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Complex dynamics in an OLG model of growth with inherited tastes

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Abstract

The aim of this article is to study the local and global dynamics of a general equilibrium closed economy with overlapping generations and inherited tastes (aspirations). It shows that the interaction between the intensity of aspirations and the elasticity of substitution of effective consumption, affects the qualitative and quantitative long-term dynamics of the model. In addition, periodic cycles and complex features emerge. It remarkably extends the literature on endogenous fluctuations showing that: 1) in an OLG model with aspirations there exists a super-critical Neimark-Sacker bifurcation, 2) endogenous fluctuations occur even when the elasticity of substitution of effective consumption is smaller than one, thus reconciling the existence of economic cycles with empirical estimates, and 3) the interaction between aspirations and inter-temporal preferences affects the steady-state equilibrium and dynamic outcomes.

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1 Introduction

Starting from the seminal articles of Becker and Murphy (1988), Abel (1990) and Becker (1992), endogenous preferences have become a topic of increasing importance in macroeconomics, from both theoretical and empirical perspectives. One of the most important objective for macroeconomists is to understand the reasons why output and other macroeconomic variables (e.g., employment, investments) fluctuate over time both in the short term (equity premium puzzle) and long term (economic growth).

When the utility of individuals depends on both own consumption and a reference level where comparing it, the possibility of the existence of consumption externalities is in place. The macroeconomic effects of phenomena known as catching-up-with-the-Joneses

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(the consumption reference of an individual is represented by past average consumption in the economy) or keeping-up-with-the-Joneses (the consumption reference of an individual is represented by current average consumption in the economy) have been widely studied (for instance, Galí, 1994; Alonso-Carrera et al., 2005). There are several articles that include consumption externalities and analyse their implications at the macroeconomic level (Alonso-Carrera et al., 2004, 2005, 2007, 2008), dealing also with phenomena related to habits and aspirations. The difference between these two concepts can be summarised as follows.

- Habits refer to the case in which preferences of an individual depend on both his own consumption and a benchmark level that weights the consumer's own past consumption experience. In an overlapping generations (OLG) framework with a representative agent, the existence of internal habits implies that individual preferences over his own consumption when old are evaluated in comparison with his own consumption when young (Alonso-Carrera et al., 2007).
- Aspirations refer to the case in which preferences of an individual depend on his own consumption as well as on a benchmark level that weights the consumption experience of others. In an OLG framework with a representative agent, the existence of external habits implies that preferences over consumption bundles by the current generation are affected by the standard of living based on the consumption experience of the past generation (parent), which represents a reference to contrast the level of current consumption (de la Croix, 1996; de la Croix and Michel, 1999).

By focusing on aspirations, de la Croix (1996) analysed the local properties of an OLG economy with capital accumulation and Cobb-Douglas utility and production functions, and found that a Neimark-Sacker bifurcation can occur if the importance of aspirations in utility is large enough. The economic mechanism through which instability is observed is the following. Aspirations tend to increase consumption of the current generation and reduce savings and capital accumulation. However, when aspirations become sufficiently low this process can be reverted and capital accumulation can increase. With this regard, de la Croix (1996) showed that the steady-state equilibrium can undergo a Neimark-Sacker bifurcation but did not show whether such a bifurcation is super-critical or sub-critical. Knowing whether a Neimark-Sacker bifurcation is super-critical (giving rise to the existence of attracting cycles) or sub-critical (giving rise to the existence of repelling cycles) is of importance in a macro-dynamic setting. In fact, only when a steady-state equilibrium undergoes a super-critical Neimark-Sacker bifurcation an economy is able to show observable and persistent oscillations in income. Later, de la Croix and Michel (1999) generalised de la Croix (1996) by considering general specification of utility and production functions. In that article, they gave sufficient conditions for the existence and uniqueness of a long-term equilibrium of an economy with aspirations, showing also the conditions with respect to which the steady-state equilibrium is a saddle point. Then, they concentrated on optimal growth and found whether and how a centralised economy with aspirations can be decentralised in the market through an adequate use of taxes and subsidies. However, they did not consider the nature of the Neimark-Sacker bifurcation they found and do not account for a global analysis of the model. Therefore, for the general class of utility and production functions used by de la Croix and Michel (1999), nothing can be said with regard to the existence of observable and persistent oscillations.

This present article aims to fill this gap. Specifically it presents a thoughtful study of local and global dynamics in a general equilibrium OLG closed economy with inherited tastes (aspirations) à la de la Croix (1996) and de la Croix and Michel (1999) by assuming a utility function with a Constant Inter-temporal Elasticity of Substitution (CIES) with respect to effective consumption. We find that the steady-state equilibrium can undergo a *super-critical* Neimark-Sacker bifurcation and thus the model is able to generate observable persistent oscillations (economic cycles). More in detail, (1) we show the condition under which a feasible region for an OLG with aspirations does actually exist. This is of importance for the global analysis presented later in this article, (2) we give *necessary and sufficient* conditions for the existence of the fixed points, while de la Croix and Michel (1999) only state sufficient conditions for a general class of utility and production functions; (3) with regard to local stability, we show for some limiting cases that the fixed point can be locally *asymptotically stable or locally unstable*, while de la Croix (1996) and de la Croix and Michel (1999) stated only conditions to have a *saddle point*.

Our model exhibits some other additional results due to the assumption of CIES preferences. Specifically, (a) endogenous fluctuations occur even when the elasticity of substitution of effective consumption is smaller than one. This reconciles the existence of economic cycles with empirical estimates. (b) There exists evidence of a different route to chaos (period doubling), while in the works of de la Croix (1996) and de la Croix and Michel (1999) the steady-state equilibrium can lose stability only through a Neimark-Sacker bifurcation. (c) Aspirations can play a stabilising role. This dramatically contrasts both de la Croix (1996) and de la Croix and Michel (1999), which showed that aspirations are always a destabilising device. Indeed, as also stressed by de la Croix and Michel (1999), the existence of a consumption externality that causes a spillover effect from two subsequent generations implies: the existence of decreasing returns in the process that transfers resource between generations, as the stock of capital currently used in production (and then wages of current workers) are financed by saving of the previous generation; the existence of constant returns in the process that generates standard-of-living aspirations from the old generation to the young generation. Then, due to the former effect since the increasing wage rate of the young workers may be not be adequately high to offset the need of higher consumption due to aspirations, which is a mechanism that shows constant returns. Therefore, saving reduces from this channel causing in turn a reduction in capital accumulation and production per workers. If this reduction is sufficiently strong, the degree of aspirations reduces either. With a low degree of aspirations, saving start increasing inverting then the process. This cyclical behaviour, of course, may generate convergence towards the steady-state equilibrium or destabilization through a Neimark-Sacker bifurcation. However, with CIES preferences the destabilizing role of aspirations can be reverted. This depends on the relative size of the elasticity of substitution, and then on how aspirations affects the interest rate and savings. Indeed, aspirations play an opposite role with respect to de la Croix (1996) and de la Croix and Michel (1999), i.e. they can work as a stabilising device, when the elasticity of substitution of effective consumption is sufficiently high.

The rest of this article is organised as follows. Section 2 builds on the model. Sections $3-4-5-6$ focus on the study of local and global dynamics of the model and underline the main economic results. More in detail, these sections describe the feasible region and show the existence of a positive fixed point. In addition, some findings about the local stability of the interior fixed point and the bifurcations it undergoes when some parameters change are presented. Numerical simulations are used to support the analysis. Section 7 outlines the conclusions.

