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# **Altruistic Overlapping Generations of Households and the Contribution of Human Capital to Economic Growth**

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# Altruistic Overlapping Generations of Households and the Contribution of Human Capital to Economic Growth \*

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## Abstract

I developed a dynamic deterministic general equilibrium model accounting for human capital accumulation through both home education and schooling. The model is characterized by an altruistic link between households of succeeding generations in the sense parents, caring about their children's welfare, freely impart them some knowledge at home in addition to helping them financially when they are schooling. The education regime is private and features distinguishing my model from related works are: (1) young households are economically active and work part-time while schooling, (2) allocating time to schooling or labor entails disutility, (3) tuition is proportional to the time allocated to schooling. I calibrated the model to some balanced growth facts observed between 1981 and 2013 in the Province of Quebec.

The model is then used to investigate the contribution of human capital to economic growth. To do that, I simulate it assuming in turn a permanent rise in the tuition rate and the household's ability to learn. Each of these two shocks reveals a positive correlation between education, human capital, and output. The predictions of the model are then used to shed a light on the student crisis Quebec witnessed in 2012 following our former Liberal government's decision to increase tuition. I predict that raising tuition will neither harm education nor negatively impact on students' ability to pay.

**Keywords:** Education, economic growth, human capital, overlapping generations.

**JEL:** I25, O31, O41

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## 1 Introduction

Becker's 1964 seminal work and subsequent studies including Lucas Jr (1988), Barro (1991, 2001), and Mankiw, Romer, and Weil (1992) shed a light on the key role played by *human capital* in economic growth. Human capital, defined as the ability to perform labor, can be acquired through education and experience. Accumulating human capital enhances households' productivity, induces additional investment in physical capital, and favors economic growth.<sup>1</sup> In the same time, some authors such as Bils and Klenow (2000) questioned, on the basis of some empirical evidence, the importance given to the contribution to economic growth of human capital accumulation through formal education. By formal education, I mean schooling. Conversely, several authors investigated the impact of economic growth on human capital accumulation. DeJong and Ingram (2001) showed that, in the US over the sample period 1970-1996, college enrolments used as a proxy for human capital accumulation through schooling was negatively correlated with output growth rate. They also found that an increase in wage induced by a positive technology shock negatively impacted on human capital accumulation. As for Fowler and Young (2004), human capital accumulation by young households is rather procyclical.

This essay specifically deals with the contribution to economic growth of human capital accumulation through both home and formal education. Home education, also known as intergenerational knowledge spill-over, is about young households inheriting without any effort some of their parents' knowledge whereas formal education involves some resource, precisely time and income, allocations.<sup>2</sup> Lucas Jr (1988) advocated the modeling of home education arguing that "human capital accumulation is a *social* activity, involving *groups* of people". In modeling households' decision to invest in education, one of the following two assumptions are often made about their life span: (1) they are infinitely-lived (Razin, 1972; Lucas Jr, 1988; DeJong and Ingram, 2001), or (2) they are finitely-lived (Tran-Nam, Truong, and Van Tu, 1995; Shimomura and Tran-Nam, 1997; Heckman, Lochner, and Taber, 1998; Sadahiro and Shimasawa, 2002). I follow the latter class of models, *viz*, households in my model are finitely-lived and, in addition, are heterogeneous in their age. This framework called overlapping generations (OLG) originated from Samuelson's 1958 and Diamond's 1965 contributions. The use of the OLG framework is motivated by the fact that: (1) education is an investment that largely takes place in the earlier stage of a household's life-cycle, (2) the financing of education could involve the contribution of older generations of households. The OLG framework helps easily represent these realities.

The contribution of older generations of households to education financing could be modeled in several ways depending on whether the education regime entertained is public or private. Under a public regime, education is free and financed out of income tax revenue (Glomm and Ravikumar, 1992; Tran-Nam, Truong, and Van Tu, 1995). Under a private regime, it costs to get educated and altruistic parents directly pay for

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<sup>1</sup>Aghion and Howitt (1998, 2009) reviewed some popular theories and empirics on this subject.

<sup>2</sup>Henceforth, whenever I mention education without any further precision, I am referring to formal education. The expressions education and schooling will therefore be used interchangeably.

their children's education (Glomm and Ravikumar, 1992) whereas selfish parents just grant them a loan they reimburse when they finish schooling. (Tran-Nam, Truong, and Van Tu, 1995; Shimomura and Tran-Nam, 1997). In the model I have developed herein, the education regime is private and young households generously receive some financial transfers from their parents. I allow the transfer a young household receives to differ from the tuition he pays. Some features distinguishing my model from that of Glomm and Ravikumar are: (1) I have allowed young households to work part-time while schooling, (2) the way I have modeled *altruism*, and (3) tuition is proportional to the time allocated to schooling. Students working part-time on or off campus to finance their needs is nowadays an overwhelming reality that I want my model to take into account. Besides, students, especially those pursuing a university degree, are economically active agents and do not necessarily live under the same roof as their parents. To take this into account, I have modeled young households' consumption decision separately from their parents' one.

In the literature, altruism means caring about one's offspring welfare. There are two possible ways of modeling it. The first way is to posit that parents value their children's welfare, as Barro (1974) did. The second way is to say parents value the quality of education passed on to their children, as Glomm and Ravikumar (1992) did. With Barro's recursive modeling of altruism, one ends up expressing a household's welfare as a weighted sum of his life-cycle utility and that of each of his descendants. In my OLG framework, altruism is in the sense of Barro. All households therefore have their preferences defined over their own consumption and leisure and their children's welfare. As a consequence, allocating time to schooling or labor entails disutility. Most growth models with endogenous accumulation of human capital that I surveyed abstracted from these disutilities for convenience reasons. I have represented the preferences over consumption and leisure by a logarithmic utility function.

There are interesting features in other models that I do not take into account. For instance, Tran-Nam, Truong, and Van Tu (1995) Shimomura and Tran-Nam (1997) introduced uncertainty in the outcome of education, *i.e.*, a student may or may not succeed in education. For convenience reasons, I have modeled deterministically the human capital production sector. The human capital accumulated by a household, in my model, positively and with certainty depends on his ability to learn, the time he has allocated to education, and the level of his parents' human capital. Similar approaches include on the one hand Lucas Jr (1988) who modeled human capital as a cumulative outcome of the time allocated each period to education and on the other hand Glomm and Ravikumar (1992) who used a Cobb-Douglas technology to model the human capital accumulation process with as inputs the time allocated to education, the educational expenses, and the human capital inherited from parents. Whereas the quality of education is held constant in the former model, in the latter one, it depends on the household's educational expenses. The household's ability to learn, in my model, is a time-dependent parameter whose motion may depend on several factors including the household's personal aptitude, the total factor productivity as Fowler and Young (2004) did, the quality of the available didactic resources as well as that of teachers.

The other production sector in my model is the one manufacturing the final output. I

have modeled this sector using a Cobb-Douglas technology with as inputs both human and physical capital. While such authors as Heckman, Lochner, and Taber (1998), Sadahiro and Shimasawa (2002), and Fowler and Young (2004) included physical capital as input other authors such as Glomm and Ravikumar (1992), Tran-Nam, Truong, and Van Tu (1995), and Shimomura and Tran-Nam (1997), seeking a tractable analytical solution, abstracted from this input. My choice to include physical capital is motivated by the fact that households' decision to accumulate human capital through schooling impacts on their savings and consequently on physical capital accumulation because schooling entails allocating less time to labor in addition to paying tuition fees.

The rest of this paper consists of four sections. In the next section, which is Section 2, my model is sketched. It is made up of two (production) sectors: the human capital sector operated by some overlapping generations of households and the final output sector operated by (business) firms. The model is characterized by an altruistic link between members of succeeding generations. Parents, in addition to educating their children at home, could financially help them while they are schooling.

