seems to indicate that additional and finer characterizations for such cooperative concepts could be obtained through non-cooperative solutions of different games, in which a player represents the objection system whereas another one is in charge of the counter-objection mechanism. This is part of our further research.

Appendix

Proof of Lemma 3.1. Let us assume that f_x is objected by (S, g) meaning that: $\int_S g(t)d\mu(t) \leq \int_S \omega(t)d\mu(t)$, $g \succsim_t f_x$ for every $t \in S$ and $\mu\left(\{t \in S | g \succ_t f_x\}\right) > 0$. Let $S_i = S \cap I_i$ and $\bar{S} = \{i \in N | \mu(S_i) > 0\}$. Since S blocks f_x via g, we have that there exists a type $k \in N$ and a set $A \subset S_k = S \cap I_k$, with $\mu(A) > 0$, such that $g(t) \succ_k f_x$, for every $t \in A$.

Let $\bar{g}$ be the allocation given by $\bar{g}_i = \frac{1}{\mu(S_i)} \int_{S_i} g(t)d\mu(t)$, for every $i \in \bar{S}$. Then, by convexity of the preferences, we have $\bar{g}_i \succsim_i x_i = f_x(t)$ for every $t \in S_i = S \cap I_i$ and $i \in \bar{S}$; and $\bar{g}_k \succ_k x_k = f_x(t)$ for every $t \in S_k$.[11] Thus, $(\bar{S}, \bar{g})$ is an objection *à la Aubin* to the allocation x in the economy $\mathcal{E}$, since we have that: *(i)* $\sum_{i \in \bar{S}} \mu(S_i)\bar{g}_i \leq \sum_{i \in \bar{S}} \mu(S_i)\omega_i$, *(ii)* $\bar{g}_i \succsim_i x_i$ for every $i \in \bar{S}$ and *(iii)* there exists $k \in \bar{S}$ such that $\bar{g}_k \succ_k x_k$.

Assume that the objection $(\bar{S}, \bar{g})$ has a counter-objection $(\bar{T}, z)$, that is, there exists $\{\lambda_i\}_{i \in \bar{T}}$ with $\lambda_i \in (0, 1]$ for every $i \in \bar{T}$, such that: *(i)* $\sum_{i \in \bar{T}} \lambda_i z_i \leq \sum_{i \in \bar{T}} \lambda_i \omega_i$, *(ii)* $z_i \succ_i \bar{g}_i$ for every $i \in \bar{T} \cap \bar{S}$ and *(iii)* $z_i \succ_i x_i$ for every $i \in \bar{T} \setminus \bar{S}$.

[11] See the Lemma in García-Cutrín and Hervés-Beloso (1993) for further details.

If $\bar{T} \cap \bar{S} = \emptyset$ then, in the associated continuum economy $\mathcal{E}_c$, any coalition $T = \bigcup_{i \in \bar{T}} T_i \subset I$ with $\mu(T_i) = \lambda_i$, counter-objects the objection (S, g) via the allocation f_z given by $f_z(t) = z_i$ for every $t \in T_i$. Otherwise (i.e., $\bar{T} \cap \bar{S} \neq \emptyset$), from the previous condition *(ii)* we can deduce that for every $i \in \bar{T} \cap \bar{S}$, there exists $A_i \subset S_i$ with $\mu(A_i) > 0$, such that $z_i \succ_i g(t)$ for every $t \in A_i$. This is again a consequence of the convexity property of preferences. Let $a = \min\{\mu(A_i), i \in \bar{T} \cap \bar{S}\}$ and take M large enough such that $\alpha_i = \frac{\lambda_i}{M} \leq a$ for every $i \in \bar{T}$.

