Common Agency with Risk-Averse Agent

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Abstract

I consider a common agency model under adverse selection with a risk averse agent. Contracting takes place ex ante when all players have symmetric, although incomplete, information. The coordination problem between principals leads to more distortion in the optimal policy from the first best compared to the case of risk neutrality. In contrast with the risk neutral case the principals are unable to screen completely the agent's preferences if she/he is sufficiently risk averse. However, if the agent is almost risk neutral the output is separating, but the transfer schedules keep track of asymmetric contractual externality. When risk aversion goes to zero the transfers become truthful as in the complete information case.

1 Introduction

In this paper I make an attempt to explore the ex ante contracting in common agency when the agent is risk-averse. The theory of common agency under complete information started from a seminal paper of Bernheim and Whinston (1986). This gave an impulse to a burgeoning literature based on this model,[1] which studied mostly complete information truthful equilibria. Truthful contributions were introduced by Bernheim and Whinston (1986) and they at margin everywhere coincide with marginal valuations of the principals. These contributions implement the first-best and essentially are the only coalition-proof equilibria of complete information common agency game. Recently, the theory of the common agency under asymmetric information

[1] See Grossman and Helpman (2001) for thorough exposition.

has been developed. The fairly optimistic result of the common agency under perfect information; the efficiency of the outcome, cannot be held in this framework. As Dixit (1996) points out there are transaction costs in the models of political influence which preclude the efficient outcome. The common agency theory under uncertainty is therefore in some sense is more realistic. Modelling political influence with conflicting principals and asymmetric information, Martimort and Semenov (2005) show that in order to support an efficient outcome for ex ante contracting with risk-neutral agent, the contributions may be negative for some realizations of uncertainty parameter.

To capture different attitudes towards risk, I consider the common agency model, where two risk-neutral principals contract with a risk-averse agent, who has *CARA* utility with parameter of absolute risk aversion σ. For most of the paper I consider small values of σ, focusing on the difference between risk-neutral and risk-averse agent. The distortion from the first best is aggravated due to coordination problem and asymmetry of information in the common agency case. The output schedule for sufficiently big value of risk aversion is no longer strictly monotonic; bunching occurs for less efficient types. An interesting lesson from the model is that the contributions keep track of truthfulness, and in the limiting case of zero risk aversion they are truthful. This is in sharp contrast with Martimort and Stole (2005). They show that when uncertainty washes out the equilibrium payoffs correspondence is not in general lower hemi-continuous.

In principal-agent literature risk aversion of the agent did not receive much attention. In the theory of the firms, the firm normally has interim knowledge of information, therefore contracting is interim. However, even in this theory there are reasons to consider ax ante contracting. Harris and Raviv (1979), consider the firm as a contract which allocates resources at the ex ante stage. They show that under strong enforceability of the contracts, the first best is implementable. Salanié (1990) points out that it may well be that the conditions of deliverance of the good between producer and distributors are established before both parties know the demand.

Martimort and Semenov (2005) consider common agency model under asymmetric information with ex post and ex ante contracting. The structure of the contract differs greatly. As in Laussel and Le Breton (1998), ex ante contracting results in efficient outcome implemented by truthful contributions. However, the contributions may be negative for some realizations of the uncertainty parameter.[2] Laussel and Le Breton (1998) extend truthful

[2]This is an important feature of the common agency game with conflicting principals. If the principals are aligned (as in Laussel and Le Breton (1998), Martimort and Stole

implementation to the common agency model under asymmetric information with ex ante contracting.

Conversely, ex post contracting is far from efficient. The structure of contributions implementing the outcome is now determined by the out of equilibrium prolongation of contributions. Martimort and Stole (2005) show that contributions may be truthful (in the case of simple equilibria) or not (for example in the case of natural equilibria).[3] The bunching result in this paper resembles Salanié's (1990) work for the one-principal case. Laffont and Rochet (1998) consider risk-averse agent with ex post contracting in context of public regulation with one principal. Baron and Besanko (1987) show that randomness in cost or noisy monitoring of supplier's deterministic cost are no longer equivalent as one considers a risk-averse supplier, contrary to the case of a risk-neutral agent.

2 The Model

Several principals $P_i,\ i = 1, ..., n$, contract with a common agent A to produce a certain amount of public good q. The principals non-cooperatively offer contingent contributions $\langle t_1(q), t_2(q), ..., t_n(q)\rangle$. I assume that the principals are risk-neutral and the surplus of the principal P_i is $v_i q - \frac{q^2}{2n}$. The principal P_i's objective function is therefore

$$V_i(q) = v_i q - \frac{q^2}{2n} - t_i.$$

The valuations $v_i,\ i = 1, 2, ..., n$, of the good by the principals are common knowledge. The agent's utility function takes into account aversion towards risk by the agent. First, the ex post material payoff of the agent

$$W(q, \theta) = \sum_{i=1}^{n} t_i(q) - \theta q.$$

The parameter θ is an efficiency parameter which is uniformly distributed on the interval $[\underline{\theta}, \overline{\theta}]$. The agent is risk-averse with $CARA$ utility function; if he/she receives s, his/her utility is

$$U^\sigma(s) = 1 - e^{-\sigma s}.$$

The parameter $\sigma \geq 0$ characterizes a different attitude towards risk by the agent; if $\sigma = 0$, the agent is risk-neutral, if $\sigma = \infty$, the agent is infinitely

(2005) then the non-negativity of equilibrium transfers is ensured.

[3]See the General Introduction and Chapter 4.

risk-averse. In the latter case, ex ante contracting is equivalent to contracting with ex post participation constraints.

The agent has an outside opportunity normalized to zero. Contracting takes place ex ante, when both principals and the agent have symmetric information about the efficiency parameter θ.[4] In the General Introduction the contracting space and the method based on Taxation and Revelation Principles are described.