2 The economy

We consider a general equilibrium OLG closed economy populated by a continuum of identical two-period lived individuals of measure one per generation. Each generation overlaps for one period with the previous generation and then overlaps for one period with the next generation. Time is discrete and indexed by $t = 0, 1, 2, ...$ Life of the typical agent is divided into youth and old age. An individual works when he is young and then retires when he is old. The young member of generation t is endowed with one unit of labour inelastically supplied to firms, and receives competitive wage $w_t > 0$ per unit of labour supplied. The budget constraint of a young individual belonging to generation t is the following:

$$c_{1,t} + s_t = w_t, \tag{1}$$

implying that working income (w_t) is divided between material consumption when young ($c_{1,t}$) and savings (s_t). When old, an individual retires and lives with the amount of resources saved when young plus the expected interest accrued from time t to time $t+1$ at rate r^e_{t+1} (which will become the realised interest rate at time $t+1$). We also assume the existence of a (perfect) market for annuities, so that the budget constraint at time $t+1$ of a young individual of generation t can be expressed as follows:

$$c_{2,t+1} = \frac{R^e_{t+1}}{p} s_t, \tag{2}$$

where $c_{2,t+1}$ is consumption when old, $R^e_{t+1} := 1 + r^e_{t+1}$ is the expected interest factor and $0 < p < 1$ is the constant inter-temporal subjective discount factor.

The typical individual of generation t draws utility from consumption when young and consumption when old. In addition, we assume that the member of generation t evaluates his own consumption when young in comparison with the level of aspirations inherited by his parent (h_t) (de la Croix, 1996; de la Croix and Michel, 1999). These are bequeathed tastes for the individual born at time t that represent a reference to compare current consumption. The lifetime utility function of generation t, therefore, is the following,

$$U_t = \begin{cases} \frac{(c_{1,t}-\gamma h_t)^{1-\sigma}}{1-\sigma} + p\frac{c_{2,t+1}^{1-\sigma}}{1-\sigma}, & if \quad \sigma > 0, \sigma \neq 1, \\ \ln(c_{1,t} - \gamma h_t) + p \ln(c_{2,t+1}), & if \quad \sigma = 1 \end{cases}, \tag{3}$$

where $0 < \gamma < 1$ captures the intensity of aspirations in utility. From (3), the elasticity of inter-temporal substitution of effective consumption is $\frac{1}{\sigma}$. In what follows, we will divide the cases $0 < \sigma < 1$ and $\sigma > 1$.

By taking the wage rate, the expected interest rate and the level of aspiration h_t as given, the individual representative of generation t chooses $c_{1,t}$ and $c_{2,t+1}$ to maximise lifetime utility function (3) subject to (1), (2) and $c_{1,t} > \gamma h_t$. Then, we get:

$$(c_{1,t} - \gamma h_t)^{-\sigma} = \lambda_t, \tag{4}$$

and

$$c_{2,t+1}^{-\sigma} = \frac{\lambda_t}{R^e_{t+1}}, \tag{5}$$

where λ_t is the Lagrange multiplier. From (4) and (5), the first order conditions for an interior solution are the following:

$$c_{2,t+1} = (R^e_{t+1})^{\frac{1}{\sigma}}(c_{1,t} - \gamma h_t,), \tag{6}$$

$$s_t = \frac{p(R^e_{t+1})^{\frac{1-\sigma}{\sigma}}}{1 + p(R^e_{t+1})^{\frac{1-\sigma}{\sigma}}}[w_t - \gamma h_t]. \tag{7}$$

Finally, by combining (6) with (1) and (2) we get:

$$c_{1,t} = \frac{w_t + p(R^e_{t+1})^{\frac{1-\sigma}{\sigma}}\gamma h_t}{1 + p(R^e_{t+1})^{\frac{1-\sigma}{\sigma}}}, \tag{8}$$

$$c_{2,t+1} = \frac{(R^e_{t+1})^{\frac{1}{\sigma}}[w_t - \gamma h_t]}{1 + p(R^e_{t+1})^{\frac{1-\sigma}{\sigma}}}. \tag{9}$$

Firms are identical and act competitively on the market. The production function of the representative firm is the standard neoclassical Cobb-Douglas technology with constant returns to scale, that is $Q_t = AK_t^{\alpha}L_t^{1-\alpha}$, where Q_t, K_t and $L_t = N_t$ are output, capital and labour input at time t respectively, $A > 0$ is a scale parameter and $0 < \alpha < 1$ is the output elasticity of capital. Defining $k_t := K_t/N_t$ and $q_t := Q_t/N_t$ as capital and output per worker, respectively, the intensive form production function is $q_t = Ak_t^{\alpha}$. By assuming that output is sold at the unit price and capital fully depreciates at the end of every period, profits maximisation implies that the interest factor and wage rate are equal to the marginal productivity of capital and marginal productivity of labour, respectively, that is:

$$R_t = \alpha A k_t^{\alpha-1}, \tag{10}$$

$$w_t = (1-\alpha)Ak_t^{\alpha}. \tag{11}$$

Following de la Croix (1996), we assume that aspirations depend on the standard of living of individuals of the previous generation when young. This implies that

$$h_t = c_{1,t-1}. \tag{12}$$

The market-clearing condition in the capital market is given by $k_{t+1} = s_t$. Then, the two-dimensional system that characterises the dynamics of the economy is the following:

$$T_1 : \begin{cases} k_{t+1} = \frac{p(R^e_{t+1})^{\frac{1-\sigma}{\sigma}}}{1+p(R^e_{t+1})^{\frac{1-\sigma}{\sigma}}}[w_t - \gamma h_t] \\ h_{t+1} = \frac{w_t + p(R^e_{t+1})^{\frac{1-\sigma}{\sigma}}\gamma h_t}{1+p(R^e_{t+1})^{\frac{1-\sigma}{\sigma}}} \end{cases}, \tag{13}$$

where $w_t = (1-\alpha)Ak_t^{\alpha}$ and $R^e_{t+1} = \alpha A k_t^{\alpha-1}$ if individuals have static expectations or $R^e_{t+1} = \alpha A k_{t+1}^{\alpha-1}$ if individuals have rational expectations. Obviously, in the case of logarithmic preferences ($\sigma = 1$) it is not important to specify whether an individual has static or rational expectations about future factor prices. With regard to CIES preferences ($\sigma \neq 1$), we study the local and global dynamics of the model under static expectations (that allows us to define an explicit expression for the accumulation of both the stock of capital and stock of aspirations), while leaving the case of rational expectations to future research.

Now, define $x' = k_{t+1}$, $x = k_t$, $y' = h_{t+1}$, $y = h_t$, $\beta := \frac{1-\sigma}{\sigma}$ and

$$A(x) = m_1 x^{\alpha\beta}, \ \ B(x) = x^{\beta}, \ \ C(x) = m_2 x^{\alpha},$$

where $m_1 := p(\alpha A)^{\beta}$ and $m_2 := (1-\alpha)A$. Then, the two-dimensional dynamic system described in (13) can be rewritten by resorting to the following continuous and differentiable map in $(0, +\infty) \times [0, +\infty)$:

$$T_1 : \begin{cases} x' = f(x, y) = \frac{A(x)[C(x)-\gamma y]}{A(x)+B(x)} \\ y' = g(x, y) = \frac{B(x)C(x)+A(x)\gamma y}{A(x)+B(x)} \end{cases}, \ \text{if} \ x > 0. \tag{14}$$

It is important to clarify the meaning of parameter β with respect to σ. If $\sigma \in (0, 1)$ then $\beta > 0$, in particular if $\sigma \to 1^-$ (resp. 0^+) then $\beta \to 0^+$ (resp. $+\infty$). This replicates the case of Cobb-Douglas (resp. Leontief) preferences. If $\sigma > 1$ then $\beta \in (-1, 0)$ and, in particular, if $\sigma \to 1^+$ (resp. $\sigma \to +\infty$) then $\beta \to 0^-$ (resp. $\beta \to -1^+$). In what follows, we will distinguish between the cases $\beta > 0$ (i.e., $\sigma \in (0, 1)$) and $\beta \in (-1, 0)$ (i.e., $\sigma > 1$).

Observe that the condition $x > 0$ should hold for system T_1 to be well-defined. Anyway, when $\beta > 0$ it is possible to verify that map T_1 can be extended to be well defined in points $(0, y)$ by posing $x' := -\gamma y$ and $y' := \gamma y$.

