Accounting for Business Cycles in Canada: I. The Role of Supply-Side Factors

Accolley, Delali

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Delali Accolley
accolleyd@aim.com

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Abstract

After documenting business cycle facts in Canada, I have used a bunch of popular models to explain them. The common features of these models are: the use of the neoclassical growth framework, the assumption that prices are flexible enough to ensure a general equilibrium, and the reliance on supply-side factors, mainly technological change, to explain business cycles. I have also assessed the ability of these models to replicate these business cycle facts.

Keywords: Macroeconomics, Business Cycles.
JEL: E10, E32,E37

Contents

Some Abbreviations and Acronymns

BGP	Balanced Growth Path
DSGE	Dynamic Stochastic General Equilibrium
FOC	First-Order Condition
GDP	Gross Domestic Product
GSS	General Social Survey
HP	Hodrick & Prescott
IRF	Impulse Response Function
LFS	Labor Force Survey
OLS	Ordinary Least Squres
RBC	Real Business Cycle
TFP	Total Factor Productivity
US	United States

Some Unfamiliar Greek Letters

Letter	Name	Letter	Name
ε	varepsilon	η	eta
ϑ	vartheta	$\varkappa$	varkappa
ν	nu	ξ	xi
ϖ	varpi	υ Υ	upsilon
ϕ Φ	phi	φ	varphi
χ	chi	ψ	psi

The prefix var in some names stands for variant

1 Introduction

Business cycles are recurrent and persistent fluctuations in aggregate economic variables. A business cycle is made up of two periods: one of expansion and one of contraction. During an expansion, gross domestic product (GDP), consumption, investment, and employment all rise. On the opposite, a contraction is the decline, over at least two consecutive quarters, of the aggregate economic activity. In between these two periods are turning points called peaks and troughs. According to the National Bureau of Economic Research (NBER), the duration of a business cycle, which is the time interval between either two neighboring peaks or two neighboring troughs, is between six and thirty-two quarters.

The business cycle research aims at: (1) documenting empirically the features of the short-run fluctuations observed in aggregate variables, (2) identifying the forces driving these fluctuations, and (3) elaborating models capable of replicating or explaining them. The modern (or postwar) business cycle models, pioneered by Lucas (1975, 1980) and Kydland and Prescott (1982), use the *neoclassical (optimal) growth* framework to explain these fluctuations. Earlier efforts to understand business cycles were led in the first half of the twentieth century by NBER economists Wesley C Mitchell, Frederick C Mills, and Simon S Kuznets.

The neoclassical growth framework is an environment that enables explaining the aggregate economic activity as the result of the optimizing behavior of rational decision makers. Thus, real variables such as aggregate consumption, investment, labor, and production result from households' utility and firms' profit maximization problems, and a simultaneous equilibrium on goods and services, and labor markets. Both long-run growth and short-run fluctuations in these real variables are driven by supply-side factors such as: technological change and shocks to human capital productivity. As far as technological change is concerned, it can be neutral or sector specific.

A plethora of models use the neoclassical growth framework to explain business cycles. The most successful of these models include

- the basic neoclassical model (King, Plosser, and Rebelo, 1988a,b; Cooley and Prescott, 1995),

- the indivisible labor model (Hansen, 1985),

- the investment-specific technological change model (Greenwood, Hercowitz, and Krusell, 1997, 2000),

- the household production model (Greenwood and Hercowitz, 1991; Benhabib, Rogerson, and Wright, 1991; Greenwood, Rogerson, and Wright, 1995),

- the human accumulation model (Einarsson and Marquis, 1997; DeJong and Ingram, 2001), and

- the time-to-build model (Kydland and Prescott, 1982).

These models that assume all markets are perfectly competitive and prices are flexible, make up a class called real business cycle (RBC) theory. According to the RBC theory, nominal variables (nominal interest rates, prices, and money) do not affect real variables. Money is thus neutral and supplied exogenously. There have been later, in the RBC theory, moves towards the recognition of the non-neutrality of money and its role in driving business cycles (Cooley and Hansen, 1989, 1995, 1998; Gavin and Kydland, 1999). But these attempts keep relying on the assumption that market are perfectly competitive and always clear.

A competing alternatives to the RBC theory are the new Keynesian theory. Unlike the former class, it posits such market's failures as imperfect competition in the setting of nominal prices and wages, which could cause nominal rigidities, especially when the latter variables are set in advance and do not costlessly adjust to clear markets. According to the new Keynesian theory, business cycles are largely driven by aggregate demand shocks such as shocks to money demand or households' preference. Money supply is thus endogenous.

Both RBC and new-Keynesian models make up a bulk called the dynamic stochastic general equilibrium (DSGE) models. The purpose of this paper is to portray and explain business cycles in Canada using the RBC theory. I investigate the role of money supply in business cyclke fluctuations in a separate work (Accolley, 2016).

The rest of this paper is organized as follows. Section 2 reviews and critically assesses some popular methods of extracting from data their cyclical components. It then portrays business cycles in Canada over the period 1981:Q1-2012:Q4 by extracting the cyclical components of some key macroeconomic variables to compute some summary statistics. These summary statistics are the standard deviation, the correlation coefficient with cyclical GDP, and the autocorrelation coefficients. Since GDP is a macroeconomic variable summarizing the overall state of an economy, the literature has identified its cyclical components as a measure of business cycle. The variables whose cyclical components increase or decrease along with cyclical GDP are said to be procyclical. Those whose cyclical components are negatively correlated or uncorrelated with cyclacal GDP are respectively said to be countercyclical or acyclical. While most key variables are procyclical, government consumption and some measures of money supply, which aims at stabilizing the economy, turn out to be countercyclical.

Sections 3 through 8, in turn, sketch each of the models listed above. Their parameters are set using observed data. The models are then simulated and their ability to replicate the business cycle fluctuations observed in Canada is assessed

comparing summary statistics.
The basic RBC model (Section 3) is the Ramsey-Cass-Koopmans growth model with a time allocation decision and aggregate disturbances to firms' technology. This model successfully explains much of the fluctuations observed in GDP and investment but when it comes to the labor market its performance is limited. It only explains about half of the volatility in cyclical hours worked and productivity and cannot replicate the observed correlation between these two variables.
The basic RBC model attributes all the fluctuations observed in the total hours worked to variations in the average hours worked by a household. As a matter of fact, more than half of the variations in the former variable comes from variations in the number of workers, *viz.*, people entering or leaving the labor market. So that changes in the total hours worked rather reflect changes in the number of workers, the indivisible labor model (Section 4) amends the basic RBC model assuming household either work full-time or not at all. It turns out that this model explains a greater proportion of cyclical GDP, investment, and hours worked than the basic RBC model do.
The investment-specific technological change model (Section 5) distinguishes between two types of capital goods: machinery & equipment (computers, communication devices, electrical appliances, furniture, vehicles . . .) and non-residential structures (offices, factories, stores . . .). This model purports to explain the quantitative role of an alternative engine of growth: the improvement in the state of the technology for producing machinery & equipment. It has transpired that although investment in machinery & equipment represents a small share of GDP, technological change lowering specifically their price explains a greater share of business cycles fluctuations.
Introducing household production into the neoclassical growth framework (Section 6) enables better explaining the volatility in the consumption of market-produced goods but does not substantially add to explaining facts observed in the labor market. On the other hand, introducing both household production and human capital accumulation, *viz* the acquisition of skills through education, (Section 7) proves successful in explaining labor market short-run dynamics.
Section 8 introduces two features missing in the previous models: gestation lags, which is more precisely the time and process it takes to build physical capital, along with preferences that depend on both current and past leisure. This helps explains all the volatility observed in hours worked and wage.
Finally, some concluding remarks and discussions are placed in Section 9.

2 Business Cycle Facts

A real time series, say Y_t, $t = 1, 2 \dots T$, is made up of: long-run growth (or trend) components Y_{gt} and short-run cyclical components Y_{ct}. The problem of portraying business cycles consists primarily in finding proper ways of filtering Y_t, *i.e.*, isolating Y_{ct} from Y_t. Three filters commonly used are: the linear, the first-difference, and the Hodrick and Prescott (HP) filters. For a review of more filtering methods, see Canova (1998a,b) and Baxter and King (1999), among others.

The Linear Filter— It defines the cyclical components as residuals from regressing the natural logarithm of Y_t on an intercept and a linear time trend t.

$$Y_t = Y_0 g^t \exp(\epsilon_t),$$

where g is the long-run gross growth rate of Y_t. Taking the natural logarithm of both sides gives

$$\begin{aligned} \ln Y_t &= \underbrace{\ln Y_0 + t\ln g}_{} + \underbrace{\varepsilon_t}_{}, \ \ \varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2) \\ y_t &= \quad y_{gt} \quad + \ y_{ct}. \end{aligned}$$

The cyclical components are assumed to be normally and independently distributed with a zero mean and a constant variance. This assumption fails when ε_t is serially correlated, particularly when $\varepsilon_t = \rho\varepsilon_{t-1} + v_t$ and $\rho \geq 1$. With $\rho \geq 1$, the mean of ε_t is no longer constant and y_t is then said to have a stochastic trend, which the linear filter does not remove.

The First-Difference Filter— When applied to the natural logarithm of Y_t, it returns its growth rate,

$$\begin{aligned} y_{ct} &= y_t - y_{t-1} \\ &\approx \frac{Y_t - Y_{t-1}}{Y_{t-1}}, \end{aligned}$$

where the second line is a Taylor series approximation around y_{t-1} of the right-hand side (rhs) elements of the first line. [1] In this filter, the trend is defined as y_{t-1}. This filter removes stochastic trend from data, unlike the linear filter. Its main drawback is that when removing the low-frequency components of y_t, it accentuates its high-frequency components (see the demonstration in Appendix A). The share of the fluctuations in y_t that are attributable to cycles lasting four quarters or less (seasonality) will more than double in Δy_t and the share attributable to cycles lasting, say, between six and thirty two quarters (business cycles) will shrink.

The HP Filter— It is the most popular filter in applied macroeconomics. It considers the growth component y_{gt} as a stochastic time-varying parameter and estimates it by least squares (see the objective function of the program below). To ensure y_{gt} moves smoothly over time, the constraint that the sum of its squared second differences be the smallest possible is imposed.

[1] A first-order Taylor series approximation of $y = f(x)$ around x_0 is $f(x_0) + (x - x_0)f\prime(x_0)$.

$$\min_{\{y_{gt}\}_{t=-1}^{T}} \left\{ \sum_{t=1}^{T} (y_t - y_{gt})^2 + \lambda \sum_{t=1}^{T} \left[(y_{gt} - y_{gt-1}) - (y_{gt-1} - y_{gt-2}) \right]^2 \right\},$$

where λ, the smoothing parameter, is set to 100, 1 600, or 14 400 depending on whether the data are annual, quarterly, or monthly. When applied to quarterly data, the HP filter produces reasonable estimates of the cyclical components. On annual data, Baxter and King (1999, pp 588-90) recommended setting the smoothing parameter to 10 instead of 100 and dropping at least the first and last three observations, otherwise the cyclical components returned would be distorted.

In the rest of this section, business cycles in Canada over the period 1981:Q1-2012:Q4 is portrayed by: (1) extracting and plotting the cyclical components of some macroeconomic variables including GDP, consumption, investment, wage, and hours worked using the HP filter, and (2) computing their standard deviation, correlation with cyclical GDP, and autocorrelation. The standard deviation measures their volatility, *viz.* how they fluctuate. Since GDP is a variable summarizing the overall state of the economy, the sign of their correlation coefficient with cyclical GDP indicates whether they are procyclical, countercyclical, or acyclical. The autocorrelation coefficient indicates how persistent the fluctuations are.

As far as volatility is concerned, it will be misleading comparing the standard deviation of, say, cyclical GDP to that of cyclical hours worked. To be able to make comparisons in order to find out which variables are the most or least volatile, the standard deviations must be scale invariant. A way of making them scale invariant is either using the percentage deviation of the variables from their trends or filtering instead their natural logarithm. The first approach is useful when there are variables that assume negative values.

$$\frac{Y_t - Y_{gt}}{Y_t} \approx \ln Y_t - \ln Y_{gt}$$
$$\frac{Y_{ct}}{Y_t} \approx y_t - y_{gt}$$

The two approaches will not necessary give identical results but are equivalent in the sense that the left-hand side element of the above relations turns out tobe the first-order Taylor series approximation of the rhs element around the trend Y_{gt}. In order to deal with some possible negative values in the actual or simulated data, I have instead used the percentage deviations from trend to compute the standard deviations, except for interest and inflation rates which are already scale invariant.

The following facts emerge from Figures 2.1 through 2.3 and the summary statistics reported in Tables 2.1 and 2.2.

1. Households' consumption of non-durable goods is less volatile than output.

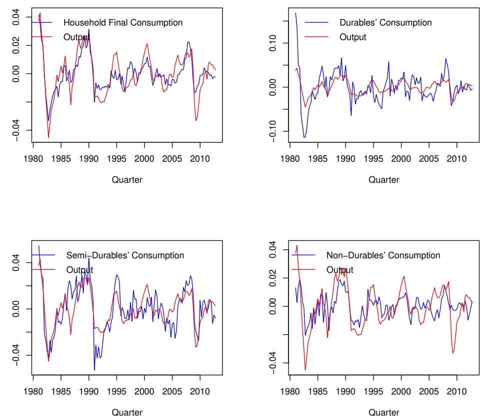

Figure 2.1: The Cyclical Behavior of Various Types of Household Consumption Compared to that of GDP, Canada, 1981:Q1-2012:Q4

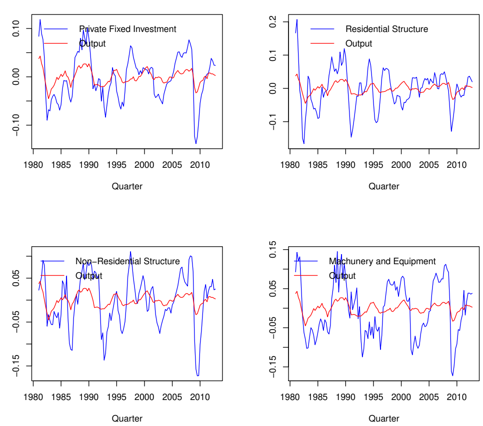

Figure 2.2: The Cyclical Behavior of Various Types of Business Investment Compared to that of GDP, Canada, 1981:Q1-2012:Q4

Table 2.1: Cyclical Behavior of the Canadian Economy, Percentage Deviation from Trend of Key Variables, 1981:Q1-2012:Q4

Variable	Percentage Standard Deviation	Correlation with Output	First-order Autocorrelation
Output (GDP)	1.51	1	.9
Consumption Expenditure	.83	.75	.82
Household Consumption	1.15	.85	.85
Durables' Consumption	3.75	.72	.74
Semi-Durables' Consumption	2	.76	.78
Non-Durables' Consumption	.82	.64	.67
Government Consumption	.97	-.19	.74
Gross Fixed Capital Formation	4.06	.8	.87
Private Fixed Investment	5.01	.79	.89
Residential Structures	6.12	.54	.84
Non-Residential Structures	6.05	.6	.85
Machinery & Equipment	7.32	.69	.86
Government Fixed Investment	3.23	-.01	.84
Actual Hours Weekly Worked			
Average	.54	.8	.71
Total	1.48	.91	.89
Hourly Earnings	1.2	-.21	.84
Productivity	.64	.25	.61
Money Supply			
Monetary Base	1.61	.03	.77
M1	4.68	.34	.85
M2	1.85	-.26	.88
M1+	3.35	.31	.9
M2+	1.81	-.38	.94
M1++	2.36	-.17	.79
M2++	1.21	-.09	.91
Interest Rates			
Bank Rate	1.38	.65	.8
Treasury Bills 1 Month	1.37	.63	.81
Treasury Bills 3 Month	1.36	.66	.8
Treasury Bills, 1 Year	1.29	.65	.77
Prices			
GDP Deflator	1.08	0	.8
Consumer Price Index	.92	-.52	.81
Inflation	.52	.19	.13

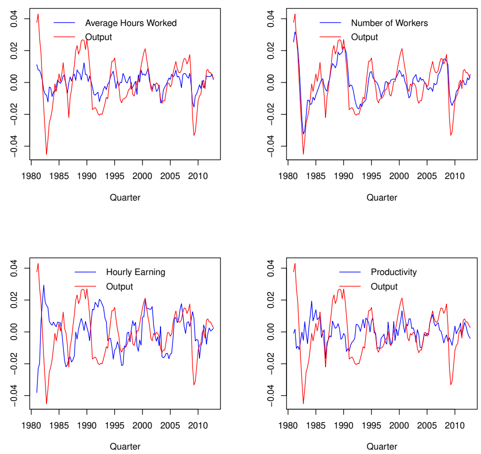

Figure 2.3: The Cyclical Behavior of the Labor Market Compared to that of GDP, Canada, 1981:Q1-2012:Q4

Table 2.2: Cyclical Behavior of the Labor Market, Canada, 1981:Q1-2012:Q4

Correlation	Average Hours	Total Hours
Productivity	-.01	-.17
Wage	-.3	-.32

2. Households' consumption of durable goods is 2.5 times as volatile as output.

3. Machinery and equipment fluctuate more than any other type of investment. They are about 5 times as volatile as output.

4. All the types of household consumption and private investment are procyclical.

5. Government's consumption is countercyclical.

6. Government's fixed investment is acyclical.

7. The average and total hours worked by households are procyclical and less volatile than output.

8. Hourly earnings fluctuate less than output.

9. Hourly earnings and productivity are both less volatile than total hours worked.

10. The correlation between cyclical productivity and average hours worked is close to zero.

11. The measures of money supply M1 and M1+ are procyclical whereas M2 and M2+ are countercyclical.

12. The volatility of the measures of money supply decreases with their broadness.

13. Interest rate is procyclical and less volatile than output.

14. Prices are countercyclical and inflation is procyclical and less volatile than output.

3 The Basic Neoclassical Model

Two types of agents make up the economy: infinitely-lived households and firms. Households maximize their expected lifetime utility defined over consumption c and leisure $1 - l$, given their budget constraints. As for firms, they maximize profits made from producing and selling goods and services.

3.1 The Households

They are all identical. The representative household's preferences are represented by the logarithmic utility

$$u(c_t, l_t) = \ln c_t + \upsilon \ln(1 - l_t), \quad \upsilon \geq 0, \tag{3.1}$$

where the variable l_t denotes the share of time devoted to labor and the parameter υ, the relative weight of leisure. He faces the following two resource constraints:

$$c_t + i_t = w_t l_t + r_t k_t \tag{3.2a}$$
$$k_{t+1} = (1 - \delta) k_t + i_t, \tag{3.2b}$$

where the variables i_t, k_t, r_t, w_t and the constant $0 < \delta < 1$ denote respectively investment, capital stock, the real interest rate, the real wage at time t, and the depreciation rate of capital. According to (3.2a), the representative household receives both labor and capital incomes, which are used to finance his consumption and investment spendings. Relation (3.2b) is the law of motion of capital stock. Given the above two constraints, he programs to maximize his expected lifetime utility $\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$, where $0 < \beta < 1$ is the discount factor. The recursive formulation of this optimization problem is [2]

$$\begin{aligned} \mathcal{V}(k_t, z_t) &= \max_{c_t, l_t, i_t, k_{t+1}} u(c_t, l_t) + \beta \mathrm{E}_t \mathcal{V}(k_{t+1}, z_{t+1}) : \\ c_t + i_t &= w_t l_t + r_t k_t \\ k_{t+1} &= (1 - \delta) k_t + i_t, \end{aligned}$$

where z_t, the total factor productivity (TFP) parameter, is the only source of uncertainty. The derived first-order condition (FOC) and Euler equations are: [3]

$$w_t (1 - l_t) = \upsilon c_t \tag{3.3a}$$
$$\beta \mathrm{E}_t \left[(1 + r_{t+1} - \delta) \frac{c_t}{c_{t+1}} \right] = 1. \tag{3.3b}$$

The condition (3.3a) relates to the intra-temporal trade-off between consumption and leisure. The Euler equation (3.3b) is the inter-temporal pattern of consumption.