The timing of the common agency game with ex ante contracting is following;

• The principals simultaneously offer to the agent menu of output contingent contributions

$$\langle t_1(q), t_2(q), ..., t_n(q)\rangle_{q\in R^+},$$

• The agent accepts some subset of offers, or rejects all,
• The parameter θ is realized by the agent,
• Production q takes place, and contributions $t_i(q)$ are paid.

The *equilibrium* of the game consists in $(n+1)-$ tuple of functions

$$\langle t_1(q), t_2(q), ..., t_n(q), q(\theta)\rangle,$$

such that the output chosen by the agent

$$q(\theta) \in \arg\max_q \left\{ \sum_{i=1}^{n} t_i(q) - \theta q \right\}, \text{ for all } \theta \in \left[\underline{\theta}, \overline{\theta}\right], \tag{1}$$

and given a profile of contribution $t_{-i}(q) = \langle t_1(q), ..., t_{i-1}(q), t_{i+1}(q), ..., t_n(q)\rangle$ offered by principals $P_{k\neq i}$, the strategy $t_i(q)$ of the principal P_i is a best response given maximizing behavior of the agent (1).

Focusing on non-negative contributions in ex post contracting greatly simplifies the exposition. For the agent it is weakly dominant strategy to accept all offers in ex ante and ex post contracting. Solving the game backward, the condition (1) under concavity of agent's objective function leads to[5]

$$\sum_{i=1}^{n} t_i'(q(\theta)) = \theta q. \tag{2}$$

[4] I assume that

$$v = v_1 + v_2 > \overline{\theta},$$

to insure provision for any value of efficiency parameter.

[5] Here and henceforth, I denote by prime derivative with respect to q, and by dote derivative w.r.t. θ.

The principals design contributions such that it is not beneficial for them to deviate from their own schedule, given the contribution schedule of the other principal.

$\star$ *First best:* The first best arises under complete information when all parties form a grand coalition. The outcome of this game may also be represented in the complete information framework when both principals merge as one. The first best level of output for the state θ is

$$q^{FB}(\theta) = v - \theta,$$

where $v = \sum_{i=1}^{n} v_i$ is a joint valuation of the good by all principals. The contributions from the principals which support this outcome are $t_i(\theta) = \theta q^{FB}(\theta)$.

3 Benchmark A: Risk-Neutrality and Infinite Risk Aversion

In this paper I describe equilibria for the cases of zero and infinite risk-aversion.

$\star$ *Risk Neutrality, $\sigma = 0$:* I consider first the common agency under ex ante contracting with a risk-neutral agent, i.e., when $\sigma = 0$. It introduces the ex ante constraint on the material payoff;

$$E_\theta W(q(\theta), \theta) \geq 0.$$

The efficient outcome is implemented via the following truthful contributions[6]

$$t_i(q) = v_i q - \frac{q^2}{2n} - C_i, \; i = 1, ..., n.$$

Importantly, in this subsection I do not impose the assumption of the non-negativity of contributions. Conceptually, the situation with *risk-neutral* agent and *ex ante* contracting is very close to the case of complete information. The technique, which is used to describe truthful equilibria is also similar to complete information framework (see Laussel and Le Breton (1998) and Martimort and Semenov (2005)). Laussel and Le Breton (2001) show that in order to study equilibrium payoffs, it is useful to connect the common agency game under ex ante contracting to the cooperative game with characteristic function

$$W_S = E_\theta W_S(\theta),$$

[6]See for example Martimort and Semenov (2005).

where $W_S(\theta)$ is the joint surplus of the principals from the set $S \in 2^N$, and the agent who has efficiency parameter θ.

Here and afterwards, to avoid cumbersome calculations without getting new insights, I will focus on the case $n = 2$. The next Proposition characterizes the set of truthful equilibria.

Proposition 1 *Any truthful equilibrium of common agency game under risk neutrality implements an efficient outcome;* $q^{FB}(\theta) = v - \theta$.

i) ***Non-coordinated Principals:*** *If the valuations of principals satisfy*

$$v < \frac{1}{\underline{\theta} + \overline{\theta}} \left((v_1 - v_2)^2 + \underline{\theta}^2 + \overline{\theta}^2 + \underline{\theta}\overline{\theta} \right),$$

then the game is subadditive and the agent gets a positive rent in the unique truthful equilibrium;

$$\int_{\underline{\theta}}^{\overline{\theta}} W(\theta) \frac{d\theta}{\Delta\theta} = \sum_{i=1}^{2} W_i - W_{12} > 0$$

The equilibrium contributions are

$$t_i(q) = v_i q - \frac{q^2}{2n} - C_i, \ i = 1, 2, \tag{3}$$

with constants $C_i = \int_{\underline{\theta}}^{\overline{\theta}} \left(-\frac{(v-\theta)^2}{2} + (v_{-i} - \theta)^2 \right) \frac{d\theta}{\Delta\theta}$. *The payoff of the principal* P_i *is* C_i.

ii) ***Coordinated Principals:*** *If* $v \geq \frac{1}{\underline{\theta}+\overline{\theta}} \left((v_1 - v_2)^2 + \underline{\theta}^2 + \overline{\theta}^2 + \underline{\theta}\overline{\theta} \right)$, *then the game is superadditive, the truthful equilibrium is not unique and the agent gets no rent in any of them. The equilibrium contributions are determined by (3).*

iii) Equilibrium contributions are non negative in expected value.

Proposition brings interesting insights. When principals are not very interested in output, or they are very different in objectives the agent has a positive rent. In this case the objectives of principals conflict sufficiently.

The more the principals value the agent's production, and the less their interests are dispersed, the more successful they are in extracting all agent's surplus. The coordination problem is not very hard in this case. It sounds somewhat unexpected, but the bigger the valuation and the less divergence, the more congruent the principals are. In fact, they act as a one merged principal who extracts all rent as in the one-principal model with ex ante contracting, and then, both principals just re-distribute the joint surplus. To support efficient outcome negative contributions close to inefficient parameter are possible. Weaker condition is imposed on this problem; contributions in expected terms are non negative.

