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Abstract. We introduce a new notion of bargaining set for finite economies and show a convergence result.

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1 Introduction

The core of an economy is defined as the set of allocations which cannot be blocked by any coalition. Thus, the veto mechanism that defines the core implicitly assumes that individuals are not forward-looking. However, one may ask whether an objection or veto is credible or, on the contrary, not consistent enough so other agents in the economy may react to it and propose an alternative or counter-objection.

The first outcome of this two-step conception of the veto mechanism was the work by Aumann and Maschler (1964), who introduced the concept of bargaining set, containing the core of a cooperative game.[1] This original concept of bargaining set was later adapted to atomless economies by Mas-Colell (1989). The main idea is to inject a sense of credibility and stability to the veto mechanism, hence permitting the implementation of some allocations which otherwise would be formally blocked, although in a non-credible way. Thus, only objections without counter-objections are considered as credible or justified, and consequently, blocking an allocation becomes more difficult.

In the case of pure exchange economies with a finite number of traders, the set of Walrasian allocations is a strict subset of the core which is also strictly contained in the bargaining set. Under conditions of generality similar to those required in Aumann's (1964) core-Walras equivalence theorem, Mas-Colell (1989) showed that the bargaining set and the competitive allocations coincide for continuum economies. These equivalence results provide foundations for the Walrasian market equilibrium and, at the same time, bring up the question of whether there are analogies in economies with a large, but finite number of agents. A classical contribution in this direction is the one by Debreu and Scarf (1963), who stated a first formalization of Edgeworth's (1881) conjecture, showing that the core and the set of Walrasian allocations become arbitrarily close whenever a finite economy is replicated a sufficiently large number of times. However, in contrast to the Debreu-Scarf core convergence theorem, the work by Anderson, Trockel and Zhou (1997), ATZ from now on, proved that the Mas-Colell bargaining set does not shrink to the set of Walrasian allocations in a sequence of replicated economies as the core does.

The non-convergence ATZ's example highlights a weakness of the adaptation of Mas-Colell's bargaining set notion for finite economies; roughly speaking, this is basically due to the fact that the concept of a justified objection is very stringent. To be precise, if a coalition with a justified objection includes only part of some type of agents then it is not possible for these agents to strictly improve at the objection. Exploiting this demanding property, that comes from the ATZ reading of Mas-Collel's (1989) bargaining set for

[1] Maschler (1976) discussed the advantages that the bargaining set has over the core.

finite economies, we revisit the aforementioned example by ATZ and state an alternative and simple non-convergence proof.

In this paper, we introduce a concept, which we call justified* objection, that avoids the above stringent requirement for an objection to be credible. In this way, we provide a reformulation of the bargaining set that allows us to obtain a convergence result under the assumption that the Walrasian correspondence is continuous. Moreover, we state an example that establishes the impossibility of dropping the continuity hypothesis.

The necessity of the continuity of the equilibrium mapping imposes a limitation to our convergence result. However, we find conditions on the primitives of the original finite economy that ensure the required continuity property holds and therefore guarantee that our convergence theorem remains true. These conditions leads to uniqueness of equilibrium for the set of economies where the continuity is assumed. We remark that this is the case of the ATZ's example for which our bargaining set shrinks to the unique Walrasian allocation.

The rest of the work is structured as follows. In Section 2 we collect notations and preliminaries. In Section 3, we introduce a notion of bargaining set and point out the main differences with Mas-Colell's definition. In Section 4, we analyze convergence properties of our bargaining set. In order to facilitate the reading of the paper, the proofs of the results are contained in a final Appendix.

2 Preliminaries and notations

Let $\mathcal{E}$ be an exchange economy with a finite set of agents $N = \{1, \dots, n\}$, who trade a finite number ℓ of commodities. Each consumer i has a preference relation $\succsim_i$ on the consumption set $\mathbb{R}^\ell_+$, with the properties of continuity, convexity[2] and strict monotonicity. Then, preferences are represented by utility functions $U_i, i \in N$. Let $\omega_i \in \mathbb{R}^\ell_{++}$ be the endowments of consumer i. So the economy is $\mathcal{E} = (\mathbb{R}^\ell_+, \succsim_i, \omega_i, i \in N)$.

An allocation x is a consumption bundle $x_i \in \mathbb{R}^\ell_+$ for each agent $i \in N$. The allocation x is feasible in the economy $\mathcal{E}$ if $\sum_{i=1}^n x_i \leq \sum_{i=1}^n \omega_i$. A price system is an element of the $(\ell-1)$-dimensional simplex of $\mathbb{R}^\ell_+$. A Walrasian equilibrium is a pair (p, x), where p is a price system and x is a feasible allocation such that, for every agent i, the bundle x_i maximizes U_i in the budget set $B_i(p) = \{y \in \mathbb{R}^\ell_+ \text{ such that } p \cdot y \leq p \cdot \omega_i\}$. We denote by

[2] The convexity of preferences we require is the following: If a consumption bundle z is strictly preferred to $\hat{z}$ so is the convex combination $\lambda z + (1-\lambda)\hat{z}$ for any $\lambda \in (0,1)$. This convexity property is weaker than strict convexity and it holds, for instance, when the utility functions are concave.

$W(\mathcal{E})$ the set of Walrasian allocations for the economy $\mathcal{E}$.

A coalition is a non-empty set of consumers. An allocation y is said to be attainable or feasible for the coalition S if $\sum_{i\in S} y_i \leq \sum_{i\in S}\omega_i$. Let $x \in \mathbb{R}_+^{\ell n}$ be a feasible allocation in $\mathcal{E}$. The coalition S blocks x if there exists an allocation y which is attainable for S, such that $y_i \succsim_i x_i$ for every $i \in S$ and $y_j \succ_j x_j$ for some $j \in S$. When S blocks x via y we say that (S, y) is an objection to x. A feasible allocation is efficient if it is not blocked by the grand coalition, formed by all the agents. The core of the economy $\mathcal{E}$, denoted by $C(\mathcal{E})$, is the set of feasible allocations which are not blocked or objected by any coalition of agents. It is known that, under the hypotheses above, the economy $\mathcal{E}$ has Walrasian equilibrium and that any Walrasian allocation belongs to the core (in particular, it is efficient).