[2] For further details on recursive methods, the interested reader is referred to Sargent (1987), Stockey, Lucas, and Prescott (1989), and Ljungovist and Sargent (2004), among others.

[3] This optimization problem is detailed in the appendix.

3.2 The Firms

They combine aggregate (physical) capital K_t and labor L_t to produce a composite numeraire commodity Y_t that can be consumed or invested. The technology is Cobb-Douglas and exhibits constant returns to scale.

$$Y_t = \exp(z_t) K_t^{\alpha} L_t^{1-\alpha}, \ 0 < \alpha < 1, \tag{3.4}$$

where α is the share of capital income in aggregate output (in short, capital share), and z_t, the TFP parameter, follows a stationary autoregressive process of order 1 –an $AR(1)$ process, in short.

$$z_t = \rho z_{t-1} + \epsilon_t, \qquad \epsilon \sim \mathcal{N}\left(0, \sigma_{\epsilon}^2\right). \tag{3.5}$$

The autoregressive parameter ρ in (3.5) is referred to as persistence parameter. Innovations ϵ_t are normally distributed with mean 0 and variance σ_{ϵ}^2.

Firms are perfectly competitive. Their profit maximization problem and the associated FOCs are

$$\max_{K_t, L_t} e^{z_t} K_t^{\alpha} L_t^{1-\alpha} - r_t K_t - w_t L_t.$$

$$K_t: \qquad r_t = \alpha \frac{Y_t}{K_t} \tag{3.6a}$$

$$L_t: \qquad w_t = (1-\alpha) \frac{Y_t}{L_t} \tag{3.6b}$$

The FOCs (3.6) say firms' profits are maximized when the inputs used are paid their marginal productivities.

3.3 General and Stationary Equilibria

The labor force is constant over time and normalized to unity. Equilibrium in all the markets is defined as follows.

Definition (Competitive equilibrium)**.** *The competitive equilibrium consists of a set of prices* $\{(r_t, w_t)\}_{t=0}^{\infty}$, *an allocation* $\{(c_t, i_t, l_t, k_{t+1})\}_{t=0}^{\infty}$ *for the representative household, and an allocation* $\{(K_t, L_t, Y_t)\}_{t=0}^{\infty}$ *for firms such that:*

i. $\{(c_t, i_t, l_t, k_{t+1})\}_{t=0}^{\infty}$ *solves the representative household's optimization problem at the stated prices,*

ii. $\{(K_t, L_t, Y_t)\}_{t=0}^{\infty}$ *solves firms' profit maximization problem,*

iii. capital and labor markets clear, i.e., $k_t = K_t$ *and* $l_t = L_t$.

At steady state, no variable grows. Therefore, the time subscripts and expectations no longer matter and can be dropped . A closed form solution can then be obtained for all the variables in terms of the baseline parameters. Some of these solutions are presented below:

$$r = \frac{1-\beta(1-\delta)}{\beta} \tag{3.7a}$$

$$w = (1-\alpha)\left[\frac{\alpha\beta}{1-\beta(1-\delta)}\right]^{\frac{\alpha}{1-\alpha}} \tag{3.7b}$$

$$\frac{i}{y} = \frac{\alpha\beta\delta}{1-\beta(1-\delta)} \tag{3.7c}$$

$$l = \frac{(1-\alpha)[1-\beta(1-\delta)]}{(1-\alpha)[1-\beta(1-\delta-\delta\upsilon)]+\upsilon(1-\beta)}. \tag{3.7d}$$

At steady state, an increase in the discount factor β, *ceteris paribus*, raises the investment-output ratio and lowers the real interest rate as (3.7c) and (3.7a) suggest. On the other hand, (3.7d) and (3.7b) suggest that this increase causes a decrease in the labor supply and raises wage. The effects of a rise in the depreciation rate δ are the opposite of those caused by a rise in the discount factor. A rise in the capital share α, *ceteris paribus*, has no effect on the real interest rate at steady state but raises the investment-output ratio. It also causes a decrease in the supply of labor and an increase in the real wage as long as $\alpha/r > 1$. When the leisure weight υ increases labor supply decreases.

3.4 Calibration

The calibration consists in assigning values to the parameters α, β, δ, υ, ρ, and σ_ϵ using sample averages computed from national accounts data so as the model economy match at steady state such key observations as the investment-output ratio and hours worked by households. The calibration procedure follows Cooley and Prescott (1995) and Gomme and Rupert (2006). The data used are quarterly, cover the period 1981-2012 (128 quarters). They are all from Statistics Canada.

The Capital Share — From (3.6a), one has

$$\alpha = \frac{r_t K_t}{Y_t} = \frac{\text{Aggregate capital incomes}}{\text{GDP}}.$$

The GDP income-based estimates identified as capital incomes are:

1. the corporate profits before tax, the interest and miscellaneous investment income, the capital depreciation allowance,

2. a share of the accrued net income of farm operator from farm production, the net income of non-farm unincorporated business including rent, the inventory

valuation adjustment, the net indirect taxes, and the statistical discrepancies.

The incomes in the first item are referred to as *unambiguous* capital incomes because they remunerate *specifically* the capital input. Those in the second item are *ambiguous* incomes in the sense they remunerate *indistinctly* both capital and labor. Given the nature of the available data, the capital share can be expressed empirically as

$$\alpha = \frac{\text{Unambiguous capital incomes } + \ \alpha\text{Ambiguous incomes}}{\text{GDP}},$$

which implies

$$\alpha = \frac{\text{Unambiguous capital incomes}}{\text{GDP} - \text{Ambiguous incomes}}.$$

The average value of α for the whole Canadian economy over the sample period is .328.

The Capital Depreciation Rate — It follows from (3.2b), the law of motion of capital, that:

$$\delta = \frac{i_t - (k_{t+1} - k_t)}{k_t}.$$

The depreciation rate is the ratio of the current period's capital depreciation to the end of previous period's capital stock k_t. The current period's depreciation is measured as the difference between the gross investment i_t and the net investment $k_{t+1} - k_t$. The average annual depreciation rate for the whole economy over the sample period is 5.49%. This implies a quarterly rate of 1.37 %.

The Discount Factor — The share of business investment in output is .165. Given this value and those assigned above to the capital share and the depreciation rate, it follows from (3.7c) that the discount factor β equals .987.

The Leisure Weight — According to Statistics Canada's labor force survey (LFS) estimates, over the sample period, the weekly average hours actually worked by households aged 15 and over is 34.39. It also emerges from the 1996, 2001, and 2006 censuses of Canada and from the 2011 national household survey that the average weeks worked in a year is 42.57. The general social survey (GSS) indicates that households allocate 10.45 hours a day to personal care. I use this information to calculate the share of time allocated to labor as

$$l = \frac{\text{Average actual weekly hours worked} \times \text{Average weeks worked}}{(24 - \text{Average time spent on personal care}) \times 7 \times 52},$$

which equals .297 and means households allocate about 29.7% of their discretionary time to labor. [4] Given the value assigned to l and the previous calibration results, the leisure weight obtained from (3.7d) is 1.904.

[4] Discretionary time is the expression used to refer to the number of hours that have not been allocated to personal care.

Table 3.1: The Parameters of the Basic RBC Model

Households	β	Discount factor	.987
	υ	Leisure weight	1.904
Firms	α	Capital share	.328
	δ	Depreciation rate	.0137
	ρ	Persistence parameter	.95
	σ_ϵ	Standard deviation of the innovation	.007

The Technology Shock — One obtains from log-differentiating (3.4), the aggregate production function,

$$\Delta z_t = \Delta \ln Y_t - \alpha \Delta \ln K_t - (1-\alpha)\Delta \ln L_t.$$

I have obtained the series on Δz_t as residuals from the above relation using (1) the value previously assigned to the parameter α, (2) quarterly real GDP as a measure for Y_t, (3) annual real capital stock, all assets and industries, as a measure for K_t, assuming no quarterly change in this series, and (4) quarterly averages of monthly series on the total actual hours worked from the LFS as a measure for L_t. Knowing Δz_t, I have computed z_t, the TFP series, as a cumulative sum of the former variable. Following Cooley and Prescott (1995), I have set the persistence parameter, to .95, which gives .007 as estimate for the standard deviation of the innovation.

3.5 Numerical Solution and Findings

The numerical solution consists in simulating the basic RBC model and computing impulse responses. The simulation is about generating, from the model, time series $\{(c_t,\ i_t,\ K_t,\ L_t,\ r_t,\ w_t,\ Y_t,\ z_t)\}_{t=1}^{T}$ assuming the economy was at steady state at time 0 and then, from time 1 to time T, is repeatedly hit by innovations of size ϵ_t. These innovations are random realizations from a normal distribution with mean 0 and variance σ_ϵ^2. As for the impulse responses, they measure deviations of the variables from their steady state values following a one-off technology shock of size σ_ϵ that has occurred at time 0. The length of the simulated series, T, equals 128, which is the length of the quarterly time series used to calibrate the model. The cyclical components of the simulated series are extracted using the HP-filter. I have then computed their percentage standard deviations and some correlation coefficients. Given the uncertainty associated with the technology shock, the simulation has been performed hundred times and the summary statistics computed are averaged. This is called a Monte Carlo experiment. The purpose of taking averages is: (1) to estimate, from the stochastic model, the true values of the summary statistics of interest and (2) to see how well the model

Table 3.2: Cyclical Behavior of the Canadian and the Basic RBC Economies, Percentage Deviation from Trend of Key Variables, 128 Observations

	Canadian Economy			Basic RBC Economy		
Variable	(1)	(2)	(3)	(1)	(2)	(3)
Output (GDP)	1.51	1	.9	1.39	1	.7
Consumption	1.15	.85	.85	.38	.91	.77
Investment	5.01	.79	.89	6.8	1	.69
Hours	1.48	.91	.89	.74	.99	.69
Wage	1.2	-.21	.84	.67	.99	.72

Columns (1) display the percentage standard deviations, columns (2) display the correlation coefficient with output, and columns (3) display the first-order autocorrelation coefficient.

replicates the business cycle statistics reported in Table 2.1. I have reported in Table 3.2, the averages of some summary statistics from the experiment. Note that the simulations have been carried out using the package Dynare. [5]

The percentage standard deviation of output from the model economy is 1.39. This means about 92 % of the observed fluctuations in output in Canada are explained by technology shock. According to evidence produced by Cooley and Prescott (1995) using the basic RBC model, technology shock explains 79 % of the observed fluctuations in the US output over the period 1954:Q1-1991:Q2. It emerges from a similar exercise carried out by King and Rebelo (2000) that it is 77 % of the fluctuations in the US output that is accounted for by this shock over the period 1947:Q1-1996:Q4. As Cooley and Prescott (1995), I have found that the volatility in the simulated hours worked is only about half of that in the actual series. Simulated hours worked are more volatile than simulated wage (productivity). The simulated investment series are highly volatile.

Consumption, investment, hours worked, and wage in the model economy are procyclical. The actual data show that cyclical consumption, investment, and hours worked are indeed positively correlated with cyclical output but that is not the case for wage, which is countercyclical. Actually, despite its success, a limitation in the basic RBC model is its failure to explain the cyclical co-movement between productivity and other variables (Hansen and Wright, 1992). The correlation between cyclical productivity and hours worked that results from the simulations is .57 whereas the actual one is close to zero as one could see in Table 2.2. Finally, the first-order autocorrelation coefficients indicate that there are more persistence in the actual series than in the simulated series. Talking about

[5] For details about Dynare, please, go to http://dynare.org.

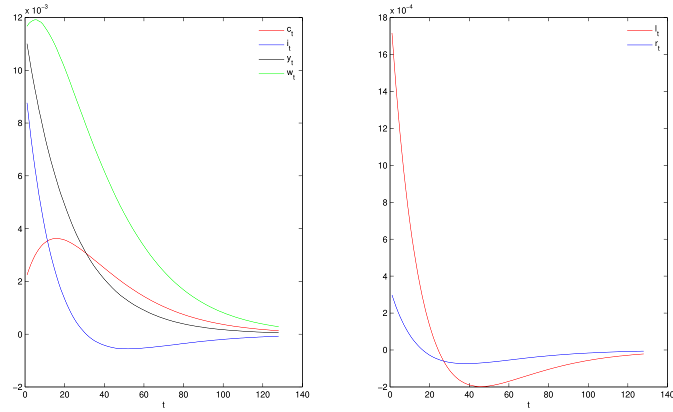

Figure 3.1: Impulse Responses, Deviation from Steady State, the Basic RBC Model

persistence, the impulse response functions (IRFs) plotted in Figure 8.1 shows how the economy returns gradually to steady state after a one-off technology shock.

Figure 8.1 shows that a one-off positive technology shock initially has positive impacts on all the variables and it takes quite long for these impacts to fade out. The technology shock raises capital stock. Output increases as a consequence. As the capital stock is increasing, interest rate, which equals the marginal product of capital, falls to reach after some periods a level below its steady state value. Due to the fall in interest rate, saving (or investment) becomes unattractive and households increase their consumption (substitution effect). Consumption further increases due to the rise in output (income effect).

4 The Indivisible Labor Model

In the basic RBC model, households choose, each period, the amount of time to allocate to labor. This is called the *intensive margin* to volatility of labor supply. Besides, households are identical and are all employed. So, labor supply does not adjust along the *extensive margin*, *viz*, no one enters or leaves the labor market. In fact, fluctuations in the total hours worked, L_t, are a combination of fluctuations in both the number of workers N_t, and the average hours worked, l_t

$$\mathrm{var}(\ln L_{ct}) = \mathrm{var}(\ln N_{ct}) + \mathrm{var}(l_{ct}) + 2\mathrm{cov}(N_{ct}, l_{ct}),$$

where the operators var and cov are respectively abbreviations for variance and covariance, and the subscript c refers to the cyclical components of the series.

Data show that labor adjustment along the intensive margin, which is the only possibility in the basic RBC model, does not account for much of the fluctuations observed in total hours worked. In Canada, over the period 1981-2012, 53% of the variance of the cyclical total hours worked is directly explained by variations in the number of workers and only 13% of this variance is directly attributable to changes in the average hours worked. Hansen (1985) found that the shares of the cyclical total hours worked directly explained by variations in N_t and l_t are respectively 55% and 20% in the US. In addition to this finding, Hansen pointed out that most people either work full-time or not at all, *viz*, labor is indivisible, hence the need for a model capable of explaining the high variability in both the total hours worked and the number of workers.

4.1 The Model

In the indivisible labor model, each household has a probability ϖ_t of working full-time and a probability $1-\varpi_t$ of not working at all. Working full-time means supplying l_0 hour labor service per time period. Unemployed household receive full employment insurance. The utility function of each household is the same as the one defined by (3.1) in the previous section. Thus the expected social utility is

$$\begin{aligned}\mathrm{E}_t u\left(c_t, l_t\right) &= \varpi_t u\left(c_t, l_0\right) + (1-\varpi_t) u\left(c_t, 0\right) \\ &= \ln c_t + \varpi_t \upsilon \ln(1-l_0),\end{aligned}$$

where the expected hours worked l_t equals $\varpi_t l_0$. It follows that $\varpi_t = l_t/l_0$. Plugging the expression of ϖ_t into the expected social utility gives

$$\mathrm{E}_t u\left(c_t, l_t\right) = \ln c_t - \Upsilon l_t, \text{ with } \Upsilon = -\upsilon \frac{\ln(1-l_0)}{l_0}.$$

The above utility is linear in labor. Given (1) preferences are rather defined over consumption and leisure, and (2) a utility function is *ordinal*, *i.e.*, invariant for any strictly increasing transformation (see for details Varian, 1992, p 95 or Mas-Colell, Winston, and Green, 1995, p 9), one can add a constant term to the above utility to have

$$\mathrm{E}_t u\left(c_t, l_t\right) = \ln c_t + \Upsilon(1-l_t). \tag{4.1}$$

The particularity of (4.1) is that the society's inter-temporal elasticity of substitution of leisure is infinite whereas this elasticity for a household is finite. [6]

[6] The inter-temporal elasticity of substitution of leisure is defined as $--\partial \ln(\lambda_{t+1}/\lambda_t)/\partial \ln\left[u_{\lambda_{t+1}}(c_{t+1}, \lambda_{t+1})/u_{\lambda_t}(c_t, \lambda_t)\right]$, where $\lambda_t = 1 - l_t$ designates leisure and $u_{\lambda_t}(c_t, \lambda_t) = \partial u(c_t, \lambda_t)/\partial \lambda_t$.

The FOC and Euler equation from the representative household's problem is

$$w_t = \Upsilon c_t \tag{4.2a}$$

$$\beta \mathrm{E}_t \left[(1 + r_{t+1} - \delta) \frac{c_t}{c_{t+1}} \right] = 1. \tag{4.2b}$$

To see the difference between the indivisible labor and the basic RBC models, compare (4.2a) to (3.3a). Unlike the latter relation, wage is independent of labor supply in the former relation.

Firms' optimal behavior is the same in the indivisible labor model as in the basic RBC model — see subsection 3.2 on page 13. The behavior of the equilibrium prices and quantities is therefore described by the FOCs from the representative household's optimization problem (4.2), the constraints he faces (3.2), firms' FOCs (3.6), the final output technology (3.4), and the law of motion of TFP (3.5).

4.2 Numerical Solution and Findings

At steady state, the closed form solution of the total hours worked in terms of the baseline parameters is

$$l = \frac{1-\alpha}{\Upsilon} \frac{1-\beta(1-\delta)}{1-\beta\left[1-\delta(1-\alpha)\right]}. \tag{4.3}$$

The values assigned to the parameters are the same as those in the Table 3.1, except for the leisure weight, Υ. Solving (4.3), gives the calibration value of Υ, which is 2.71.

I first undertook the Monte Carlo experiments using the value of σ_ϵ, the standard deviation of the innovation, reported in Table 3.1. The percentage standard deviation of output generated by the indivisible labor model exceeded that of the actual economy. I have then reduced σ_ϵ by 25%, to have the indivisible labor model replicate exactly the actual economy's output percentage standard deviation. Table 4.1 reports the summary statistics describing the cyclical behavior of the Canadian economy and those describing the behavior of both the basic RBC and the indivisible labor models, for $\sigma_\epsilon = .0056$.

It transpires that the indivisible labor model generates more fluctuations than the basic RBC model. With a lower σ_ϵ, it exactly replicates the percentage standard deviation of cyclical GDP. It also generates 80% of the true value of the standard deviation of cyclical hours worked. The percentage standard deviation it generates for investment is much higher than its true value but closer to that of machinery and equipment reported in Table 2.1. The percentage standard deviation of cyclical wage it generates is lower than the one from the basic RBC model. The indivisible labor also failed to replicate the negative correlation between cyclical productivity and hours worked but the correlation coefficient it generated, .4, is lower than that generated by the basic RBC model.