Remarkably, this Proposition resembles the case of ex-ante contracting in Martimort and Semenov (2005). The equilibrium contributions are non-negative if the parameter β, which characterizes the strength of ideological bias of the agent or informally reverse of his/her marginal utility for money, is small enough.

$\star$ *Infinite Risk Aversion, $\sigma = \infty$:* The case with infinite risk - aversion is conceptually different. Now, the truthful equilibria play less of a role as in the risk-neutral case. Since the agent is infinitely averse towards the risk, he/she does not accept any negative payoff. The contract includes ex post participation constraint

$$W\left(q_{\infty}^{CA}(\theta), \theta\right) \geq 0, \text{ for all } \theta. \tag{4}$$

To receive the optimal output, I use the technique described in the introduction of this dissertation. The output is determined by the equation

$$\sum_{i=1}^{n} S_i'\left(q_{\infty}^{CA}(\theta)\right) = \frac{\partial C\left(q_{\infty}^{CA}(\theta), \theta\right)}{\partial q} + n\frac{F(\theta)}{f(\theta)}\frac{\partial^2 C\left(q_{\infty}^{CA}(\theta), \theta\right)}{\partial\theta\partial q},$$

which in the current framework gives[7]

$$q_{\infty}^{CA}(\theta) = v - \theta - 2\left(\theta - \underline{\theta}\right). \tag{5}$$

The output $q_{\infty}^{CA}(\theta)$ is no longer efficient, but still fully revealing. Equilibrium contributions which support (5) may have very different structure.[8] Since I consider only the natural equilibria, they can be obtained from General Introduction:

$$t_i(q) = \max\left\{S_i(q) - \int \frac{F(\theta(q))}{f(\theta(q))}\frac{\partial^2 C(q, \theta(q))}{\partial\theta\partial q}dq, 0\right\} =$$

$$\max\left\{\frac{2v_1 - v_2 + \underline{\theta}}{3}q - \frac{q^2}{12} - C_i, 0\right\}.$$

For simplicity, I focus on symmetric case: $v_1 = v_2 = \frac{v}{2}$.

[7]This is a special case of Martimort and Stole (2005).

[8]Truthful equilibria may or may not arise endogenously in the limit when $\Delta\theta \to 0$. It depends on how contributions are determined for the out of equilibrium policies (see Martimort and Stole (2005) and Martimort and Semenov (2005)).

Proposition 2 *In the common agency game with ex post contracting and risk-neutral agent;*

i) The equilibrium output is linear;

$$q_{\infty}^{CA}(\theta) = v - 3\theta + \underline{\theta},$$

ii) The equilibrium contributions are

$$t_i(q) = \max\left\{\frac{v/2+\underline{\theta}}{3}q - \frac{q^2}{12} - C_i, 0\right\},$$

with constants (C_1, C_2) *determined by*

$$W\left(\overline{\theta}\right) = 0, \text{ and } t_i\left(\overline{\theta}\right) = 0.$$

The assumption that the principals have the same marginal valuation pins down the "coordinated equilibria" in which the principals are aligned. In this case the ex post constraint (4) is binding, it reflects the "merging nature" of this common agency game.[9]

The optimal policies for polar cases of risk neutrality $q^{FB}(\theta)$ and infinite risk aversion $q_{\infty}^{CA}(\theta)$ determine the range where all equilibrium policies for intermediate values of risk aversion are situated. The feature of these polar policies is that they are separating for all $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$; more explicitly, they are strictly decreasing over the whole range of θ. Distortion from the first best is bigger in common agency game than in the case with one principal $q_{\infty}^{M}(\theta)$:

$$q_{\infty}^{M}(\theta) > q_{\infty}^{CA}(\theta), \text{ for all } \theta \in (\underline{\theta}, \overline{\theta}].$$

4 Benchmark B: One-Principal Case

In this Section I consider two principals who perfectly coordinate their contribution strategies. It is equivalent to the situation where one principal P^M who aggregates preferences of both principals contracts ex ante with risk-averse agent.[10] The principal compensates the agent with monetary contribution t. The case of risk neutrality of the agent is a standard problem in incentive literature. The utility of the principal is

$$V(q) = vq - \frac{q^2}{2} - t(q),$$

[9]If valuations are sufficiently different, then the non-coordinated equilibria arise. In these equilibria the ex post participation constraint is not binding. The binding constraints are full participation constraints.

[10]This model is similar to Salanié (1990).

and he/she offers the contract $\langle t(q)\rangle_{q\in R^{+}}$ to the agent who has to decide about acceptance of this contract *ex ante*.

The program of the principal (P^{M});[11]

$$\underset{q(\theta),t(\theta)}{Max}\int_{\underline{\theta}}^{\overline{\theta}}\left[vq(\theta)-\frac{q^{2}(\theta)}{2}-t(\theta)\right]\frac{d\theta}{\Delta\theta},$$

subject to

$$\dot{W}(\theta)=-q(\theta), \tag{6}$$

$$\dot{q}(\theta)\leq 0, \text{ and} \tag{7}$$

$$E_{\theta}U^{\sigma}(W(\theta))=\int_{\underline{\theta}}^{\overline{\theta}}\left(1-e^{-\sigma W(\theta)}\right)\frac{d\theta}{\Delta\theta}\geq 0. \tag{8}$$

Constraints (6) and (7) guarantee implementability of the output $q(\theta)$. To solve the problem (P^{M}) I apply the optimal control techniques (see for example Seierstad and Sydsaeter (1987)). The state variables of the problem are: $q(\theta)$, $W(\theta)$ and $z(\theta)=\int_{\underline{\theta}}^{\theta}\left(1-e^{-\sigma W(\theta)}\right)\frac{d\theta}{\Delta\theta}$ with co-states $\nu(\theta)$, $\lambda(\theta)$, and $\mu(\theta)=\mu$ correspondingly. The control variable is $c(\theta)=\dot{q}(\theta)$. The problem is now transformed to

$$\underset{q(\theta),W(\theta),z(\theta),c(\theta)}{Max}\int_{\underline{\theta}}^{\overline{\theta}}\left[vq(\theta)-\frac{q^{2}(\theta)}{2}-\theta q(\theta)-W(\theta)\right]\frac{d\theta}{\Delta\theta},$$

subject to (6), and $c(\theta)\leq 0$.