Along this paper, we will refer to sequences of replicated economies. For each positive integer r, the r-fold replica economy $r\mathcal{E}$ of $\mathcal{E}$ is a new economy with rn agents indexed by ij, $j = 1,\ldots,r$, such that each consumer ij has a preference relation $\succsim_{ij}=\succsim_i$ and endowments $\omega_{ij} = \omega_i$. That is, $r\mathcal{E}$ is a pure exchange economy with r agents of type i for every $i \in N$. Given a feasible allocation x in $\mathcal{E}$ let rx denote the corresponding equal treatment allocation in $r\mathcal{E}$, which is given by $rx_{ij} = x_i$ for every $j \in \{1,\ldots,r\}$ and $i \in N$.

In addition, we will use the fact that, regarding Walrasian equilibria, a finite economy $\mathcal{E}$ with n consumers is equivalent to a continuum economy $\mathcal{E}_c$ with n-types of agents as we specify below. Given the finite economy $\mathcal{E}$, let $\mathcal{E}_c$ be the associated continuum economy, where the set of agents is $I = [0,1] = \bigcup_{i=1}^n I_i,$ with $I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right)$ if $i \neq n$; $I_n = \left[\frac{n-1}{n}, 1\right]$; and all the agents in the subinterval I_i are of the same type i. In this case, $x = (x_1,\ldots,x_n)$ is a Walrasian allocation in $\mathcal{E}$ if and only if the step function f_x (defined by $f_x(t) = x_i$ for every $t \in I_i$) is a competitive allocation in $\mathcal{E}_c$.

3 Bargaining sets for finite economies

The core does not assess the "credibility" of the objections; any attainable allocation which is blocked by a coalition is dismissed. The argument that objections might be met with counter-objections leads to bargaining set notions that depend on the way justified or credible objections are defined.

Since the original bargaining set notion was introduced by Aumann and Maschler (1964) for cooperative games, several versions have been defined and studied. More specifically, Mas-Colell's (1989) bargaining set may be defined for finite economies and, in this case, it can be larger than the core (see example in Section 6 in Mas-Colell, 1989).

Next, we present both Mas Colell's definition of bargaining set for a finite economy $\mathcal{E}$ and ours (respectively named $B_{MC}(\mathcal{E})$ and $B(\mathcal{E})$ hereafter), highlighting the main differences between both concepts. For it, given $A, B \subseteq N$, we denote by $A \setminus B$ the set of agents that are in A and not in B.

3.1 Mas-Colell's bargaining set

An objection (S, y) to the allocation x has a counter-objection in the economy $\mathcal{E}$ if there exists a coalition T and an attainable allocation z for T such that
(i) $z_i \succ_i y_i$ for every $i \in T \cap S$ and
(ii $z_i \succ_i x_i$ for every $i \in T \setminus S$.

An objection which cannot be counter-objected is said to be justified.

$B_{MC}(\mathcal{E})$ is the set of all the feasible allocations in the economy $\mathcal{E}$ which, if they are objected (or blocked), could also be counter-objected.

3.2 Our bargaining set

An objection (S, y) to the allocation x in the initial economy $\mathcal{E}$ is counter-objected in the replicated economy $r\mathcal{E}$ if there exist a set of types $\mathcal{T} \subset N$, an equal treatment allocation $(z_i, i \in \mathcal{T})$ and natural numbers $n_i \leq r, i \in \mathcal{T}$, such that
(i) $\sum_{i \in \mathcal{T}} n_i z_i \leq \sum_{i \in \mathcal{T}} n_i \omega_i$ and
(ii) $z_i \succ_i y_i$ for every $i \in \mathcal{T} \cap S$ and $z_i \succ_i x_i$ for every $i \in \mathcal{T} \setminus S$.

We say that an objection is justified* if it is not counter-objected in any replicated economy. *A feasible allocation belongs to $B(\mathcal{E})$ if it has no justifed* objection.*

Consider that a coalition S blocks the allocation x in the economy $\mathcal{E}$ via y. Note that if y is not in the core of the economy restricted to S then there is a subcoalition $\hat{S} \subset S$ that is able to attain an allocation z such that $z_i \succ_i y_i$ for every $i \in \hat{S}$. Then, without loss of generality, we can consider only objections that assign the same utility level to members of the same type. This point allows us to consider our definition of bargaining set in the sequence of replicated economies. To simplify, in the sequence of replicated economies we restrict the objecting mechanism to equal-treatment allocations.[3] That is, an objection to rx in the replicated economy $r\mathcal{E}$ is justied* if it has the equal-treatment

[3]We remark that restricting the objection process to equal treatment allocations makes more difficult to have justified objections and then the convergence of the bargaining sets we define implies the convergence when objections are not required to be equal treatment allocations.

property and it is not counter-objected in any replicated economy. Also by convexity, we consider without loss of generality equal-treatment allocations for counter-objecting.

Next we characterize justified* objections as Walrasian objections.[4] To be precise, an objection (S, y) to the allocation x in the economy $\mathcal{E}$ is said to be Walrasian if there exists a price system p such that (i) $p \cdot v \geq p \cdot \omega_i$ if $v \succsim_i y_i$, $i \in S$ and (ii) $p \cdot v \geq p \cdot \omega_i$ if $v \succsim_i x_i$, $i \notin S$.