Some theoretical results emerging from households and firms' optimizing behavior are presented in Section 3. It appears that a young household substitutes education for labor at a rate greater than unity. Furthermore, within an altruistic economy, a young household always allocates more time to leisure than his parents do.

In Section 4, the dynamic deterministic general equilibrium (DDGE) model that I have built is calibrated to some balanced growth facts observed in the Province of Quebec between 1981 and 2013. It is then solved numerically and simulated assuming in turn a permanent rise in the tuition rate and the household's ability to learn. The purpose of these simulations is to show the *implied* long-run relationship between education, human capital, output, and some other variables. These two shocks reveal a positive correlation between output and education and human capital. However, the simulated correlation coefficients generated from the shock on the household's ability to learn are higher than those generated from the shock on the tuition rate.

Finally, in Section 5, the predictions from the model are used to investigate the student crisis referred to as Maple Spring that Quebec witnessed in 2012.<sup>3</sup> As a matter of fact, on March 17, 2011, Quebec's former Liberal Finance Minister, Mr Raymond Bachand, announced in his 2011-2012 budget speech an increase in university tuition. From Fall 2012 till 2017, tuition would increase each year by \$ 325 to reach \$ 3 793 in 2017. To protest against this decision, students started on February 13, 2012 what became the longest student strike in Quebec's history. My model predicts that students should not worry too much about the rise in tuition since it will induce a rise in the educational transfer they receive along with an increase in their human capital stock. I also made some policy recommendations in that final section.

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<sup>3</sup>The reference Maple Spring was made in relation to the popular uprisings named Arab spring that were going on in the Arab world at the same time. Maple, which is called *Érable* and is a homophone of *Arabe* in French, is a common tree in Quebec that produces syrup in the beginning of Spring

## 2 The Model

The economy consists of two agents: (1) some overlapping generations of households and (2) firms.

*The households* – Each generation of households lives for three periods of time. During the first period where they are young, households invest some time in formal education to accumulate human capital and, at the same time, work part-time. They then become mature, are full-time employed, and procreate. They are pensioned off during the third period and pass away later on. Population grows exponentially at the exogenous rate  $0 < n < 1$ . Households are altruistic and have preferences defined over consumption, leisure, and their children's welfare. Therefore, following Barro (1974), the welfare of a household born at time  $t = 0, 1, 2, \dots$  can be defined as

$$u_t = \sum_{g=0}^2 \frac{1}{(1+\rho)^g} [\ln c_{g,t+g} + \sigma \ln \lambda_{g,t+g}] + \frac{1+n}{(1+\phi)(1+\rho)} u_{t+1},$$

hence the following (linear-in-life-cycle-utility) social welfare

$$u_t = \sum_{t=0}^{\infty} \left[ \frac{1+n}{(1+\phi)(1+\rho)} \right]^t \sum_{g=0}^2 \frac{1}{(1+\rho)^g} [\ln c_{g,t+g} + \sigma \ln \lambda_{g,t+g}], \quad (2.1)$$

with  $0 < (1+n)/(1+\phi)(1+\rho) < 1$ . The parameters  $-1 < \phi < 1$ ,  $0 < \rho < 1$ , and  $\sigma > 0$  are respectively the selfishness parameter, the time preference rate, and leisure weight. When  $\phi < 0$ , one has  $\rho > \rho + \phi(1+\rho)$ , *viz.*,  $\rho$ , the time preference rate used by the household to discount his own life-cycle utility, is greater than  $\rho + \phi(1+\rho)$ , the intergenerational time preference rate. He is then said to be altruistic. On the other hand, a positive  $\phi$  is a sign of selfishness. The variables  $c_{g,t+g}$  and  $0 < \lambda_{g,t+g} < 1$  are respectively the consumption and leisure at time  $t+g$  of a household aged  $g$ . A household faces the following constraints

$$\lambda_{gt} = 1 - e_{gt} - l_{gt}, \quad 0 \leq e_{gt}, l_{gt}, \lambda_{gt} \leq 1 \quad (2.2a)$$

$$h_{1t+1} = (1 + \psi_t e_{0t}) h_{0t}, \quad \psi_t > 0 \quad (2.2b)$$

$$h_{0t} = \frac{\gamma}{1+n} h_{1t}, \quad 0 < \gamma < 1 \quad (2.2c)$$

$$a_{1t+1} = w_t l_{0t} h_{0t} + \epsilon_{1t} - c_{0t} - f_t e_{0t} \quad (2.2d)$$

$$a_{2t+2} = w_{t+1} l_{1t+1} h_{1t+1} + (1 + r_{t+1}) a_{1t+1} - c_{1t+1} - (1+n) \epsilon_{1t+1} \quad (2.2e)$$

$$c_{2t+2} = (1 + r_{t+2}) a_{2t+2}. \quad (2.2f)$$

Constraint (2.2a) states a household shares his time endowment normalized to unity between leisure, education  $e_{0t}$ , and labor  $l_{g,t+g}$ . Constraint (2.2b) relates  $h_{1t+1}$ , the human capital stock after graduation, to the amount of time he allocated to education. The parameter  $\psi_t$  in that constraint is the household's ability to learn. Relation (2.2c) is about home education. It says, parents by raising their children hand down to them

some of their knowledge.<sup>4</sup> The share  $\gamma$  is referred to as the intergenerational knowledge spill-over coefficient. Relations (2.2d) through (2.2f) are the household's life cycle budget constraints. During the first period of his life, he receives both labor income and an educational transfer  $\epsilon_{1t}$  from his parents. The variable  $w_t$  denotes the real wage at time  $t$ . The labor income depends on both his hours worked and human capital stock. Out of his incomes, he finances his consumption and tuition; what is left is invested in financial assets. The tuition paid,  $f_t e_{0t}$ , is proportional to the time allocated to education. During the second period, the household receives both labor and financial assets incomes. The variable  $r_{t+1}$  denotes the real interest rate at time  $t + 1$ . From the budget constraint (2.2f) one could see that, during retirement, the household lives only on his financial assets income and leaves no bequest at the end of his life. It is worth noting that households have perfect foresight and both formal education, *viz* attending a university, and home education are the only ways of accumulating human capital. The tuition rate  $f_t$  is set exogenously. The representative household maximizes (2.1), the social welfare, subject to constraints (2.2a) through (2.2f).

*The firms* – The aggregate production technology operated by firms is Cobb-Douglas and defined by

$$Y_t = K_t^\alpha [\exp(xt)H_t]^{1-\alpha}, \quad 0 < \alpha < 1. \quad (2.3)$$

The variables  $Y_t$ ,  $K_t$ , and  $H_t$  are respectively the aggregate output, physical capital, and effective labor. The parameters  $x$  and  $\alpha$  are respectively the exogenous labor-augmenting technological progress growth rate and the share of physical capital income in the aggregate output. They maximize their profit defined as

$$\max_{K_t, H_t} K_t^\alpha [\exp(xt)H_t]^{1-\alpha} - (1 + r_t)K_t - w_t H_t.$$

In the above expression,  $1 + r_t$ , the rental price of the aggregate physical capital, indicates that this input completely depreciates or becomes out-of-date at the end of each time period.

### 3 The General and Balanced Growth Equilibria

Before defining the *general equilibrium*, *i.e.*, the equilibrium on all the markets, the equations derived from households and firms' optimizing behavior are presented. The first order conditions (FOCs) from households' optimization problem are<sup>5</sup>

$$\sigma c_{0t} = w_t h_{0t}(1 - e_{0t} - l_{0t}) \quad (3.1a)$$

$$\sigma c_{1t} = w_t h_{1t}(1 - l_{1t}) \quad (3.1b)$$

$$(1 + r_{t+1})c_{0t} = (1 + \rho)c_{1t+1} \quad (3.1c)$$

$$(1 + r_{t+1})c_{1t} = (1 + \rho)c_{2t+1} \quad (3.1d)$$

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<sup>4</sup>The expressions children, students, and young households are interchangeably used herein. *Idem* for the expressions mature households and parents. Given, on the one hand, the relationship between the households and, on the other hand, their occupations, it has been difficult to stick to one expression.