Let $\beta > 0$. Then the final two-dimensional, continuous and differentiable system $(T, \mathbb{R}^2_+)$ is given by $T := T_1 \cup T_2$ where T_1 is given by (14) while T_2 is defined as follows:

$$T_2 : \begin{cases} x' = -\gamma y \\ y' = \gamma y \end{cases}, \ \text{if} \ x = 0. \tag{15}$$

The dynamics of system T are quite difficult to be handled in a neat analytical form. Therefore, in Appendix A we transform system T in a simpler form by taking into account that map T_1 can be rewritten (by applying an opportune transformation) in a one-dimensional, second order difference equation. As a consequence, its dynamics can be carried out by investigating a two-dimensional dynamic system of first order difference equations, which is simpler than the initial one. Therefore, we obtain system $S := S_1 \cup S_2$, where

$$S_1 : \begin{cases} x' = F(x, z) = \frac{A(x)\{C(x)+\gamma x-\gamma C(z)\}}{A(x)+B(x)} \\ z' = G(x) = x \end{cases}, \ \text{if} \ x > 0. \tag{16}$$

and

$$S_2 : \begin{cases} x' = -\gamma C(z) \\ z' = 0 \end{cases}, \ \text{if} \ x = 0, \tag{17}$$

describing the time evolution of the capital per worker x, while the dynamics of aspirations y are obtained as $y = C(z) - x$. In the rest of the article we will deal with the study of the dynamics generated by system S for any $\sigma > 0$, that is $\beta > -1$ holds. Knowing that $m_1 = p(\alpha A)^{\beta}$, $m_2 = (1-\alpha)A$, we note that map S_1 can be written as follows:

$$S_1^* : \begin{cases} x' = F(x, z) = \frac{p(\alpha A)^{\beta}[(1-\alpha)Ax^{\alpha}+\gamma x-\gamma(1-\alpha)Az^{\alpha}]}{p(\alpha A)^{\beta}+x^{(1-\alpha)\beta}} \\ z' = G(x) = x \end{cases}, \ \text{if} \ x > 0, \tag{18}$$

In addition, we observe that if $x = 0$ and $\beta > 0$ then $S_2^+ = S_2$ is given by (17), while if $\beta \in (-1, 0]$ then, for any fixed value of z, $\lim_{x\to 0^+} F(x, z) = 0$ and $\lim_{x\to 0^+} G(x) = 0$, thus obtaining:

$$S_2^- : \begin{cases} x' = 0 \\ z' = 0 \end{cases}, \ \text{if} \ x = 0. \tag{19}$$

By taking into account the previous considerations, we will focus on the study of system $S^* := S_1^* \cup S_2^*$, where

$$S_2^* : \begin{cases} S_2^+ & \text{if } \beta > 0 \\ S_2^- & \text{if } \beta \in (-1, 0] \end{cases} .$$

3 The feasible region

Before starting with the discussion of the dynamics generated by system S^*, it is important to show that any attractor at finite distance of system S^* (if it exists) cannot be globally attracting in $\mathbb{R}_+^2$.

To prove this result, we first observe that, when considering the evolution of the two state variables x and z, system S^* may produce trajectories that exit from set $\mathbb{R}_+^2$. We now recall the following definition.

Definition 1. *Let* ${S^*}^t(x(0), z(0))$ *denote the* $t-th$ *iterate of system* S^* *for a given initial condition (i.c.)* $(x(0), z(0)) \in \mathbb{R}_+^2$*. Then the sequence* $\psi_t = \{(x(t), z(t))\}_{t=0}^{\infty}$ *is called trajectory. A trajectory* ψ_t *is feasible for system* S^* *if* $(x(t), z(t)) \in \mathbb{R}_+^2$ *for all* $t \in \mathbb{N}$*, otherwise it is unfeasible.*

About the existence of unfeasible trajectories the following proposition holds (see Appendix B for the proof).

Proposition 2. *System* S^* *always admits unfeasible trajectories.*

From Proposition 2, it follows that if S^* admits feasible trajectories then set D containing all initial conditions $(x(0), z(0))$ that generate feasible trajectories is a subset of $\mathbb{R}_+^2$. We call set D the feasible region. Furthermore, observe that $S^*(0,0) = (0,0)$ for all parameter values so that D is non-empty. In order to better characterise the structure of set D, a preliminary consideration is the following. From the proof of Proposition 2, we observe that the function

$$z = \tilde{h}(x) = \left(\frac{m_2 x^\alpha + \gamma x}{\gamma m_2}\right)^{\frac{1}{\alpha}}, \tag{20}$$

defines a curve in the (x, z) plane which is strictly increasing and convex and such that $\lim_{x\to 0^+} \tilde{h}(x) = 0$ and $\lim_{x\to +\infty} \tilde{h}(x) = +\infty$. As is previously shown, $\forall x > 0$ trajectories starting above this curve are not feasible. On the other hand, the dynamics embedded into the $z-$axis are governed by system S_2^*, and it can immediately be seen that if $\beta > 0$ all initial conditions $(0, z(0))$, $z(0) > 0$, generate unfeasible trajectories, while if $\beta \in (-1, 0]$ then all initial conditions $(0, z(0))$ generate trajectories converging to the origin.

In Figures 1 (a) and (b) we fix the key parameters of the model and depict the feasible region in white for a positive value of β (Figure 1 (a)) and a negative values of β (Figure 1 (b)). We also represent curve $\tilde{h}(x)$ in yellow. Observe that the set of initial conditions that generates unfeasible trajectories is also given by the grey points lying below curve $\tilde{h}(x)$, representing initial conditions that generate trajectories that exit from the set $\mathbb{R}_+^2$ after the first iterate.

The following proposition concerning the structure of the feasible region holds (see Appendix B for the proof).

Proposition 3. *Let* $\beta > 1$ *hold and system* S^* *given by (17) and (18). Then* $\exists I(\underline{0}, r)$ *such that all initial conditions* $(x(0), z(0)) \in \{\mathbb{R}_+^2 - I(\underline{0}, r)\}$ *generate unfeasible trajectories.*

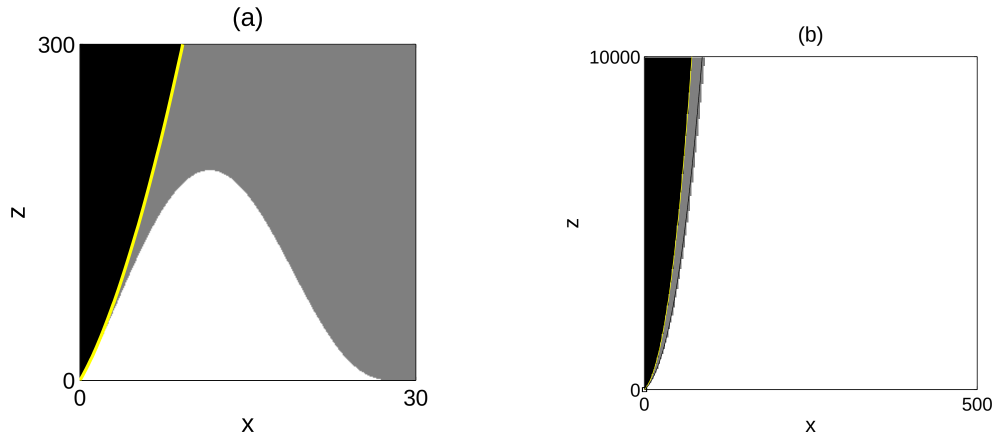

Figure 1: The feasible set is depicted in white, the grey and black regions represent the set of initial conditions generating unfeasible trajectories; the curve $\tilde{h}(x)$ defined in (20) is depicted in yellow. (a) Parameter values: $\alpha = 0.33$, $A = 10$, $\beta = 2$, $p = 0.4$ and $\gamma = 0.4$. (b) The feasible region is depicted for $\beta = -0.4$.

From the previous Proposition it follows that if $\beta > 1$ then system S^* may be able to produce feasible trajectories only if the initial condition is taken into an opportune subset of $I(\underline{0}, r)$. By taking into account the relationship $y = C(z) - x$, this means that at the initial state, given a positive initial value of the capital per worker $x(0)$, the initial value of aspirations $y(0)$ should not be too high.

By using numerical techniques, it is possible to verify that the result proved in Proposition 3 for $\beta > 1$ still holds for $\beta \in (0, 1]$. Furthermore, it can be observed that the size of the feasible region increases as β decreases (that is σ increases) and that if $\beta \in (-1, 0]$ set D is unbounded (Figure 1 (b)), i.e. with Cobb-Douglas preferences ($\beta = 0$) or if $\beta \in (-1, 0)$ (that is $\sigma > 1$).