Table 4.1: Cyclical Behavior of the Canadian, the Basic RBC, and the Indivisible Labor Economies, Percentage Deviation from Trend of Key Variables, 128 Observations

	Canadian Economy			Basic RBC			Indivisible Labor		
Variable	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
Output (GDP)	1.51	1	.9	1.11	1	.7	1.51	1	.7
Consumption	1.15	.85	.85	.31	.91	.77	.38	.88	.79
Investment	5.01	.79	.89	5.42	.99	.69	7.6	.99	.69
Hours	1.48	.91	.89	.6	.99	.69	1.19	.99	.69
Wage	1.2	-.21	.84	.53	.99	.72	.38	.88	.79

Columns (1) display the percentage standard deviations, columns (2) display the correlation coefficient with output, and columns (3) display the first-order autocorrelation coefficient.

Finally, Figure 4.1 compares the IRFs from both the indivisible labor and the basic RBC models. The first response of consumption, investment, labor, and output to a one-off shock of size $\sigma_\epsilon = .0056$ is stronger in the economy with indivisible labor than in the basic RBC economy.

5 The Investment-Specific Technological Change

Capital stock is made up of: (1) structures and (2) machinery and equipment (equipment, in short). Structures are the value of constructions (offices, plants, shopping centers...), additions, and renovations. As for equipment, it consists of movables such as: computers, means of communication and transportation, and tooling. Data in Canada show

- An inverse long-run (equilibrium) relationship between the *relative price of new equipment* and their share in GDP (see the first panel of Figure 5.1). The relative price of new equipment is the ratio their price index to the price index of non-durable goods. This relative price is declining over time at a quarterly rate of .86% (3.44% per annum).

- An inverse short-run relationship between the relative price of new equipment and investment in new equipment (see the second panel of Figure 5.1). The correlation coefficient between the cyclical components of these two variables is -.37.

Earlier, Greenwood, Hercowitz, and Krusell (1997, 2000) made the same observations using US data. They found that: (1) the relative price of new equipment

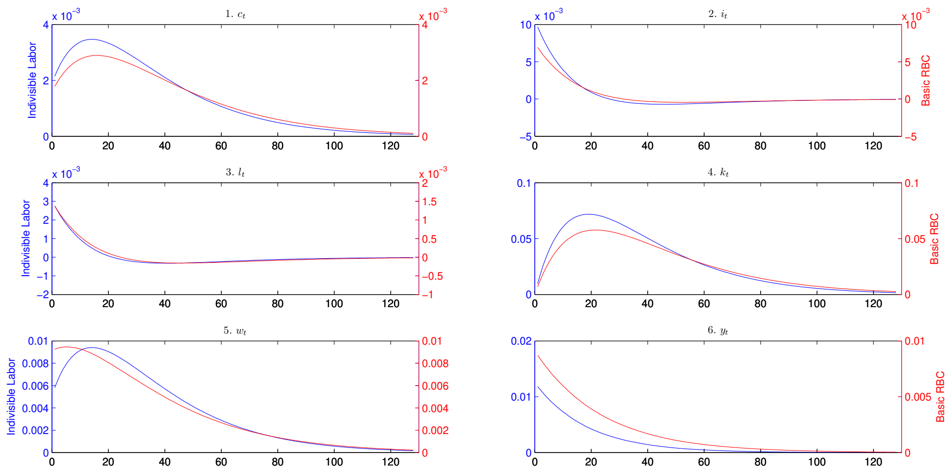

Figure 4.1: Impulse Responses, Deviation from Steady State, the Indivisible Labor and Basic RBC Models

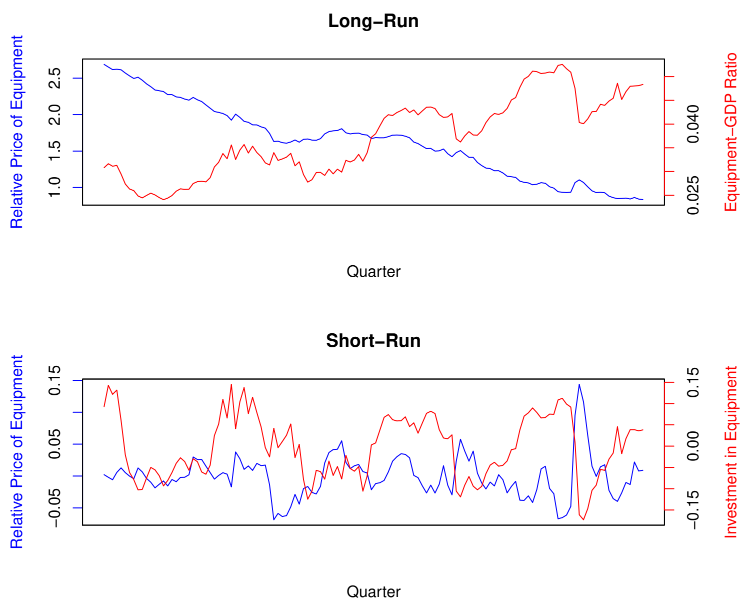

Figure 5.1: Investment in Machinery and Equipment and their Relative Price, Long-Run Relationship and Short-Run Fluctuations, Canada, 1981:Q1-2012:Q4

declined at an average annual rate of 3.2% while their share in the aggregate output was increasing, (2) the correlation coefficient between the cyclical relative price of new equipment and investment in new equipment was -.46. They attribute the decline in the relative price of new equipment to a type of technological advances called *investment-specific* technological change as opposed to neutral technological change. The former type of technological change improves over time the efficiency in the production of equipment whereas the latter improves the aggregate productivity.

Their economy is made up of three types of agents: households, firms, and the government.

5.1 The Households

Their preferences is described by relation (3.1). They face the following three resource constraints.

$$c_t + i_{et} + i_{st} = (1-\tau_l)w_t l_t + (1-\tau_k)(r_{et}\hbar_t k_{et} + r_{st}k_{st}) + \tau_t$$
$$- \exp(\tilde{q}_t)\phi_e \left(\frac{k_{e,t+1}}{q_t} - \kappa_e \frac{k_{et}}{q_t}\right)^2 \frac{q_t}{k_{et}} - \phi_s \frac{(k_{s,t+1} - \kappa_s k_{st})^2}{k_{st}} \quad (5.1a)$$
$$k_{e,t+1} = \left(1 - \frac{b}{\omega}\hbar_t^{\omega}\right) k_{et} + i_{et} q_t \quad (5.1b)$$
$$k_{s,t+1} = (1-\delta_s)k_{st} + i_{st}, \quad (5.1c)$$

The first relation is the representative household's budget constraint. It says he receives both capital and labor incomes. He pays taxes on these incomes. The parameters τ_l and τ_k are respectively the labor and capital income tax rates. The variables r_{et} and r_{st} are the interest rates on investing in equipment and structures. Observe that the subscripts e and s respectively refer to equipment and structures. The representative household also receives a tax return τ_t from the government.

He uses his incomes to finance both his consumption and new investment in equipment i_{et} and new investment in structures i_{st}. There is a cost associated with changing the level of investment. In the literature, this cost is referred to as investment adjustment cost. The adjustment costs here are quadratic to ensure they are either positive or nil. The parameters ϕ_e and ϕ_s are the adjustment cost parameters. As for the parameters κ_e and κ_s, they are respectively the growth factors of the stocks of equipment and structures along the *balanced growth path* (BGP). [7]

The constraints (5.1b) and (5.1c) are respectively the law of motion of equipment and structures. Whereas the depreciation rate of structure δ_s is constant,

[7] When the variables in a model are not stationary, one talks about BGP instead of steady state to refer to the situation where they grow at a constant rate over time.

the depreciation of equipment, which is faster, rather depends on, $\hbar_t$ its utilization rate. The parameter $\omega > 1$ in the equipment depreciation rate is is the proportionality coefficient between the after tax equipment income and equipment depreciation. Another distinguishing feature between (5.1b) and (5.1c) is the factor q_t. This parameter, which is the inverse of the relative price of equipment, measures the productivity of new vintage of equipment. This productivity grows at the average rate γ_q

$$
\begin{aligned}
q_t &= \gamma_q^t \exp(\tilde{q}_t), \\
\tilde{q}_t &= \rho_q \tilde{q}_{t-1} + \epsilon_{qt} \quad \epsilon_{qt} \sim \mathcal{N}\left(0; \sigma_q^2\right).
\end{aligned} \tag{5.2}
$$

The FOCs and Euler equations from households' optimizing behavior are (see details in Appendix B.2)

$$
\upsilon c_t = (1-\tau_l) w_t (1-l_t) \tag{5.3a}
$$

$$
\begin{aligned}
&\beta\left[(1-\tau_k) r_{e,t+1} \hbar_{t+1} q_{t+1} + \left(1 - \frac{b}{\omega} \hbar_{t+1}^{\omega}\right)\right] + \beta \exp(\tilde{q}_{t+1}) \times \\
&\phi_e \frac{k_{e,t+2} - \kappa_e k_{e,t+1}}{k_{e,t+1}} \left(2\kappa_e + \frac{k_{e,t+2} - \kappa_e k_{e,t+1}}{k_{e,t+1}}\right) = \frac{c_{t+1}}{c_t} \frac{q_{t+1}}{q_t} \times \\
&\left[1 + 2\exp(\tilde{q}_t) \phi_e \frac{k_{et+1} - \kappa_e k_{et}}{k_{et}}\right]
\end{aligned} \tag{5.3b}
$$

$$
\begin{aligned}
&\beta\left[(1-\tau_k) r_{s,t+1} + (1-\delta_s)\right] + \beta \phi_s \frac{k_{s,t+2} - \kappa_s k_{s,t+1}}{k_{s,t+1}} \times \\
&\left(2\kappa_s + \frac{k_{s,t+2} - \kappa_s k_{s,t+1}}{k_{s,t+1}}\right) = \frac{c_{t+1}}{c_t} \times \\
&\left(1 + 2\phi_s \frac{k_{st+1} - \kappa_s k_{st}}{k_{st}}\right)
\end{aligned} \tag{5.3c}
$$

$$
(1-\tau_k) r_{et} q_t = b \hbar_t^{\omega - 1}. \tag{5.3d}
$$

The expectation operator E_t is deliberately omitted in'(5.3b) and (5.3c), which are the relations governing the trade-off between consumption and investment. Relation (5.3d) governs the choice of the optimal utilization rate.

5.2 The Firms

The final good is produced using equipment, structures,and labor as inputs. As already said, unlike structures, the rate of utilization of equipment is variable.

$$
y_t = z_t \left(\hbar_t k_{et}\right)^{\alpha_e} k_{st}^{\alpha_s} l_t^{1-\alpha_e-\alpha_s}, \; 0 < \alpha_e, \alpha_s, \alpha_e + \alpha_s < 1 \tag{5.4}
$$

The TFP parameter grows exponentially at the average rate γ_z

$$\begin{aligned} z_t &= \gamma_z^t \exp(\tilde{z}_t) \\ \tilde{z}_t &= \rho \tilde{z}_{t-1} + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}\left(0; \sigma^2\right). \end{aligned} \tag{5.5}$$

Firms face the following profit maximization problem

$$\max_{k_{et}, k_{st}, l_t} z_t \left(\hbar_t k_{et}\right)^{\alpha_e} k_{st}^{\alpha_s} l_t^{1-\alpha_e-\alpha_s} - r_{et} \hbar k_{et} - r_{st} k_{st} - w_t l_t,$$

with as FOCs

$$k_{et}: \quad \alpha_e \frac{y_t}{k_{et}} = r_{et} \hbar_t \tag{5.6a}$$

$$k_{st}: \quad \alpha_s \frac{y_t}{k_{st}} = r_{st} \tag{5.6b}$$

$$l_t: \quad (1 - \alpha_e - \alpha_s) \frac{y_t}{l_t} = w_t. \tag{5.6c}$$

According to (5.6), all inputs are paid their marginal products.

5.3 The Government

Its budget is always balanced. The taxes it raises on both capital and labor incomes are entirely returned to households.

$$\tau_k \left(r_{et} h_t k_{et} + r_{st} k_{st}\right) + \tau_l w_{lt} l_t = \tau_t \tag{5.7}$$

The DSGE model is fully described by the following relations: (5.3) the FOCs from the representative household's utility maximization problem, (5.1) his resource constraints, (5.4) firms' production technology, (5.6) the FOCs from their profit maximization problem, (5.7) the government budget, and both (5.2) and (5.5), which are respectively the law of motions of investment-specific and neutral technological changes.

5.4 The Balanced Growth and Calibration

The time allocated to labor and the utilization rate of equipment are stationary variables, which means they are constant along the BGP. So are the innovations ϵ and ϵ_q. All the other variables are trended.

If g designates the (gross) rate of growth along the BGP of i_{et}, the investment in equipment, (5.1b) suggests that k_{et}, the stock of equipment, is constrained to grow at the rate $g\gamma_q$. According to constraint (3.2a), c_t, i_{st}, k_{st}, w_t, and τ_t all have to grow at the rate g. Finally, (5.4), the production technology, suggests that

$$g = \left(\gamma_z \gamma_q^{\alpha_e}\right)^{1/(1-\alpha_e-\alpha_s)}. \tag{5.8}$$

Evaluating the model along the BGP, one also has

$$g\gamma_q = \beta\left[1 + \left(1 - \frac{1}{\omega}\right) b\hbar^{\omega}\right] \tag{5.9a}$$

$$g = \beta\left[(1-\tau_k)\alpha_s \frac{\hat{y}}{\hat{k}_s} + 1 - \delta_s\right], \tag{5.9b}$$

where $\hat{y}$ and $\hat{k}_s$ are respectively y and k_s divided by their long-run growth component. What transpires from the above two relations is that the after-tax gross return on capital is the same for equipment and structures. This return equals g/β. If the return on one of the two types of capital were higher than the other, households would specialize in investing in the one giving the highest return.

The Parameters g and γ_q— These parameters are estimated by regressing in turn the logarithm of GDP and the logarithm of the inverse of the relative price of equipment on a constant and time trend using the ordinary least squares (OLS) method.

$$\begin{aligned} \widehat{\ln y_t} &= \underset{(2380)}{13.34} + \underset{(7.65)}{.006t} \\ \bar{R}^2 &= .983, \qquad t_{2.5\%}(126) = 1.98 \end{aligned} \tag{5.10a}$$

$$\begin{aligned} \widehat{\ln q_t} &= \underset{(-70.49)}{-.989} + \underset{(45.41)}{.0086t} \\ \bar{R}^2 &= .942, \qquad t_{2.5\%}(126) = 1.98 \end{aligned} \tag{5.10b}$$

In the above equations, $\widehat{\ln y_t}$ and $\widehat{\ln q_t}$ refer respectively to the fitted values of the logarithm of GDP and the inverse of the relative price of equipment. The statistics $\bar{R}^2$ is the adjusted *coefficient of determination.* It gives the proportion of the fluctuations in the data that the linear model explains. Comparing the values in parentheses, the t-ratios, to the critical value $t_{2.5\%}(126) = 1.98$ tells whether the parameters are significantly different from zero. It turns out that the linear models explain respectively 98.3% and 94.2% of the observed variations in the data and all the estimated parameters are significantly different from 0. The slope parameters (or the time trends) are respectively the quarterly growth rate of output and that of investment-specific technological change. Therefore, $g = 1.006$ and $\gamma_q = 1.0086$.

Regressing the residuals from (5.10b) on their first lag gives the estimates of ρ_q.

$$\begin{aligned} \widehat{\tilde{q}}_t &= \underset{(36.48)}{.96\tilde{q}_{t-1}} \\ \bar{R}^2 = .913, &\quad t_{2.5\%}(126) = 1.98, \quad \sigma_q = .023 \end{aligned}$$

The Discount Factor— Relations (5.9) show that the return on capital employed, *viz.* the ratio of the after-tax interest payments to the capital used, is $(g/\beta) - 1$. The quarterly average return on capital employed in Canada over the period 1988-2012 is 6.74% per annum, which is equivalent to 1.64% over a quarter. Solving then the equation gives $\beta = .99$.

The Depreciation Rates— These parameters are computed as shown in subsection 3.4 using data on the stocks and depreciation of equipment and structures. It follows that $\frac{b}{\omega}h^{\omega} = .022$ and $\delta_h = .008$.

The shares of equipment and structures in output are respectively .037 and .052. It follows from the law of motion of equipment and structures that along the BGP, the capital-output ratios are:

$$\frac{\tilde{k}_e}{\hat{y}} = \frac{1}{g\gamma_q + \frac{b}{\omega}\hbar^{\omega} - 1}\frac{\hat{i}_e}{\hat{y}} = .997$$
$$\frac{\hat{k}_s}{\hat{y}} = \frac{1}{g + \delta_s - 1}\frac{\hat{i}_s}{\hat{y}} = 3.693,$$

where $\hat{i}_e$, $\hat{i}_s$, and $\tilde{k}_e$ are i_e, i_s, and k_e divided by their respective (gross) growth rates along BGP.

The Capital and Labor Income Shares and Tax Rates— The sum of α_e and α_s are set equal to .328, the value of the capital share obtained in subsection 3.4. The labor income tax rate is set to the average *implicit tax rate* of all family units, which is 17.75%. [8] The average implicit tax rate is the average income tax expressed as a percentage of the average total income. Solving (5.9) along with the above two equalities, one has $\alpha_e = .114$, $\alpha_s = .214$, $\tau_k = .59$, and $\omega = 2.14$.

The TFP Growth Rate— There are two alternative ways of getting γ_z: either using directly relation (5.8) or doing a growth accounting.
Given that the values of g, α_e, α_s, and γ_q are known, solving (5.8) gives $\gamma_z = 1.003$.
The growth accounting aims at generating values of $\ln z_t$ after differentiating the logarithm of (5.4)

$$\begin{aligned}\Delta \ln z_t &= \Delta \ln y_t - \alpha_e \Delta \ln(\hbar k_{et}) - \alpha_s \Delta \ln k_{st} - (1 - \alpha_e - \alpha_s)\Delta \ln l_t \\ &= \ln \gamma_z + \Delta \tilde{z}_t. \end{aligned} \tag{5.12}$$

The series used in the growth accounting exercise are the quarterly real GDP, quarterly industrial capacity utilization rate, and annual real stocks of equipment and structures. To transform the annual series of stocks of equipment and structures into quarterly data, I have assumed there is no change in the stocks within

[8] Greenwood, Hercowitz, and Krusell (1997, 2000) used instead the marginal tax rate of 40% in their calibration exercise. The problem with the marginal tax rate is that it is not a flat rate as specified in the model. It is the tax rate applied only to the last dollar of taxable inome.

each year. Regressing the estimates for $\Delta \ln z_t$ on a constant as the last relation in (5.12) suggests, gives an estimate for $\ln \gamma_z$. [9]

$$\begin{aligned} \widehat{\ln z_t} = \quad & .002 \\ & (4.23) \\ t_{2.5\%}(126) = 1.98 & \end{aligned}$$

The TFP growth rate turns out to be 1.002, which is almost the same as what I got using directly relation (5.8). The correlation between the TFP shock ϵ_t and the shock to q_t, ϵ_{qt}, turns out to be .19.

Besides, taking the logarithm of relation (5.8) gives the shares of the long-run growth rate explained by the TFP and investment-specif technological change.

$$\ln g = \frac{\ln \gamma_z}{1 - \alpha_e - \alpha_s} + \frac{\alpha_e}{1 - \alpha_e - \alpha_s} \ln \gamma_q \tag{5.13}$$

TPF growth accounts for 78% of the long-run growth rate and the investment-specific technological change explains 22 % of the long-run growth rate. Greenwood, Hercowitz, and Krusell (1997) found that investment-specific technological change explained 58 % of long-run growth in the US. The magnitude of the difference in my results and theirs may be due to the fact that I have used quarterly data whereas they used annual data. In the next subsection, I investigate the contribution of the investment-specific technological change to short-run fluctuations.