<sup>5</sup>These results are detailed in the Appendix.



$$c_{1t} = (1 + \phi)c_{0t} \quad (3.1e)$$

$$\gamma \left[ l_{0t+1} + \left( 1 + \frac{f_{t+1}}{w_{t+1}h_{0t+1}} \right) e_{0t+1} \right] + l_{1t+1} = \frac{1 + r_{t+1}}{\psi_t} \frac{w_t}{w_{t+1}} \left( 1 + \frac{f_t}{w_t h_{0t}} \right) - \frac{\gamma}{\psi_{t+1}} \left( 1 + \frac{f_{t+1}}{w_{t+1}h_{0t+1}} \right). \quad (3.1f)$$

The conditions (3.1a) and (3.1b) are respectively the trade-off made between consumption and leisure during the first and second stages of the life cycle. The Euler equations (3.1c) and (3.1d) are about the households' inter-temporal consumption choice. Relation (3.1e) explains the difference between a household and his children's contemporaneous consumption in terms of altruism/selfishness. As for (3.1f), it is about the intra-temporal trade-off between education and labor. Some theoretical evidence that emerges from the model are now highlighted.

**Proposition 3.1** (Education-Labor Trade-off). *When the tuition is proportional to the time allocated to schooling, a unit increase (decrease) in the latter activity, ceteris paribus, results in a greater decrease (increase) in the time allocated to labor.*

*Proof.* From (3.1f), it transpires that the marginal rate of substitution of education for labor is greater than one in absolute value

$$\frac{\partial l_{0t+1}}{\partial e_{0t+1}} = - \left( 1 + \frac{f_{t+1}}{w_{t+1}h_{0t+1}} \right).$$

□

Proposition 3.1 comes from the fact that increasing the time allocated to schooling occasions two costs: the forgone wage and the additional tuition. If tuition were lump-sum or formal education were free, the household would instead decrease the time allocated to labor on a one-one basis when the time allocated to schooling increased.

**Proposition 3.2** (Altruistic Parents and their Children's Leisure). *Within an altruistic economy, children allocate more time to leisure than their parents do.*

*Proof.* Consider (3.1a) describing the intra-temporal trade-off between consumption and leisure made by a young household. Divide this by (3.1b) to get

$$\frac{c_{0t}}{c_{1t}} = \frac{h_{0t} \lambda_{0t}}{h_{1t} \lambda_{1t}}$$

Using (3.1e) to replace  $c_{1t}$  in the above relation by  $(1 + \phi)c_{0t}$  and then calling on (2.2c), one gets

$$\frac{\lambda_{0t}}{\lambda_{1t}} = \frac{1 + n}{\gamma(1 + \phi)} > 1, \quad \text{for } -1 < \phi < 0, \quad (3.2)$$

which establishes the claim made in Proposition 3.2. □

Note that even if parents were selfish, *i.e.*,  $\phi > 0$ , (3.2) would still hold so long as  $\phi < (1+n)/\gamma - 1$ .

At equilibrium, both human and physical capital are remunerated at their marginal productivity.

$$\alpha \frac{Y_t}{K_t} = 1 + r_t \quad (3.3a)$$

$$(1 - \alpha) \frac{Y_t}{H_t} = w_t \quad (3.3b)$$

**Definition 3.3** (General Equilibrium). *It consists of prices  $\{(f_t, r_t, w_t)\}_{t=0}^{\infty}$ , an aggregate state of the world  $\{(\psi_t, \exp(xt))\}_{t=0}^{\infty}$ , and allocations:*

- $\{(a_{1t+1}, c_{0t}, e_{0t}, l_{0t})\}_{t=0}^{\infty}$  for young households,
- $\{(a_{2t+1}, c_{1t}, \epsilon_{1t}, h_{1t}, l_{1t})\}_{t=0}^{\infty}$  for mature households,
- $\{(c_{2t})\}_{t=0}^{\infty}$  for elderly households, and
- $\{(H_t, K_t, Y_t)\}_{v=0}^{\infty}$  for firms,

solving simultaneously:

1. the households' optimization problem, *i.e.*, relations (3.1a) through (3.1f) along with the constraints (2.2a) through (2.2f),
2. the firms' optimization problem, *i.e.*, relations (2.3), (3.3b) and (3.3a),

and clearing

3. the financial and labor markets, *i.e.*,

$$K_t = \left( a_{1t} + \frac{a_{2t}}{1+n} \right) \frac{N_{0t}}{1+n}$$

$$H_t = (\gamma l_{0t} + l_{1t}) \frac{N_{0t}}{1+n} h_{1t},$$

where  $N_{0t}$  is the number of households born at time  $t$ .

The time households allocate to education and labor, respectively  $e_{0t}$ ,  $l_{0t}$ , and  $l_{1t}$ , are stationary variables, *i.e.*, they fluctuate around their constant means  $e_0$ ,  $l_0$ , and  $l_1$ . Following the literature (DeJong and Ingram 2001 and Fowler and Young 2004, among others), I consider the household's ability to learn  $\psi_t$  as a stationary parameter.

It follows from (2.2b) and (2.2c) that  $h_{1t+1} = \gamma(1 + \psi_t e_{0t})h_{1t}/(1+n)$ . I then assume the autoregressive parameter  $\gamma(1 + \psi_t e_{0t})/(1+n)$  is greater than one, which implies human capital stock after graduation,  $h_{1t}$ , is a trended variable. The parameter  $\nu$  will now denote this latter autoregressive parameter along the balanced growth path (BGP). Then, in Definition 3.3, the equilibrium condition on the labor market suggests the gross growth rate of  $H_t$  along the BGP is  $(1+n)\nu$ .

Since technological progress is labor-augmenting and given the growth rate of effective labor, aggregate output and physical capital are constrained to grow each period by the factor  $(1+n)\nu \exp(x)$  along the BGP. It then turns out that the real wage  $w_t$  grows at the same rate a technological progress whereas the real interest rate is stationary. The tuition rate  $f_t$  and all the other per capita variables grow at the gross rate  $\nu \exp(x)$ .

I have removed the trend from the non-stationary variables by dividing them by their growth components, which gives rise to the variables  $\hat{y}_t = Y_t/[N_{0t}\nu^t \exp(xt)]$ ,  $\hat{k}_t = K_t/[N_{0t}\nu^t \exp(xt)]$ ,  $\bar{h}_t = H_t/(1+n)\nu^t$ ,  $\tilde{w}_t = w_t/\exp(xt)$ ,  $\bar{h}_{1t} = h_{1t}/\nu^t$ , and  $\hat{\mathbf{z}}_t = \mathbf{z}_t/[\nu^t \exp(xt)]$  with  $\mathbf{z}_t = \{a_{1t}, a_{2t}, c_{0t}, c_{1t}, c_{2t}, f_t, \epsilon_{1t}\}$ . The expressions that follow define some variables and ratios along the BGP.

$$r = (1 + \phi)(1 + \rho)\nu \exp(x) - 1 \quad (3.4a)$$

$$\tilde{w} = (1 - \alpha) \left[ \frac{\alpha}{(1 + \phi)(1 + \rho)\nu \exp(x)} \right]^{\frac{\alpha}{1-\alpha}} \quad (3.4b)$$

$$\frac{\hat{k}}{\hat{y}} = \frac{\alpha}{(1 + \phi)(1 + \rho)\nu \exp(x)} \quad (3.4c)$$

$$\frac{\hat{k}}{\bar{h}} = \left[ \frac{\alpha}{(1 + \phi)(1 + \rho)\nu \exp(x)} \right]^{\frac{1}{1-\alpha}} \quad (3.4d)$$

$$e_0 = \frac{1}{\psi} \left( \frac{1+n}{\gamma} \nu - 1 \right) \quad (3.4e)$$

$$\bar{h} = \frac{\nu}{\psi} \left[ \frac{(1 + \phi)(1 + \rho)}{1 + n} - 1 \right] \left( \bar{h}_1 + \frac{1+n}{\gamma} \frac{\hat{f}}{\tilde{w}} \right) \quad (3.4f)$$

According to (3.4a) and (3.4b), an increase in the technological progress growth rate  $x$ , the selfishness parameter  $\phi$ , the time preference rate  $\rho$ , or the growth rate of human capital  $\nu$ , *ceteris paribus*, raises the BGP interest rate and has the opposite effect on the real wage. This is due to the induced decrease in physical capital as (3.4c) or (3.4d) shows and the induced increase in labor supply as (3.4f) suggests.