This arguments and Propositions 2 and 3 trivially prove the following result.

Proposition 4. *System S^* always admits an attractor at finite distance which is not globally attracting in $\mathbb{R}^2_+$.*

The results herewith obtained show that while the unique equilibrium in the Diamond's model is globally stable and thus all trajectories converge to such an equilibrium, the model extended with aspirations is able to produce feasible trajectories only whether the initial conditions belong to an appropriate set. In particular, for any initial value of capital per worker, the initial value of aspirations, must be sufficiently low. Furthermore, numerical evidences have shown that the size of the feasible region increases when σ increases. With Cobb-Douglas preferences, set D becomes unbounded and it remains unbounded also for $\sigma > 1$. More in detail the following two cases occur.

(i) If $\sigma \in (0, 1)$ (high elasticity of substitution of effective consumption), irrespective of the stock of aspirations, an economy may be located in a region that generates unfeasible trajectories even if it starts with high values of the capital stock per worker (developed countries). The standard OLG model extended with aspirations is able to produce feasible trajectories for intermediate values of the capital stock per worker and stock of aspirations. The economic reason for this result is twofold: (1) when the economy starts, the generation living at the initial state of the world must not have consumed too much when young to adequately save to allow the future generation to avoid to inherits a level of aspirations that generates unfeasible trajectories (negative savings); (2) however, saving should not be at too high a level to avoid unfeasible trajectories as well. The second point is relevant especially with regard to the effects of positive shocks on physical capital (i.e., capital transfers from foreign countries, capital donations from external donors), which may therefore be detrimental in an economy with aspirations because they may be a source of unfeasible trajectories.

(*ii*) If $\sigma > 1$ (low elasticity of substitution with respect to effective consumption), an initial capital stock per worker great enough always guarantees that an economy lies in a region that generates feasible trajectories, even if the initial value of the stock of aspirations is small. With this kind of preferences, only economies that starts with a small stock of capital per worker (developing or underdeveloped countries) may be entrapped in a region that generates unfeasible trajectories. Whether an economy lies in a region that generates feasible or unfeasible trajectories is an empirical matter with σ.

4 Existence and number of fixed points

We now consider the question of the existence and number of fixed points (or steady states) of system S^*. The steady states of system S^* are all solutions of the system $S^*(x,z) = (x,z)$.

The following Proposition proved in Appendix B, holds.

Proposition 5. *System S^* admits two fixed points for all parameter values: the origin $E_0 = (0,0)$, and the interior fixed point $E^* = (x^*, x^*)$.*

The position of the unique interior fixed point E^* on the plane depends on the parameters of the model; in particular, it depends on the two key parameters γ and β, that measure the intensity of aspirations and the inter-temporal elasticity of substitution with respect to effective consumption, respectively. By taking into account the proof of Proposition 5 presented in Appendix B, we note that $g(\omega)$ is a strictly decreasing function and it does not depend on γ, while $f(\omega)$ is a strictly increasing function and it depends on γ. More precisely, for any given value of $\omega > 0$ and $\beta > -1$ we have that

$$\frac{\omega^{\beta+1}}{1-\gamma_1} < \frac{\omega^{\beta+1}}{1-\gamma_2}, \quad \forall\, 0 < \gamma_1 < \gamma_2 < 1.$$

As a consequence, x^* is decreasing with respect to γ. On the one hand, this implies that the steady-state capital stock per worker is lower in the economy with bequeathed tastes than in the standard Diamond economy, as in de la Croix (1996). On the other hand, the role of β on E^* can be ambiguous as it depends also on the value of x^*. The following proposition proved in Appendix B holds.

Proposition 6. *Let $E^* = (x^*, x^*)$ be the interior fixed point of system S^*. Then: (i) if $p(1-\gamma)\left(\frac{1-2\alpha}{\alpha}\right) < 1$, $\frac{\partial x^*}{\partial \beta} > 0$; (ii) if $p(1-\gamma)\left(\frac{1-2\alpha}{\alpha}\right) > 1$, $\frac{\partial x^*}{\partial \beta} < 0$; (iii) if $p(1-\gamma)\left(\frac{1-2\alpha}{\alpha}\right) = 1$, $\frac{\partial x^*}{\partial \beta} = 0$.*

According to the previous result the effect of a change in β on the position of the interior fixed point is ambiguous. Observe also that the condition $p(1-\gamma)\left(\frac{1-2\alpha}{\alpha}\right) < 1$ corresponds to $\omega_1^* = \frac{(x^*)^{1-\alpha}}{\alpha A} < 1$ and consequently to $x^* < (\alpha A)^{\frac{1}{1-\alpha}} = x_\infty$. Hence, from Proposition

6, it follows that if $p(1-\gamma)\left(\frac{1-2\alpha}{\alpha}\right) < 1$ (resp. $p(1-\gamma)\left(\frac{1-2\alpha}{\alpha}\right) > 1$) then $x^* < x_\infty$(resp. $x^* > x_\infty$), and if β increases, then x^* increases (resp. decreases) up to the limit value x_∞ to which x^* converges when $\beta \to +\infty$, i.e. x^* is upper (resp. lower) bounded.

Now, let

$$x^*_{-1} = \left[A\frac{p(1-\gamma)(1-\alpha)-\alpha}{p(1-\gamma)}\right]^{\frac{1}{1-\alpha}}.$$

Then, according to Proposition 6 it can be observed that if x^* is increasing (resp. decreasing) with respect to β, then x^* converges to its minimum (resp. maximum) value as $\beta \to -1^+$, that is given by 0 (resp. x^*_{-1}). Finally, in the Cobb-Douglas case ($\beta = 0$) one gets

$$x^* = x^*_0 = \left[\frac{(1-\gamma)p(1-\alpha)A}{1+(1-\gamma)p}\right]^{1/(1-\alpha)}.$$

The previous results can be summarised in the following remark.

Remark 7. *Let $E^* = (x^*, x^*)$ be the interior fixed point of system S^*.*
(i) If $\beta \to \infty$ (i.e. $\sigma \to 0^+$) then $x^* \to x_\infty = (\alpha A)^{1/(1-\alpha)}$ and $\forall \beta > -1$ if $p(1-\gamma)\left(\frac{1-2\alpha}{\alpha}\right) <(>)1$ then $x^* < (>) x_\infty$;
(ii) if $\beta = 0$ (i.e. $\sigma = 1$) then $x^* = x^*_0 = \left[\frac{(1-\gamma)p(1-\alpha)A}{1+(1-\gamma)p}\right]^{1/(1-\alpha)}$;
(iii) if $\beta \to -1^+$ and $p(1-\gamma)\left(\frac{1-2\alpha}{\alpha}\right) > (<) 1$ then $x^* \to x^*_{-1} (\to 0)$.

Figure 2 (b) shows - for two different values of α (the output elasticity of capital) - that the effect of a change in β on the position of the interior fixed point is ambiguous. If α is sufficiently high (resp. low) then condition (i) (resp. (ii)) of Proposition 6 holds and when $\beta \to -1^+$ the steady-state stock of capital is the smallest (resp. largest) one with respect to other values of the individual degree of substitution of consumption over time. Observe that a sufficient condition for x^* to be increasing in β is $\alpha > 1/3$, or γ (resp. p) is sufficiently high (resp. low). This result sheds new light on the role of preference parameters (the aspiration intensity and the inter-temporal discount factor in this context) on steady-state income (neoclassical economic growth). For any given value of β, the economy may converge towards a long-term high or low income level depending on technology and preference parameters. In particular, the lower the capital share in production and aspiration intensity, and the higher the inter-temporal subjective discount factor, the more likely an economy converges towards a steady state with low income (as is shown in Figure 2 panels (c) and (d)).