Proposition 3.1 *Let x be a feasible allocation in the finite economy $\mathcal{E}$. Then, any objection to x is justified* if and only if it is a Walrasian objection.*

We stress that the proof of this result also shows that an equal treatment objection in a replicated economy is justified if and only if it is a Walrasian objection. Observe that the notion of Walrasian objection in a replicated economy does not depend explicitly on the number of members of each type that form the coalition that objects an allocation; what does become important is the set of types which are involved in the objection.

From the above characterization we can also deduce that when the objection $(\mathcal{S}, y)$ involves all the types then it is justified* if and only if y is a competitive allocation in the economy restricted to $\mathcal{S}$. However, note that in general being a Walrasian objection is much more demanding.

3.3 $B_{MC}(\mathcal{E})$ vs. $B(\mathcal{E})$: a comparison

Note that the only relevant difference is the way a justified objection is defined, and this fact has the following consequences when adapting the definition by Mas-Colell to a sequence of replicated economies.

In our definition, we consider equal-treatment allocations and whenever an agent of type i is assigned the commodity bundle y_i within a coalition involved in an objection, any individual of the same type i that joins a coalition for a counter-objection necessarily needs to be assigned a bundle that improves her upon y_i, independently of the number of members of type i in the coalition.

Moreover, if rx has a justified* objection in $r\mathcal{E}$, then the same objection is also justified* in $\hat{r}\mathcal{E}$ for any $\hat{r} \geq r$. Thus, as it happens with the core, our bargaining set

[4]The concept of Walrasian objection requires the introduction of a price system p, and is based on a self selection property: members that participate in a coalition in a Walrasian objection against an allocation are those who would trade at the price vector p rather than get the consumption bundle they receive by such an allocation.

shrinks under replication, i.e., $B((r+1)\mathcal{E}) \subseteq B(r\mathcal{E})$ for any natural number r. This is not the case for the Mas-Colell's bargaining set.[5]

In addition, from the characterization of justified* objections as Walrasian objections we can deduce that the fact that $(\mathcal{S}, y)$ is a justified* objection to rx in $r\mathcal{E}$ and $y_i \succ_i x_i$ does not imply that the all the agents of type i are members of $\mathcal{S}$. This is in contrast to Mas-Colell's notion for which if a coalition with a justified objection includes only part of some type of agents then it is not possible for these agents to strictly improve at the objection.

3.4 ATZ's non-convergence example revisited

In this section, we analyze the aforementioned example by ATZ highlighting the main differences between $B_{MC}(\mathcal{E})$ and $B(\mathcal{E})$ that we have pointed out previously and the implications regarding convergence properties. The analysis provides a different way to prove the non-convergence of the Mas-Colell bargaining set when we replicate the economy. On the other hand, we stress that in this example our bargaining set converges to the set of Walrasian allocations.

Consider an economy with two consumers and two commodities. The endowments are $\omega_1 = (3,1)$ and $\omega_2 = (1,3)$. Both consumers have the same utility function $U(a,b) = \sqrt{ab}$. Given $\alpha \in [0,4]$, let $h(\alpha)$ be the allocation that gives (α,α) to agents of type 1 and $(4-\alpha, 4-\alpha)$ to agents of type 2. Then, $\mathcal{H} = \left\{h(\alpha),\ \alpha \in \left[\sqrt{3}, 4-\sqrt{3}\right]\right\}$ is the set of individually rational, Pareto optimal and equal-treatment allocations. ATZ showed that the measure of the set of allocations in $\mathcal{H}$ which are not in the Mas-Colell and Zhou bargaining sets tends to zero as the economy is replicated. Therefore, they provide a non-convergence example for the Mas-Colell bargaining set.

In which follows we state an alternative non-convergence proof. For it, let a be numeraire and let p denote de price of b. For each $\tau = r_1/r_2 \in \mathbb{R}_+$, let $\mathcal{E}_{|\tau}$ be the economy restricted to r_1 agents of type 1 and r_2 of type 2. Some calculations show that the Walrasian equilibrium for $\mathcal{E}_{|\tau}$ is given by the price $p(\tau) = \frac{3\tau+1}{\tau+3}$, and the allocation which assigns $x_1(\tau) = \left(\frac{3\tau+5}{\tau+3}, \frac{3\tau+5}{3\tau+1}\right)$ and $x_2(\tau) = \left(\frac{5\tau+3}{\tau+3}, \frac{5\tau+3}{3\tau+1}\right)$ to agents of type 1 and 2, respectively. Let $V_i(\tau) = \left(U(x_i(\tau))\right)^2$, for $i = 1,2$. The function V_1 is decreasing and convex whereas V_2 is increasing and concave.

Consider the non-Walrasian allocation $\hat{x}$ given by $\hat{x}_1 = (4,4) - x_2(\sqrt{2})$ and $\hat{x}_2 = x_2(\sqrt{2})$. We find a unique positive number $\hat{\tau}$ such that $(U(\hat{x}_1))^2 = V_1(\hat{\tau})$. Consider the

[5] Note that, following Mas-Colell's approach, if we have a justified objection in the economy $r\mathcal{E}$ it cannot be justified in the economy $(r+1)\mathcal{E}$.

two types associated economy where agents of type 1 are represented by the interval [0,1] and agents of type 2 by (1,2]. Since V_1 is decreasing and $\hat{x}$ is individually rational, the set of all potential justified objections (in the sense of Mas-Colell) is given by the interval $[\sqrt{2}, \hat{\tau}]$ (see figure below). Any coalition $S \subset [0,2]$ such that $\mu(S \cap [0,1]) = 1$ and $\mu(S \cap (1,2]) = 1/\sqrt{2}$ blocks $f_{\hat{x}}$ (the step function given by $\hat{x}$) via the allocation that assigns $x_1(\sqrt{2})$ to agents in $S \cap [0,1]$ and $x_2(\sqrt{2})$ to agents in $S \cap (1,2]$. This implies that the only coalitions able to make a justified objection are those with measure $1 + 1/\sqrt{2}$. In other words, although every $\tau \in [\sqrt{2}, \hat{\tau}]$ defines an objection to $f_{\hat{x}}$, the unique which is (Mas-Colell) justified is given by $\tau = \sqrt{2}$.[6] Thus we conclude that there is no justified objection in any replicated economy, that is, $r\hat{x}$ belongs to the Mas-Colell bargaining set of $r\mathcal{E}$ for every r, which proves the non-convergence.