$$z(\underline{\theta})=0,\ z\left(\overline{\theta}\right)\geq 0,\ z\left(\overline{\theta}\right)\mu\left(\overline{\theta}\right)=0. \tag{9}$$

The Hamiltonian of the problem is

$$H=\left[vq-\frac{q^{2}}{2}-\theta q-W\right]\frac{1}{\Delta\theta}-\lambda q-\mu\frac{e^{-\sigma W}}{\Delta\theta}+\nu c.$$

The maximization of the Hamiltonian with respect to control c leads to

$$\nu c=0,\ \nu\geq 0,\ c\leq 0. \tag{10}$$

Other necessary conditions are

$$\dot{\lambda}(\theta)=-\frac{\partial H}{\partial W}=\frac{1}{\Delta\theta}-\frac{\mu\sigma}{\Delta\theta}e^{-\sigma W(\theta)}, \tag{11}$$

[11] In what follows the transfers are written with the same notation: $t(q)$ and $t(\theta)$. To avoid confusion the derivative of transfers with respect to q is denoted as $t'(q)$, whereas the derivative with respect to θ is denoted as $\dot{t}(\theta)$.

$$\dot{\nu}(\theta) = -\frac{\partial H}{\partial q} = -(v - q(\theta) - \theta)\frac{1}{\Delta\theta} + \lambda(\theta), \tag{12}$$

$$\lambda(\underline{\theta}) = \lambda(\overline{\theta}) = 0, \tag{13}$$

$$z(\underline{\theta}) = 0,\ z(\overline{\theta}) \geq 0,\ z(\overline{\theta})\mu = 0,\ \text{and } \nu(\underline{\theta}) = \nu(\overline{\theta}) = 0.$$

Co-state variables $\lambda(\theta)$ and $\nu(\theta)$ are continuous and piecewise differentiable. From (11), $\mu = \frac{\Delta\theta}{\sigma}\left(\int_{\underline{\theta}}^{\overline{\theta}} e^{-\sigma W(t)}dt\right)^{-1} > 0$. Therefore from transversality conditions (13), $z(\overline{\theta}) = 0$, and thus $\mu = \frac{1}{\sigma}$. If $c(\theta) < 0$ on the interval, then on this interval $\nu(\theta) \equiv 0$, and from (12)

$$\frac{\dot{q}(\theta) + 1}{\Delta\theta} + \dot{\lambda}(\theta) \equiv 0.$$

This using (11) leads to

$$e^{-\sigma W(\theta)} = \dot{q}(\theta) + 2, \tag{14}$$

which yields $\dot{q}(\theta) > -2$, and

$$\ddot{q}(\theta) = -\sigma\dot{W}(\theta)e^{-\sigma W(\theta)} = \tag{15}$$

$$= \sigma q(\theta)\left(\dot{q}(\theta) + 2\right).$$

Thus the equilibrium output is a convex function of θ. The equation (15) is a necessary for output $q(\theta)$ to be a part of the optimal contract in the intervals of separating. It turned out that separation prevails for the most efficient values of the cost parameter. For less efficient values the principal exhibits some bunching. The exact form of the optimal contract is established in the following

Proposition 3 *There exist $\widehat{\sigma}_M \geq 0$ such that the equilibrium in one-principal case has the form:*

i) If $\sigma \leq \widehat{\sigma}_M$, then the equilibrium output $q(\theta)$ is the unique solution of the differential equation

$$\ddot{q}(\theta) = \sigma q(\theta)\left(\dot{q}(\theta) + 2\right),$$

$$q(\underline{\theta}) = q^{FB}(\underline{\theta}),\ q(\overline{\theta}) = q^{FB}(\overline{\theta}),\ \textit{and}$$

$$0 > q(\theta) > -2.$$

Output $q(\theta)$ is implemented via contributions

$$t(\theta) = \theta q(\theta) - \frac{1}{\sigma} Log\left(\dot{q}(\theta) + 2\right).$$

ii) If $\sigma > \widehat{\sigma}_M$, then in the equilibrium there is bunching; there are $\widetilde{\theta}_M(\sigma)$ and $\widetilde{q}_M(\sigma)$ such that

$\cdot$ *for $\theta \in \left[\widetilde{\theta}_M(\sigma), \overline{\theta}\right]$ the output $q(\theta) = \widetilde{q}_M(\sigma)$,*

$\cdot$ *for $\theta \in \left[\underline{\theta}, \widetilde{\theta}_M(\sigma)\right]$ the output $q(\theta)$ is the unique solution of*

$$\ddot{q}(\theta) = \sigma q(\theta)\left(\dot{q}(\theta) + 2\right),$$

$$q(\underline{\theta}) = q^{FB}(\underline{\theta}),\ q\left(\widetilde{\theta}_M(\sigma)\right) = \widetilde{q}_M(\sigma),\ \textit{and}$$

$$0 > q(\theta) > -2.$$

Bunching exists for sufficiently large values of the absolute risk aversion. However, for infinite risk-averse agent separating again occurs for all θ. The equation (15) in the limit when $\sigma \to 0$ converges to the first best output $q^{FB}(\theta)$. The equilibrium contributions are negative for some values of θ. This resembles the ex ante contracting with risk-neutral agent.