5 Local stability of fixed points

In order to study the local stability of the two fixed points of system S^*, consider the Jacobian matrix associated to S^*_1, representing the linearization of the dynamic system S^*_1, given by:

$$JS^*_1(x,z) = \begin{pmatrix} F_x(x,z) & F_z(x,z) \\ 1 & 0 \end{pmatrix}. \tag{21}$$

About the local stability of E_0 it can be observed that, since

$$det(JS^*_1(x,z)) = \frac{\gamma m_1 m_2 \alpha}{(m_1 + x^{\beta(1-\alpha)})z^{(1-\alpha)}}$$

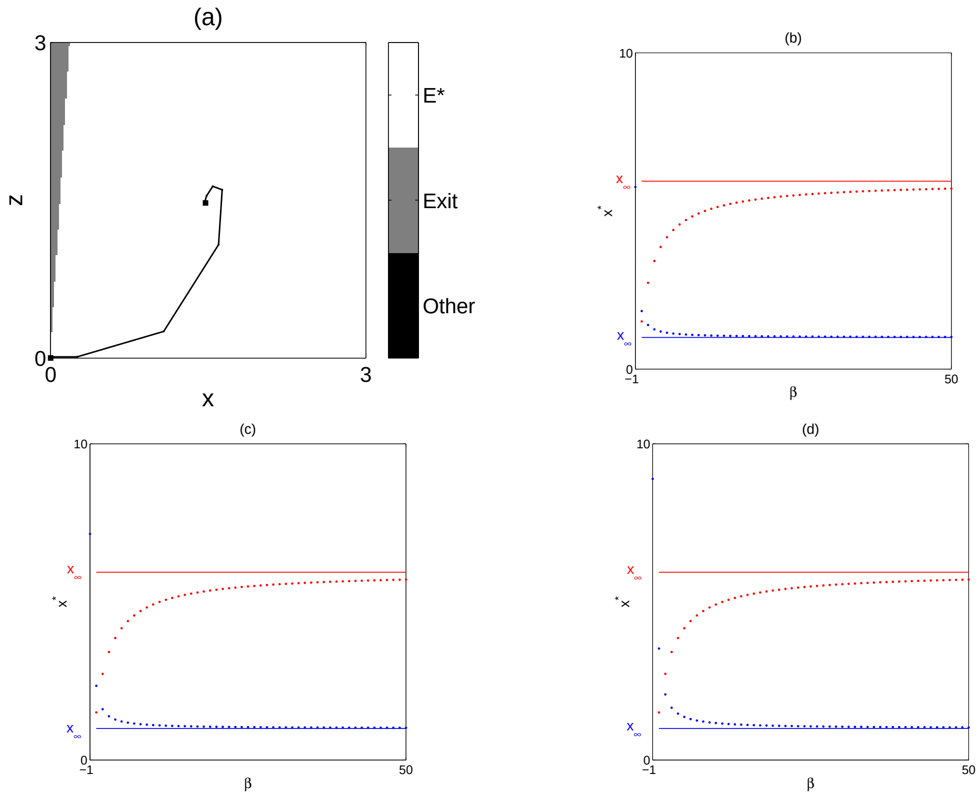

Figure 2: (a) A feasible trajectory starting from $(0.001, 0.001)$ and converging to E^* is depicted for $A = 10$, $p = 0.4$, $\gamma = 0.4$ and $\beta = 0$, $\alpha = 0.33$. (b) Long term capital per capita equilibrium value as β increases for different values of α: in red $\alpha = 0.33$ (and the sequence is increasing) while in blue $\alpha = 0.1$ (and the sequence is decreasing). (c) Long term capital per capita equilibrium value as β increases for different values of γ: in red $\gamma = 0.4$ (and the sequence is increasing) while in blue $\gamma = 0.2$ (and the sequence is decreasing). (d) Long term capital per capita equilibrium value as β increases for different values of p: in red $p = 0.4$ (and the sequence is increasing) while in blue $p = 0.9$ (and the sequence is decreasing).

then if $\beta > 0$ and $x \to 0^+$, $z \to 0^+$ we have that $det(JS_1^*(x,z)) \to +\infty$; if $\beta \in (-1,0]$ and $x \to 0^+$, $z \to 0^+$ then $det(JS_1^*(x,z))$ does not admit the limit (observe, for instance, that if $z = kx$ and $k > 0$ $\lim_{x\to 0^+} det(JS_1^*(x,kx)) = +\infty$). In both cases a condition for the local stability is violated (see Medio and Lines 2001). These considerations prove the following Proposition.

Proposition 8. *The origin is a locally unstable fixed point of system S^*.*

The previous Proposition holds for all parameter values, i.e. E_0 is locally unstable also with Cobb-Douglas utility. In Figure 2 (a) we depict a feasible trajectory starting from an initial condition close to the origin and converging to E^* when $\beta = 0$.

In order to consider the Jacobian matrix evaluated at the interior fixed point E^*, observe that $x^* = z^* > 0$ and that the following relation holds (see the proof of Proposition 5 in Appendix B):

$$(x^*)^{\beta(1-\alpha)} = m_1 m_2 (1-\gamma)(x^*)^{\alpha-1} - m_1(1-\gamma). \tag{22}$$

As a consequence, the Jacobian matrix evaluated at E^* can be written as follows:

$$JS_1^*(x^*,x^*) = \begin{pmatrix} F_x(x^*,x^*) & F_z(x^*,x^*) \\ 1 & 0 \end{pmatrix}$$

where

$$F_x(x^*,x^*) = \frac{m_1\left\{m_2[\alpha + (1-\gamma)(\alpha\beta - \beta)]x^{*(\beta-1)(1-\alpha)} + m_1 m_2[\gamma(1-\beta+\alpha\beta)+\alpha]x^{*(\alpha-1)} + \gamma^2 m_1\right\}}{(m_1 m_2(1-\gamma)x^{*(\alpha-1)} + \gamma)^2}, \tag{23}$$

and

$$F_z(x^*,x^*) = -\frac{\gamma m_2 \alpha}{m_2(1-\gamma) + \gamma(x^*)^{1-\alpha}}. \tag{24}$$

Since we cannot explicitly obtain the coordinates of fixed point E^*, the local stability analysis cannot be carried out for all parameter values. However, we can find some results concerning the stability of the interior fixed point in some limit cases related to the parameters of interest γ and β. The following Proposition holds (the proof is in Appendix B).

Proposition 9. *Consider system S^*. (i) If $\gamma \to 0^+$ and $\beta \to 0$ then E^* is locally stable; (ii) if $\gamma \to 0^+$ and $\beta \to +\infty$ then E^* is locally unstable; (iii) if $\gamma \to 1^-$ then E^* is locally unstable.*

We now want to consider the local stability of E^* for negative values of parameter β, that is the other limit case $\beta \to -1^+$ and $\gamma \to 0^+$. A preliminary consideration is that, according to the proof of Proposition 6 in Appendix B, if $\beta = -1$ then $f(\omega_1) = 1$ and consequently the interior fixed point still exists if and only if $\frac{p(1-\gamma)(1-\alpha)}{\alpha} > 1$. This last inequality holds for $\gamma = 0$ iff $p > \frac{\alpha}{1-\alpha}$. Then, the following Proposition holds (see Appendix B for the proof).