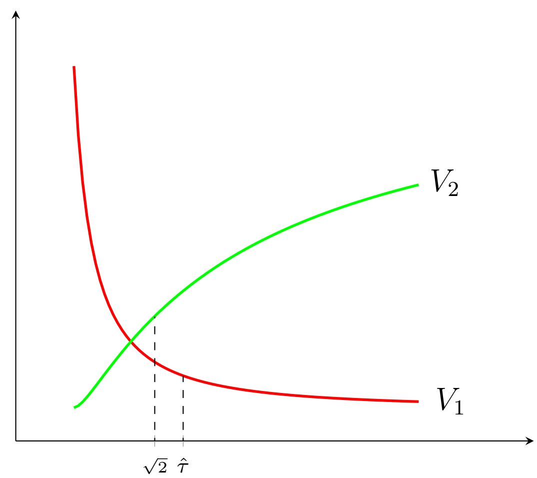

Fig 1: $(U(\hat{x}_1))^2 = V_1(\hat{\tau})$ and $(U(\hat{x}_2))^2 = V_2(\sqrt{2})$.

Let us now analyze the previous example under our notion of bargaining set. For it, we remark that any rational number $\tau \in \left[\sqrt{2}, \hat{\tau}\right]$ leads to a justified* objection for the allocation $r\hat{x}$ for some replicated economy $r\mathcal{E}$. This implies that $r\hat{x}$ does not belong to our bargaining set for any large enough replicated economy.

Furthermore, for each $\alpha \in (\sqrt{3}, 4 - \sqrt{3})$, there exist τ_α and τ^α such that $V_1(\tau_\alpha) = \alpha^2$ and $V_2(\tau^\alpha) = (4-\alpha)^2$. Note that $\alpha = 2$ defines the Walrasian allocation and $V_1(1) = V_2(1) = 4$. However, for any $\alpha \neq 2$, we have $\tau^\alpha < \tau_\alpha$. Let $\alpha \in (\sqrt{3}, 2) \cup (2, 4 - \sqrt{3})$. Then, $V_1(\tau) > \alpha^2$ and $V_2(\tau) > (4-\alpha)^2$, for any $\tau \in (\tau^\alpha, \tau_\alpha)$. For each rational number $\tau \in (\tau^\alpha, \tau_\alpha)$, let $r_1(\tau), r_2(\tau)$ be natural numbers such that $\tau = r_1(\tau)/r_2(\tau)$. Note that the coalition formed by $r_1(\tau)$ consumers of type 1 and $r_2(\tau)$ of type 2 with the

[6] This is so because if a coalition with a Mas-Colell justified objection includes only part of some type of agents, then it is not possible for these agents to strictly improve with the objection.

allocation $x(\tau)$ is a Walrasian objection to $rh(\alpha)$ for any replicated economy $r\mathcal{E}$ with $r \geq \max\{r_1(\tau), r_2(\tau)\}$. Applying Proposition 3.1 we obtain that the objection we have constructed is justified*. Therefore, we conclude that the counterexample by ATZ does not lead to a non-convergence result for the notion of bargaining set we have proposed. Actually, since we deal with a set of economies where the equilibrium is unique, the convergence result we state in the next section guarantees that, in this example, our bargaining set shrinks to the Walrasian allocation when the economy is replicated.

4 A convergence result

In this section, we analyze convergence properties of our bargaining set. First we show that under a continuity property of the equilibrium price correspondence, the Walrasian allocations of a finite economy are characterized as allocations that belong to the bargaining set of every replicated economy. Then, we state an example which shows that such a continuity is a necessary condition.

Starting from the initial economy $\mathcal{E}$, we construct auxiliary continuum economies with a finite number of types and use the following notation. Consider a non-null vector $\alpha = (\alpha_i, i \in N) \in [0,1]^n$ such that $\sum_{i \in N} \alpha_i = 1$. Let $N_\alpha = \{i \in N | \alpha_i > 0\}$, n_α denotes the cardinality of N_α and $m_\alpha = \max\{i | i \in N_\alpha\}$. For each $i \in N_\alpha$, let $I_i(\alpha) = [\bar{\alpha}_{i-1}, \bar{\alpha}_i)$ if $i \neq m_\alpha$ and $I_i(\alpha) = [\bar{\alpha}_{m_\alpha - 1}, 1]$ if $i = m_\alpha$, where $\bar{\alpha}_i = \sum_{h=0}^{i-1} \alpha_h$, with $\alpha_0 = 0$. Finally, $\mathcal{E}_c(\alpha)$ denotes the continuum economy with n_α types of agents, where consumers in the subinterval $I_i(\alpha)$ are of type i (i.e, have endowments ω_i and preferences $\succsim_i$).

Note that when determining the market-clearing prices of an economy, it is sufficient to consider only the excess demand mappings. Let $\mathcal{Z}$ denote the set of excess demand correspondences. Given a continuum economy, where the set of agents is represented by the interval $I = [0,1]$, the measure which describes it is given by $\upsilon(F) = \mu(\{t \in I | Z_t \in F\})$ for each Borel set $F \subset \mathcal{Z}$, being Z_t the excess demand correspondence of the agent $t \in I$.[7] We endow the set of measures describing the economies with the weak convergence topology. The following continuity assumption allows us to state a convergence result for the bargaining set we have defined.

(C) The equilibrium price mapping is continuous at the measures η_α defining the auxiliary continuum economies $\mathcal{E}_c(\alpha)$ with a finite number of types.