5 Common Agency

In this Section I consider common agency with a risk-averse agent. The agent chooses $q(\theta)$ as the solution of the maximization problem

$$q(\theta) \in \arg\max_q \left\{ \sum_{i=1}^{2} t_i(q) - \theta q \right\},$$

where $t_i(q)$ are equilibrium contributions. The material payoff of the agent is then

$$W(\theta) = \sum_{i=1}^{2} t_i(q(\theta)) - \theta q(\theta). \tag{16}$$

The best response program (P_i) of the principal P_i to offer $\{t_{-i}(q)\}$ can be written as

$$\underset{q(\theta), t_i(\theta)}{Max} \int_{\underline{\theta}}^{\overline{\theta}} \left[v_i q(\theta) - \frac{q^2(\theta)}{4} - t_i(\theta) \right] \frac{d\theta}{\Delta\theta},$$

subject to (6), (7), (8), and

$$E_\theta U^\sigma \left(W\left(\theta\right)\right) \geq E_\theta U^\sigma \left(W_{-i}\left(\theta\right)\right) \tag{17}$$

Using the expression (16) for information rent $W(\theta)$, the program (P_i) is transformed into the following optimal control problem:

$$\underset{q(\theta),W(\theta),z(\theta),c(\theta)}{Max} \int_{\underline{\theta}}^{\overline{\theta}} \left[v_i q\left(\theta\right) - \frac{q^2\left(\theta\right)}{4} - \theta q\left(\theta\right) - W\left(\theta\right) + t_{-i}\left(\theta\right)\right] \frac{d\theta}{\Delta\theta},$$

The constraint (17) is a new feature of the model, it will be taken with co-state p_i. The Hamiltonian of the P_i's program,

$$H_i = \left[v_i q - \frac{q^2}{4} - \theta q - W + t_{-i}\left(q\right)\right] \frac{1}{\Delta\theta} - \lambda_i q - \mu_i \frac{e^{-\sigma W}}{\Delta\theta} - p_i \frac{e^{-\sigma W}}{\Delta\theta} + \nu_i c,$$

The necessary conditions are;

$$\dot{W}\left(\theta\right) = -q\left(\theta\right), \text{ with co-state variable } \lambda_i\left(\theta\right), \tag{18}$$

$$\dot{z}\left(\theta\right) = \frac{1 - e^{-\sigma W(\theta)}}{\Delta\theta} \text{ - co-state } \mu_i, \tag{19}$$

$$\dot{q}\left(\theta\right) = c\left(\theta\right) \text{ - co-state } \nu_i\left(\theta\right), \tag{20}$$

$$\nu_i\left(\theta\right) c\left(\theta\right) = 0,\ c\left(\theta\right) \leq 0,\ \nu_i\left(\theta\right) \geq 0, \tag{21}$$

$$\dot{\lambda}_i\left(\theta\right) = -\frac{\partial L_i}{\partial W} = \frac{1}{\Delta\theta} - \frac{\mu_i \sigma}{\Delta\theta} e^{-\sigma W(\theta)} - \frac{p_i \sigma}{\Delta\theta} e^{-\sigma W}, \tag{22}$$

$$\dot{\nu}_i\left(\theta\right) = -\frac{\partial H_i}{\partial q} = -\left(v_i - \frac{q\left(\theta\right)}{2} - \theta + t_i'\left(q\left(\theta\right)\right)\right) \frac{1}{\Delta\theta} + \lambda_i\left(\theta\right), \tag{23}$$

$$\begin{gathered} \lambda_i\left(\underline{\theta}\right) = \lambda_i\left(\overline{\theta}\right) = 0,\ z\left(\underline{\theta}\right) = 0,\ z\left(\overline{\theta}\right) \geq 0, \\ \nu_i\left(\underline{\theta}\right) = \nu_i\left(\overline{\theta}\right) = 0,\ \mu_i \geq 0,\ \mu_i z\left(\overline{\theta}\right) = 0, \end{gathered} \tag{24}$$

$$\begin{gathered} p_i\left(E_\theta U^\sigma W\left(\theta\right) - E_\theta U^\sigma W_{-i}\left(\theta\right)\right) = 0, \\ p_i \geq 0,\ E_\theta U^\sigma W\left(\theta\right) \geq E_\theta U^\sigma W_{-i}\left(\theta\right), \end{gathered} \tag{25}$$

with the continuous co-state variables as in the one-principal program.[12] Again assume that on the non-degenerate interval $c(\theta) < 0$, then maximization of Hamiltonians H_i, $i = 1, 2$, yields

$$\dot{\nu}_i\left(\theta\right) = -\frac{\partial H_i}{\partial q} = -\left(v_i - \frac{q}{2} - \theta + t_{-i}'\left(q\right)\right) \frac{1}{\Delta\theta} + \lambda_i \equiv 0,$$

[12] In Martimort and Semenov (2005) the discontinuity of the co-state variable is allowed. They consider ex post participation constraint and ex post full participation constraint.

on the interval where $q(\theta)$ is strictly decreasing. Adding up these conditions and using the first order condition

$$t_1'(q(\theta)) + t_2'(q(\theta)) = \theta,$$

lead to

$$q(\theta) = v - \theta - \Delta\theta(\lambda_1(\theta) + \lambda_2(\theta)). \tag{26}$$

Since $\lambda_i(\theta) > 0$ on $\left(\underline{\theta}, \overline{\theta}\right)$,[13] the common agency's output is always sub-optimal inside the support. Differentiating expression (26) and using (22) yield

$$\sigma(\mu_1 + p_1 + \mu_2 + p_2) e^{-\sigma W(\theta)} = \frac{1}{2}\left(\dot{q}(\theta) + 3\right).$$

The optimal outcome in the intervals of separating is, therefore, the solution of the differential equation

$$\ddot{q}(\theta) = \sigma q(\theta)\left(\dot{q}(\theta) + 3\right). \tag{27}$$