All the calibrated parameters except ϕ, the adjustment cost parameter, are reported in Table 5.1. Since the adjustment costs are nil along the BGP, ϕ can assume many values. Its values will be set in the next subsection so that the model matches some standard deviation and cross-correlation observed in Canada.

5.5 Numerical Solution and Findings

Monte Carlo simulation has been performed assuming investment-specific technical change (q shock) occurs randomly each period. Two extreme values have been assigned in turn to ϕ: 3.3 and 3.8. Setting the adjustment cost parameter to 3.3 yields the highest percentage standard deviation for investment in equipment while making sure investment in structures is pro-cyclical. Setting $\phi = 3.8$ generates the lowest possible volatility in investment in equipment while making sure that the simulated correlation between cyclical consumption and output does not exceed the one observed in Canada. In short, the higher the adjustment cost parameter, the lower the percentage standard deviations of cyclical investment in equipment and output. Thus, $\phi = [3.3, 3.8]$ will produce a sort of confidence interval for the contribution of investment-specific technological change to business cycle.

[9] Regressing a variable on a constant is equivalent to taking its average.

Table 5.1: The Parameters of the Investment-Specific Technological Model

Households	β	Discount factor	.99
	τ_k	Capital income tax rate	.587
	τ_l	Labor income tax rate	.178
	υ	Leisure weight	1.436
Firms	g	Output Growth Rate	1.006
	α_e	Equipment share	.114
	α_s	Structures share	.214
	γ_z	TFP growth rate	1.003
	$\frac{b}{\omega}h^{\omega}$	Equipment depreciation rate	.022
		ω	2.141
		b	.072
	δ_s	Structures depreciation rate	.008
	ρ	Persistence parameter	.95
	σ	Standard deviation of the innovation	.005
Investment-Specific	γ_q	Growth rate	1.009
Technological	ρ_q	Persistence parameter	.96
Change	σ_q	Standard deviation of the innovation	.023

In addition to the divisible labor assumption that is implicit in the preferences used, I have also carried out other Monte Carlo simulations assuming labor is indivisible. Assuming labor is indivisible means using (4.1) instead of (3.1) to describe households' preferences. In this particular case, the intra-temporal trade-off between consumption and leisure will be (4.2a) and the calibrated leisure weight will be $\Upsilon = 2.043$.

Tables 5.2 and 5.3 report some summary statistics describing the cyclical behavior of the Canadian and artificial economies respectively for the two extreme values of ϕ.

It turns out that investment-specific technological change explains 75% to 86% of the fluctuations in investment in equipment observed in Canada. Obviously, it contributes much less to the fluctuations in structures— between 5% and 6%. Its contribution to output fluctuations is between 17% and 21%, which is almost similar to its contribution to long-run growth, 22%. As far as the labor market is concerned, the q shock generates less volatility in hours worked and real wage.

A limitation in the investment-specific technological change model is that it exaggerates the negative correlation between the relative price of equipment and output.

Figure 5.2 plots some impulse responses to a one-off q shock of one standard deviation. The q shock means the relative price of equipment has become temporarily low and gradually returns to its initial level. The one-off q shock has

Table 5.2: Cyclical Behavior of the Canadian, the Investment-Specific Technological Change Model (Divisible and Indivisible Labor), $\phi = 3.3$, Percentage Deviation from Trend of Key Variables, 128 Observations

	Canadian Economy			Divisible Labor			Indivisible Labor		
Variable	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
Output (GDP)	1.51	1	.9	.27	1	.72	.32	1	.71
Consumption	1.15	.85	.85	.11	.79	.87	.12	.9	.81
Equipment	7.32	.69	.86	6.15	.94	.69	6.29	.95	.69
Structures	6.05	.6	.85	.35	.05	.92	.38	.69	.92
Hours	1.48	.91	.89	.14	.95	.7	.22	.96	.7
Relative Price	3.25	-.05	.76	2.97	-.95	.7	2.97	-.96	.7
Wage	1.2	-.21	.84	.14	.96	.78	.12	.9	.81

Columns (1) display the percentage standard deviations, columns (2) display the correlation coefficient with output, and columns (3) display the first-order autocorrelation coefficient.

Table 5.3: Cyclical Behavior of the Canadian, the Investment-Specific Technological Change Model (Divisible and Indivisible Labor), $\phi = 3.8$, Percentage Deviation from Trend of Key Variables, 128 Observations

	Canadian Economy			Divisible Labor			Indivisible Labor		
Variable	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
Output (GDP)	1.51	1	.9	.26	1	.72	.31	1	.71
Consumption	1.15	.85	.85	.1	.85	.84	.13	.92	.79
Equipment	7.32	.69	.86	5.49	.95	.69	5.61	.96	.69
Structures	6.05	.6	.85	.28	.4	.96	.38	.84	.86
Hours	1.48	.91	.89	.13	.95	.7	.2	.96	.7
Relative Price	3.25	-.05	.76	2.97	-.95	.7	2.97	-.96	.7
Wage	1.2	-.21	.84	.14	.97	.77	.13	.92	.79

Columns (1) display the percentage standard deviations, columns (2) display the correlation coefficient with output, and columns (3) display the first-order autocorrelation coefficient.

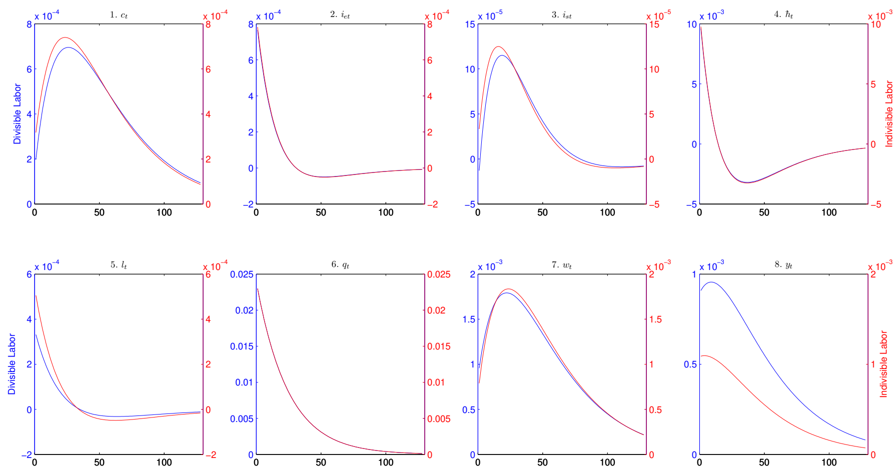

Figure 5.2: Impulse Responses to a q-shock, Deviation from Steady State, the Investment-Specific Technological Change Model, Divisible and Indivisible Labor, $\phi = 3.3$

an initial positive impact on all the variables but the magnitude of the response varies. The impact on investment in equipment fades out along with its utilization rate as the relative price of equipment starts increasing. On the other hand, consumption and investment in structures, which are alternative uses of households' income, rise as the relative price of equipment starts increasing. What also appears in Figure 5.2 is that assuming indivisible labor does not change the response of investment in equipment, its price, and utilization rate to a q shock.

6 The Household Production Model

This model is based on the observation that all the economic activity does not take place in the market. As a matter of fact,

- According to the LFS and the GSS of Statistics Canada, a worker allocates 29.7% of his discretionary time to a paid work and 18.8% to *household work*. By household work, I mean cooking, washing up, housekeeping, and maintenance and repair.
- Investment in household capital, which consists of consumer durables and residential structures, exceeds by about 41% investment in market capital, which comprises non-residential structures and machinery and equipment (see Figure 6.1).

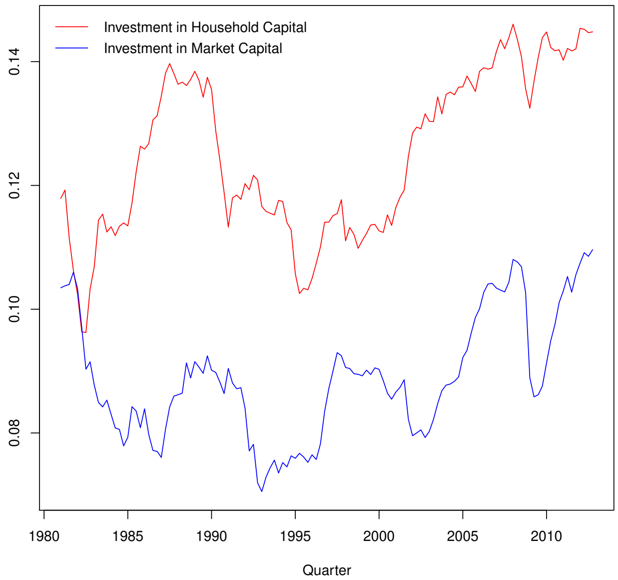

Figure 6.1: Investment in Household and Market Capital as a Share of GDP, Canada, 1981:Q1-2012:Q4

Benhabib, Rogerson, and Wright (1991) and Greenwood, Rogerson, and Wright (1995) reported that, in the US, an average married couple allocated between 25% and 28% of its discretionary time to household work. As for investment in household capital, it exceeded by about 15% investment in market capital. They noted that, despite its importance, household production was absent from models of aggregate economic activity. But the behavior of this latter sector was not independent of the market as it turned out, for instance, that individuals who had a paid job spent less time on household work than unemployed individuals. They then came to the conclusion that, given its size, household production was an important missing element in existing models of the aggregate economy. Greenwood and Hercowitz (1991) and Benhabib, Rogerson, and Wright (1991) thus elaborated a model including household production that was *observationally equivalent* to one without any. Their economy consists of: households, firms, and a government.

6.1 The Households

They are all identical and infinitely lived. They have preferences defined over consumption and leisure. Consumption C_t is a composite of two goods: market-produced goods, c_{mt}, and home-produced goods, c_{ht}. Leisure, $1 - l_{mt} - l_{ht}$, is the time that has not been allocated to market and household work. Their instantaneous utility is defined as follows

$$\begin{aligned} u\left(c_{mt}, c_{ht}, l_{mt}, l_{ht}\right) &= \ln\left(C_t\right) + \upsilon \ln\left(1 - l_{mt} - l_{ht}\right) \\ C_t &= \left[a c_{mt}^{e} + (1-a) c_{ht}^{e}\right]^{1/e}. \end{aligned} \tag{6.1}$$

The parameter a is the share of the market-produced good in households' consumption. The elasticity of substitution between market- and home-produced goods is $1/(1-e)$, $e \leq 1$. [10] Both types of goods are said to be independent, for $e = 0$. For $0 < e \leq 1$, they are substitutes, and for $e < 0$ they are complementary goods.

Households face the following four resource constraints

$$c_{mt} + i_{mt} + i_{ht} = (1-\tau_l) w_t l_{mt} + (1-\tau_k) r_t k_{mt} + \tau_t \tag{6.2a}$$

$$k_{mt+1} = (1-\delta) k_{mt} + i_{mt} \tag{6.2b}$$

$$k_{ht+1} = (1-\delta) k_{ht} + i_{ht} \tag{6.2c}$$

$$c_{ht} = k_{ht}^{\eta} (z_{ht} l_{ht})^{1-\eta}. \tag{6.2d}$$

According to constraint (6.2a), the representative household uses his disposable income to finance his consumption of market-produced goods and his investment in both market and household capital. Constraints (6.2b) and (6.2c) are the laws of motion of market and household capital. Both types of capitals depreciate at the same rate δ. The technology of the home-produced good, described by (6.2d), is Cobb-Douglas and exhibits a labor-augmenting technological change. The law of motion of technological change in the household sector is

$$\begin{aligned} z_{ht} &= \gamma^t \exp(\tilde{z}_{ht}) \\ \tilde{z}_{ht} &= \rho \tilde{z}_{h,t-1} + \epsilon_{ht}, \ \ \epsilon_{ht} \sim \mathcal{N}(0, \sigma^2) \end{aligned} \tag{6.3}$$

where the innovation ϵ_{ht} is normally and independently distributed with a zero mean and a constant variance. The parameter γ is the expected (or deterministic) gross growth rate of z_{ht}. The FOCs and Euler equations from households' optimization problem are the following (see details in Appendix B.3).

$$\frac{\upsilon}{a} C_t^{e} c_{mt}^{1-e} = (1-\tau_l) w_t (1 - l_{mt} - l_{ht}) \tag{6.4a}$$

$$(1-a)(1-\eta)\left(\frac{c_{ht}}{C_t}\right)^{e} = \upsilon \frac{l_{ht}}{(1 - l_{mt} - l_{ht})} \tag{6.4b}$$

[10] The elasticity of substitution is defined as $\mathrm{d}\ln(c_{mt}/c_{ht}) \big/ \mathrm{d}(\ln(u_{c_{ht}}/u_{c_{mt}}))$, where u_{c_m} and u_{c_h} are respectively the marginal utility of the market- and home-produced goods.

$$\beta \mathrm{E}_t \left\{ [(1-\tau_k) r_{t+1} + (1-\delta)] \left(\frac{c_{mt}}{c_{m,t+1}} \right)^{1-e} \left(\frac{C_t}{C_{t+1}} \right)^e \right\} = 1 \tag{6.4c}$$

$$\beta \mathrm{E}_t \left\{ \left[\eta \frac{1-a}{a} \frac{c_{h,t+1}}{k_{h,t+1}} \left(\frac{c_{mt}}{c_{h,t+1}} \right)^{1-e} + (1-\delta) \left(\frac{c_{mt}}{c_{m,t+1}} \right)^{1-e} \right] \left(\frac{C_t}{C_{t+1}} \right)^e \right\} = 1 \tag{6.4d}$$

Relations (6.4a) and (6.4b) are the intra-temporal substitution between leisure and respectively the consumption of market- and home-produced goods. As for (6.4c) and (6.4d), they show the inter-temporal substitution of the market- and home-produced goods. Comparing (6.4a) to (3.3a) and (6.4c) to (3.3b), one could observe that, for $e = \eta = 0$, the household production model generates exactly the same time path as a standard model. That is why the household production model is said to be observationally equivalent to a standard model.

6.2 The Firms

The market production technology is

$$y_t = k_{mt}^{\alpha} \left(z_{mt} l_{mt} \right)^{1-\alpha}, \tag{6.5}$$

with

$$\begin{aligned} z_{mt} &= \gamma^t \exp(\tilde{z}_{mt}) \\ \tilde{z}_{m,t} &= \rho \tilde{z}_{m,t-1} + \epsilon_{mt},\ \epsilon_{mt} \sim \mathcal{N}(0, \sigma^2). \end{aligned} \tag{6.6}$$

Note that the persistence parameter, ρ, and the standard deviation of innovation, σ, are the same in both the household and market sectors in order to have z_{ht} mimicking z_{mt}. Besides, innovations in both sectors are contemporaneously correlated, $\mathrm{cov}(\epsilon_{ht}, \epsilon_{mt}) = \sigma_{\epsilon_h, \epsilon_m}$. [11]

Firms' problem is to maximize their profit, $k_{mt}^{\alpha} \left(z_{mt} l_{mt} \right)^{1-\alpha} - r_t k_{mt} - w_t l_{mt}$, which has as FOCs

$$k_{mt}: \quad r_t = \alpha \frac{y_t}{k_{mt}} \tag{6.7}$$

$$l_{mt}: \quad w_t = (1-\alpha) \frac{y_t}{l_{mt}}. \tag{6.8}$$

[11] The correlation between ϵ_{mt} and ϵ_{ht} is $\sigma_{\epsilon_h, \epsilon_m} / \sigma^2$.

6.3 The Government

Its revenue consists of taxes on capital and labor incomes and its expenses are made up of lump-sum transfers to households and consumption (purchases) G_t.

$$\tau_k r_t k_{mt} + \tau_l w_t l_{mt} = \tau_t + G_t \tag{6.9}$$

The law of motion of government consumption is

$$\begin{aligned} G_t &= \bar{G}\gamma^t \exp(\tilde{G}_t) \\ \tilde{G}_t &= \rho_G \tilde{G}_{t-1} + \epsilon_{Gt}, \quad \epsilon_{Gt} \sim \mathcal{N}(0, \sigma_G^2). \end{aligned} \tag{6.10}$$

Relations (6.2) trough (6.10) describe the DSGE model.

6.4 The Balanced Growth Path and Calibration

Along the BGP, labor and interest rate are stationary. All the other variables grow at the rate g. Evaluating (6.4c) along the BGP shows

$$g = \gamma = \beta \left[(1-\tau_k)\alpha \frac{\hat{y}}{\hat{k}_m} + 1 - \delta \right], \tag{6.11}$$

where $\hat{k}_{mt} = k_{mt}/\gamma^t$ and $\hat{y}_t = y_t/\gamma^t$.

The Growth Rate and the Discount Factor— The quarterly gross growth rate of output in Canada is 1.006– see (5.10). This value is assigned to γ in (6.11). The average quarterly return on capital employed is 1.64 %. Equating this figure to the expression, $\gamma/\beta - 1$, which is the return on capital implied by (6.11), yields $\beta = .99$.

The Capital Income Tax Rate— The law of motion of market capital along the BGP is

$$(\gamma + \delta - 1)\frac{\hat{k}_m}{\hat{y}} = \frac{\hat{i}_m}{\hat{y}} = .089.$$

The depreciation rate of market capital is set to .0137 as in Subsection 3.4. This implies the market capital-output ratio is 4.4. Setting the market capital share to .328 as in Subsection 3.4 and solving (6.11) for τ_k yields .596.

The Share of of Capital in Household Production— Along the BGP, (6.4d) becomes

$$\gamma = \beta \left(\eta \frac{1-a}{a} \frac{\hat{c}_h^e \hat{c}_m^{1-e}}{\hat{k}_h} + 1 - \delta \right). \tag{6.12}$$

One can get rid of the parameter a and the variables $\hat{c}_h$ and $\hat{c}_m$ in (6.12) using both (6.4a) and (6.4b). Actually, combining (6.4a) and (6.4b) gives along the BGP

$$\hat{c}_h^e = \frac{a}{(1-a)(1-\eta)}(1-\tau_l)\hat{w} l_h \hat{c}_m^{e-1},$$

which one can plug into (6.12) to have

$$\gamma = \beta\left[\frac{\eta}{1-\eta}\frac{(1-\tau_l)\hat{w}l_h}{\hat{k}_h} + 1 - \delta\right]. \tag{6.13}$$

The average value of the implicit income tax rate, which is 17.7%, is assigned to τ_l. The average value of the implicit government transfer rate over the period 1981-2012, which is 12.21%, is assigned to the ratio $\hat{\tau}/\hat{y}$. The time allocated to paid work and household work are respectively set to .297 and .188. The share in GDP of investment in household capital is .125, which implies that the household capital-output ratio is 6.19. Given this information, one can solve (6.13) for η and get .348.

The Persistence Parameters and Standard Deviations— As in Subsection 3.4, ρ is set to .95 and σ to .007. For the government, I have regressed the natural logarithm of its consumption expenditure on an intercept and a time trend to extract the residuals $\tilde{G}_t$.

$$\begin{aligned} \widehat{\ln G_t} &= \underset{(1778)}{12.19} + \underset{(45.71)}{.004t} \\ \bar{R}^2 &= .943, \qquad t_{2.5\%}(126) = 1.98 \end{aligned} \tag{6.14a}$$

$$\begin{aligned} \widehat{\tilde{G}_t} &= \underset{(55.99)}{.98\tilde{G}_{t-1}} \\ \bar{R}^2 &= .96, \qquad \sigma_G = .008 \end{aligned} \tag{6.14b}$$

The above regression results show that the quarterly rate of growth of government consumption expenditure, .004%, is lower than the rate of growth of the economy, .006%. But I will constrain this variable to grow at the same rate as the other trended variables in the model. This implies a correlation of -.02 between ϵ_{mt} and ϵ_{Gt}.