[7] In particular, each economy $\mathcal{E}_c(\alpha)$ is described by the measure η_α on $\mathcal{Z}$ defined by $\eta_\alpha(F) = \sum_{i \in T_F} \mu(I_i(\alpha))$, for each $F \subseteq \mathcal{Z}$, where $T_F = \{i \in N | Z_i \in F\}$ and Z_i is excess demand correspondence for consumer i in the economy $\mathcal{E}$.

Theorem 4.1 *Assume that the continuity property (C) holds. Then, an allocation is Walrasian in the finite economy $\mathcal{E}$ if and only if it belongs to the bargaining set of every replicated economy. That is,*

$$W(\mathcal{E}) = \bigcap_{r \in \mathbb{N}} B(r\mathcal{E}).$$

Recall that the set of economies (described by measures, as above) on which the equilibrium price correspondence is continuous is open and dense. (See Dierker, 1973, or Hildenbrand, 1974). Regardless of the continuity of the equilibrium mapping holds for a residual set of economies, we know from the Debreu-Mantel-Sonnenschein theorem that we cannot hope to get more structure on the set of equilibria unless we state additional strong assumptions.

Then, the continuity property we require to get convergence is a limitation for our result. In spite of this, since the Walrasian correspondence is upper semicontinuous (see Hildenbrand and Mertens, 1972), uniqueness of Walrasian equilibrium guarantees that our convergence theorem holds. Therefore, as we detail below, we find assumptions on the primitives of the original economy $\mathcal{E}$ that ensure condition (C) holds and, in turn convergence of our bargainind set.

If the aggregate excess demand function satisfies the gross substitutes property[8], the Walrasian equilibrium is unique (up to normalization of the equilibrium price vector) and globally stable (see Arrow and Hahn, 1971, for additional details). Note that if the individual demand satisfies the gross substitute property[9] (an increase in the price of good increases the demand for every other good) for all consumers, then individual and aggregate excess demand functions also satisfy it.

Moreover, a sufficient condition for satisfaction of the gross substitutes condition aforementioned is that agents have C^2 and separable utility functions with the Arrow-Pratt measure of relative risk aversion that is everywhere less than one (see Varian, 1985). Furthermore, it follows from a result of Mityushin and Polterovich (1978) that if agents have strictly concave utility functions with relative risk aversion that is everywhere less than four[10] and if their endowments are collinear (that is, the distribution of income is price-independent), then equilibrium is unique (see Mas-Colell, 1991, for a discussion

[8] Z satisfies the gross substitute if $p \geq q$ and $p_h = q_h$ for some h, then $Z_h(p) \geq Z_h(q)$ and, if $Z(p) = Z(q)$, then $p = q$. Other related property is the weak axiom of revealed preference that says that if $pZ(q) \leq 0$ and $qZ(p) \leq 0$, then $Z(q) = Z(p)$. If the economy is regular, this implies that the equilibrium is unique.

[9] The standard example is a demand that comes from the maximization of a Cobb-Douglas utility function subject to a budget constraint with strictly positive endowments. A generalization is the utility function $U(x) = \prod_{h=1}^{\ell} (x_h - \beta_h)^{\gamma_h}$, with $\beta_h \leq 0,\ \gamma_h > 0$ and $\sum_{h=1}^{\ell} \gamma_h = 1$.

[10] This property can be looked at as a concrete expression of the idea that the substitution effects

of this result and of uniqueness generally). In addition, Fisher (1972) showed that if all goods are normal (income elasticities nonnegative), then a necessary and sufficient condition for an utility function to yield individual demand functions with the gross substitute property is that, for every pair of commodities, the Allen-Uzawa elasticity of substitution exceeds the greater of the two income elasticities. Thus, in a two-commodity world with homothetic preferences, gross substitution obtains if the elasticity of substitution is everywhere larger than 1. Finally, we also remark that, under strictly convexity and monotonicity of preferences, if the initial endowment is a Walrasian allocation, then this is the unique equilibrium allocation.

If the economy $\mathcal{E}$ verifies the aforementioned conditions, we have uniqueness of equilibrium for all the auxiliary economies $\mathcal{E}_c(\alpha)$; and therefore we have convergence of the bargaining set. Next, we state an example that illustrates why the continuity assumption (C) is required and shows the impossibility of obtaining a convergence result if we allow for discontinuities of the equilibrium correspondence.

Counterexample. Let $\mathcal{E}$ be an exchange economy with two commodities and two agents, endowed with $\omega_1 = (2,1)$ and $\omega_2 = (1,2)$ respectively, who have the same utility function U, defined as follows:

$$U(x,y) = \begin{cases} \frac{1}{2^{1/4}}\sqrt{x} + \sqrt{y} & \text{if } \; x > \sqrt{2}\, y, \text{ and} \\ \sqrt{x} + (2 - 2^{1/4})\sqrt{y} & \text{if } \; x \leq \sqrt{2}\, y. \end{cases}$$

Let x be the numeraire, let p denote the price of y and let $d_i(p)$ be the demand function for each agent i. The equilibrium price for this economy is $p^* = 2 - 2^{1/4}$.