The structure of the optimal contract is described in

Proposition 4 *There is $\widehat{\sigma}_{CA} \geq 0$ such that in the equilibrium of the common agency game;*

i) If $\sigma \leq \widehat{\sigma}_{CA}$, then the equilibrium output $q(\theta)$ is the unique solution of the differential equation

$$\ddot{q}(\theta) = \sigma q(\theta)\left(\dot{q}(\theta) + 3\right),$$

$$q(\underline{\theta}) = q^{FB}(\underline{\theta}),\ q\left(\overline{\theta}\right) = q^{FB}\left(\overline{\theta}\right),\ \text{and}$$

$$0 > q(\theta) > -3.$$

The aggregated contribution schedule

$$t_1(\theta) + t_2(\theta) = \theta q(\theta) - \frac{1}{\sigma} Log \frac{1}{2}\left(\dot{q}(\theta) + 3\right)$$

and the contributions of the principal P_i,

$$t_i(\theta) = v_i q(\theta) - \frac{q^2(\theta)}{4} - \Delta\theta \int_{\underline{\theta}}^{\theta} \lambda_i(\theta)\dot{q}(\theta)\, d\theta - C_i.$$

[13] $\lambda_i(\theta)$ are strictly concave on $\left(\underline{\theta}, \overline{\theta}\right)$, and using transversality conditions; $\lambda_i(\underline{\theta}) = \lambda_i\left(\overline{\theta}\right) = 0$, lead to $\lambda_i(\theta) > 0$ on $\left(\underline{\theta}, \overline{\theta}\right)$.

The constants (C_1, C_2) *are determined by*

$$-C_1 - C_2 = -\frac{1}{\sigma} Log \frac{1}{2} \left(\dot{q}\left(\underline{\theta}\right) + 3 \right) - \frac{q^2\left(\underline{\theta}\right)}{2}.$$

ii) If $\sigma > \widehat{\sigma}_{CA}$*, then in the equilibrium there is bunching; there are* $\widetilde{\theta}_{CA}(\sigma)$ *and* $\widetilde{q}_{CA}(\sigma)$ *such that*

$\cdot$ *for* $\theta \in \left[\widetilde{\theta}_{CA}(\sigma), \overline{\theta}\right]$ *the output* $q(\theta) = \widetilde{q}_{CA}(\sigma)$*,*

$\cdot$ *for* $\theta \in \left[\underline{\theta}, \widetilde{\theta}_{CA}(\sigma)\right]$ *the output* $q(\theta)$ *is the unique solution of*

$$\ddot{q}(\theta) = \sigma q(\theta) \left(\dot{q}(\theta) + 3 \right),$$

$$q\left(\underline{\theta}\right) = q^{FB}\left(\underline{\theta}\right), \; q\left(\widetilde{\theta}_{CA}(\sigma)\right) = \widetilde{q}_{CA}(\sigma), \; \textit{and}$$

$$0 > q(\theta) > -3.$$

The equilibrium for common agency game for small σ resembles one for one-principal case. The coordination problem distorts output downward. The contributions for principal P_i keep track of truthful contributions;

$$t_i(\theta, \sigma) = v_i q(\theta, \sigma) - \frac{q^2(\theta, \sigma)}{4} - \Delta\theta \int_{\underline{\theta}}^{\theta} \lambda_i(\theta, \sigma)\, \dot{q}(\theta, \sigma)\, d\theta - C_i. \tag{28}$$

If the coefficient of risk-aversion goes to zero: $\sigma \to 0$, the contributions

$$t_i(\theta, \sigma) \to v_i q(\theta, 0) - \frac{q^2(\theta, 0)}{4} - C_i,$$

i.e., the contributions in the limit are truthful. However, even a small risk aversion distorts truthfulness of contributions. This is an important consequence of the model.

Corollary 5 *The set of equilibrium payoffs achieved as limit* $\sigma \to 0$ *of natural equilibria is a whole subset of equilibrium payoffs achieved with truthful equilibria under complete information.*

By contrast, Martimort and Stole (2005) show in the case of diminishing support of distribution of θ, the equilibrium schedules of natural equilibria always keep track of contractual externality and do not converge to truthful contributions. The set of payoffs achieved with natural equilibria in the limit is a strict subset of the equilibrium payoffs achieved with truthful equilibria under complete information. In the present case, it is not true and contributions are truthful in the limit. However, in the limit truthful equilibria considered in the case of ex-ante contracting with risk neutrality, not complete information.

The solution of the equation (27) with boundary conditions $q\left(\underline{\theta}\right) = v - \underline{\theta}$, and $q\left(\overline{\theta}\right) = v - \overline{\theta}$ gives the equilibrium output of the common agency game. Comparing this solution with the output for one-principal case yields

Proposition 6 *The output produced by the common agent is always lower than the output produced for the merged principal:*

$$q_{CA}\left(\theta\right) < q_M\left(\theta\right), \ \forall \theta \in (\underline{\theta}, \overline{\theta}].$$

The Propositions 4 and 6 describe solution of the common agency problem for small levels of risk-aversion. Another interesting issue which can be proceeded is to explore the extent of bunching in the common agency game relative to one-principal case. In principle, the necessary and sufficient conditions are sufficient to answer this question. However, the conditions are quite complicated and perhaps computer simulations will be needed.

Proofs

- **Proof of Proposition 1:**

Let $S \in 2^N$. The joint surplus $W_S\left(\theta\right)$ of the coalition S and the agent at state θ,

$$W_S\left(\theta\right) = \max_q \left\{ \left(\sum_{i \in S} v_i \right) q - s\frac{q^2}{2n} - \theta q \right\} = \frac{n}{2s} \left(\left(\sum_{i \in S} v_i \right) - \theta \right)^2, \text{ and}$$

where s is the cardinality of S. Particularly, for $n = 2$;

$$W_i\left(\theta\right) = \left(v_i - \theta\right)^2, \text{ and } W_{12}\left(\theta\right) = \frac{\left(v - \theta\right)^2}{2}.$$

Expected surpluses are

$$W_i = \int_{\underline{\theta}}^{\overline{\theta}} W_i\left(\theta\right) \frac{d\theta}{\Delta\theta}, \text{ and } W_{12} = \int_{\underline{\theta}}^{\overline{\theta}} W_{12}\left(\theta\right) \frac{d\theta}{\Delta\theta}.$$

Taking the difference

$$W_1 + W_2 - W_{12} = \int_{\underline{\theta}}^{\overline{\theta}} \left\{ \left(v_1 - \theta\right)^2 + \left(v_i - \theta\right)^2 - \frac{\left(v - \theta\right)^2}{2} \right\} \frac{d\theta}{\Delta\theta} > 0, \text{ if}$$

$$\int_{\underline{\theta}}^{\overline{\theta}} \left\{(v_1 - v_2)^2 + 3\theta^2 - 2v\theta\right\} d\theta > 0.$$

The Proposition 1 then follows.