All that remains to be done before simulating the model is to assign values to the parameters a, e, υ, and $\sigma_{\epsilon_h,\epsilon_m}$. The value of a will depend on the value assigned to e. The higher the elasticity of substitution parameter, the higher the share of market-produced good. The elasticity of substitution parameter is used to compute the leisure weight υ but the value of the latter parameter is insensitive to the value assigned to the former.

6.5 Numerical Solution and Findings

I have chosen e and $\sigma_{\epsilon_h,\epsilon_m}$ so that the simulated standard deviation of the cyclical market consumption matches observations. This is achieved with $e = 3/4$ and setting $\sigma_{\epsilon_h,\epsilon_m}$ so that the correlation coefficient between ϵ_{mt} and ϵ_{ht} equals 2/3.

Table 6.1: The Parameters of the Household Production Model

Households	β	Discount factor	.99
	η	Capital share	.348
	τ_k	Capital income tax rate	.596
	τ_l	Labor income tax rate	.178
Firms	α	Capital share	.328
	γ	Growth rate	1.006
	δ	Depreciation rate	.0137
	ρ	Persistence parameter	.95
	σ	Standard deviation of innovation	.007
Government	ρ_G	Persistence parameter	.98
	σ_G	Standard deviation of innovation	.008

Table 6.2: Cyclical Behavior of the Canadian and the Household Production Economies, Percentage Deviation from Trend of Key Variables, 128 Observations

	Canadian Economy			Household Production		
Variable	(1)	(2)	(3)	(1)	(2)	(3)
Output (GDP)	1.51	1	.9	1.22	1	.81
Market Consumption	.94	.76	.75	.93	.87	.81
Investment	4.3	.8	.88	3.66	.94	.75
Market Hours	1.48	.91	.89	.67	.96	.81
Wage	1.2	-.21	.84	.59	.96	.77

Columns (1) display the percentage standard deviations, columns (2) display the correlation coefficient with output, and columns (3) display the first-order autocorrelation coefficient.
Market consumption is made up of non-durable and semi-durable goods.
Investment is the sum of the investment in market and household capital.
$e = 3/4$, $a = .596$, $\upsilon = .848$, and $\text{cor}(\epsilon_{mt}, \epsilon_{ht}) = 2/3$

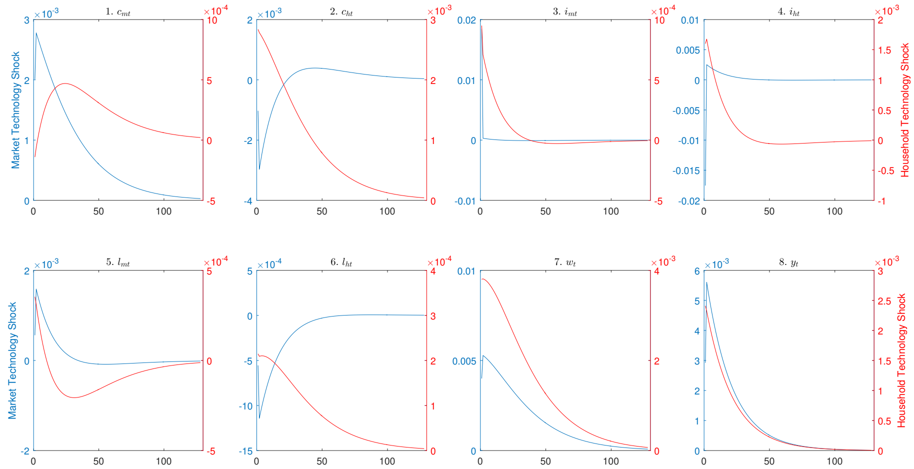

Figure 6.2: Impulse Responses, Deviation from Steady State, the Household Production Model, Market and Household Technology Shocks, $e = 3/4$

The model is simulated using both market and non-market technology shocks. The simulated standard deviations of cyclical output and investment reported in Table 6.2 represent respectively 81% and 85% of the actual observations. The model outperforms the basic, indivisible labor, and investment-specific technological changes models in explaining the volatility in consumption. But as far as hours worked in the market and wage are concerned, it has not substantially contributed to explaining their volatility.

Some weaknesses of the household production model are:

- It does not capture the negative correlation between productivity (or wage) and hours worked in the market. The correlation coefficient simulated is high and positive (.85).

- The actual correlation between cyclical investment in market and household capital is positive and low (.11). On the contrary, the simulated statistic is negative and high (-.96).

- The simulated investment in market and household capital series (that I have not reported here) are much more volatile than the actual series.

As far as the correlation between productivity and hours worked is concerned, adding a shock to government consumption does not help resolve this problem using the calibrated parameters.

Figure (6.2) shows the response of the economy to one-off shocks of size σ to the market and household technologies. The look and magnitude of the response of the economy are not the same in the two cases. A positive shock to z_{mt} initially raises wage and considerably shifts labor, consumption, and investment from the household production sector to the market. On the other hand, a positive shock to z_{ht} initially raises labor supply to both sectors, increases the consumption of home-produced goods and lowers the consumption of market-produced goods. While market investment crowds out home investment when a shock to z_{mt} takes place that is not the case when a shock to z_{ht} occurs. The reason why is that the household sector only produces consumption goods and hinges on the market to produce the capital it uses. An increase in the demand for household capital induced by a shock to z_{ht} will therefore boost activities in the market to meet this demand. The market sector will thus need more capital and labor inputs to produce the additional household capital.

7 The Household Production and Human Capital Accumulation Model

In the household production model, the correlation between cyclical investment in market and household capital is highly negative whereas the actual data indicate a positive correlation. A way to resolve this problem put forward by Einarsson and Marquis (1997) is to endogenize growth by introducing human capital accumulation in the model. Human capital accumulation is about allocating time to *education*, *viz.* schooling, training and skill development, in order to acquire knowledge. Therefore, along the intensive margin, a household could allocate his time between leisure, paid and household work, and education.

Human capital accumulation and macroeconomic fluctuations are interrelated. It impacts on the time households allocate to labor, on their income and physical capital accumulation as well as on their future productivity. Meanwhile, the overall state of an economy also impacts on human capital accumulation. During contractions, households take advantage of the fact that wage, the opportunity cost of not working, is low to improve their skills or develop new ones.

Einarsson and Marquis (1997) found that when the household production model is augmented with a human capital accumulation sector, one shock, only a shock to the market technology, instead of two shocks is enough to have the model replicate the positive correlation between cyclical investment in market and household capital. In their model, households do not suffer any disutility from allocating time to education and labor. Following DeJong and Ingram (2001) and Benhabib, Rogerson, and Wright (1991), I include this feature.

7.1 The Households

Households derive utility from the consumption of market- and home-produced goods, and leisure. Leisure is now defined as the share of time that has not been allocated to education, paid and household work. The representative household instantaneous utility is

$$\begin{aligned} u\left(c_{mt}, c_{ht}, e_t, l_{mt}, l_{ht}\right) &= \ln\left(C_t\right) + \upsilon \ln\left(1 - e_t - l_{mt} - l_{ht}\right) \\ C_t &= \left[a c_{mt}^{e} + (1-a) c_{ht}^{e}\right]^{1/e}, \end{aligned} \tag{7.1}$$

where e_t is the share of time he allocates to education.

He faces the following resource constraints

$$c_{mt} + i_{mt} + i_{ht} = (1-\tau_l) w_t h_t l_{mt} + (1-\tau_k) r_t k_{mt} + \tau_t \tag{7.2a}$$

$$k_{mt+1} = (1-\delta) k_{mt} + i_{mt} \tag{7.2b}$$

$$k_{ht+1} = (1-\delta) k_{ht} + i_{ht} \tag{7.2c}$$

$$c_{ht} = k_{ht}^{\eta} (z_{ht} h_t l_{ht})^{1-\eta} \tag{7.2d}$$

$$h_{t+1} = (1 + \psi_t e_t) h_t. \tag{7.2e}$$

Constraints (7.2a) through (7.2d) are the same as in the previous section except that the effective units of labor supplied are now a combination of both the time share supplied by the household and his human capital, h_t. Constraint (7.2e) indicates human capital is accumulated through allocating time to education. Education is publicly provided. The parameter $\psi_t > 0$ is the household's ability to learn also known as human capital productivity coefficient. It follows a stationary random process

$$\begin{aligned} \psi_t &= \bar{\psi} \exp(\tilde{\psi}_t) \\ \tilde{\psi}_t &= \rho \tilde{\psi}_{t-1} + \epsilon_{\psi t} \quad \epsilon_{\psi t} \sim \mathcal{N}\left(0, \sigma_{\psi}^2\right). \end{aligned} \tag{7.3}$$

The FOCs and Euler equations from the household optimization problem are the following. See details in Appendix B.4.

$$\frac{\upsilon}{a} C_t^{e} c_{mt}^{1-e} = (1-\tau_l) w_t h_t (1 - e_t - l_{mt} - l_{ht}) \tag{7.4a}$$

$$(1-a)(1-\eta) \left(\frac{c_{ht}}{C_t}\right)^{e} = \upsilon \frac{l_{ht}}{(1 - e_t - l_{mt} - l_{ht})} \tag{7.4b}$$

$$\beta \mathrm{E}_t \left\{ \left[(1-\tau_k) r_{t+1} + (1-\delta)\right] \left(\frac{c_{mt}}{c_{m,t+1}}\right)^{1-e} \left(\frac{C_t}{C_{t+1}}\right)^{e} \right\} = 1 \tag{7.4c}$$

$$\beta \mathrm{E}_t \left\{ \left[\eta \frac{1-a}{a} \frac{c_{h,t+1}}{k_{h,t+1}} \left(\frac{c_{mt}}{c_{h,t+1}} \right)^{1-e} + (1-\delta) \left(\frac{c_{mt}}{c_{m,t+1}} \right)^{1-e} \right] \left(\frac{C_t}{C_{t+1}} \right)^e \right\} = 1 \tag{7.4d}$$

$$\mathrm{E}_t \left\{ \left[(e_{t+1} + l_{m,t+1} + l_{h,t+1}) \psi_t + \frac{\psi_t}{\psi_{t+1}} \right] \frac{w_{t+1}}{(1-\tau_k) r_{t+1} + 1 - \delta} \right\} = w_t \tag{7.4e}$$

Relations (7.4a) and (7.4b) govern labor supply. Conditions (7.4a) and (7.4b) govern the trade-off between current consumption and investment in market and household capital. Relations (7.4e) compares the opportunity cost of allocating an additional unit of time to education to the present value of the expected gain that results from this investment.

7.2 The Firms

The market good is produced using physical capital and effective units of labor.

$$\begin{aligned} y_t &= k_{mt}^{\alpha} (z_{mt} h_t l_{mt})^{1-\alpha} \\ z_{mt} &= \gamma_z^t \exp(\tilde{z}_{mt}) \\ \tilde{z}_{m,t} &= \rho \tilde{z}_{m,t-1} + \epsilon_{mt}, \;\; \epsilon_{mt} \sim \mathcal{N}(0, \sigma^2) \end{aligned} \tag{7.5}$$

Firms maximize their profit, $k_{mt}^{\alpha} (z_{mt} h_t l_{mt})^{1-\alpha} - r_t k_{mt} - w_t h_t l_{mt}$ paying capital and labor their marginal products.

$$k_{mt}: \quad r_t = \alpha \frac{y_t}{k_{mt}} \tag{7.6}$$

$$l_{mt}: \quad w_t h_t = (1-\alpha) \frac{y_t}{l_{mt}}. \tag{7.7}$$

7.3 The Government

It returns part of its revenue made up of capital and labor income taxes to households as lump-sum transfers and uses the rest to finance its expenses G_t, which includes providing free education.

$$\tau_k r_t k_{mt} + \tau_l w_t h_t l_{mt} = \tau_t + G_t \tag{7.8}$$

The DSGE model consists of relations (7.2) through (7.8).

7.4 The Balanced Growth Path and Calibration

Since the autoregressive parameter in (7.2e) is greater than unity, human capital is trended. Along the BGP, it grows at the gross rate $\nu = 1 + \psi e$. It follows from (7.5) that output grows at the rate $g = \gamma\nu$ So do market and household physical capital, investment, and consumption. Hours worked and interest rate are stationary. Wage grows at the rate γ.

The output growth rate, g, is 1.006 and the labor-augmenting technological change expected growth rate, γ, is 1.002. The values of $\alpha, \beta, \delta, \eta, \tau_k$, and τ_l are borrowed from the previous section. If follows from the relation $g = \gamma\nu$ that ν equals 1.004. Evaluating (7.4e) along the BGP gives $\psi = \nu(1-\beta)/\beta(l_m + l_h)$. The shares of time allocated to paid and household work being respectively .297 and .188, ψ turns out to be equal to .02 and the share of time allocated to education equals .22. The share of time allocated to education implied by the model matches observations. According to the GSS of Statistics Canada,, a student allocates, on average, 5.25 hours a day to education and related activities. This represents about 21% of his discretionary time over over an academic year.

Since the value assigned to the normalized human capital, *i.e.* h_t/ν^t, along the BGP does not not impact on the model's parameter, I have set it to unity.

7.5 Numerical Solution and Findings

Table 7.1 shows some summary statistics from the Monte Carlo simulations considering the economy is hit by both market technology and human capital productivity shocks. Both shocks are contemporaneously uncorrelated and the standard deviation of the human capital productivity shock is set to .616 times that of the market technology. Figure (7.1) plots some impulse responses to both shocks.

The model explains all the fluctuations in output in addition to being able to replicate the volatility in hours worked and the observed correlation between the latter variable and cyclical productivity. However, it is still unable to replicate the observed positive correlation between investment in market and household capital. This tells us that introducing human capital accumulation might be a necessary but not a sufficient condition to generate a positive correlation between both types of investments. To be able to replicate this positive correlation, one has, in addition to introducing human capital accumulation, to drop the assumption that capital and labor in the household sector are independent inputs. This implies replacing (7.2d), the Cobb-Douglas production function Benhabib, Rogerson, and Wright (1991) and Greenwood, Rogerson, and Wright (1995) used, with the more general production function Greenwood and Hercowitz (1991)

$$c_{ht} = \left[\eta k_{ht}^s + (1-\eta)(z_{ht} h_t l_{ht})^s\right]^{1/s}.$$

Gangopadhyay and Hatchondo (2009) reviewed some models that successfully resolve the issue of simultaneity of market and household investment over the business cycle. One of these models is the time-to-build.

Table 7.1: Cyclical Behavior of the Canadian and the Household Production and Human Capital Accumulation Economies, Percentage Deviation from Trend of Key Variables, 128 Observations

	Canadian Economy			Household Production		
Variable	(1)	(2)	(3)	(1)	(2)	(3)
Output (GDP)	1.51	1	.9	1.51	1	.8
Market Consumption	.94	.76	.75	.41	.89	.73
Study Hours				2.89	-.84	.77
Human Capital				.06	-.16	.96
Investment	4.3	.8	.88	6.14	..99	.81
Market Hours	1.48	.91	.89	1.4	.92	.8
Wage	1.2	-.21	.84	.57	.36	.73

Columns (1) display the percentage standard deviations, columns (2) display the correlation coefficient with output, and columns (3) display the first-order autocorrelation coefficient.
Market consumption is made up of non-durable and semi-durable goods.
Investment is the sum of the investment in market and household capital.
$e = -.9$, $a = .434$, $\upsilon = .49$, $\mathrm{cor}(\epsilon_{mt}, \epsilon_{\psi t}) = 0$, $\sigma_\psi = .616 * \sigma$

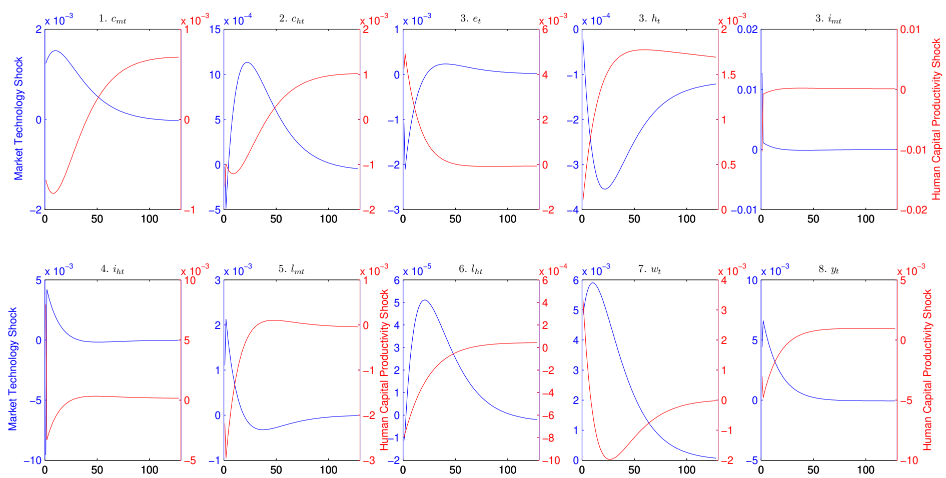

Figure 7.1: Impulse Responses, Deviation from Steady State, the Household Production and Human Capital Accumulation Model, Market and Human Capital Technology Shocks, $e = 0$

8 The Time-to-Build Model

Rome was not build in one day says a Frech proverb. As for this model proposed by Kydland and Prescott (1982), it says no production facility is set up within one quarter. It takes several quarters to design and complete an investment project. Thus, at any time, one can distinguish between three main types of physical capital: the stocks of inventory and productive capital, and non-inventory investments. Inventories are inputs just like labor and productive capital. The economy consists of infinitely-lived households and firms.

8.1 The Households

The representative household is endowed with preferences defined over consumption and leisure. These preferences are said to be non-time-separable because they depend on current and past leisure,

$$u\left[c_t, \Phi(\mathrm{L})(1-l_t)\right] = \frac{1}{e} c_t^{\frac{e}{3}} \left[\Phi(\mathrm{L})(1-l_t)\right]^{\frac{2}{3}e} \tag{8.1}$$

where e is a substitution parameter also known as risk aversion parameter and $\Phi(\mathrm{L})$ is a lag polynomial defined as follows

$$\Phi(\mathrm{L}) = \sum_{i=0}^{\infty} \Phi_i \mathrm{L}^i.$$

Assuming $\sum_{i=0}^{\infty} \Phi_i = 1$ and that $\Phi_i = (1-\chi)^{i-1}\Phi_1$ for $i \geq 1$, one then has

$$\begin{aligned}\Phi(\mathrm{L})(1-l_t) &= 1 - \sum_{i=0}^{\infty} \Phi_i \mathrm{L}^i l_t \\ &= 1 - \Phi_0 l_t - \Phi_1 \sum_{i=1}^{\infty} (1-\chi)^{i-1} \mathrm{L}^i l_t.\end{aligned}$$

Further assuming that $0 \leq \chi \leq 1$, one has summing the Φ_is, $\Phi_0 + \Phi_1/\chi = 1$, which we plug into the above relation to get

$$\Phi(\mathrm{L})(1-l_t) = 1 - \Phi_0 l_t - (1-\Phi_0)\chi \sum_{i=1}^{\infty} (1-\chi)^{i-1} \mathrm{L}^i l_t.$$

Finally, defining the variable $\varphi_t = \sum_{i=1}^{\infty} (1-\chi)^{i-1} \mathrm{L}^i l_t$, one ends up with the following recursive representation of current and past leisure

$$\begin{aligned}\Phi(\mathrm{L})(1-l_t) &= 1 - \Phi_0 l_t - (1-\Phi_0)\chi\varphi_t \\ \varphi_{t+1} &= (1-\chi)\varphi_t + l_t.\end{aligned} \tag{8.2}$$

The parameters Φ_0 and χ determine the degree of inter-temporal substitutability of leisure. The lower are these parameters, the higher is the effect of past leisure choices on current and future utility.