Consider a coalition $T \subset I$ in the continuum economy $\mathcal{E}_c$ with $T = \cup_{i \in \bar{T}} T_i$, such that $T_i \subset A_i$, if $i \in \bar{T} \cap \bar{S}$; $T_i \subset I_i$, if $i \in \bar{T} \setminus \bar{S}$ and $\mu(T_i) = \alpha_i$, for every $i \in \bar{T}$. Then, defining the step function h as $h(t) = z_i$ if $t \in T_i$, we have that: *(i)* $\int_T h(t) d\mu(t) = \sum_{i \in \bar{T}} \alpha_i z_i \leq \sum_{i \in \bar{T}} \alpha_i \omega_i = \int_T \omega(t) d\mu(t)$, *(ii)* $h(t) \succ_i g(t)$ for every $t \in T_i$ with $i \in \bar{T} \cap \bar{S}$; and *(iii)* $h(t) \succ_i x_i = f_x(t)$ for every $t \in T_i$ with $i \in \bar{T} \setminus \bar{S}$.

Note that *(ii)* and *(ii)* mean $h(t) \succ_t g(t)$ for every $t \in T \cap S$ and $h(t) \succ_t f_x(t)$ for every $t \in T \setminus S$, respectively. In other words, we have constructed a counter-objection (T, h) for the objection (S, g), which concludes the proof.

Q.E.D.

Proof of Theorem 3.1. Since the Aubin core coincides with the set of Walrasian allocations for the economy $\mathcal{E}$ (see Aubin, 1979), we have that any Walrasian allocation has no objection in the sense of Aubin and therefore belongs to the bargaining set of $\mathcal{E}$.

Let us show that $B(\mathcal{E}) \subseteq W(\mathcal{E})$. Consider an allocation $x \in B(\mathcal{E})$ and the step function[12] f_x which is a feasible allocation in the associated n-types continuum economy $\mathcal{E}_c$. It suffices to show that f_x belongs to the Mas-Colell bargaining set of $\mathcal{E}_c$.[13] Let us assume that f_x is blocked by the coalition S via the allocation g in $\mathcal{E}_c$ and that (S, g) is a justified objection to f_x in the sense of Mas-Colell. By Lemma 3.1 we can ensure that $(\bar{S}, \bar{g})$ is a justified objection to x in $\mathcal{E}$, where $\bar{g}_i = \frac{1}{\mu(S_i)} \int_{S_i} g(t) d\mu(t)$, for every $i \in \bar{S} = \{i \in N \mid \mu(S \cap I_i) > 0\}$. This is in contradiction to the fact that $x \in B(\mathcal{E})$ and concludes the proof.

Q.E.D.

Proof of Proposition 3.1. Let (S, y) be an objection *à la Aubin* to x. Assume

[12]For every $t \in [0, 1]$, $f_x(t) = x_i$ if $t \in I_i$

[13]This is so because the Mas-Colell bargaining set of $\mathcal{E}_c$ equals the set of competitive allocations (Mas-Colell, 1989), which is also equivalent to the core (Aumann, 1964), and f_x is competitive in $\mathcal{E}_c$ if and only if x is Walrasian in $\mathcal{E}$.

(T, z) is a counter-objection in the sense of Aubin to (S, y). Then, there exist coefficients $\lambda_i \in (0,1]$ for each $i \in T$, such that: $\sum_{i\in T} \lambda_i z_i \le \sum_{i\in T} \lambda_i \omega_i$; $z_i \succ_i y_i$ for every $i \in T \cap S$ and $z_i \succ_i x_i$ for every $i \in T \setminus S$. Since (S, y) is a Walrasian objection at prices p we have that $p \cdot z_i > p \cdot \omega_i$, for every $i \in T \cap S$ and $p \cdot z_i > p \cdot \omega_i$, for every $i \in T \setminus S$. This implies $p \cdot \sum_{i\in T} \lambda_i z_i > p \cdot \sum_{i\in T} \lambda_i \omega_i$, which contradicts that z is attainable by T with weights $\lambda_i, i \in T$. Thus, we conclude that (S, y) is a justified objection.

To show the converse, let (S, y) be a justified objection to x and let $a = (a_1, \ldots, a_n)$ be an allocation (not necessarily feasible) such that $a_i = y_i$ if $i \in S$ and $a_i = x_i$ if $i \notin S$. For every consumer i define $\Gamma_i = \{z \in \mathbb{R}^\ell | z + \omega_i \succsim_i a_i\} \bigcup \{0\}$ and let Γ be the convex hull of the union of the sets $\Gamma_i, i \in N$.