Consider r_i agents of type $i = 1, 2$ and let $\tau = r_1/r_2$. The Walrasian equilibrium prices for this restricted replicated economy, $\mathcal{E}(\tau)$, are

$$p(\tau) = \begin{cases} 2^{1/4}\sqrt{\frac{2\tau+1}{\tau+2}} & \text{if } \; \tau > \tau^*, \\ \left[\underline{p}, \overline{p}\right] & \text{if } \; \tau = \tau^*, \text{ and} \\ (2 - 2^{1/4})\sqrt{\frac{2\tau+1}{\tau+2}} & \text{if } \; \tau < \tau^*, \end{cases}$$

where $\tau^* = 1 + \frac{3}{2}\sqrt{2}$, $\underline{p} = 2^{1/4}(2 - 2^{1/4})$ and $\overline{p} = \sqrt{2}$. Note that there is a continuum of equilibria for the restricted economy $\mathcal{E}(\tau^*)$ and a unique equilibrium for any other economy $\mathcal{E}(\tau)$ with $\tau \neq \tau^*$. For each $\tau \in \mathbb{R}_+$, the utility levels which can be attained for each type of consumers at a Walrasian allocation of the economy $\mathcal{E}(\tau)$ are given by the mappings $V_i(\tau) = U(d_i(p(\tau)))$, $i = 1, 2$, whose graphical representations are shown in the following figure, where $\alpha_i = \min\{V_i(\tau^*)\}$ and $\beta_i = \max\{V_i(\tau^*)\}$:

dominate the income effects.

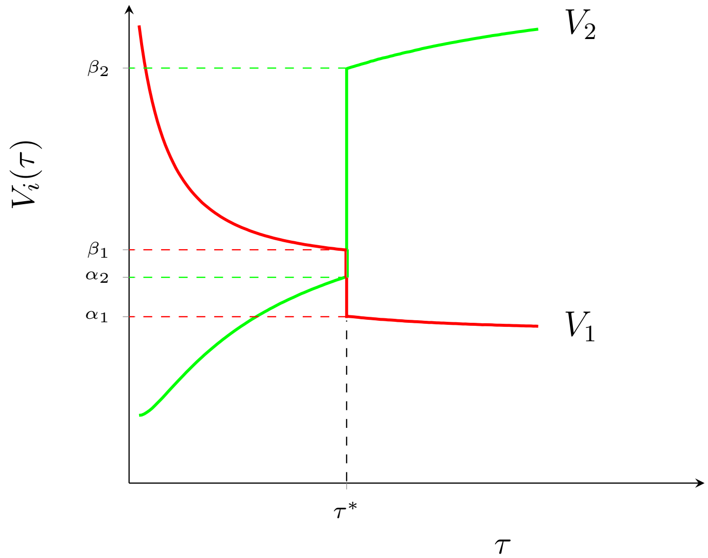

Fig. 2: V_1 and V_2 are not lower semicontinuous at τ^*.

Consider a feasible allocation $h = (h_1, h_2)$ such that $U(h_i) \in (\alpha_i, \beta_i)$.[11] Since h is individually rational, in order to block it in a replicated economy, both types need to be present. In addition, there is no justified* objection for h whenever $\tau > \tau^*$ or $\tau < \tau^*$. It is possible, though, to find justified* objections in $\mathcal{E}(\tau^*)$. Let p_i be the equilibrium price for $\mathcal{E}(\tau^*)$ such that $U(d_i(p_i)) = U(h_i)$. As illustrated in the figure below, any price in $[p_2, p_1] \subset \left[\underline{p}, \overline{p}\right]$ leads to a justified* objection. However, since τ^* is an irrational number, such set of justified* objections cannot be attained in any replicated economy, which proves the non-convergence.

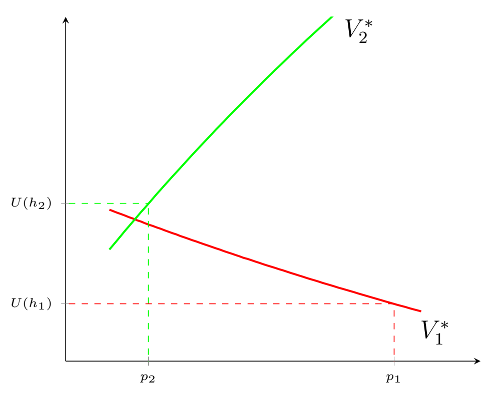

Fig. 3: We get $V_i^*(p) = U(d_i(p))$, with $p \in p(\tau^*)$ by "zooming in" on the Fig. 2 when $\tau = \tau^*$.

[11]For instance, we can take $h_1 = \left(\frac{11^2}{5^2\left(3-2^{1/4}\right)^2}, \frac{11^2}{5^2\left(3-2^{1/4}\right)^2}\right)$ and $h_2 = (3,3) - h_1$.

Appendix

Proof of Proposition 3.1. Let (S, y) be a Walrasian objection to x. Assume that it is counter-objected in some replicated economy $r\mathcal{E}$. That is, there exist $\mathcal{T} \subseteq N$ and natural numbers $r_i \leq r$ for each $i \in \mathcal{T}$, such that: $\sum_{i\in\mathcal{T}} r_i z_i \leq \sum_{i\in\mathcal{T}} r_i \omega_i$; $z_i \succ_i y_i$ for every $i \in \mathcal{T} \cap S$ and $z_i \succ_i x_i$ for every $i \in \mathcal{T} \setminus S$. Since (S, y) is a Walrasian objection at prices p we have that $p \cdot z_i > p \cdot \omega_i$, for every $i \in \mathcal{T} \cap S$ and $p \cdot z_i > p \cdot \omega_i$, for every $i \in \mathcal{T} \setminus S$. This implies $p \cdot \sum_{i\in\mathcal{T}} r_i z_i > p \cdot \sum_{i\in\mathcal{T}} r_i \omega_i$, which is a contradiction. Thus, we conclude that (S, y) is a justified* objection.

To show the converse, let (S, y) be a justified* objection to x and let $a = (a_1, \ldots, a_n)$ be an allocation such that $a_i = y_i$ if $i \in S$ and $a_i = x_i$ if $i \notin S$. For every i define $\Gamma_i = \{z \in \mathbb{R}^\ell | z + \omega_i \succsim_i a_i\} \bigcup \{0\}$ and let Γ be the convex hull of the union of the sets $\Gamma_i, i \in N$. A similar proof to the limit theorem on the core by Debreu and Scarf (1963) shows that $\Gamma \bigcap (-\mathbb{R}^\ell_{++})$ is empty, which implies that 0 is a frontier point of Γ. Then, there exists a price system p such that $p \cdot z \geq 0$ for every $z \in \Gamma$. Therefore, we conclude that (S, y) is a Walrasian objection.