In (8.1), the relative weight of leisure is set to 2, which is almost the calibrated value of υ (1.904) in Subsection 3.4.

The representative household faces the following resource constraints

$$c_t + i_t = w_t l_t + r_t \varkappa_t + q_t(r_t + \delta)k_t \tag{8.3a}$$

$$k_{t+1} = (1-\delta)k_t + s_{1t} \tag{8.3b}$$

$$s_{j,t+1} = s_{j+1,t}, \ \ j = 1,2,3 \tag{8.3c}$$

$$i_t = \frac{1}{4}\sum_{j=1}^{4} s_{jt} + \varkappa_{t+1} - \varkappa_t. \tag{8.3d}$$

Constraint (8.3a) says he finances both his consumption and investment out of labor and capital incomes. The capital stocks that generate incomes are: the the inventory stock $\varkappa_t$ and the productive capital k_t. The inventory stock is made up of six cash flows $1/4\sum_{j=2}^{4}\sum_{v=t-(j-1)}^{t-1} s_{jv}$. Its rental price is r_t. The price of productive capital is q_t and its rental price is $q_t(r_t+\delta)$.

It takes four quarters to build capital. According to (8.3b), s_{1t}, which denotes an investment project that is currently one quarter away from completion, will be part of next period's productive capital. Constraint (8.3c) states that a project that is $j+1$ quarters from completion today will, by all means, be j periods from completion next quarter. Constraint (8.3d) says, each period, one-fourth of the values of the projects are put in place and investment is the sum of non-inventory and inventory investments.

The Euler equations from the optimization problem are:

$$\beta \mathrm{E}_t\left[(1+r_{t+1})c_{t+1}^{\frac{e}{3}-1}\left[\Phi(\mathrm{L})(1-l_{t+1})\right]^{\frac{2}{3}e}\right] = c_t^{\frac{e}{3}-1}\left[\Phi(\mathrm{L})(1-l_t)\right]^{\frac{2}{3}e} \tag{8.4a}$$

$$\begin{aligned}\mathrm{E}_t\left\{\left[2(\Phi_0-\chi)-(1-\chi)\Phi(\mathrm{L})(1-l_{t+1})\frac{w_{t+1}}{c_{t+1}}\right]c_{t+1}^{\frac{e}{3}}\left[\Phi(\mathrm{L})(1-l_{t+1})\right]^{\frac{2}{3}e-1}\right\}\\ = \frac{1}{\beta}\left[2\Phi_0-\Phi(\mathrm{L})(1-l_t)\frac{w_t}{c_t}\right]c_t^{\frac{e}{3}}\left[\Phi(\mathrm{L})(1-l_t)\right]^{\frac{2}{3}e-1}\end{aligned} \tag{8.4b}$$

$$q_t = \frac{1}{4}\left[\sum_{j=1}^{3}\prod_{v=t-(4-j)}^{t-1}(1+r_{v+1})+1\right] \tag{8.4c}$$

Equations (8.4a) governs the inter-temporal substitution of consumption and equation (8.4b) governs the inter-temporal substitution of leisure. Equation (8.4c) states the price of one unit of productive capital as the sum of the shares of investment made during each of the four stages of completion augmented with the compound interests they generate as inventories.

8.2 The Firms

The aggregate production technology is

$$y_t = \exp(z_t)\left[(1-\xi)k_t^{-\vartheta} + \xi \varkappa_t^{-\vartheta}\right]^{\frac{-\alpha}{\vartheta}} l_t^{1-\alpha}, \tag{8.5}$$

where $0 < \xi < 1$ is the share of inventories in physical capital stock and $\vartheta > 0$ is the parameter of the elasticity of substitution between productive capital and the stock of inventory. The TFP z_t is made up of a transitory component $\tilde{z}_t$ and a white noise ϵ_{2t}, with

$$\begin{aligned} \tilde{z}_t &= \rho \tilde{z}_{t-1} + \epsilon_{1t} \\ z_t &= \tilde{z}_t + \epsilon_{2t}. \end{aligned} \tag{8.6}$$

Households could not directly observe the TFP because of a corrupting noise ϵ_{3t}. The indicator of the state of technology they observe is

$$Z_t = z_t + \epsilon_{3t}, \quad \epsilon_{\imath t} \sim \mathcal{N}\left(0, \sigma_\imath^2\right), \ \imath = 1, 2, 3. \tag{8.7}$$

The FOCs from firms' profit maximization problem are:

$$k_t: \quad q_t(r_t + \delta) = \alpha \frac{(1-\xi)k_t^{-(\vartheta+1)}}{(1-\xi)k_t^{-\vartheta} + \xi\varkappa_t^{-\vartheta}} y_t \tag{8.8a}$$

$$\varkappa_t: \quad r_t = \alpha \frac{\xi\varkappa_t^{-(\vartheta+1)}}{(1-\xi)k_t^{-\vartheta} + \xi\varkappa_t^{-\vartheta}} y_t \tag{8.8b}$$

$$l_t: \quad w_t = (1-\alpha)\frac{y_t}{l_t} \tag{8.8c}$$

Equilibrium in all markets is described by relations (8.2) through (8.8).

8.3 Calibration

At steady state, the TFP parameter z and its indicator Z are nil because there is no innovation. As a consequence, no variable grows. One gets the following three relations evaluating (8.4a), (8.4b), and (8.4c) at steady state

$$r = \frac{\beta}{1-\beta} \tag{8.9a}$$

$$(1-\alpha)\frac{1-l}{l} = 2\frac{c}{y}\left(\Phi_0 + \frac{1-\Phi_0}{r+\chi}\chi\right) \tag{8.9b}$$

Table 8.1: The Parameters of the Time-to-Build Model

Households	β	Discount factor	.958
	Φ_0	Share of current leisure	.25
	χ	Leisure inter-temporal substitution parameter	.65
	e	Risk aversion parameter	-.5
Firms	α	Capital share	.328
	δ	Depreciation rate	.055
	ϑ	Capital substitution parameter	3.5
	ξ	Share of inventories in capital	.000005
	ρ	Persistence parameter	.95
	σ_1	Standard deviation of the innovation	.007
	σ_2	Standard deviation of the innovation	.007
	σ_3	Standard deviation of the innovation	.007

$$q = \frac{(1+r)^4 - 1}{4r}. \tag{8.9c}$$

The values of the parameters α, δ, ρ, and σ_1 are from the calibration exercise done in Subsection 3.4. The parameters e, Φ_0, χ, ϑ, σ_2, and σ_3 are free parameters, *i.e.*, their values will be set without using any data. I have set e and ϑ, respectively to -.5 and 3.5. Only one of the two parameters determining the inter-temporal substitutability of leisure, either Φ_0 or χ, has to be fixed. The other one will be determined from the model using (8.9b). As for the parameters β and ξ, they will be set evaluating the model at steady state and using observed data.

It follows from (8.3c) and (8.3d) that, at steady state, $i = s_1 = s_2 = s_3 = s_4$. The investment-output ratio is .165. The stock of inventory, which consists of six cash flows, should therefore be one-fourth of GDP ($\frac{\varkappa}{y} = \frac{6}{4} \times \frac{i}{y}$). The implied capital-output ratio, $\frac{k}{y} = \frac{i}{\delta y}$, equals 3. Note that the depreciation rate used, .055, is annual instead of quarterly. The reason is that it takes four quarters (one year) to build capital. Solving (8.5), (8.8a), (8.8b), along with the capital- and inventories-output ratios, one gets ξ, which equals .000005, as well as the steady state values of output, interest rate, the stocks of capital and inventories. Knowing the steady state interest rate, one computes β using (8.9a). Finally, setting χ to .65, Φ_0 turns out to be .25.

The values of the parameters are reported in Table 8.1.

8.4 Findings

Table 8.2 displays some business cycle summary statistics from the Canadian economy and the time-to-build model.

Table 8.2: Cyclical Behavior of the Canadian and the Time-to Build Economies, Percentage Deviation from Trend of Key Variables, 128 Observations

	Canadian Economy			Time-to-Build Economy		
Variable	(1)	(2)	(3)	(1)	(2)	(3)
Output (GDP)	1.51	1	.9	.85	1	.72
Consumption	1.15	.85	.85	.76	.22	.19
Investment	5.01	.79	.89	5.69	.75	.38
Hours	1.48	.91	.89	1.55	.43	.17
Wage	1.2	-.21	.84	1.42	.13	.12

Columns (1) display the percentage standard deviations, columns (2) display the correlation coefficient with output, and columns (3) display the first-order autocorrelation coefficient.

The time-to-build model turns out to be successful in explaining all the cyclical fluctuations observed in investment, hours, and wage. Furthermore, it replicates the near-zero correlation between cyclical productivity and hours worked. The time-to-build model explains all the cyclical fluctuations in output in the US and only half of the fluctuations in hours worked (Kydland and Prescott, 1982).

Figure 8.1 shows that whereas hours worked monotonically decrease in response to a one-off technology shock, wage (or productivity) temporarily increases.

9 Discussion

Some striking features of business cycles are: (1) the substantial persistence in the fluctuations of the aggregate economic variables, (2) the high positive correlation between total hours worked and output, and (3) the absence of correlation between average hours worked and productivity.

The ability of RBC models to replicate the persistence in the dynamics of macroeconomic variables depends on both how they generate and propagate fluctuations. The basic RBC model, which relies only on TFP shocks to generate business cycle fluctuations and on both the inter-temporal substitution of consumption and capital accumulation to propagate them, has not matched the data. The indivisible labor model by attributing all the fluctuations in total hours worked to variations in the number of workers outperformed the basic RBC model in generating volatility. When it comes to the persistence in the dynamics of output, investment, and hours worked, it has not done a better job. On the other hand, the home production and the home production coupled with the human capital accumulation models that rely on more than one shocks and several propagation mechanisms display higher persistence.

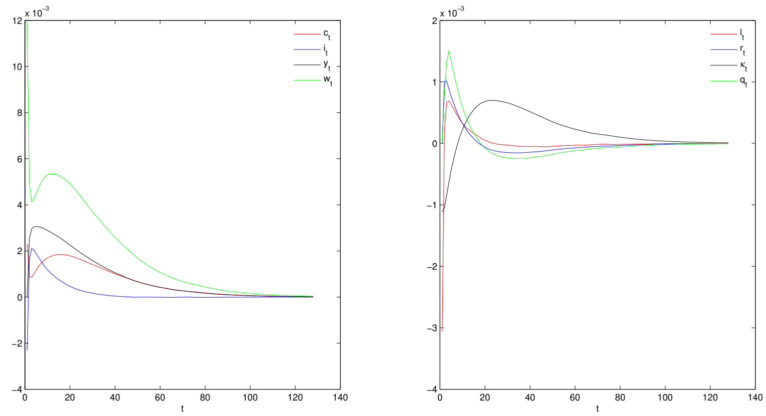

Figure 8.1: Impulse Responses to a $\tilde{z}$-shock, Deviation from Steady State, the Time-to-Build Model

All the models have generated high positive correlation between cyclical total hours worked and output but the one closest to observations is from the household production coupled with human capital accumulation model.

The other test most RBC models failed is their ability to replicate the near-zero correlation between productivity and hours worked over the business cycle. The features that have enabled replicating this correlation are: the introduction of human capital accumulation and non-time-separable preferences. Another feature that may work is the introduction of government consumption.

Gali (1999), among others, showed that using instead a monetary model with monopolistic competition and sticky prices, TFP shocks generate the near-zero correlation observed between the two latter variables. He also sustained that shocks other than those emphasized by the RBC theory are instrumental in explaining business cycles. As an example, he showed that much of the high positive correlation observed between cyclical output and hours stems from monetary shocks. The second part of this paper is dedicated to models that emphasize the role of non-technology shocks in business cycle fluctuations in Canada (see Accolley, 2016).

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Appendices

A The First-Difference Filter

Assume a stationary variable y_t following a general autoregressive moving average process of order p and q, $ARMA(p,q)$

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}.$$

Letting L denotes the lag operator, the $ARMA(p,q)$ process can be written as follows

$$\left(1 - \phi_1 \mathrm{L} - \phi_2 \mathrm{L}^2 - \cdots - \phi_p \mathrm{L}^p\right) y_t = (1 + \theta_1 \mathrm{L} + \theta_2 \mathrm{L}^2 + \cdots + \theta_q \mathrm{L}^q)\varepsilon_t,$$

The auto-covariance generating function of y_t, *i.e.*, the function expressing the sequence of the covariances of y_t with all of its lags and leads, is

$$\begin{aligned} \mathrm{g}_y(z) &= \sum_{j=-\infty}^{+\infty} \mathrm{cov}(y_t, y_{t-j}) z^j \\ &= \sigma^2 \frac{(1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q)(1 + \theta_1 z^{-1} + \theta_2 z^{-2} + \cdots + \theta_q z^{-q})}{(1 - \phi z - \phi_2 z^2 - \cdots - \phi_p z^p)(1 - \phi_1 z^{-1} - \phi_2 z^{-2} - \cdots - \phi_p z^p)} \end{aligned}$$

where σ^2 is the variance of ε_t.

It follows that the auto-covariance generating function of the first-difference filter $\Delta y_t = (1 - L) y_t$ is

$$\begin{aligned} \mathrm{g}_{\Delta y}(z) &= (1 - z)(1 - z^{-1})\mathrm{g}_y(z) \\ &= (2 - z - z^{-1})\mathrm{g}_y(z). \end{aligned}$$

For $z = \cos(\omega) - ß \sin(\omega)$, $z^{-1} = \cos(\omega) + ß \sin(\omega)$ and the above relation becomes

$$\mathrm{s}_{\Delta y}(\omega) = 2[1 - \cos(\omega)]\mathrm{s}_y(\omega),$$

where s_y, the population spectrum of y_t, equals g_y divided by 2π. For any $0 \leq \bar{\omega} \leq \pi$, the integral of $\mathrm{s}_y(\omega)$ between $-\bar{\omega}$ and $\bar{\omega}$ gives the portion of the variance y_t that can be attributed to cycle with frequencies less than or equal to $\bar{\omega}$. If a cycle is of frequency $\bar{\omega}$, its duration is $2\pi/\bar{\omega}$.

It turns out that $\mathrm{s}_{\Delta y}(0) = 0$, $s_{\Delta y}(\pi/2) = 2 s_y(\pi/2)$, and $\mathrm{s}_{\Delta y}(\pi) = 4 s_y(\pi)$. This means the first-difference filter removes the low-frequency components and accentuates the high-frequency components. For further details, see Hamilton (1994, pp 61-3 and 170-1).

B The Optimization Problems

B.1 The Basic RBC Model

$$\begin{aligned}\mathcal{V}(k_t, z_t) = \max_{c_t, l_t, i_t, k_{t+1}} & \ln c_t + \upsilon \ln(1-l_t) + \beta \mathrm{E}_t \mathcal{V}(k_{t+1}, z_{t+1}) \\ & + \mu_{1t}\left[w_t l_t + r_t k_t - c_t - i_t\right] \\ & + \mu_{2t}\left[(1-\delta)k_t + i_t - k_{t+1}\right] \end{aligned} \tag{B.1}$$

The First-Order Conditions (FOCs)

$$c_t : \quad \frac{1}{c_t} = \mu_{1t} \tag{B.2a}$$

$$l_t : \quad \frac{\upsilon}{1-l_t} = \mu_{1t} w_t \tag{B.2b}$$

$$i_t : \quad \mu_{1t} = \mu_{2t} \tag{B.2c}$$

$$k_{t+1} : \quad \beta \frac{\partial \mathrm{E}_t \mathcal{V}(k_{t+1}, z_{t+1})}{\partial k_{t+1}} = \mu_{2t} \tag{B.2d}$$

$$\begin{aligned} k_t : \quad \frac{\partial \mathcal{V}(k_t, z_t)}{\partial k_t} &= r_t \mu_{1t} + (1-\delta)\mu_{2t} \Rightarrow \\ \frac{\partial \mathcal{V}(k_{t+1}, z_{t+1})}{\partial k_{t+1}} &= r_{t+1}\mu_{1t+1} + (1-\delta)\mu_{2t+1} \end{aligned} \tag{B.2e}$$

It follows from the above conditions that:

$$\upsilon c_t = (1-l_t) w_t \tag{B.3a}$$

$$\beta \mathrm{E}_t \left[(1 + r_{t+1} - \delta)\frac{c_t}{c_{t+1}}\right] = 1. \tag{B.3b}$$

Relation (B.3a) is a linear combination of (B.2a) and (B.2b). To get (B.3b), first, plug (B.2e) into (B.2d) to have

$$r_{t+1}\mu_{1t+1} + (1-\delta)\mu_{2t+1} = \mu_{2t}.$$

Then use the equality in (B.2c) along with (B.2a).