Let us show that $\Gamma \bigcap (-\mathbb{R}^\ell_{++})$ is empty. Assume that $\delta \in \Gamma \bigcap (-\mathbb{R}^\ell_{++})$. Then, there is $\lambda = (\lambda_i, i \in N) \in [0,1]^n$, with $\sum_{i=1}^n \lambda_i = 1$, such that $\delta = \sum_{i=1}^n \lambda_i z_i \in \Gamma$. This implies that the coalition $T = \{j \in N \mid \lambda_j > 0\}$ counter-objects (S, y) via the allocation $\hat{z}$ where $\hat{z}_i = z_i + \omega_i - \delta$ for each $i \in T$. Indeed, $\sum_{j\in T} \lambda_j \hat{z}_j = \sum_{j\in T} \lambda_j \omega_j$. Moreover, since $z_i \in \Gamma_i$ for every $i \in T$ and $\delta \ll 0$, by monotonicity of preferences, $\hat{z}_i \succ_i y_i$ for every $i \in T \cap S$ and $\hat{z}_i \succ_i x_i$ for every $i \in T \setminus S$. This is a contradiction.

Thus, $\Gamma \bigcap (-\mathbb{R}^\ell_{++}) = \emptyset$, which implies that 0 is a frontier point of Γ. Therefore, there exists a hyperplane that supports Γ at 0. That is, there exists a price system p such that $p \cdot z \ge 0$ for every $z \in \Gamma$. This means that $p \cdot v \ge p \cdot \omega_i$, if $v \succsim_i a_i$. Therefore, we conclude that (S, y) is a Walrasian objection.

Q.E.D.

Proof of Lemma 4.1. Let an allocation y be attainable for a coalition S with participation rates $\lambda_i, i \in S$. That is, $\sum_{i\in S} \lambda_i y_i \le \sum_{i\in S} \lambda_i \omega_i$. It suffices to note that there exists $(\alpha_i, i \in S)$, with $\alpha_i \le \delta$ for every $i \in S$ such that $\sum_{i\in S} \alpha_i y_i \le \sum_{i\in S} \alpha_i \omega_i$. To see this, let M be large enough so that $\alpha_i = \lambda_i / M \le \delta$, for every $i \in S$. Thus, the same allocation y is also attainable for the same coalition S with participation rates arbitrarily small. The same reasoning holds for the case of both objections and counter-objections.

Q.E.D.

Proof of Lemma 4.2. Let x be a feasible allocation and (S, y) an objection to x. Let (T, z) be a counter-objection to (S, y). This means that there exist coefficients α_i, $i \in T$, such that (i) $\sum_{i\in T} \alpha_i z_i = \sum_{i\in T} \alpha_i \omega_i$ and (ii) $z_i \succ_i y_i$ for every $i \in T \cap S$, and $z_i \succ_i x_i$ for every $i \in T \setminus S$.

For every natural $k \in \mathbb{N}$, we define a_i^k, $i \in T$, as the smallest integer greater than or equal to $k\alpha_i$. Let us denote $z_i^k = \dfrac{k\alpha_i}{a_i^k}(z_i - \omega_i) + \omega_i$. Since $\lim_{k\to\infty} z_i^k = z_i$ for every $i \in T$, by continuity of preferences, we have that $z_i^k \succ_i y_i$ for every $i \in T \cap S$ and $z_i^k \succ_i x_i$ for every $i \in T \setminus S$, for all k large enough.

By construction, we have $\sum_{i\in T} a_i^k(z_i^k - \omega_i) = 0$. Denoting $q_i^k = \frac{a_i^k}{\sum_{i\in T} a_i^k}$ we obtain (i) $\sum_{i\in T} q_i^k z_i^k = \sum_{i\in T} q_i^k \omega_i$ and (ii) $z_i^k \succ_i y_i$ for every $i \in T \cap S$, and $z_i^k \succ_i x_i$ for every $i \in T \setminus S$, for all k large enough.

Q.E.D.

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