An increase in the population growth rate  $n$ , *ceteris paribus*, lessens the hours worked by households, as (3.4f) suggests. As for the time a household allocates to schooling, according to (3.4e), it increases as  $n$  increases. A rise in the intergenerational knowledge spill-over coefficient  $\gamma$  means young households are inheriting more human capital from their parents, which causes them to reduce the time they allocate to schooling. An improvement in  $\psi$ , the household's ability to learn, will also result in a decrease in the time allocated to education. This latter impact as well as that of an increase in the tuition rate are furthered in the next section.

## 4 The Numerical Solution

First, I have normalized the life span of a household to nine years and each of the three stages of the life-cycle lasts three years. Fowler and Young (2004) took a similar approach in solving numerically their model. The rest of the model is calibrated to

match some balanced growth facts observed in Quebec over the sample period 1981-2013.

*The population growth rate*– The population  $N_t$  is made up of young, mature, and retired households. It grows exponentially at the rate  $n$ , *i.e.*,  $N_t = N_0(1 + n)^t$ . To have an estimate of  $n$ , I have regressed the natural logarithm of the population aged 15 and over on an intercept and a linear time trend. The data used are from Statistics Canada and the ordinary least squares estimate of the annual population growth rate is .009.<sup>6</sup> The equivalent compound triennial rate is 2.7%.

*The physical capital's share*– I have used Statistics Canada's income based gross domestic product (GDP) to compute  $\alpha$ , the share of physical capital in the aggregate output, following Cooley and Prescott (1995) and Gomme and Rupert (2005). This share is defined as the ratio of the *unambiguous physical capital income* to the total unambiguous incomes

$$\alpha = \frac{\text{unambiguous physical capital income}}{\text{GDP-ambiguous income}}.$$

The unambiguous capital income is made up of the income-based GDP estimates that are considered as remunerating specifically the physical capital input. It consists of corporate profits before tax, interest and miscellaneous investment income, and the capital depreciation allowance. As for the ambiguous income, it remunerates indistinctly both labor and capital. It comprises the net income of farm operators and unincorporated business, the taxes on factors of production and products, and the statistical discrepancies. The average of this share is .313.

*The technological progress growth rate*– I have computed this parameter doing some growth accounting. Log-differentiating (2.3) gives

$$x = \frac{\Delta \ln Y_t}{1 - \alpha} - \frac{\alpha}{1 - \alpha} \Delta \ln K_t - \Delta \ln H_t.$$

I have used real GDP, real gross capital stock, and the total hours worked as respective measures of  $Y_t$ ,  $K_t$ , and  $H_t$ . The average annual growth rate of technological progress turns out to be .01, which is equivalent to .03 over three years.

*The human capital growth rate*– The long-run growth rate of the aggregate output is  $(1 + n)\nu \exp(x) - 1$ . I have run a log-linear regression of the real output on an intercept and a time trend using annual data from Statistics Canada. From now on, I define output as the sum of household final consumption expenditure and business gross fixed capital formation. This econometric model fits 98% of the observed data. The slope parameter, which equals .027, gives an estimate of the annual long-run output growth rate. This growth rate is equivalent to 8.1% over three years. One solves for  $\nu$ , the human capital growth rate, equating this latter estimate to the expression  $(1 + n)\nu \exp(x) - 1$ , which yields  $\nu = 1.02$ .

*The other parameters and ratios* – According to the labor force survey by Statistics Canada, over the period 1981-2013, households in Quebec aged 15-24 allocated, on

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<sup>6</sup>The log-linear model,  $\ln N_t = \ln N_0 + t \ln(1 + n)$ , explains 99.5% of the observed variability in Quebec's population, with all the coefficients being statistically significant.

average, 15.2 hours a week to part-time work while those aged 25 and over worked, on average, 36.7 hours a week. It emerges from the 2001 and 2006 census of Canada that the average weeks worked mostly part-time and full-time in Quebec are respectively 35.4 and 45.4. Besides, according to the Statistics Canada's general social survey (GSS), students and workers in Quebec allocated, on average between 1986 and 2010, respectively 10.78 and 10.44 hours a day to personal care, which include night sleep and meals at home. I have therefore computed  $l_0$  and  $l_1$  as the ratios of the total hours worked by young and mature households to their respective total discretionary time, which gives .112 and .337 respectively.<sup>7</sup>

The GSS data also reveals that, between 1986 and 2010, students in Quebec allocated, on average, 6.3 hours a day to professional activities, *i.e.*, both labor, education and related activities. Subtracting the weekly hours worked by young households from the time they allocated weekly to their professional activities enables computing the share of time they allocated to education  $e_0$ , which equals .234.

The average investment-output ratio is .2. Recall that output is defined as the sum of consumption and investment. Since physical capital completely depreciates from one period to the other, one has  $K_{t+1} = I_t$ , where  $I_t$  denotes the aggregate investment.<sup>8</sup> As the model constrains both variables to grow at the same rate, one has  $(1+n)\nu \exp(x)\hat{k}/\hat{y} = \hat{i}/\hat{y}$  along the BGP. This implies the capital-output ratio  $\hat{k}/\hat{y}$  is .19.

The value of a household's human capital stock only affects the level of variables and has no impact on parameters and ratios. I have therefore normalized  $\bar{h}_1$  to unity. As for the intergenerational knowledge spill-over coefficient  $\gamma$ , the highest value it can assume is .8. Otherwise tuition rate would be negative, meaning one has to pay a household before he forgo labor or leisure to allocate time to education because he is already well instructed. For  $\gamma$  lower than .8, the share of tuition fees in the aggregate output implied by the model will be much higher than the actual one, which is about .001. The value assigned to  $\gamma$  impacts on such parameters as the time preference rate  $\rho$ , the leisure weight  $\sigma$ , and the selfishness parameter  $\phi$ . Whereas  $\rho$  increases along with  $\gamma$ , both  $\sigma$  and  $\phi$  decrease.

All the other parameters and the initial position of the economy are deduced from what preceded. Table 4.1 displays the calibrated parameters.

The calibration exercise points out a high selfishness. Given the high values of  $\phi$  and  $\rho$  displayed in Table 4.1, .3 and .21 respectively, it appears that households put more weight on their own current consumption than on their children's. As a result, even though the educational transfer exceed the tuition fees, students still have to work and borrow money to finance their consumption. One can interpret the educational transfer as a lump-sum tax raised on mature households' income and entirely transferred to the younger generation as de la Croix and Michel (2002, pp 129-30) did. In this particular case, the analysis of the BGP indicates that students cannot pay their tuition and live

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<sup>7</sup>The discretionary time is the number of hours that are not allocated to personal care.

<sup>8</sup>The annual depreciation rate of physical capital is 5.4 %, which means it lasts about eighteen-and-a-half years. Thus, assuming physical capital completely depreciates within one time interval primarily means a complete depreciation over one generation.