■

• **Proof of Proposition 3:** First I prove that the optimal output may not have interval of bunching and then separation. At the same time the proof shows that there should not be distortion at the top.

• Suppose to the contrary that there exist $\theta_1,\ \theta_2,\ \theta_3$ such that $q(\theta)$ is constant on $[\theta_1, \theta_2]$ and strictly decreasing on $[\theta_2, \theta_3]$. Consider

$$\dot{\nu}(\theta) = -(v - q(\theta) - \theta)\frac{1}{\Delta\theta} + \lambda(\theta).$$

$\dot{\nu}(\theta)$ is continuous and piecewise differentiable function. Since $c(\theta) < 0$ on $[\theta_2, \theta_3]$, $\dot{\nu}(\theta) = 0$ on this interval. Therefore

$$\ddot{\nu}(\theta) = \left(\dot{q}(\theta) + 2\right)\frac{1}{\Delta\theta} - \frac{1}{\Delta\theta}e^{-\sigma W(\theta)} = 0,\ \forall\theta \in (\theta_2, \theta_3).$$

By continuity of $W(\theta)$ this leads to

$$\ddot{\nu}\left(\theta_2^+\right) = 0 < \frac{2}{\Delta\theta} - \frac{1}{\Delta\theta}e^{-\sigma W(\theta_2)} = \ddot{\nu}\left(\theta_2^-\right).$$

Since $\dddot{\nu}(\theta) = \frac{\sigma}{\Delta\theta}\dot{W}(\theta)e^{-\sigma W(\theta)} < 0$ on $[\theta_1, \theta_2]$ then $\ddot{\nu}\left(\theta_1^+\right) > 0$. Continuity of $\dot{\nu}(\theta)$ implies that $q(\theta)$ cannot be strictly decreasing on the interval to the left of θ_1. This yields that $q(\theta)$ is constant on the interval $[\underline{\theta}, \theta_2]$ and $\ddot{\nu}(\theta) > 0$ on $(\underline{\theta}, \theta_2)$. This means that $\dot{\nu}(\theta) < 0$ on $(\underline{\theta}, \theta_2)$.[14] But $\nu(\theta)$ is convex on $[\underline{\theta}, \theta_2]$ and if $\nu(\underline{\theta}) = \nu\left(\theta_2^+\right) = 0$ that leads that $\dot{\nu}(\theta)$ must be positive on some interval belonging to $[\underline{\theta}, \theta_2]$. Contradiction. This gives a structure of the optimal contract in Proposition 1.

• Since $c > 0$ on $\left[\underline{\theta}, \widetilde{\theta}\right]$ for some $\widetilde{\theta}$ and $\dot{\nu}(\underline{\theta}) = 0$ then $q(\underline{\theta}) = v - \underline{\theta}$.

Lemma 7 $\frac{\partial q(\theta,\sigma)}{\partial\sigma} \leq 0.$

Proof. Since $q(\underline{\theta}, \sigma) = v - \underline{\theta}$ and $q\left(\overline{\theta}, \sigma\right) = v - \overline{\theta}$, $\frac{\partial q(\underline{\theta},\sigma)}{\partial\sigma} = \frac{\partial q\left(\overline{\theta},\sigma\right)}{\partial\sigma} = 0$. Suppose to the contrary that there is a pair $\left(\widehat{\theta}, \widehat{\sigma}\right)$ such that $\frac{\partial q\left(\widehat{\theta},\widehat{\sigma}\right)}{\partial\sigma} > 0$.

[14] Since $\dot{\nu}(\theta) = 0$ on $[\theta_2, \theta_3]$.

Then for the function $\psi(\theta) = \frac{\partial q(\theta,\widehat{\sigma})}{\partial \sigma}$ there exists a local interior maximum $\theta_{\max}$ such that

$$\psi(\theta_{\max}) > 0,$$
$$\psi'(\theta_{\max}) = 0, \text{ and}$$
$$\psi''(\theta_{\max}) \le 0$$

Differentiating equation $\frac{\partial^2 q(\theta,\widehat{\sigma})}{\partial \theta^2} = \sigma q(\theta,\widehat{\sigma})\left[\frac{\partial q(\theta,\widehat{\sigma})}{\partial \sigma} + \alpha\right]$ with respect to σ and evaluating the result at $(\theta_{\max},\widehat{\sigma})$ one gets

$$0 \ge \frac{\partial^3 q(\theta_{\max},\widehat{\sigma})}{\partial \theta^2 \partial \sigma} = q(\theta_{\max},\widehat{\sigma})\left[\frac{\partial q(\theta_{\max},\widehat{\sigma})}{\partial \sigma} + \alpha\right] +$$

$$+\sigma \frac{\partial q(\theta_{\max},\widehat{\sigma})}{\partial \sigma}\left[\frac{\partial q(\theta_{\max},\widehat{\sigma})}{\partial \theta} + \alpha\right] + \sigma q(\theta_{\max},\widehat{\sigma})\frac{\partial^2 q(\theta_{\max},\widehat{\sigma})}{\partial \theta \partial \sigma}$$

The third term on the R.H.S. is zero, the first and the second ones are positive. Therefore contradiction. ■

⋆ Consider now the class of equations which determine the solution in the separating intervals;

$$\ddot{q}(\theta,\alpha) = \sigma q(\theta,\alpha)\left(\dot{q}(\theta,\alpha) + \alpha\right)$$

$$q(\underline{\theta},\alpha) = v - \underline{\theta},\ q(\overline{\theta},\alpha) = v - \overline{\theta}.$$

$$-\alpha < \frac{\partial q(\theta,\alpha)}{\partial \theta} < 0$$

It can be shown that the solution of the equation is unique and it is strictly decreasing if and only if $\sigma < \widehat{\sigma}_M$ for some $\widehat{\sigma}_M > 0$.