B.2 The Investment-Specific Technological Change Model

$$\mathcal{V}(k_{et}, k_{st}, z_t, q_t) = \max_{c_t, l_t, i_{et}, i_{st}, k_{e,t+1}, k_{s,t+1}, \hbar_t} \ln c_t + \upsilon \ln(1-l_t) + \beta \mathrm{E}_t \mathcal{V}\left(k_{e,t+1}, k_{s,t+1}, z_{t+1, q_{t+1}}\right)$$

$$
\begin{aligned}
&+\mu_{1t}\left[(1-\tau_l)w_t l_t + (1-\tau_k)(r_{et}\hbar_t k_{et} + r_{st}k_{st}) + \tau_t - c_t - i_{et} - i_{st}\right] \\
&-\mu_{1t}\left[\exp(\tilde{q}_t)\phi_e \frac{\left(k_{e,t+1}-\kappa_e k_{et}\right)^2}{q_t k_{et}} + \phi_s \frac{\left(k_{s,t+1}-\kappa_s k_{st}\right)^2}{k_{st}}\right] \\
&+\mu_{2t}\left[\left(1-\frac{b}{\omega}\hbar_t^{\omega}\right)k_{et} + i_{et}q_t - k_{e,t+1}\right] \\
&+\mu_{3t}\left[(1-\delta_s)k_{st} + i_{st} - k_{s,t+1}\right]
\end{aligned}
\tag{B.4}
$$

The FOCs

$$
\begin{aligned}
c_t: &\quad \frac{1}{c_t} = \mu_{1t} && \text{(B.5a)} \\
l_t: &\quad \frac{\upsilon}{1-l_t} = \mu_{1t}(1-\tau_l)w_t && \text{(B.5b)} \\
i_{et}: &\quad \mu_{1t} = \mu_{2t}q_t && \text{(B.5c)} \\
i_{st}: &\quad \mu_{1t} = \mu_{3t} && \text{(B.5d)} \\
k_{e,t+1}: &\quad \beta\frac{\partial \mathrm{E}_t \mathcal{V}\left(\bullet_{t+1}\right)}{\partial k_{e,t+1}} = 2\mu_{1t}\exp(\tilde{q}_t)\phi_e \frac{k_{et+1}-\kappa_e k_{et}}{q_t k_{et}} + \mu_{2t} && \text{(B.5e)} \\
k_{s,t+1}: &\quad \beta\frac{\partial \mathrm{E}_t \mathcal{V}\left(\bullet_{t+1}\right)}{\partial k_{s,t+1}} = 2\mu_{1t}\phi_s \frac{k_{st+1}-\kappa_s k_{st}}{k_{st}} + \mu_{3t} && \text{(B.5f)} \\
\hbar_t: &\quad \mu_{1t}(1-\tau_k)r_{et} = \mu_{2t} b \hbar_t^{\omega-1} && \text{(B.5g)}
\end{aligned}
$$

The Envelope Conditions

$$
\begin{aligned}
k_{e,t}: &\quad \frac{\partial \mathcal{V}\left(\bullet_t\right)}{\partial k_{et}} = \mu_{1t}\exp(\tilde{q}_t)\phi_e \frac{k_{e,t+1}-\kappa_e k_{et}}{q_t k_{et}}\left(2\kappa_e + \frac{k_{e,t+1}-\kappa_e k_{et}}{k_{et}}\right) \\
&\qquad + \mu_{1t}(1-\tau_k)r_{et}\hbar_t + \mu_{2t}\left(1-\frac{b}{\omega}\hbar_t^{\omega}\right) && \text{(B.6a)} \\
k_{st}: &\quad \frac{\partial \mathcal{V}\left(\bullet_t\right)}{\partial k_{st}} = \mu_{1t}\phi_s \frac{k_{s,t+1}-\kappa_s k_{st}}{k_{st}}\left(2\kappa_s + \frac{k_{s,t+1}-\kappa_s k_{st}}{k_{st}}\right) \\
&\qquad + \mu_{1t}(1-\tau_k)r_{st} + \mu_{3t}(1-\delta_s) && \text{(B.6b)}
\end{aligned}
$$

Plugging now the first leads of (B.6a) and (B.6b) into (B.5e) and (B.5f) then getting rid of the Lagrange multipliers using the FOCs (B.5a), (B.5c) and (B.5d) yields

$$\upsilon c_t = (1-\tau_l) w_t (1-l_t) \tag{B.7a}$$

$$\begin{aligned}
&\beta \left[(1-\tau_k) r_{e,t+1} \hbar_{t+1} q_{t+1} + \left(1 - \frac{b}{\omega} \hbar_{t+1}^{\omega}\right)\right] + \beta \exp(\tilde{q}_{t+1}) \times \\
&\phi_e \frac{k_{e,t+2} - \kappa_e k_{e,t+1}}{k_{e,t+1}} \left(2\kappa_e + \frac{k_{e,t+2} - \kappa_e k_{e,t+1}}{k_{e,t+1}}\right) = \frac{c_{t+1}}{c_t} \frac{q_{t+1}}{q_t} \times \\
&\left[1 + 2\exp(\tilde{q}_t) \phi_e \frac{k_{et+1} - \kappa_e k_{et}}{k_{et}}\right]
\end{aligned} \tag{B.7b}$$

$$\begin{aligned}
&\beta \left[(1-\tau_k) r_{s,t+1} + (1-\delta_s)\right] + \beta \phi_s \frac{k_{s,t+2} - \kappa_s k_{s,t+1}}{k_{s,t+1}} \times \\
&\left(2\kappa_s + \frac{k_{s,t+2} - \kappa_s k_{s,t+1}}{k_{s,t+1}}\right) = \frac{c_{t+1}}{c_t} \times \\
&\left(1 + 2\phi_s \frac{k_{st+1} - \kappa_s k_{st}}{k_{st}}\right)
\end{aligned} \tag{B.7c}$$

$$(1-\tau_k) r_{et} q_t = b \hbar_t^{\omega - 1} \tag{B.7d}$$

The Normalized Equations

Let's define: $\hat{\mathbf{x}}_t = \frac{\mathbf{x}_t}{g^t}$ with $\mathbf{x}_t = (c_t, i_{et}, i_{st}, k_{st}, w_t, \tau_t)$, $\tilde{k}_{et} = \frac{k_{et}}{(g\gamma_q)^t}$, $\bar{q}_t = \frac{q_t}{\gamma_q}$, $\bar{z}_t = \frac{z_t}{\gamma_z}$, $\check{r}_{et} = \gamma_q^t r_{et}$, $\kappa_e = g\gamma_q$, and $\kappa_s = g$.
The adjustment cost parameters ϕ_e and ϕ_s are related as follows $g_e^2 \phi_e = g^2 \phi_s$. Let's set $\phi_e = \phi$ to have $\phi_s = \gamma_q^2 \phi$.

$$\upsilon \hat{c}_t = (1-\tau_l) \hat{w}_t (1-l_t) \tag{B.8a}$$

$$\begin{aligned}
&\frac{\beta}{g\gamma_q} \left[(1-\tau_k) \check{r}_{e,t+1} \hbar_{t+1} \bar{q}_{t+1} + \left(1 - \frac{b}{\omega} \hbar_{t+1}^{\omega}\right)\right] + \beta \exp(\tilde{q}_{t+1}) \times \\
&\phi g\gamma_q \frac{\tilde{k}_{e,t+2} - \tilde{k}_{e,t+1}}{\tilde{k}_{e,t+1}} \left(2 + \frac{\tilde{k}_{e,t+2} - \tilde{k}_{e,t+1}}{\tilde{k}_{e,t+1}}\right) = \frac{\hat{c}_{t+1}}{\hat{c}_t} \frac{\bar{q}_{t+1}}{\bar{q}_t} \times \\
&\left[1 + 2\exp(\tilde{q}_t) \phi g\gamma_q \frac{\tilde{k}_{et+1} - \tilde{k}_{et}}{\tilde{k}_{et}}\right]
\end{aligned} \tag{B.8b}$$

$$\begin{aligned}
&\frac{\beta}{g} \left[(1-\tau_k) r_{s,t+1} + (1-\delta_s)\right] + \beta \phi g \gamma_q^2 \frac{\hat{k}_{s,t+2} - \hat{k}_{s,t+1}}{\hat{k}_{s,t+1}} \times \\
&\left(2 + \frac{\hat{k}_{s,t+2} - \hat{k}_{s,t+1}}{\hat{k}_{s,t+1}}\right) = \frac{\hat{c}_{t+1}}{\hat{c}_t} \times \\
&\left(1 + 2\phi g \gamma_q^2 \frac{\hat{k}_{st+1} - \hat{k}_{st}}{\hat{k}_{st}}\right)
\end{aligned} \tag{B.8c}$$

$$(1-\tau_k)\check{r}_{et}\bar{q}_t = b\hbar_t^{\omega-1} \tag{B.8d}$$

$$\begin{aligned}\hat{c}_t + \hat{i}_{et} + \hat{i}_{st} &= (1-\tau_l)\hat{w}_t l_t \\ +(1-\tau_k)(\check{r}_{et}\hbar_t\tilde{k}_{et} + r_{st}\hat{k}_{st}) &+ \hat{\tau}_t - \exp(\tilde{q}_t)\times \\ \phi g\gamma_q\left(\frac{\tilde{k}_{e,t+1}}{\bar{q}_t} - \frac{\tilde{k}_{et}}{\bar{q}_t}\right)^2 \frac{\bar{q}_t}{\tilde{k}_{et}} &- \phi g\gamma_q^2 \frac{\left(\hat{k}_{s,t+1} - \hat{k}_{st}\right)^2}{\hat{k}_{st}}\end{aligned} \tag{B.8e}$$

$$\left(1 - \frac{b}{\omega}\hbar_t^{\omega}\right)\tilde{k}_{et} + \hat{i}_{et}q_t = g\gamma_q\tilde{k}_{e,t+1} \tag{B.8f}$$

$$(1-\delta_s)\hat{k}_{st} + \hat{i}_{st} = g\hat{k}_{s,t+1} \tag{B.8g}$$

$$\bar{q}_t = \exp(\tilde{q}_t) \tag{B.8h}$$

$$\tilde{q}_t = \rho_q\tilde{q}_{t-1} + \epsilon_{qt} \tag{B.8i}$$

$$\bar{z}_t\left(\hbar_t\tilde{k}_{et}\right)^{\alpha_e}\hat{k}_{st}^{\alpha_s} l_t^{1-\alpha_e-\alpha_s} = \hat{y}_t \tag{B.8j}$$

$$\bar{z}_t = \exp(\zeta_t) \tag{B.8k}$$

$$\alpha_e\frac{\hat{y}_t}{\tilde{k}_{et}} = \check{r}_{et}\hbar_t \tag{B.8l}$$

$$\alpha_s\frac{\hat{y}_t}{\hat{k}_{st}} = r_{st} \tag{B.8m}$$

$$(1-\alpha_e-\alpha_s)\frac{\hat{y}_t}{l_t} = \hat{w}_t \tag{B.8n}$$

$$\tau_k\left(\check{r}_{et}h_t\tilde{k}_{et} + r_{st}\hat{k}_{st}\right) + \tau_l\hat{w}_{lt}l_t = \hat{\tau}_t \tag{B.8o}$$

B.3 The Household Production Model

$$\begin{aligned}\mathcal{V}\left(k_{mt}, k_{ht}, z_{mt}, z_{ht}\right) = \max\ & \frac{1}{e}\ln\left[ac_{mt}^e + (1-a)c_{ht}^e\right] + \upsilon\ln\left(1 - l_{mt} - l_{ht}\right) \\ & + \beta \mathrm{E}_t\mathcal{V}\left(k_{m,t+1}, k_{h,t+1}, z_{m,t+1}, z_{h,t+1}\right) \\ & + \mu_{1t}\left[1-\tau_l)w_t l_{mt} + (1-\tau_k)r_t k_{mt} + \tau_t - c_{mt} - i_{mt} - i_{ht}\right] \\ & + \mu_{2t}\left[(1-\delta)k_{m()t} + i_{mt} - k_{mt+1}\right] \\ & + \mu_{3t}\left[(1-\delta)k_{h()t} + i_{ht} - k_{ht+1}\right] \\ & + \mu_{4t}\left[k_{ht}^{\eta}(z_{ht}l_{ht})^{1-\eta} - c_{ht}\right]\end{aligned} \tag{B.9}$$

The FOCs

$$c_{mt}: \quad a\frac{c_{mt}^{e-1}}{ac_{mt}^e + (1-a)c_{ht}^e} = \mu_{1t} \tag{B.10a}$$

$$c_{ht}: \quad (1-a)\frac{c_{ht}^{e-1}}{ac_{mt}^e + (1-a)c_{ht}^e} = \mu_{4t} \tag{B.10b}$$

$$l_{mt}: \quad \frac{\upsilon}{1-l_{mt}-l_{ht}} = \mu_{1t}(1-\tau_l)w_t \tag{B.10c}$$

$$l_{ht}: \quad \frac{\upsilon}{1-l_{mt}-l_{ht}} = \mu_{4t}(1-\eta)\frac{c_{ht}}{l_{ht}} \tag{B.10d}$$

$$i_{mt}: \quad \mu_{1t} = \mu_{2t} \tag{B.10e}$$

$$i_{ht}: \quad \mu_{1t} = \mu_{3t} \tag{B.10f}$$

$$k_{m,t+1}: \quad \beta\frac{\partial \mathrm{E}_t \mathcal{V}(\bullet_{t+1})}{\partial k_{m,t+1}} = \mu_{2t} \tag{B.10g}$$

$$k_{h,t+1}: \quad \beta\frac{\partial \mathrm{E}_t \mathcal{V}(\bullet_{t+1})}{\partial k_{h,t+1}} = \mu_{3t} \tag{B.10h}$$

The Envelope Conditions

$$k_{mt}: \quad \frac{\partial \mathcal{V}(\bullet_t)}{\partial k_{mt}} = \mu_{1t}(1-\tau_k)r_t + (1-\delta)\mu_{2t} \tag{B.11a}$$

$$k_{ht}: \quad \frac{\partial \mathcal{V}(\bullet_t)}{\partial k_{ht}} = (1-\delta)\mu_{3t} + \mu_{4t}\eta\frac{c_{ht}}{k_{ht}} \tag{B.11b}$$

Plugging the leads of the envelope conditions into the FOCs and rearranging gives

$$\frac{\upsilon}{a}C_t^e c_{mt}^{1-e} = (1-\tau_l)w_t(1-l_{mt}-l_{ht}) \tag{B.12a}$$

$$(1-a)(1-\eta)\left(\frac{c_{ht}}{C_t}\right)^e = \upsilon\frac{l_{ht}}{(1-l_{mt}-l_{ht})} \tag{B.12b}$$

$$\beta \mathrm{E}_t\left\{[(1-\tau_k)r_{t+1} + (1-\delta)]\left(\frac{c_{mt}}{c_{m,t+1}}\right)^{1-e}\left(\frac{C_t}{C_{t+1}}\right)^e\right\} = 1 \tag{B.12c}$$

$$\beta \mathrm{E}_t\left\{\left[\eta\frac{1-a}{a}\frac{c_{h,t+1}}{k_{h,t+1}}\left(\frac{c_{mt}}{c_{h,t+1}}\right)^{1-e} + (1-\delta)\left(\frac{c_{mt}}{c_{m,t+1}}\right)^{1-e}\right]\left(\frac{C_t}{C_{t+1}}\right)^e\right\} = 1, \tag{B.12d}$$

with $C_t = [ac_{mt}^e + (1-a)c_{ht}^e]^{1/e}$.

B.4 The Household Production and Human Capital Accumulation Model

$$
\begin{aligned}
\mathcal{V}\left(h_t, k_{mt}, k_{ht}, z_{mt}, z_{ht}\right) = \max\ & \frac{1}{e}\ln\left[ac_{mt}^{e} + (1-a)c_{ht}^{e}\right] + \upsilon \ln\left(1 - e_t - l_{mt} - l_{ht}\right) \\
& + \beta \mathrm{E}_t \mathcal{V}\left(h_{t+1}, k_{m,t+1}, k_{h,t+1}, z_{m,t+1}, z_{h,t+1}\right) \\
& + \mu_{1t}\left[1-\tau_l)w_t h_t l_{mt} + (1-\tau_k)r_t k_{mt} + \tau_t - c_{mt} - i_{mt} - i_{ht}\right] \\
& + \mu_{2t}\left[(1-\delta)k_{m()t} + i_{mt} - k_{mt+1}\right] \\
& + \mu_{3t}\left[(1-\delta)k_{h()t} + i_{ht} - k_{ht+1}\right] \\
& + \mu_{4t}\left[k_{ht}^{\eta}(z_{ht}h_t l_{ht})^{1-\eta} - c_{ht}\right] + \mu_{5t}\left[(1+\psi_t e_t)h_t - h_{t+1}\right]
\end{aligned}
\tag{B.13}
$$

The FOCs

$$c_{mt}: \quad a\frac{c_{mt}^{e-1}}{ac_{mt}^{e} + (1-a)c_{ht}^{e}} = \mu_{1t} \tag{B.14a}$$

$$c_{ht}: \quad (1-a)\frac{c_{ht}^{e-1}}{ac_{mt}^{e} + (1-a)c_{ht}^{e}} = \mu_{4t} \tag{B.14b}$$

$$e_t: \quad \frac{\upsilon}{1 - e_t - l_{mt} - l_{ht}} = \mu_{5t}\psi_t h_t \tag{B.14c}$$

$$l_{mt}: \quad \frac{\upsilon}{1 - e_t - l_{mt} - l_{ht}} = \mu_{1t}(1-\tau_l)w_t h_t \tag{B.14d}$$

$$l_{ht}: \quad \frac{\upsilon}{1 - e_t - l_{mt} - l_{ht}} = \mu_{4t}(1-\eta)\frac{c_{ht}}{l_{ht}} \tag{B.14e}$$

$$i_{mt}: \quad \mu_{1t} = \mu_{2t} \tag{B.14f}$$

$$i_{ht}: \quad \mu_{1t} = \mu_{3t} \tag{B.14g}$$

$$h_{t+1}: \quad \beta\frac{\partial \mathrm{E}_t \mathcal{V}(\bullet_{t+1})}{\partial h_{t+1}} = \mu_{5t} \tag{B.14h}$$

$$k_{m,t+1}: \quad \beta\frac{\partial \mathrm{E}_t \mathcal{V}(\bullet_{t+1})}{\partial k_{m,t+1}} = \mu_{2t} \tag{B.14i}$$

$$k_{h,t+1}: \quad \beta\frac{\partial \mathrm{E}_t \mathcal{V}(\bullet_{t+1})}{\partial k_{h,t+1}} = \mu_{3t} \tag{B.14j}$$

The Envelope Conditions

$$h_t: \quad \frac{\partial \mathcal{V}(\bullet_t)}{\partial h_t} = \mu_{1t}(1-\tau_l)w_t l_{mt} + (1-\eta)\mu_{4t}\frac{c_{ht}}{h_t} + \mu_{5t}(1+\psi_t e_t) \tag{B.15a}$$

$$k_{mt}: \quad \frac{\partial \mathcal{V}(\bullet_t)}{\partial k_{mt}} = \mu_{1t}(1-\tau_k)r_t + (1-\delta)\mu_{2t} \tag{B.15b}$$

$$k_{ht}: \quad \frac{\partial \mathcal{V}(\bullet_t)}{\partial k_{ht}} = (1-\delta)\mu_{3t} + \mu_{4t}\eta\frac{c_{ht}}{k_{ht}} \tag{B.15c}$$

Plugging the leads of the envelope conditions into the FOCs and rearranging gives

$$\frac{\upsilon}{a}C_t^e c_{mt}^{1-e} = (1-\tau_l)w_t h_t(1-e_t-l_{mt}-l_{ht}) \tag{B.16a}$$

$$(1-a)(1-\eta)\left(\frac{c_{ht}}{C_t}\right)^e = \upsilon\frac{l_{ht}}{(1-e_t-l_{mt}-l_{ht})} \tag{B.16b}$$

$$\beta \mathrm{E}_t\left\{[(1-\tau_k)r_{t+1}+(1-\delta)]\left(\frac{c_{mt}}{c_{m,t+1}}\right)^{1-e}\left(\frac{C_t}{C_{t+1}}\right)^e\right\} = 1 \tag{B.16c}$$

$$\beta \mathrm{E}_t\left\{\left[\eta\frac{1-a}{a}\frac{c_{h,t+1}}{k_{h,t+1}}\left(\frac{c_{mt}}{c_{h,t+1}}\right)^{1-e} + (1-\delta)\left(\frac{c_{mt}}{c_{m,t+1}}\right)^{1-e}\right]\left(\frac{C_t}{C_{t+1}}\right)^e\right\} = 1 \tag{B.16d}$$

$$\mathrm{E}_t\left\{\left[(e_{t+1}+l_{m,t+1}+l_{h,t+1})\,\psi_t + \frac{\psi_t}{\psi_{t+1}}\right]\frac{w_{t+1}}{(1-\tau_k)r_{t+1}+1-\delta}\right\} = w_t, \tag{B.16e}$$

with $C_t = [ac_{mt}^e + (1-a)c_{ht}^e]^{1/e}$.