Table 4.1: The Parameters of the Model

Households	$n$	Population growth rate	.027
	$\gamma$	Knowledge spill-over coefficient	.8
	$\nu$	Human capital growth rate	1.02
	$\rho$	Time preference rate	.21
	$\sigma$	Leisure weight	4.145
	$\phi$	Selfishness parameter	.3
	$\psi$	Household ability to learn	1.347
Firms	$x$	Technological progress growth rate	.01
	$\alpha$	Physical capital share	.313

on the grants or loans they receive from the government. They end up with debts after their education.

It is also worth noting that schooling accounts for 28.4%, *i.e.*,  $(1 + n)/\gamma - 1$ , of a household's human capital and the estimated value for the ability to learn of households, 1.35, as it appears in Table 4.1, is very high.

The model is now solved numerically assuming in turn changes brought about by a shock to the tuition rate and the household's ability to learn. These parameter and exogenous variable are influential in households' decision to invest in education. In each of the two cases investigated, the economy is hit by an expected and permanent exogenous shock occurring the first period. The model is simulated over ten periods of time and the transitional dynamics of such key variables as the time allocated to education and labor, the human and physical capital stocks, the educational transfer, wage, and output are sketched in Figure 4.1. The simulations are implemented in Matlab using the package Dynare.

How a new BGP is reached after a one percent permanent increase in the tuition rate is plotted in blue line in Figure 4.1. When education becomes less affordable, students initially decrease the time they allocate to education (panel 1) to increase the time they allocate to labor (panel 3). At the same time, mature households decrease the time they allocate to labor (panel 4) because wage has decreased (panel 5) and interest rate has increased. The rise in interest rate raises mature households' wealth (panel 6), which induces a rise in the educational transfer they grant their children (panel 8). This then encourages students to work less and allocate more time to education. When the new BGP is reached, the time allocated to education returns to its initial level.<sup>9</sup> So do the hours worked and wage but the stocks of human capital, the aggregate output, and the educational transfer remain higher. This enables us to conclude that the increase in the tuition rate favors education and growth.

The red lines in Figure 4.1 show the transitional dynamics after a .1% rise in the household' ability to learn. The shapes of these transitional dynamics are, in most cases, the opposite of those generated by the rise in the tuition rate. The increase

<sup>9</sup>According to relation (3.4e), the time allocated to education along the BGP does not depend on the tuition rate.

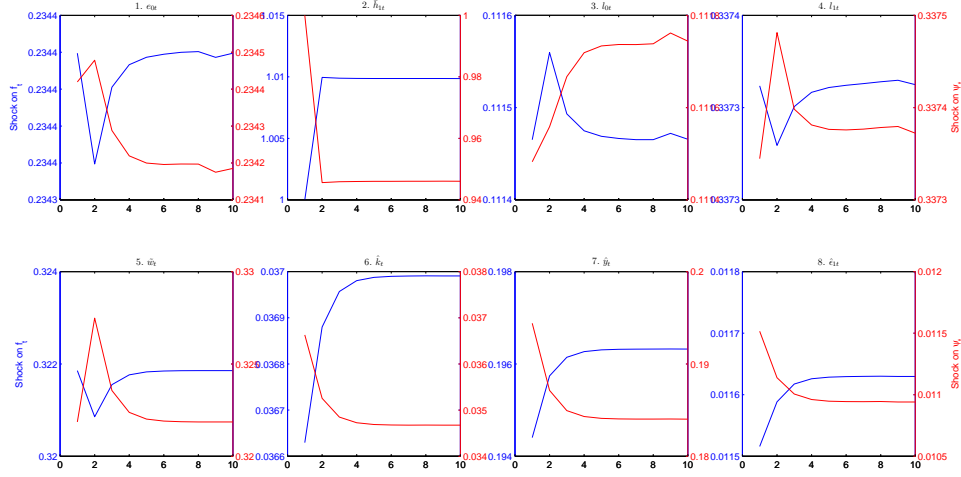


Figure 4.1: Estimates of the Dynamic Paths after a Permanent 1% Rise in the Tuition Rate and a .1% Rise in the Household’s Ability to Learn

Table 4.2: Simulated Correlation with Output

	$e_{0t}$	$h_{1t}$	$\hat{c}_{1t}$
Shock on $\hat{f}_t$	.138	.951	.998
Shock on $\psi_t$	.761	.953	.999

in the household’s ability to learn occasions an economic downturn. Households have ended up allocating less time to education and the level of human capital stock, output, and educational transfer reached are lower than in the previous case.

For each of the two shocks, Table 4.2 displays the correlation coefficients of the simulated output series with the time allocated to education, the stock of human capital, and the educational transfer. All the two shocks reveal a positive and high correlation between the aggregate output and the variables of interest. The correlation coefficients are higher in the case of a rise in the household’s ability to learn.

## 5 Discussion

I have used herein an altruistic OLG model in which young households receive from their parents some home education and financial support. They then add to their human capital stock by completing some formal education while working part-time. Some theoretical evidence emerge from the model: (1) when the tuition is proportional to the time allocated to education, the marginal rate of substitution of education for labor is greater than unity in absolute value, (2) within an altruistic economy, young households allocate more time to leisure than their parents do. The DDGE model has

then been calibrated to match some balanced growth facts observed in the Province of Quebec over the sample period 1981-2013. The calibration results indicate *inter alia* that parents are selfish towards their children and education accounts for 28.4% of a household's human capital. Thereafter the calibrated model has been used to investigate the contribution to economic growth of human capital accumulation through education. To do that I have first identified a parameter and an exogenous variable that could affect a household's decision to invest in education. This parameter is the household's ability to learn and the exogenous variable is the tuition rate. Then, the model has been solved numerically and simulated assuming in turn a permanent increase in the tuition rate and the household's ability to learn. It turns out that human capital accumulation through education promotes growth.

After these simulations, I now address some topical issues in Quebec. Should the government increase tuition as the former Liberal Finance Minister Mr Raymond Bachang announced in his 2011-2012 budget speech? Were students right to protest against this decision, were their worries justified? My model predicts that an increase in the tuition rate will not at all affect, in the *long-run*, the time allocated to education. The time allocated to education and labor as well as real wage will move back to their initial levels after falling. Human capital stocks and output will increase.

Would education become less affordable after the increase in tuition as some of the protesters sustained? My model predicts that the rise in the tuition rate will not harm students' ability to pay for their education since during the transition to the new BGP, parents (or the society) will adjust accordingly the financial support they grant them.

I will now use the model to assess three education policy measures: (1) free university education in Quebec as proposed by some students union leaders, (2) indexing tuition to the rate of growth of households' disposable income, as the Parti Québécois led by Mrs Pauline Marois recommended in February 2013 during the submit on higher education, and (3) investing in cultural and sportive activities on campuses and acquiring up-to-date didactic resources.

Free education means a negative shock to the tuition rate. The effects of this measure will be the opposite of those illustrated in the previous section. At first, education will become popular insofar as the share of time allocated to this activity will rise. After some periods, this share will move down to its initial level. Output and human capital will fall.

On the other hand, indexing the tuition rate is equivalent to maintaining the *status quo* in the *normalized* model. The economy will keep moving along its initial BGP.

As for investing in extra-curricular activities on campuses or in new didactic resources, this will improve students' ability to learn. With less effort, they could accumulate the same amount of knowledge as previous cohorts. But, a substantial reduction in the time allocated to education will end up reducing the level of human capital stock. A way of avoiding this fall in human capital stock, when implementing this policy, would be not to reduce the financial help granted to students.

To finish with, I address a modeling issue. A household's human capital depends on both the time he has allocated to education and his ability to learn. I have not



included tuition or educational expenses as some authors did. A reason why I have chosen not to is that, since the time the decision to increase tuition the way the Liberal Party proposed was abolished by the Parti Québécois, universities have been able to cut their budget to deliver the same quality of education. I have therefore concluded it would not be, at this stage, efficient including tuition in the human capital production function. Including, later, tuition in the human capital production function will be quite straightforward. One has just to define the actual household ability to learn as the product of tuition rate and a human capital productivity parameter that has been so far unobserved.