Proposition 10. *Let $p > \frac{\alpha}{1-\alpha}$. Then if $\gamma \to 0^+$ and $\beta \to -1^+$ E^* is locally stable.*

The results concerning the local stability of the unique interior fixed point in the limit cases studied above are confirmed by looking at the cycle cartogram depicted in Figure 3 (a). It shows a two-parameter bifurcation diagram, each color describes a long-run dynamic behaviour for a given combination of γ and β and for an initial condition close to E^*. A

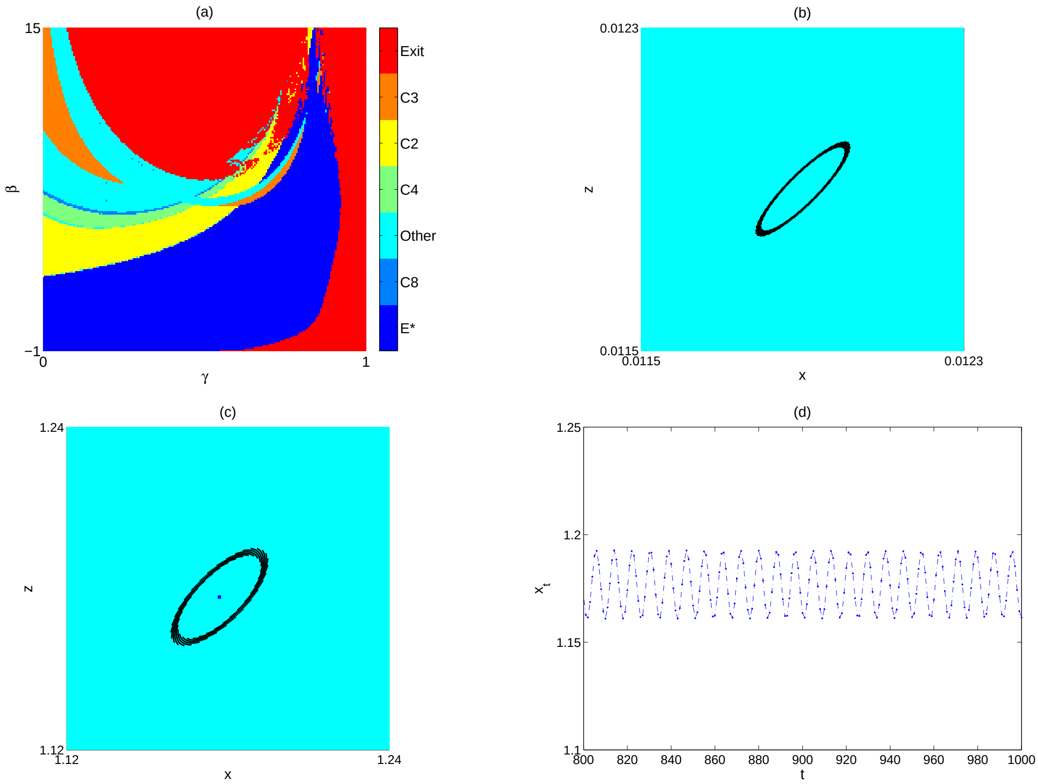

Figure 3: (a) Two dimensional bifurcation diagrams of system S^* in the plane (γ, β) for the following parameter values: $A = 10$, $p_0 = 0.5$; $\alpha = 0.27$ and the initial condition is close to E^*. (b) Closed attracting invariant curve of system S^* for the following parameter values: $A = 10$, $\alpha = 0.27$, $p = 0.5$, $\beta = -0.7$ and $\gamma = \gamma_\beta = 0.7908$. (c) Closed attracting invariant curve of system S^* for the following parameter values: $A = 10$, $\alpha = 0.27$, $p = 0.8$, $\beta = 0.9$ and $\gamma = \gamma_\beta = 0.8962$. (d) The last 200-values of x resulting for the case depicted in (c) are plotted over time.

large diversity of cycles of different order is exhibited. The red region indicates parameter values such that an unfeasible trajectory is produced.

By taking into account the results of Propositions 9 and 10, and looking at Figure 3 (a), we note that if β is not too large, i.e. $\beta \in I(0, \epsilon)$, $\epsilon > 0$, then a threshold value $\gamma_\beta \in (0, 1)$ exists such that, if γ crosses γ_β, the fixed point E^* undergoes a bifurcation. In the following Proposition, we prove that a super-critical Neimark-Sacker bifurcation occurs, thus extending the result of de la Croix (1996). The proof is in Appendix B.

Proposition 11. *Let E^* be the interior fixed point of system S^*. Then a $\epsilon > 0$ does exist such that $\forall \beta \in I(0, \epsilon)$ there exists a $\gamma = \gamma_\beta \in (0, 1)$ at which a super-critical Neimark-Sacker bifurcation occurs.*

The previous Proposition shows that if β belongs to an opportune neighbourhood of the origin (hence moving away from the Cobb-Douglas case), it is possible to find a threshold value $\gamma = \gamma_\beta$ such that E^* becomes an unstable focus and an attracting closed invariant curve is created. Observe that the magnitude of the neighbourhood of β for which the Neimark-Sacker bifurcation occurs depends on the parameter values of the model. Through numerical simulations, it is possible to find the value of γ_β for different choices of $\beta \in I(0, \epsilon)$ while fixing the other parameter values. To this purpose, we perform an algorithm and numerically find out that: (i) γ_β is increasing in β, i.e. if β increases the value of γ_β at which the Neimark-Sacker bifurcation occurs increases as well, (ii) ϵ is increasing in p i.e. the amplitude of the β-interval such that the Neimark-Sacker bifurcation occurs increases if p increases, and (iii) after the Neimark-Sacker bifurcation unfeasible trajectories are produced. In Figure 3 (b) and (c) two closed attracting invariant curves created via the Neimark-Sacker bifurcation are depicted for negative and positive values of β. In panel (d) the diagram versus time of the cycle depicted in panel (c) is shown. Proposition 11, therefore, shows that endogenous fluctuations can actually be observed under Cobb-Douglas preferences (de la Croix, 1996).

It is important to observe that while de la Croix (1996) has verified the existence of a Neimark-Sacker bifurcation in an OLG economy with Cobb-Douglas preferences and aspirations, but has not clarified whether it is either a super-critical bifurcation or sub-critical bifurcation, we have shown that increasing aspirations in an economy with CIES preferences may cause the existence of a super-critical Neimark-Sacker bifurcation and the appearance of endogenous business cycle. Therefore, endogenous fluctuations (alternatively, business cycles) emerge also when the elasticity of substitution of effective consumption is less than 1. This fact reconciles the existence of business cycles with the empirical evidence that argues that the elasticity of substitution ranges amongst relatively small values (Hall, 1988; Blundell-Wignall et al., 1995; Lund and Engsted, 1996). As is known, the preceding established literature with overlapping generations à la Diamond (1965) (i.e., Michel and de la Croix, 2000; de la Croix and Michel, 2002; Chen et al., 2008; Fanti and Spataro, 2008; Fanti and Gori, 2013) has shown that the emergence of endogenous fluctuations necessarily requires a value of the elasticity of substitution of effective consumption larger than 1. For instance, de la Croix and Michel (2002) show in an OLG model à la Diamond (1965) without aspirations that the dynamics is always globally stable when the elasticity of substitution of effective consumption is smaller than $1/(1 - \alpha)$, where α is the output elasticity of capital in the Cobb-Douglas production function (also used in the present article). Thus, the emergence of endogenous fluctuations (business cycle) necessarily requires an elasticity of substitution of effective consumption noticeably larger than 1. In order to show the quantitative relevance of the occurrence of business cycles in our model, in Figure 3 we depict

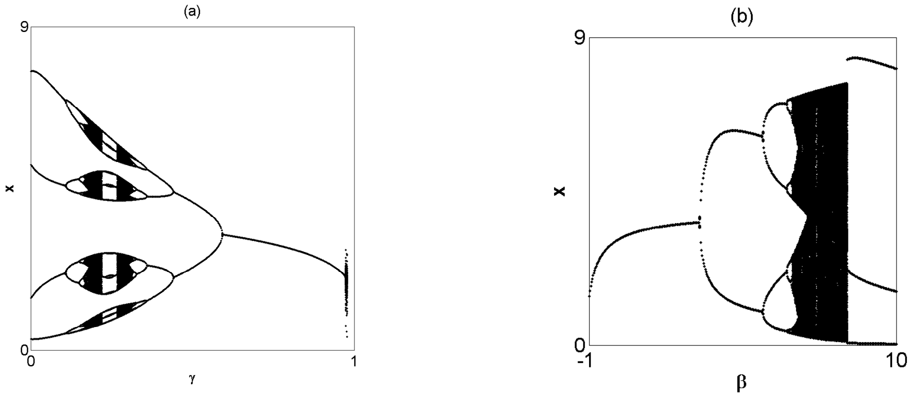

Figure 4: (a) One dimensional bifurcation diagram w.r.t. γ if $\alpha = 0.27$ and $\beta = 6$, $A = 10$ and $p = 0.5$. (b) One dimensional bifurcation diagram w.r.t. β if $\alpha = 0.27$ and $\gamma = 0.1$; $A = 10$ and $p = 0.5$.

the closed invariant curves and the shapes of the corresponding economic fluctuations for two different values of the elasticity of substitution of effective consumption that covers a wide range of the (controversial) empirical estimates: $\beta = -0.7$ and $\beta = 0.9$ (corresponding to an elasticity of substitution of effective consumption of 0.3 and 1.9, respectively). In the next section we perform some numerical experiments by considering the mutual relationship between the intensity of aspirations and the inter-temporal preferences of individuals.