■

• **Proof of Proposition 4:**

The proof is the same as for Proposition 3 with $\nu_1(\theta) + \nu_2(\theta)$ taken instead of $\nu(\theta)$. Then using first order condition one gets

$$\frac{d(\nu_1(\theta) + \nu_2(\theta))}{d\theta} = -(v - q(\theta) - \theta)\frac{1}{\Delta\theta} + \lambda_1(\theta) + \lambda_2(\theta).$$

From maximization of Hamiltonians H_i one receives that if $c(\theta) < 0$ then $\nu_1(\theta) + \nu_2(\theta) \equiv 0$.

⋆ Co-state equations and $p_i(\theta) = 0$ lead to

$$\frac{d(\lambda_1(\theta) + \lambda_2(\theta))}{d\theta} = \frac{2}{\Delta\theta} - \frac{2}{\Delta\theta}e^{-\sigma W(\theta)},$$

Then the proof is proceeded as in Proposition 3.

$\star$ Conditions (23) imply

$$\dot{t}_{-i}(\theta) = -v_i \dot{q}(\theta) + \frac{q(\theta)\dot{q}(\theta)}{2} + \theta\dot{q}(\theta) + \lambda_i(\theta)\dot{q}(\theta)\Delta\theta.$$

Taking into account maximization by the agent

$$\dot{t}_1(\theta) + \dot{t}_2(\theta) = \theta\dot{q}(\theta),$$

one gets

$$t_i(\theta) = v_i q(\theta) - \frac{q^2(\theta)}{4} - \Delta\theta \int_{\underline{\theta}}^{\theta} \lambda_i(\theta)\dot{q}(\theta)\, d\theta - C_i.$$

$\star$ Multiplying (23) by $\dot{q}(\theta)$ yields

$$\dot{t}_i(\theta) = \dot{q}(\theta)\left[v_i - \frac{q(\theta)}{2} - \lambda_i(\theta)\Delta\theta\right].$$

The output $q(\theta)$ is decreasing. The derivative of $\frac{q(\theta)}{2} + \lambda_i(\theta)\Delta\theta$ is equal to

$$\frac{2e^{-\sigma W(\theta)} - 3}{2} + 1 - e^{-\sigma W(\theta)} = -\frac{1}{2} < 0.$$

Hence if $v_i - \frac{q(\underline{\theta})}{2} - \lambda_i(\underline{\theta})\Delta\theta \geq 0$, the contributions $t_i(\theta)$ are always decreasing. This is true if

$$|v_1 - v_2| \leq \underline{\theta}.$$

$\blacksquare$

- **Proof of Corollary 5:**

Consider the function

$$\int_{\underline{\theta}}^{\theta} \lambda_i(\theta,\sigma)\dot{q}(\theta,\sigma)\, d\theta.$$

Co-state $\lambda_i(\theta,\sigma) = \frac{1}{\Delta\theta}\int_{\underline{\theta}}^{\theta}\left(1 - e^{-\sigma W(\theta,\sigma)}\right) d\theta \underset{\sigma\to 0}{\to} 0$ by continuity of $W(\theta,\sigma)$.

On the other hand $|\, \dot{q}(\theta,\sigma)\, |$ is bounded[15]

$$m < |\, \dot{q}(\theta,\sigma)\, | < M.$$

[15] In the case of two principals $m = 0,\ M = 3$.

This implies

$$\lim_{\sigma\to 0}\int_{\underline{\theta}}^{\theta}\lambda_i(\theta,\sigma)\,\dot{q}(\theta,\sigma)\,d\theta=0.$$

This and Proposition 4 leads to conclusion.

■

- **Proof of Proposition 6:**

Fix σ and consider the equation

$$\frac{\partial^2 q(\theta,\alpha)}{\partial\theta^2}=\sigma q(\theta,\alpha)\left[\frac{\partial q(\theta,\alpha)}{\partial\theta}+\alpha\right] \tag{29}$$

From the border conditions one gets: $\frac{\partial q(\underline{\theta},\alpha)}{\partial\alpha}=\frac{\partial q(\overline{\theta},\alpha)}{\partial\alpha}=0$. Suppose that there exists $\left(\widehat{\theta},\widehat{\alpha}\right)$ such that $\frac{\partial q\left(\widehat{\theta},\widehat{\alpha}\right)}{\partial\alpha}>0$. There has to exists a local maxima of $\frac{\partial q(\theta,\widehat{\alpha})}{\partial\alpha}$ at point $(\theta_{\max},\widehat{\alpha})$ with $\frac{\partial^2 q(\theta_{\max},\widehat{\alpha})}{\partial\theta\partial\alpha}=0$ and $\frac{\partial q(\theta_{\max},\widehat{\alpha})}{\partial\alpha}>0$. Then equation (29) yields

$$0\geq\frac{\partial^3 q(\theta_{\max},\widehat{\alpha})}{\partial\theta^2\partial\alpha}=\sigma\frac{\partial q(\theta_{\max},\widehat{\alpha})}{\partial\alpha}\left[\frac{\partial q(\theta_{\max},\widehat{\sigma})}{\partial\theta}+\alpha\right]+$$

$$+\sigma q(\theta_{\max},\widehat{\alpha})>0.$$

Contradiction. Therefore $\frac{\partial q(\theta,\alpha)}{\partial\alpha}\leq 0$ and from this; $q_{CA}(\theta)<q_M(\theta)$, $\forall\theta\in(\underline{\theta},\overline{\theta}]$. ■

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