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# Appendices

## A The Households' Optimization Problem

As it appears on page 6, the household born at time  $t$  seeks to maximize his life-cycle utility

$$u_t = \sum_{g=0}^{\infty} \left[ \frac{1+n}{(1+\phi)(1+\rho)} \right]^g \sum_{g=0}^2 \frac{1}{(1+\rho)^g} [\ln c_{gt+g} + \sigma \ln \lambda_{gt+g}].$$

subject to:

$$\begin{aligned} \lambda_{gt} &= 1 - e_{gt} - l_{gt}, \quad 0 \leq e_{gt}, l_{gt}, \lambda_{gt} \leq 1 \\ h_{1t+1} &= (1 + \psi_t e_{0t}) h_{0t}, \quad \psi > 0 \\ h_{0t} &= \frac{\gamma}{1+n} h_{1t}, \quad 0 < \gamma < 1 \\ a_{1t+1} &= w_t l_{0t} h_{0t} + \epsilon_{1t} - c_{0t} - f_t e_{0t} \\ a_{2t+2} &= w_{t+1} l_{1t+1} h_{1t+1} + (1 + r_{t+1}) a_{1t+1} - c_{1t+1} - (1+n) \epsilon_{1t+1} \\ c_{2t+2} &= (1 + r_{t+2}) a_{2t+2}, \end{aligned}$$

given  $f_t, r_t$  and  $w_t$ .

This optimization problem can be solved either for a single generation of households over the three periods of his life-cycle using the method of Lagrange or cross-sectionally for the three contemporaneous generations of households using the equation of Bellman. We have derived the FOCs and the Euler equations using in turn each of these two methods. But beforehand, I have reduced the number of constraints from eight to four by substituting the three time constraints in the objective function and replacing  $h_{0t}$ , the human capital stock inherited at birth, by its expression in the rest of the constraints.

### A.1 Solving the Problem Using the Method of Lagrange

$$\begin{aligned} \mathcal{L}_t = \max & \sum_{t=0}^{\infty} \left[ \frac{1+n}{(1+\phi)(1+\rho)} \right]^t [\ln c_{0t} + \sigma \ln(1 - e_{0t} - l_{0t})] \\ & \sum_{t=0}^{\infty} \left[ \frac{1+n}{(1+\phi)(1+\rho)} \right]^t \left\{ \frac{1}{(1+\rho)} [\ln c_{1t+1} + \sigma \ln(1 - l_{1t})] + \frac{1}{(1+\rho)^2} \ln c_{2t+2} \right\} \\ & + \mu_{1t} \left[ \frac{\gamma}{1+n} w_t h_{1t} l_{0t} + \epsilon_{1t} - c_{0t} - f_t e_{0t} - a_{1t+1} \right] \\ & + \mu_{2t} [w_{t+1} h_{1t+1} l_{1t+1} + (1 + r_{t+1}) a_{1t+1} - c_{1t+1} - (1+n) \epsilon_{1t+1} - a_{2t+2}] \\ & + \mu_{3t} [(1 + r_{t+2}) a_{2t+2} - c_{2t+2}] + \mu_{4t} \left[ (1 + \psi_t e_{0t}) \frac{\gamma}{1+n} h_{1t} - h_{1t+1} \right], \end{aligned}$$

where the variables  $\mu_{1t}, \dots, \mu_{4t}$  are the Lagrange multipliers or shadow prices and the variables  $f_t, r_t, w_t$ , and  $\psi_t$  are given.

The FOCs

$$c_{0t} : \left[ \frac{1+n}{(1+\phi)(1+\rho)} \right]^t \frac{1}{c_{0t}} = \mu_{1t} \quad (\text{A.1a})$$

$$c_{1t+1} : \left[ \frac{1+n}{(1+\phi)(1+\rho)} \right]^t \frac{1}{1+\rho} \frac{1}{c_{1t+1}} = \mu_{2t} \quad (\text{A.1b})$$

$$c_{2t+2} : \left[ \frac{1+n}{(1+\phi)(1+\rho)} \right]^t \frac{1}{(1+\rho)^2} \frac{1}{c_{2t+2}} = \mu_{3t} \quad (\text{A.1c})$$

$$e_{0t} : \left[ \frac{1+n}{(1+\phi)(1+\rho)} \right]^t \frac{\sigma}{1-e_{0t}-l_{0t}} + f_t \mu_{1t} = \psi_t \frac{\gamma}{1+n} h_{1t} \mu_{4t} \quad (\text{A.1d})$$

$$l_{0t} : \left[ \frac{1+n}{(1+\phi)(1+\rho)} \right]^t \frac{\sigma}{1-e_{0t}-l_{0t}} = \frac{\gamma}{1+n} w_t h_{1t} \mu_{1t} \quad (\text{A.1e})$$

$$l_{1t+1} : \left[ \frac{1+n}{(1+\phi)(1+\rho)} \right]^t \frac{1}{1+\rho} \frac{\sigma}{1-l_{1t+1}} = w_{t+1} h_{1t+1} \mu_{2t} \quad (\text{A.1f})$$

$$h_{1t+1} : w_{t+1} l_{1t+1} \mu_{2t} + \frac{\gamma}{1+n} w_{t+1} l_{0t+1} \mu_{1t+1} + (1 + \psi_{t+1} e_{0t+1}) \frac{\gamma}{1+n} \mu_{4t+1} = \mu_{4t} \quad (\text{A.1g})$$

$$a_{1t+1} : (1+r_{t+1}) \mu_{2t} = \mu_{1t} \quad (\text{A.1h})$$

$$a_{2t+2} : (1+r_{t+2}) \mu_{3t} = \mu_{2t} \quad (\text{A.1i})$$

$$\epsilon_{1t+1} : (1+n) \mu_{2t} = \mu_{1t+1} \quad (\text{A.1j})$$

$$\mu_{1t} : \frac{\gamma}{1+n} w_t h_{1t} l_{0t} + \epsilon_{1t} - c_{0t} - f_t e_{0t} = a_{1t+1} \quad (\text{A.1k})$$

$$\mu_{2t} : w_{t+1} h_{1t+1} l_{1t+1} + (1+r_{t+1}) a_{1t+1} - c_{1t+1} - (1+n) \epsilon_{1t+1} = a_{2t+2} \quad (\text{A.1l})$$

$$\mu_{3t} : (1+r_{t+2}) a_{2t+2} = c_{2t+2} \quad (\text{A.1m})$$

$$\mu_{4t} : (1 + \psi_t e_{0t}) \frac{\gamma}{1+n} h_{1t} = h_{1t+1} \quad (\text{A.1n})$$

The leads in the second and third elements on the left-hand side of (A.1g) come from the fact that a fraction of the level of human capital stock  $h_{1t+1}$  chosen by a young household at time  $t$  is inherited by his children during the next time period. The lead of the Lagrange multiplier on the right-hand side of (A.1j) points to the fact that the educational transfer is granted by a household born at time  $t$  and cashed by households born at  $t+1$ .