Q.E.D.

To proof Theorem 4.1 we show the following lemma.

Lemma. *Let x be a non-Walrasian feasible allocation in the economy $\mathcal{E}$. Then, the following statements hold:*

(i) For each i, there exist a sequence of rational numbers $r_i^k \in (0, 1]$ converging to 1 and a sequence of allocations $(x^k, k \in \mathbb{N})$ that converges to x such that: (a) $\sum_{i=1}^n r_i^k x_i^k \leq \sum_{i=1}^n r_i^k \omega_i$, (b) $x_i^k \succ_i x_i$ for every i, and (c) $x_i^k \succ_i x_i^{k+1}$ for every k and every i.

Let $r^k = \sum_{i\in N} r_i^k$ and $\alpha^k = (r_i^k / r^k, i \in N) \in [0, 1]^n$. Let f^k be the step function given by $f^k(t) = x_i^k$ for every $t \in I_i(\alpha^k)$ in the continuum economy $\mathcal{E}_c(\alpha^k)$.

(ii) If x belongs to the bargaining set of every replicated economy, then for every k, there is a justified objection (S^k, g^k), in the sense of Mas-Colell, to f^k in the economy $\mathcal{E}_c(r^k)$.

Let $\gamma^k = \left(\gamma_i^k = \mu(S^k \cap I_i(r^k))/\mu(S^k), i \in N\right) \in [0, 1]^n$. Let ν^k be the measure describing the auxiliary continuum economy $\mathcal{E}_c(\gamma^k)$.

(iii) There exists a subsequence of measures of ν^k which converges weakly to a measure ν describing a limit economy $\hat{\mathcal{E}}_c$.

Proof of (i). Observe that if x^k converges to x and $x_i^k \succ_i x_i$, for every i and k, then, under continuity of preferences, condition (c) holds by taking a subsequence if necessary.

If x is a feasible allocation that is not Pareto optimal, then, for every i, there exists y_i such that $\sum_{i=1}^n y_i \leq \sum_{i=1}^n \omega_i$ and $y_i \succ_i x_i$. The sequence given by $x_i^k = \frac{1}{k} y_i + (1 - \frac{1}{k}) x_i$ fulfills the requirements in (a) with $r_i^k = 1$ for all i and k.

Let x be a non-Walrasian feasible allocation which is efficient. Then, there exist rational numbers $a_i \in (0,1]$ (with $a_j < 1$ for some j) and bundles y_i for all $i = 1, \ldots, n$, such that $\sum_{i=1}^n a_i(y_i - \omega_i) = -\delta$, with $\delta \in \mathbb{R}_{++}^\ell$ and $y_i \succ_i x_i$, for every i (see Hervés-Beloso and Moreno-García, 2001, for details). Let $a = \sum_{i=1}^n a_i$. Given $\varepsilon \in (0,1]$, let $y_i^\varepsilon = \varepsilon y_i + (1-\varepsilon)x_i$. By convexity of preferences, $y_i^\varepsilon \succ_i x_i$ for every i. Consider $x_i^\varepsilon = x_i + \frac{\varepsilon\delta}{a^\varepsilon}$, where $a^\varepsilon = (1-\varepsilon)(n-a)$. By monotonicity, $x_i^\varepsilon \succ_i x_i$ for every i. Take a sequence of rational numbers ε_k converging to zero and, for each k and i, let $a_i^k = (1-\varepsilon_k)(1-a_i)$, $r_i^k = a_i + a_i^k \in (0,1]$, and define $x_i^k = \frac{a_i}{r_i^k} y_i^{\varepsilon_k} + \frac{a_i^k}{r_i^k} x_i^{\varepsilon_k}$. By construction, the sequences r_i^k and x_i^k ($i = 1, \ldots, n$ and $k \in \mathbb{N}$) verify the required properties.

Proof of (ii). Let q^k be a natural number such that $r_i^k = b_i^k / q^k$, with $b_i^k \in \mathbb{N}$ for each i. Since $x \in \bigcap_{r\in\mathbb{N}} B(r\mathcal{E})$, x^k cannot be a Walrasian allocation for the economy formed by b_i^k agents of type i; otherwise, the coalition formed by b_i^k members of each type i joint with x^k would define a justified* objection in the q^k-replicated economy.[12] Then, f^k cannot be a competitive allocation in $\mathcal{E}_c(r^k)$. By Mas-Colell's (1989) equivalence result, f^k is blocked by a justified objection (S^k, g^k) in $\mathcal{E}_c(r^k)$. By convexity of preferences, we can consider without loss of generality that g^k is an equal-treatment allocation.

Proof of (iii). Since the number of types of consumers we deal with is finite, without loss of generality we can consider, taking a subsequence if necessary, that $T^k = \{i \in N | \gamma_i^k > 0\} = T$ for every k. We use the same notation for such a subsequence and write γ_i^k converges to γ_i for every $i \in T$ and $\sum_{i\in T} \gamma_i = 1$. Consider $\mathcal{E}_c(\gamma)$ and let $(Z_i, i \in T)$ be the excess demand correspondences of the types that are actually present in every economy $\mathcal{E}_c(\gamma^k)$. Let us define $\tau(F) = \{i \in T | Z_i \in F\}$ for each $F \subset \mathcal{Z}$. We deduce that

$$\lim_{k\to\infty} \nu^k(F) = \lim_{k\to\infty} \sum_{i\in\tau(F)} \hat{\gamma}_i^k = \sum_{i\in\tau(F)} \gamma_i = \nu(F),$$

Therefore, ν^k converges weakly to ν that is the measure describing the economy $\mathcal{E}_c(\gamma)$.