6 Numerical evidences

We now perform some numerical simulations that are useful to inquire about the qualitative dynamics occurring when γ and β vary. First of all, we consider the role of aspirations, i.e. how the qualitative dynamics of the model changes with γ. In this analysis, we have to distinguish between different fixed values of β.

If β is sufficiently close to zero, then, as proved in Proposition 11, E^* loses stability through a Neimark-Sacker bifurcation occurring at γ_β, at which an attracting closed invariant curve is created (see Figure 3 (b) and (c)). This evidence extends the result found in de la Croix (1996) proved only for $\beta = 0$. After this bifurcation, almost all trajectories become unfeasible, as shown in Figure 3 (a).

If β is sufficiently high, then depending on the initial condition the final dynamics becomes simpler when γ increases up to a given value of $\bar{\gamma}$, after which the produced trajectory becomes unfeasible. In Figure 4 (a) $\beta = 6$ and the initial condition is close to E^*: a period doubling and halving bifurcation cascade can be observed providing that the economic cycle may be produced in an additional way with respect to the Neimark-Sacker bifurcation discussed in de la Croix (1996). In sharp contrast with his work, several numerical computations have shown evidence that aspirations play a stabilising role at intermediate values (i.e. $\gamma < \bar{\gamma}$) as the unique interior fixed point E^* is locally stable if aspirations are not too low.

From an economic point of view, aspirations play an opposite role with respect to de la Croix (1996) when β is sufficiently high (low values of σ). From a mathematical point of view, we have shown that changing the value of γ may produce a local destabilization of

E^* not only through a Neimark-Sacker bifurcation (as is shown in the previous section) but also via a period-doubling bifurcation.

About the role of individual preferences on the asymptotic dynamics of the model, it can be observed that when γ is fixed at a sufficiently low value, then a sequence of period-doubling bifurcations occur as β increases. This fact can be better understood by considering that - when β increases - the trace of the Jacobian matrix evaluated at the fixed point passes from α to $-\infty$ thus crossing -1, and a 2-period cycle is created. The sequence of period doubling bifurcations can be observed in figure 4 (b), where it is also shown that the long-term evolution of the capital per worker in an economy with aspirations increases in complexity as β increases (i.e. σ decreases). This result represents a new evidence of a different route to chaos with respect to de la Croix (1996) due to the presence of CIES preferences.

Finally, we observe that if β and γ increase together up to some limit values, the local stability of E^* can be preserved, that is bifurcations that lead to cycles of higher order are prevented (see Figure 3 (a)).

7 Conclusions

This article has concerned with the study of a general equilibrium model with overlapping generations and inherited tastes (aspirations), as in de la Croix (1996) and de la Croix and Michel (1999). It has extended them by considering individual preferences with a constant inter-temporal elasticity of substitution with respect to effective consumption. The interaction between the intensity of aspirations and the elasticity of substitution in utility affects the qualitative and quantitative long-term dynamics of the model. First, in order to avoid unfeasible trajectories the stock of aspirations should not be fixed at too high a level and the size of the stock of capital plays a different role depending on whether the elasticity of substitution is low or high. While in the former case by fixing a not too low stock of capital as an initial condition always allows feasible trajectories, in the latter case low and high stock of capital may produce unfeasible trajectories. This constitutes a warning on the perils that may be generated by a high economic growth when aspirations exist and the elasticity of substitution of effective consumption is sufficiently high. Moreover, it is shown that periodic cycles and complex features may emerge in that case. Our findings contribute the OLG literature on endogenous fluctuations by showing that: 1) the Neimark-Sacker bifurcation found by de la Croix (1996) and de la Croix and Michel (1999) is super-critical; 2) endogenous fluctuations occur even when the elasticity of substitution of effective consumption is smaller than one (in contrast with the preceding literature), thus reconciling the existence of business cycles with the widespread empirical estimates on the topic; 3) the interaction between aspirations and inter-temporal preferences affects both long-term outcomes and dynamic outcomes. In particular, with non-Cobb-Douglas utility aspirations play a stabilising role.

Our future research agenda includes the study a model with inherited tastes (aspirations) and endogenous longevity as in Chakraborty (2014) and Fanti and Gori (2014). Indeed, the relationship between these two elements may have important consequences on economic growth and development.

Appendix A

Topological transformation of system T into system S.

Let system T_1 be given by (14). Then

$$x' + y' = f(x,y) + g(x,y) = C(x) \Rightarrow y' = C(x) - x'. \tag{25}$$

From the first equation of T_1 we have $x'' = f(x', y')$ at time $t+2$, that is by taking into account equation (25),

$$x'' = f(x', C(x) - x') = F(x', x). \tag{26}$$

Equation (26) is a one-dimensional, second order difference equation. Then, given an initial condition $(x(0), x(1))$ (i.e., given the values of the capital per capita x at time $t = 0$ and $t = 1$) the trajectory obtained by applying function F is given by the following sequence $x(0), x(1), x(2), ..., x(n)$, such that $x(i) = F(x(i-1), x(i-2))$, $\forall i = 2, 3, ..., n$. It is important to underline that the sequence of the other state variable of T_1, i.e. the evolution of aspirations $y(1), y(2), ..., y(n)$, is given by $y(i) = C(x(i-1)) - x(i)$, $\forall i = 1, 2, 3, ..., n$. Finally, let

$$z' = G(x) = x, \tag{27}$$

then (26) can be written as follows

$$x'' = F(x', z') \Rightarrow x' = F(x, z), \tag{28}$$

and consequently the following system of two first order difference equations is obtained

$$S_1 : \begin{cases} x' = F(x,z) = \frac{A(x)\{C(x)+\gamma x - \gamma C(z)\}}{A(x)+B(x)} \\ z' = G(x) = x \end{cases}, \quad \text{if } x > 0. \tag{29}$$

Similarly, system S_2, that describes the dynamics of state variables x and z when $x = 0$ and $\beta > 0$, is given by:

$$S_2 : \begin{cases} x' = -\gamma C(z) \\ z' = 0 \end{cases}, \quad \text{if } x = 0. \tag{30}$$

Appendix B

Proof of Proposition 2

Let $x(0) > 0$ and $z(0) > \left(\frac{m_2 x(0)^\alpha + \gamma x(0)}{\gamma m_2}\right)^{\frac{1}{\alpha}}$. Then the first iteration of S^* gives a negative value of x', that is $x(1) < 0$. This means that $(x(1), z(1))$ exits from the set $\mathbb{R}^2_+$, hence the obtained trajectory is unfeasible.

Proof of Proposition 3

Let $x(0) = 0$ then all initial conditions $(0, z(0))$, $z(0) > 0$, generate unfeasible trajectories. Let $x(0) > 0$ and define

$$D_1 = \{(x(0), z(0)) \in (0, +\infty) \times [0, +\infty) : m_2 x(0)^\alpha - \gamma(m_2 z(0)^\alpha - x(0)) < 0\}.$$

Then, by taking into account the proof of Proposition 2, $S^*(D_1)$ exits the set $\mathbb{R}^2_+$ (black points above the yellow curve represented in Figure 1 (a)). Consider now all the preimages of first rank of set D_1, i.e. the set

$$D_2 = \{(x(-1), z(-1)) \in (0, +\infty) \times [0, +\infty) : S^*(x(-1), z(-1)) \in D_1\}.$$

From the first equation of system S_1^* we have that $x(0) = F(x(-1), z(-1))$ while, from the second equation, we have that $z(0) = x(-1)$. The inequality $m_2 x(0)^\alpha - \gamma(m_2 z(0)^\alpha - x(0)) < 0$ may then be rewritten in terms of $x(-1)$ and $z(-1)$ thus obtaining, after some algebra, the following:

$$m_2 \left(\frac{F(x(-1), z(-1))}{x(-1)} \right)^\alpha + \gamma \frac{F(x(-1), z(-1))}{x(-1)^\alpha} < \gamma m_2. \qquad (31)$$

Observe that (31) defines the set of points generating trajectories which exit $(0, +\infty) \times [0, +\infty)$ at the second iteration, i.e. the set D_2. Since $\lim_{x(-1)\to+\infty, z(-1)\to+\infty} \left(\frac{F(x(-1),z(-1))}{x(-1)} \right)^\alpha = 0$ and, if $\beta > 1$, also $\lim_{x(-1)\to+\infty, z(-1)\to+\infty} \left(\frac{F(x(-1),z(-1))}{x(-1)^\alpha} \right) = 0$ then (31) holds and consequently $S^{*2}(D_2)$ exits from $\mathbb{R}^2_+$ (D_2 is given by the gray points in Figure 1 (a)).