Now, let us get rid of the Lagrange multipliers in the FOCs (A.1a) through (A.1j) by substituting some equations or their first lead into others and rearranging to get

$$\sigma c_{0t} = \frac{\gamma}{1+n} w_t h_{1t} (1 - e_{0t} - l_{0t}) \quad (\text{A.2a})$$

$$\sigma c_{1t+1} = w_{t+1} h_{1t+1} (1 - l_{1t+1}) \quad (\text{A.2b})$$

$$(1+r_{t+1}) c_{0t} = (1+\rho) c_{1t+1} \quad (\text{A.2c})$$

$$(1 + r_{t+2})c_{1t+1} = (1 + \rho)c_{2t+2} \quad (\text{A.2d})$$

$$c_{1t+1} = (1 + \phi)c_{0t+1} \quad (\text{A.2e})$$

$$\begin{aligned} \gamma \left[ l_{0t+1} + \left( 1 + \frac{1+n}{\gamma} \frac{f_{t+1}}{w_{t+1}h_{1t+1}} \right) e_{0t+1} \right] + l_{1t+1} &= \frac{1+r_{t+1}}{\psi_t} \frac{w_t}{w_{t+1}} \left( 1 + \frac{1+n}{\gamma} \frac{f_t}{w_t h_{1t}} \right) \\ &\quad - \frac{\gamma}{\psi_{t+1}} \left( 1 + \frac{1+n}{\gamma} \frac{f_{t+1}}{w_{t+1}h_{1t+1}} \right) \end{aligned} \quad (\text{A.2f})$$

Relation (A.2a), for instance, is obtained after plugging (A.1a) into (A.1e) and rearranging. We now focus on showing how one gets (A.2f). Combine (A.1d) and (A.1e), to express  $\mu_{4t}$  as a function of  $\mu_{1t}$

$$\mu_{4t} = \left( w_t + \frac{1+n}{\gamma} \frac{f_t}{h_{1t}} \right) \frac{\mu_{1t}}{\psi_t}.$$

Plugging this latter expression and its first lead into (A.1g) yields

$$\begin{aligned} w_{t+1}l_{1t+1}\mu_{2t} + \frac{\gamma}{1+n}w_{t+1}l_{0t+1}\mu_{1t+1} \\ + \frac{\gamma}{1+n}(1 + \psi_{t+1}e_{0t+1}) \left( w_{t+1} + \frac{1+n}{\gamma} \frac{f_{t+1}}{h_{1t+1}} \right) \frac{\mu_{1t+1}}{\psi_{t+1}} &= \left( w_t + \frac{1+n}{\gamma} \frac{f_t}{h_{1t}} \right) \frac{\mu_{1t}}{\psi_t}. \end{aligned}$$

Replacing in the above equation  $\mu_{1t}$  and  $\mu_{1t+1}$  by their expressions as they respectively appear in (A.1h) and (A.1j) helps get rid of all the Lagrange multipliers

$$\begin{aligned} \left[ w_{t+1}l_{1t+1} + \gamma w_{t+1}l_{0t+1} + \frac{\gamma}{\psi_{t+1}}(1 + \psi_{t+1}e_{0t+1}) \left( w_{t+1} + \frac{1+n}{\gamma} \frac{f_{t+1}}{h_{1t+1}} \right) \right] \mu_{2t} \\ = \frac{1+r_{t+1}}{\psi_t} \left( w_t + \frac{1+n}{\gamma} \frac{f_t}{h_{1t}} \right) \mu_{2t}. \end{aligned}$$

Rearranging thereafter, one gets (A.2f).

## A.2 Solving the Problem Using the Equation of Bellman

The *state* variables in our dynamic model are  $a_{1t}$ ,  $a_{2t}$ , and  $h_{1t}$ . These three variables are enough to determine the level of the social welfare at time  $t$ . Likewise, their future values determine the future level of the social welfare. The Bellman equation enables to link recursively the social welfare of the three generations of households coexisting

at time  $t$  to that of the next coexisting generations of households.

$$\begin{aligned}
\mathcal{V}(a_{1t}, a_{2t}, h_{1t}) = & \max \ln c_{0t} + \sigma \ln(1 - e_{0t} - l_{0t}) + \frac{1 + \phi}{1 + n} [\ln c_{1t} + \sigma \ln(1 - l_{1t})] \\
& + \left( \frac{1 + \phi}{1 + n} \right)^2 \ln c_{2t} + \frac{1 + n}{(1 + \phi)(1 + \rho)} \mathcal{V}(a_{1t+1}, a_{2t+1}, h_{1t+1}) \\
& + \tilde{\mu}_{1t} \left[ \frac{\gamma}{1 + n} w_t h_{1t} l_{0t} + \epsilon_{1t} - c_{0t} - f_t e_{0t} - a_{1t+1} \right] \\
& + \tilde{\mu}_{2t} [w_t h_{1t} l_{1t} + (1 + r_t) a_{1t} - c_{1t} - (1 + n) \epsilon_{1t} - a_{2t+1}] \\
& + \tilde{\mu}_{3t} [(1 + r_t) a_{2t} - c_{2t}] + \tilde{\mu}_{4t} \left[ (1 + \psi_t e_{0t}) \frac{\gamma}{1 + n} h_{1t} - h_{1t+1} \right]
\end{aligned}$$

The FOCs

$$c_{0t} : \frac{1}{c_{0t}} = \tilde{\mu}_{1t} \quad (\text{A.3a})$$

$$c_{1t} : \frac{1 + \phi}{1 + n} \frac{1}{c_{1t}} = \tilde{\mu}_{2t} \quad (\text{A.3b})$$

$$c_{2t} : \left( \frac{1 + \phi}{1 + n} \right)^2 \frac{1}{c_{2t}} = \tilde{\mu}_{3t} \quad (\text{A.3c})$$

$$e_{0t} : \frac{\sigma}{1 - e_{0t} - l_{0t}} + f_t \tilde{\mu}_{1t} = \psi_t \frac{\gamma}{1 + n} h_{1t} \tilde{\mu}_{4t} \quad (\text{A.3d})$$

$$l_{0t} : \frac{\sigma}{1 - e_{0t} - l_{0t}} = \frac{\gamma}{1 + n} w_t h_{1t} \tilde{\mu}_{1t} \quad (\text{A.3e})$$

$$l_{1t} : \frac{1 + \phi}{1 + n} \frac{\sigma}{1 - l_{1t}} = w_t h_{1t} \tilde{\mu}_{2t} \quad (\text{A.3f})$$

$$h_{1t+1} : \frac{1 + n}{(1 + \phi)(1 + \rho)} \frac{\partial \mathcal{V}(a_{1t+1}, a_{2t+1}, h_{1t+1})}{\partial h_{1t+1}} = \tilde{\mu}_{4t} \quad (\text{A.3g})$$

$$a_{1t+1} : \frac{1 + n}{(1 + \phi)(1 + \rho)} \frac{\partial \mathcal{V}(a_{1t+1}, a_{2t+1}, h_{1t+1})}{\partial a_{1t+1}} = \tilde{\mu}_{1t} \quad (\text{A.3h})$$

$$a_{2t+2} : \frac{1 + n}{(1 + \phi)(1 + \rho)} \frac{\partial \mathcal{V}(a_{1t+1}, a_{2t+1}, h_{1t+1})}{\partial a_{2t+1}} = \tilde{\mu}_{2t} \quad (\text{A.3i})$$

$$\epsilon_{1t} : (1 + n) \tilde{\mu}_{2t} = \tilde{\mu}_{1t} \quad (\text{A.3j})$$

We dropped the FOCs with respect to the shadow prices  $\tilde{\mu}_{1t}, \dots, \tilde{\mu}_{4t}$ . The derivatives  $\partial \mathcal{V}() / \partial h_{1t+1}$ ,  $\partial \mathcal{V}() / \partial a_{1t+1}$ ,  $\partial \mathcal{V}() / \partial a_{2t+1}$  respectively in the FOCs (A.3g), (A.3h), and (A.3i) are unknown and will be found using the envelope conditions.