I now show how the Euler equation (B.16e) has been derived. First, express μ_{4t} and μ_{55t} as a function of μ_{1t} using (B.14c) through (B.14e) to get

$$\mu_{4t} = \mu_{1t}\frac{1-\tau_l}{1-\eta}\frac{w_t h_t l_{ht}}{c_{ht}}$$
$$\mu_{5t} = \mu_{1t}(1-\tau_l)\frac{w_t}{\psi_t}.$$

Replace then μ_{4t} and μ_{55t} with the above expressions in (B.15a) toget

$$\frac{\partial \mathcal{V}(\bullet_t)}{\partial h_t} = \left(e_t + l_{mt} + l_{ht} + \frac{1}{\psi_t}\right)(1-\tau_l)w_t\mu_{1t}.$$

Plugging the first lead of the above relation into B.14h yields after rearranging

$$\beta \mathrm{E}_t\left\{\left[(e_{t+1}+l_{m,t+1}+l_{h,t+1})\,\psi_t + \frac{\psi_t}{\psi_{t+1}}\right]\frac{\mu_{1,t+1}}{\mu_1 t}w_{t+1}\right\} = w_t. \tag{B.17}$$

From (B.15b) and (B.14f), one has

$$\mu_{1t} = \frac{\frac{\partial \mathcal{V}(\bullet_t)}{\partial k_{mt}}}{(1-\tau_k)r_t + 1 - \delta}.$$

Replacing $\mu_{1,t+1}$ in (B.17) with the first lead of the above relation and replacing μ_{1t} with (B.14i), one finally get (B.16e).

I now solve the representative household optimization problem replacing in (B.13) the home-produced good technology with a constant elasticity of substitution production function.

$$\begin{aligned}\mathcal{V}(h_t, k_{mt}, k_{ht}, z_{mt}, z_{ht}) = \max\ & \frac{1}{e}\ln\left[ac_{mt}^e + (1-a)c_{ht}^e\right] + \upsilon \ln(1 - e_t - l_{mt} - l_{ht}) \\ & + \beta \mathrm{E}_t \mathcal{V}(h_{t+1}, k_{m,t+1}, k_{h,t+1}, z_{m,t+1}, z_{h,t+1}) \\ & + \mu_{1t}\left[1-\tau_l)w_t h_t l_{mt} + (1-\tau_k)r_t k_{mt} + \tau_t - c_{mt} - i_{mt} - i_{ht}\right] \\ & + \mu_{2t}\left[(1-\delta)k_{m()t} + i_{mt} - k_{mt+1}\right] \\ & + \mu_{3t}\left[(1-\delta)k_{h()t} + i_{ht} - k_{ht+1}\right] \\ & + \mu_{4t}\left\{\left[\eta k_{ht}^s + (1-\eta)(z_{ht}h_t l_{ht})^s\right]^{1/s} - c_{ht}\right\} \\ & + \mu_{5t}\left[(1+\psi_t e_t)h_t - h_{t+1}\right]\end{aligned} \tag{B.18}$$

The FOCs

$$c_{mt}: \quad a\frac{c_{mt}^{e-1}}{ac_{mt}^e + (1-a)c_{ht}^e} = \mu_{1t} \tag{B.19a}$$

$$c_{ht}: \quad (1-a)\frac{c_{ht}^{e-1}}{ac_{mt}^e + (1-a)c_{ht}^e} = \mu_{4t} \tag{B.19b}$$

$$e_t: \quad \frac{\upsilon}{1 - e_t - l_{mt} - l_{ht}} = \mu_{5t}\psi_t h_t \tag{B.19c}$$

$$l_{mt}: \quad \frac{\upsilon}{1 - e_t - l_{mt} - l_{ht}} = \mu_{1t}(1-\tau_l)w_t h_t \tag{B.19d}$$

$$l_{ht}: \quad \frac{\upsilon}{1 - e_t - l_{mt} - l_{ht}} = \mu_{4t}(1-\eta)\frac{(z_{ht}h_t l_{ht})^s}{l_{ht}} \times \left[\eta k_{ht}^s + (1-\eta)(z_{ht}h_t l_{ht})^s\right]^{1/s-1} \tag{B.19e}$$

$$i_{mt}: \quad \mu_{1t} = \mu_{2t} \tag{B.19f}$$

$$i_{ht}: \quad \mu_{1t} = \mu_{3t} \tag{B.19g}$$

$$h_{t+1}: \quad \beta\frac{\partial \mathrm{E}_t \mathcal{V}(\bullet_{t+1})}{\partial h_{t+1}} = \mu_{5t} \tag{B.19h}$$

$$k_{m,t+1}: \quad \beta \frac{\partial \mathrm{E}_t \mathcal{V}(\bullet_{t+1})}{\partial k_{m,t+1}} = \mu_{2t} \tag{B.19i}$$

$$k_{h,t+1}: \quad \beta \frac{\partial \mathrm{E}_t \mathcal{V}(\bullet_{t+1})}{\partial k_{h,t+1}} = \mu_{3t} \tag{B.19j}$$

The Envelope Conditions

$$h_t: \quad \frac{\partial \mathcal{V}(\bullet_t)}{\partial h_t} = \mu_{1t}(1-\tau_l)w_t l_{mt} + \mu_{5t}(1+\psi_t e_t)$$
$$\mu_{4t}(1-\eta)\frac{(z_{ht}h_t l_{ht})^s}{h_t}\left[\eta k_{ht}^s + (1-\eta)(z_{ht}h_t l_{ht})^s\right]^{1/s-1} \tag{B.20a}$$

$$k_{mt}: \quad \frac{\partial \mathcal{V}(\bullet_t)}{\partial k_{mt}} = \mu_{1t}(1-\tau_k)r_t + (1-\delta)\mu_{2t} \tag{B.20b}$$

$$k_{ht}: \quad \frac{\partial \mathcal{V}(\bullet_t)}{\partial k_{ht}} = (1-\delta)\mu_{3t} + \mu_{4t}\eta k_{ht}^{s-1}\left[\eta k_{ht}^s + (1-\eta)(z_{ht}h_t l_{ht})^s\right]^{1/s-1} \tag{B.20c}$$

Getting rid of the Lagrange multipliers

$$\frac{\upsilon}{a}C_t^e c_{mt}^{1-e} = (1-\tau_l)w_t h_t(1-e_t-l_{mt}-l_{ht}) \tag{B.21a}$$

$$(1-a)(1-\eta)\left(\frac{c_{ht}}{C_t}\right)^e = \upsilon\frac{l_{ht}}{(1-e_t-l_{mt}-l_{ht})}\times$$
$$\frac{\eta k_{ht}^s + (1-\eta)(z_{ht}h_t l_{ht})^s}{(z_{ht}h_t l_{ht})^s} \tag{B.21b}$$

$$\beta \mathrm{E}_t\left\{\left[(1-\tau_k)r_{t+1} + (1-\delta)\right]\left(\frac{c_{mt}}{c_{m,t+1}}\right)^{1-e}\left(\frac{C_t}{C_{t+1}}\right)^e\right\} = 1 \tag{B.21c}$$

$$\beta \mathrm{E}_t\left[\eta\frac{1-a}{a}\frac{c_{h,t+1}}{k_{h,t+1}}\left(\frac{c_{mt}}{c_{h,t+1}}\right)^{1-e}\left(\frac{C_t}{C_{t+1}}\right)^e\frac{k_{h,t+1}^s}{\eta k_{h,t+1}^s + (1-\eta)(z_{h,t+1}h_{t+1}l_{h,t+1})^s}\right]$$
$$+\beta \mathrm{E}_t\left[(1-\delta)\left(\frac{c_{mt}}{c_{m,t+1}}\right)^{1-e}\left(\frac{C_t}{C_{t+1}}\right)^e\right] = 1 \tag{B.21d}$$

$$\mathrm{E}_t\left\{\left[\left(e_{t+1}+l_{m,t+1}+l_{h,t+1}\right)\psi_t + \frac{\psi_t}{\psi_{t+1}}\right]\frac{w_{t+1}}{(1-\tau_k)r_{t+1}+1-\delta}\right\} = w_t, \tag{B.21e}$$

B.5 The Time-to-Build Model

$$
\begin{aligned}
\mathcal{V}\left(\mathcal{S}_t\right) = \frac{1}{e}\left\{c_t^{1/3}\left[1-\Phi_0 l_t-(1-\Phi_0)\chi\varphi_t\right]^{2/3}\right\}^e & + \beta \mathrm{E}_t \mathcal{V}\left(\mathcal{S}_{t+1}\right) \\
& + \mu_{1t}\left[(1-\chi)\varphi_t + l_t - \varphi_{t+1}\right] \\
& + \mu_{2t}\left[w_t l_t + r_t \varkappa_t + q_t(r_t+\delta)k_t - c_t - i_t\right] \\
& + \mu_{3t}\left[(1-\delta)k_t + s_{1t} - k_{t+1}\right] \\
& + \mu_{4t}\left(s_{2t} - s_{1,t+1}\right) + +\mu_{5t}\left(s_{3t} - s_{2,t+1}\right) \\
& + \mu_{6t}\left(s_{4t} - s_{3,t+1}\right) \\
& + \mu_{7t}\left(\varkappa_t + i_t - \frac{1}{4}\sum_{j=1}^{4} s_{jt} - \varkappa_{t+1}\right),
\end{aligned} \tag{B.22}
$$

with $\mathcal{S}_t = (k_t, \varkappa_t, s_{1t}, s_{2t}, s_{3t}, Z_t, \varphi_t)$

The FOCs

$$c_t: \quad c_t^{\frac{e}{3}-1}\left[\Phi(\mathrm{L})(1-l_t)\right]^{\frac{2}{3}e} = 3\mu_{2t} \tag{B.23a}$$

$$l_t: \quad \frac{2\Phi_0}{3} c_t^{\frac{e}{3}}\left[\Phi(\mathrm{L})(1-l_t)\right]^{\frac{2}{3}e-1} = \mu_{1t} + \mu_{2t} w_t \tag{B.23b}$$

$$i_t: \quad \mu_{2t} = \mu_{7t} \tag{B.23c}$$

$$\varphi_{t+1}: \quad \beta \mathrm{E}_t \frac{\partial \mathcal{V}(\mathcal{S}_{t+1})}{\partial \varphi_{t+1}} = \mu_{1t} \tag{B.23d}$$

$$\varkappa_{t+1}: \quad \beta \mathrm{E}_t \frac{\partial \mathcal{V}(\mathcal{S}_{t+1})}{\partial \varkappa_{t+1}} = \mu_{7t} \tag{B.23e}$$

$$k_{t+1}: \quad \beta \mathrm{E}_t \frac{\partial \mathcal{V}(\mathcal{S}_{t+1})}{\partial k t + 1} = \mu_{3t} \tag{B.23f}$$

$$s_{1,t+1}: \quad \beta \mathrm{E}_t \frac{\partial \mathcal{V}(\mathcal{S}_{t+1})}{\partial s_{1,t+1}} = \mu_{4t} \tag{B.23g}$$

$$s_{2,t+1}: \quad \beta \mathrm{E}_t \frac{\partial \mathcal{V}(\mathcal{S}_{t+1})}{\partial s_{2,t+1}} = \mu_{5t} \tag{B.23h}$$

$$s_{3,t+1}: \quad \beta \mathrm{E}_t \frac{\partial \mathcal{V}(\mathcal{S}_{t+1})}{\partial s_{3,t+1}} = \mu_{6t} \tag{B.23i}$$

$$s_{4t}: \ \mu_{6t} = \frac{1}{4}\mu_{7t} \tag{B.23j}$$

The Envelope Conditions

$$\frac{\partial \mathcal{V}(\mathcal{S}_t)}{\partial \varphi_t} = -\frac{2}{3}(1-\Phi_0)\chi c_t^{\frac{e}{3}} \left[\Phi(\mathrm{L})(1-l_t)\right]^{\frac{2}{3}e-1} + (1-\chi)\mu_{1t} \tag{B.24a}$$

$$\frac{\partial \mathcal{V}(\mathcal{S}_t)}{\partial \varkappa_t} = \mu_{2t} r_t + \mu_{7t} \tag{B.24b}$$

$$\frac{\partial \mathcal{V}(\mathcal{S}_t)}{\partial k_t} = \mu_{2t} q_t (r_t + \delta) + (1-\delta)\mu_{3t} \tag{B.24c}$$

$$\frac{\partial \mathcal{V}(s_{1t})}{\partial s_{1t}} = \mu_{3t} - \frac{1}{4}\mu_{7t} \tag{B.24d}$$

$$\frac{\partial \mathcal{V}(s_{1t})}{\partial s_{2t}} = \mu_{4t} - \frac{1}{4}\mu_{7t} \tag{B.24e}$$

$$\frac{\partial \mathcal{V}(s_{1t})}{\partial s_{3t}} = \mu_{5t} - \frac{1}{4}\mu_{7t} \tag{B.24f}$$

The Euler Equations

$$\beta \mathrm{E}_t \left[(1+r_{t+1}) c_{t+1}^{\frac{e}{3}-1} \left[\Phi(\mathrm{L})(1-l_{t+1})\right]^{\frac{2}{3}e}\right] = c_t^{\frac{e}{3}-1} \left[\Phi(\mathrm{L})(1-l_t)\right]^{\frac{2}{3}e} \tag{B.25a}$$

$$\beta \mathrm{E}_t \left[q_{t+1}(1+r_{t+1}) c_{t+1}^{\frac{e}{3}-1} \left[\Phi(\mathrm{L})(1-l_{t+1})\right]^{\frac{2}{3}e}\right] = q_t c_t^{\frac{e}{3}-1} \left[\Phi(\mathrm{L})(1-l_t)\right]^{\frac{2}{3}e} \tag{B.25b}$$

$$\begin{aligned} &\beta \mathrm{E}_t \left\{ \left[\frac{2}{3}(\Phi_0-\chi) - \frac{1}{3}(1-\chi)\Phi(\mathrm{L})(1-l_{t+1})\frac{w_{t+1}}{c_{t+1}}\right] c_{t+1}^{\frac{e}{3}} \left[\Phi(\mathrm{L})(1-l_{t+1})\right]^{\frac{2}{3}e-1} \right\} \\ &\qquad = \left[\frac{2}{3}\Phi_0 - \frac{1}{3}\Phi(\mathrm{L})(1-l_t)\frac{w_t}{c_t}\right] c_t^{\frac{e}{3}} \left[\Phi(\mathrm{L})(1-l_t)\right]^{\frac{2}{3}e-1} \end{aligned} \tag{B.25c}$$

To get (B.25c), one first solves (B.23a) and (B.23b) for μ_{1t}, which gives

$$\mu_{1t} = \left[\frac{2}{3}\Phi_0 - \frac{1}{3}\Phi(\mathrm{L})(1-l_t)\frac{w_t}{c_t}\right] c_t^{\frac{e}{3}} \left[\Phi(\mathrm{L})(1-l_t)\right]^{\frac{2}{3}e-1}.$$

Then, plug this relation into the envelope condition (B.24a) to get

$$\frac{\partial \mathcal{V}(\mathcal{S}_t)}{\partial \varphi_t} = \left[\frac{2}{3}(\Phi_0-\chi) - \frac{1}{3}(1-\chi)\Phi(\mathrm{L})(1-l_t)\frac{w_t}{c_t}\right] c_t^{\frac{e}{3}} \left[\Phi(\mathrm{L})(1-l_t)\right]^{\frac{2}{3}e-1}.$$

Finally, plug the first lead of the above relation into (B.23d).

The Euler equation (B.25a) is obtained plugging the FOCs (B.23a) and (B.23c) into the envelope condition (B.24b) to have

$$\frac{\partial \mathcal{V}(\mathcal{S}_t)}{\partial \varkappa_t} = \frac{1+r_t}{3} c_t^{\frac{e}{3}-1} \left[\Phi(\mathrm{L})(1-l_t)\right]^{\frac{2}{3}e}$$

Then using (B.23e), one gets (B.25a).

The Euler equation (B.25b) is derived as follows. The FOCs (B.23c), (B.23i), and (B.23j) give

$$\begin{aligned}\beta \mathrm{E}_t \frac{\partial \mathcal{V}(\mathcal{S}_{t+1})}{\partial s_{3,t+1}} &= \frac{\mu_{2t}}{4} \\ &= \frac{\beta}{4}\ \mathrm{E}_t \frac{\partial \mathcal{V}(\mathcal{S}_{t+1})}{\partial \varkappa_{t+1}}.\end{aligned}$$

The above relation implies that

$$\begin{aligned}\frac{\partial \mathcal{V}(\mathcal{S}_t)}{\partial s_{3t}} &= \frac{1}{4}\frac{\partial \mathcal{V}(\mathcal{S}_t)}{\partial \varkappa_t} \\ \mu_{5t} - \frac{1}{4}\mu_{2t} &= \frac{1+r_t}{4}\mu_{2t} \Rightarrow \mu_{5t} = \frac{1+(1+r_t)}{4}\mu_{2t}\end{aligned}$$

Then using (B.23h) and the above relation, one has

$$\begin{aligned}\beta \mathrm{E}_t \frac{\partial \mathcal{V}(\mathcal{S}_{t+1})}{\partial s_{2,t+1}} &= \frac{1+(1+r_t)}{4}\mu_{2t} \\ &= \frac{1+(1+r_t)}{4}\beta\ \mathrm{E}_t \frac{\partial \mathcal{V}(\mathcal{S}_{t+1})}{\partial \varkappa_{t+1}},\end{aligned}$$

which implies

$$\begin{aligned}\frac{\partial \mathcal{V}(\mathcal{S}_t)}{\partial s_{2t}} &= \frac{1+(1+r_{t-1})}{4}\frac{\partial \mathcal{V}(\mathcal{S}_t)}{\partial \varkappa_t} \\ \mu_{4t} - \frac{1}{4}\mu_{2t} &= \frac{(1+r_t)+(1+r_{t-1})(1+r_t)}{4}\mu_{2t} \Rightarrow \\ \mu_{4t} &= \frac{1+(1+r_t)+(1+r_{t-1})(1+r_t)}{4}\mu_{2t}.\end{aligned}$$

Using (B.23g) and the above relation, one has

$$\begin{aligned}\beta \mathrm{E}_t \frac{\partial \mathcal{V}(\mathcal{S}_{t+1})}{\partial s_{2,t+1}} &= \frac{1+(1+r_t)+(1+r_{t-1})(1+r_t)}{4}\mu_{2t} \\ &= \frac{1+(1+r_t)+(1+r_{t-1})(1+r_t)}{4}\beta\ \mathrm{E}_t \frac{\partial \mathcal{V}(\mathcal{S}_{t+1})}{\partial \varkappa_{t+1}},\end{aligned}$$

which implies

$$
\begin{aligned}
\frac{\partial \mathcal{V}(\mathcal{S}_t)}{\partial s_{1t}} &= \frac{1+(1+r_{t-1})+(1+r_{t-2})(1+r_{t-1})}{4}\frac{\partial \mathcal{V}(\mathcal{S}_t)}{\partial \varkappa_t} \\
\mu_{3t}-\frac{1}{4}\mu_{2t} &= \frac{(1+r_t)+(1+r_{t-1})(1+r_t)+(1+r_{t-2})(1+r_{t-1})(1+r_t)}{4}\mu_{2t} \Rightarrow \\
\mu_{3t} &= \frac{1+(1+r_t)+(1+r_{t-1})(1+r_t)+(1+r_{t-2})(1+r_{t-1})(1+r_t)}{4}\mu_{2t} \\
&= \underbrace{\frac{1}{4}\left[\sum_{j=1}^{3}\prod_{v=t-(4-j)}^{t-1}(1+r_{v+1})+1\right]}_{q_t}\mu_{2t}.
\end{aligned}
$$

Finally, use the above relation along with the envelope condition (B.24c) and the FOC (B.23f) to get

$$
\beta \mathrm{E}_t\left[(1+r_{t+1})q_{t+1}\mu_{2,t+1}\right] = q_t\mu_{2t}.
$$