Proof of Proposition 5

Trivially, the origin is a fixed point since $S_2^+(0,0) = S_2^-(0,0) = (0,0)$ and no other fixed points exist on the z-axis. Now, let $x > 0$. Then a fixed point of (18) must solve equation $x^* = F(x^*, x^*)$. After some algebra one gets

$$-(x^*)^{\beta(1-\alpha)} + m_1 m_2 (1-\gamma)(x^*)^{\alpha-1} - m_1(1-\gamma) = 0$$

so that, it must be

$$\frac{\omega^{\beta+1}}{1-\gamma} = m_1 m_2 - m_1 \omega$$

where we posed $\omega = (x^*)^{1-\alpha}$. Taking into account the geometrical properties of functions $f(\omega) = \frac{\omega^{\beta+1}}{1-\gamma}$ and $g(\omega) = m_1 m_2 - m_1 \omega$ it can easily be shown that they intersect each other only once, i.e. there exists a unique $\omega^* < m_2$ such that $f(\omega^*) = g(\omega^*)$, and consequently system S_1^* always admits a unique fixed point given by $E^* = (x^*, x^*)$, where $x^* = (\omega^*)^{\frac{1}{1-\alpha}}$.

Proof of Proposition 6

From the proof of Proposition 5, we have that, at the steady state, the following equality holds

$$\omega_1^{\beta+1} = \frac{p(1-\gamma)(1-\alpha)}{\alpha} - p(1-\gamma)\omega_1$$

where $\omega_1 = \frac{x^{1-\alpha}}{\alpha A}$ and ω_1 is strictly increasing with respect to x. Notice that $f(\omega_1) = \omega_1^{\beta+1}$ depends on β while $g(\omega_1) = \frac{p(1-\gamma)(1-\alpha)}{\alpha} - p(1-\gamma)\omega_1$ does not depend on β. Let $\omega_1^* > 0$ such that $f(\omega_1^*) = g(\omega_1^*)$, then it can be easily observed that if $\omega_1^* < 1$ then ω_1^* is strictly increasing w.r.t. β and $\lim_{\beta\to+\infty} \omega_1^* = 1^-$, while if $\omega_1^* > 1$ then ω_1^* is strictly decreasing w.r.t. β and $\lim_{\beta\to+\infty} \omega_1^* = 1^+$. Finally if $\omega_1^* = 1$ then it does not change as β changes. Observe that condition $\omega_1^* < 1$ (resp. $\omega_1^* > 1$) corresponds to condition $g(1) < 1$ (resp. $g(1) > 1$) that is given by

$$p(1-\gamma)\left(\frac{1-2\alpha}{\alpha}\right) < (resp. >) 1.$$

Proof of Proposition 9

(i) If $\gamma \to 0^+$ and $\beta \to 0$ then $x^* \to \left(\frac{pm_2}{1+p}\right)^{\frac{1}{1-\alpha}}$ and consequently $det(JS_1^*(E^*)) \to 0$ while $tr(JS_1^*(E^*)) \to \alpha$ hence all conditions for the local stability hold.

(*ii*) If $\beta \to +\infty$ then $x^* \to (\alpha A)^{\frac{1}{1-\alpha}}$. It can be also verified that if $\gamma \to 0^+$ then $det(JS_1^*(E^*)) \to 0$ while $tr(JS_1^*(E^*)) \to -\infty$ hence conditions for the local stability cannot hold.

(*iii*) If $\gamma \to 1^-$ then $x^* \to 0^+$, and consequently $det(JS_1^*(E^*)) \to +\infty$ hence conditions for the local stability cannot hold.

Proof of Proposition 10

Observe that if $\gamma \to 0^+$ then $det(JS_1^*(x^*, x^*)) \to 0$. Assume also that $\beta \to -1^+$, then we have to distinguish between two cases: (*i*) if $\frac{\alpha}{1-\alpha} < p < \frac{\alpha}{1-2\alpha}$ then $x^* \to 0$ and $tr(JS_1^*(x^*, x^*)) \to \frac{\alpha}{p(1-\alpha)} < 1$; (*ii*) if $p \geq \frac{\alpha}{1-2\alpha}$ then $x^* \to x^*_{-1}$ and $tr(JS_1^*(x^*, x^*)) \to \alpha\frac{1+p}{p} < 1$. Hence E^* is locally stable.

Proof of Proposition 11

Let $\beta = 0$, then at the steady state,

$$x^* = x_0^* = \left(\frac{p(1-\alpha)A(1-\gamma)}{1+p(1-\gamma)}\right)^{\frac{1}{1-\alpha}}$$

and

$$det(JS_1^*(x_0^*, x_0^*)) = \frac{\gamma\alpha(1+p(1-\gamma))}{(1-\gamma)(1+p)}.$$

In this case, if $\gamma = \gamma_0$, where $\gamma_0 = \frac{(\alpha+1)(p+1)-\sqrt{(\alpha+1)^2(p+1)^2-4p\alpha(1+p)}}{2p\alpha} \in (0,1)$, then (*i*) $det(JS_1^*(x_0^*, x_0^*)) = 1$, (*ii*) $F_x(x_0^*, x_0^*)$ is positive and less then 2 (i.e. the trace of the Jacobian matrix belongs to the interval $(-2,2)$), (*iii*) the two non-real eigenvalues cross the unit circle at a non-zero speed when γ changes and (*iv*) none of them may be one of the first four roots of unity (excluding cases of weak resonance). These conditions identify a Neimark-Sacker bifurcation occurring at $\gamma = \gamma_0$ when $\beta = 0$.

Consider now $x^* = x^*(\beta, \gamma)$, $\beta > -1$, $\gamma \in (0,1)$. Since

$$det(JS_1^*(x^*(\beta,\gamma), x^*(\beta,\gamma))) \text{ and } F_x(x^*(\beta,\gamma), x^*(\beta,\gamma))$$

are both continuous w.r.t. β and γ then

$$det(JS_1^*(x^*(\beta,\gamma), x^*(\beta,\gamma))) \to 1 \text{ if } \beta \to 0 \text{ and } \gamma \to \gamma_0.$$

Hence, $\forall \epsilon_1 > 0\ \exists I(0, \gamma_0, \epsilon_1)$ such that if $(\beta, \gamma) \in I(0, \gamma_0, \epsilon_1)$ then

$$1 - \epsilon_1 < det(JS_1^*(x^*(\beta,\gamma), x^*(\beta,\gamma))) < 1 + \epsilon_1$$

and in particular, inside this neighborhood, there exists a $\gamma_\beta < 1$ such that

$$det(JS_1^*(x^*(\beta,\gamma_\beta), x^*(\beta,\gamma_\beta))) = 1.$$

Furthermore, there exists $I(0, \gamma_0, \epsilon_2)$ such that if $(\beta, \gamma) \in I(0, \gamma_0, \epsilon_2)$ then

$$F_x(x^*(\beta,\gamma), x^*(\beta,\gamma)) < 2.$$

Similar arguments can be used to prove that also conditions (*iii*) and (*iv*) hold thus showing that the Neimark-Sacker bifurcation occurs at $\gamma = \gamma_\beta$ if β is close to zero. Finally, since at the Neimark-Sacker bifurcation E^* loses its local stability, then a closed attracting invariant curve is created, i.e. the bifurcation is super-critical.

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Conflict of Interest: The authors declare that they have no conflict of interest.

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