The envelope conditions

$$\begin{aligned}
\frac{\partial \mathcal{V}(a_{1t}, a_{2t}, h_{1t})}{\partial h_{1t}} &= w_t l_{1t} \tilde{\mu}_{2t} + \frac{\gamma}{1 + n} w_t l_{0t} \tilde{\mu}_{1t} + (1 + \psi_t e_{0t}) \frac{\gamma}{1 + n} \tilde{\mu}_{4t} \Rightarrow \\
\frac{\partial \mathcal{V}(a_{1t+1}, a_{2t+1}, h_{1t+1})}{\partial h_{1t+1}} &= w_{t+1} l_{1t+1} \tilde{\mu}_{2t+1} + \frac{\gamma}{1 + n} w_{t+1} l_{0t+1} \tilde{\mu}_{1t+1} \\
&+ (1 + \psi_{t+1} e_{0t+1}) \frac{\gamma}{1 + n} \tilde{\mu}_{4t+1}
\end{aligned} \quad (\text{A.4a})$$

$$\begin{aligned} \frac{\partial \mathcal{V}(a_{1t}, a_{2t}, h_{1t})}{\partial a_{1t}} &= (1 + r_t) \tilde{\mu}_{2t} \Rightarrow \\ \frac{\partial \mathcal{V}(a_{1t+1}, a_{2t+1}, h_{1t+1})}{\partial a_{2t+1}} &= (1 + r_{t+1}) \tilde{\mu}_{2t+1} \end{aligned} \quad (\text{A.4b})$$

$$\begin{aligned} \frac{\partial \mathcal{V}(a_{1t}, a_{2t}, h_{1t})}{\partial a_{2t}} &= (1 + r_t) \tilde{\mu}_{3t} \Rightarrow \\ \frac{\partial \mathcal{V}(a_{1t+1}, a_{2t+1}, h_{1t+1})}{\partial a_{2t+1}} &= (1 + r_{t+1}) \tilde{\mu}_{3t+1} \end{aligned} \quad (\text{A.4c})$$

The envelope conditions (A.4a) through (A.4c) can now be plugged respectively into the FOCs (A.3g), (A.3h), and (A.3i). After that, one gets rid of the shadow prices in the FOCs to have

$$\sigma c_{0t} = \frac{\gamma}{1+n} w_t h_{1t} (1 - e_{0t} - l_{0t}) \quad (\text{A.5a})$$

$$\sigma c_{1t} = w_t h_{1t} (1 - l_{1t}) \quad (\text{A.5b})$$

$$(1 + r_{t+1}) c_{0t} = (1 + \rho) c_{1t+1} \quad (\text{A.5c})$$

$$(1 + r_{t+1}) c_{1t} = (1 + \rho) c_{2t+1} \quad (\text{A.5d})$$

$$c_{1t} = (1 + \phi) c_{0t} \quad (\text{A.5e})$$

$$\begin{aligned} \gamma \left[ l_{0t+1} + \left( 1 + \frac{1+n}{\gamma} \frac{f_{t+1}}{w_{t+1} h_{1t+1}} \right) e_{0t+1} \right] + l_{1t+1} &= \frac{1+r_{t+1}}{\psi_t} \frac{w_t}{w_{t+1}} \left( 1 + \frac{1+n}{\gamma} \frac{f_t}{w_t h_{1t}} \right) \\ &\quad - \frac{\gamma}{\psi_{t+1}} \left( 1 + \frac{1+n}{\gamma} \frac{f_{t+1}}{w_{t+1} h_{1t+1}} \right) \end{aligned} \quad (\text{A.5f})$$

## B The General Equilibrium Model

Our dynamic deterministic general equilibrium model is made up of fifteen equations and has seventeen variables. Two of these seventeen variables,  $f_t$  and  $\psi_t$  are *exogenous*, *i.e.*, they are set outside the model. The other fifteen variables are *endogenous* in the sense they are determined by the behavior of optimizing agents. The fifteen equations are listed in the next subsection. The non-stationary variables are normalized, *i.e.*, they are divided by their growth components. This gives rise to the variables  $\hat{y}_t = Y_t/[N_{0t}\nu^t \exp(xt)]$ ,  $\hat{k}_t = K_t/[N_{0t}\nu^t \exp(xt)]$ ,  $\bar{h}_t = H_t/(1+n)\nu^t$ ,  $\tilde{w}_t = w_t/\exp(xt)$ ,  $\bar{h}_{1t} = h_{1t}/\nu^t$ , and  $\hat{\mathbf{z}}_t = \mathbf{z}_t/[\nu^t \exp(xt)]$  with  $\mathbf{z}_t = \{a_{1t}, a_{2t}, c_{0t}, c_{1t}, c_{2t}, f_t, \epsilon_{1t}\}$ . Note that in solving numerically the model, the variables  $\hat{a}_{1t}$ ,  $\hat{a}_{2t}$ ,  $h_t$ , and  $\hat{k}_t$  are considered as *predetermined*, *i.e.*, their values at  $t+1$  is chosen at time  $t$  by agents.

$$\sigma \hat{c}_{0t} = \frac{\gamma}{1+n} \tilde{w}_t \bar{h}_{1t} (1 - e_{0t} - l_{0t}) \quad (\text{B.1a})$$

$$\sigma \hat{c}_{1t} = \tilde{w}_t \bar{h}_{1t} (1 - l_{1t}) \quad (\text{B.1b})$$

$$(1 + r_{t+1}) \hat{c}_{0t} = (1 + \rho) \nu \exp(x) \hat{c}_{1t+1} \quad (\text{B.1c})$$

$$(1 + r_{t+1})\hat{c}_{1t} = (1 + \rho)\nu \exp(x)\hat{c}_{2t+1} \quad (\text{B.1d})$$

$$\hat{c}_{1t} = (1 + \phi)\hat{c}_{0t} \quad (\text{B.1e})$$

$$\begin{aligned} \gamma \left[ l_{0t+1} + \left( 1 + \frac{1+n}{\gamma} \frac{\hat{f}_{t+1}}{\tilde{w}_{t+1}\bar{h}_{1t+1}} \right) e_{0t+1} \right] + l_{1t+1} &= \frac{1+r_{t+1}}{\exp(x)\psi_t} \frac{\tilde{w}_t}{\tilde{w}_{t+1}} \left( 1 + \frac{1+n}{\gamma} \frac{\hat{f}_t}{\tilde{w}_t\bar{h}_{1t}} \right) \\ &\quad - \frac{\gamma}{\psi_{t+1}} \left( 1 + \frac{1+n}{\gamma} \frac{\hat{f}_{t+1}}{\tilde{w}_{t+1}\bar{h}_{1t+1}} \right) \end{aligned} \quad (\text{B.1f})$$

$$(1 + \psi_t e_{0t}) \frac{\gamma}{1+n} \bar{h}_{1t} = \bar{h}_{1t+1} \quad (\text{B.1g})$$

$$\frac{\gamma}{1+n} \tilde{w}_t l_{0t} \bar{h}_{1t} + \hat{e}_{1t} - \hat{c}_{0t} - \hat{f}_t e_{0t} = \nu \exp(x) \hat{a}_{1t+1} \quad (\text{B.1h})$$

$$\tilde{w}_t l_{1t} \bar{h}_{1t} + (1 + r_t) \hat{a}_{1t} - \hat{c}_{1t} - (1+n) \hat{e}_{1t} = \nu \exp(x) \hat{a}_{2t+1} \quad (\text{B.1i})$$

$$(1 + r_t) \hat{a}_{2t} = \hat{c}_{2t} \quad (\text{B.1j})$$

$$\hat{k}_t^\alpha \bar{h}_t^{1-\alpha} = \hat{y}_t \quad (\text{B.1k})$$

$$\alpha \frac{\hat{y}_t}{\hat{k}_t} = 1 + r_t \quad (\text{B.1l})$$

$$(1 - \alpha) \frac{\hat{y}_t}{\bar{h}_t} = \tilde{w}_t \quad (\text{B.1m})$$

$$\frac{\hat{a}_{1t}}{1+n} + \frac{\hat{a}_{2t}}{(1+n)^2} = \hat{k}_t \quad (\text{B.1n})$$

$$(\gamma l_{0t} + l_{1t}) \frac{\bar{h}_{1t}}{1+n} = \bar{h}_t \quad (\text{B.1o})$$