Q.E.D.

Proof of Theorem 4.1 Since $W(\mathcal{E}) \subseteq C(r\mathcal{E})$, it is immediate that $W(\mathcal{E}) \subseteq \bigcap_{r\in\mathbb{N}} B(r\mathcal{E})$.

To show the converse, assume that x is a non-Walrasian allocation that belongs to the bargaining set of every replicated economy. By the previous lemma, for each natural number k, there is a subset T of types and a competitive equilibrium (p^k, g^k) in $\mathcal{E}_c(r^k)$ such that:

(i) $g_i^k \succsim_i x_i^k$ for every $i \in T$, with $g_j^k \succ_j x_j^k$ for some $j \in T$, and

(ii) $g_i^k \in d_i(p^k)$ for every $i \in T$, and $x_i^k \succsim_i d_i(p^k)$ for every $i \in N \setminus T$.[13]

Let $A_k = \left\{i \notin T | x_i \succsim_i d_i(p^k)\right\}$, $B_k = \left\{i \notin T | x_i \prec_i d_i(p^k)\right\}$. Since the number of types is finite, without loss of generality we can consider, taking a subsequence if it is necessary, that $A_k = A$ and $B_k = B$ for every k.

[12]We remark that any objecting coalition involving all types along with a Walrasian allocation for such a coalition defines a justified* objection. This is not the case for the corresponding Mas-Colell's notion.

[13]Note that, given a price vector p, all the bundles in $d_i(p)$ are indifferent; thus, when we write $z \succsim_i d_i(p)$ it means $z \succsim_i d$ for every $d \in d_i(p)$.

Consider the sequence of economies $\mathcal{E}_c(\gamma^k)$, as constructed in the previous lemma, described by the sequence of measures ν^k that converges weakly to ν. Let us choose a sequence of numbers $\delta_k \in (0,1)$ converging to 1 and let $\varepsilon^k = 1 - \delta_k$, which converges to zero. For each $i \in B$ take $\varepsilon_i^k > 0$ such that $\varepsilon^k = \sum_{i \in B} \varepsilon_i^k$. Let $T_1 = T \cup B$ and for each $i \in T_1$ define $\widetilde{\gamma}_i^k \in (0,1)$ as follows:

$$\widetilde{\gamma}_i^k = \begin{cases} \delta_k \gamma_i^k & \text{if } i \in T \\ \\ \varepsilon_i^k & \text{if } i \in B \end{cases}$$

Note that $\sum_{i \in T_1} \widetilde{\gamma}_i^k = 1$. For each k, consider the continuum economy $\mathcal{E}_c(\widetilde{\gamma}^k)$ and let $\widetilde{\nu}^k$ denote the measures on $\mathcal{Z}$ describing it. Note that $\lim_{k \to \infty} \widetilde{\gamma}_i^k = \lim_{k \to \infty} \hat{\gamma}_i^k = \gamma_i$ for every $i \in T$ and $\widetilde{\gamma}_i^k$ goes to zero as k increases for every $i \in B$. Then, the economy $\mathcal{E}_c(\widetilde{\gamma}^k)$ differs from $\mathcal{E}_c(\gamma^k)$ only in at most a finite set of types of agents whose measure goes to zero when k increases. Therefore, the sequence of measures $\widetilde{\nu}^k$ also converges weakly to ν.

Now, for each k and for each $i \in T_1 = T \cup B$, take a sequence of positive rational numbers r_i^{km} converging to $\widetilde{\gamma}_i^k$ when m increases and such that $\sum_{i \in T_1} r_i^{km} = 1$ for every m. In this way, for each k, let us consider the sequence of continuum economies $\mathcal{E}_c(r^{km})$. To simplify notation, let $\mathcal{E}_c^{kk} = \mathcal{E}_c(r^{kk})$. Note that $\lim_{k \to \infty} r_i^{kk} = \lim_{k \to \infty} \gamma_i^k$ for every $i \in T$ and $\lim_{k \to \infty} r_i^{kk} = 0$ for every $i \in B$. Then, the sequence of measures measures ν^{kk} that describes the diagonal sequence of economies $\mathcal{E}_c^{kk}$ converges weakly to ν as well.

Then, by the continuity of the equilibrium mapping at ν and the continuity of preferences, we deduce that for every k large enough there is an equilibrium price $\widetilde{p}_1^k$ for the economy $\mathcal{E}_c^{kk}$ such that $d_i(\widetilde{p}_1^k) \succ_i x_i$ for every $i \in T_1$. If $x_i \succsim_i d_i(\widetilde{p}_1^k)$ for every $i \in A$, we have found a Walrasian objection to x in a replicated economy, which is in contradiction to the fact that x belongs to the bargaining set of every replicated economy. Otherwise, let $\widetilde{A}_k = \left\{ i \notin T_1 | x_i \succsim_i d_i(\widetilde{p}_1^k) \right\}$, $\widetilde{B}_k = \left\{ i \notin T_1 | x_i \prec_i d_i(\widetilde{p}_1^k) \right\}$. As before, without loss of generality, taking a subsequence if it is necessary, we can consider $\widetilde{A}_k = \widetilde{A}$ and $\widetilde{B}_k = \widetilde{B}$ for every k. Let $T_2 = T_1 \cup \widetilde{B}$ and repeat the analogous argument. In this way, after a finite number h of iterations, we have either *(i)* $T_h = N = \{1, \ldots, n\}$ or *(ii)* $N \setminus T_h \neq \emptyset$ but $\left\{ i \notin T_h | x_i \prec_i d_i(\widetilde{p}_h^k) \right\} = \emptyset$. If *(i)* occurs we find a justified* objection to x in a replicated economy which involves all the types of agents. If *(ii)* is the case, there is also a justified* objection to x in a replicated economy but involving only a strict subset of types. In any situation we obtain a contradiction.

Q.E.D.

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