Bertrand-Edgeworth games under triopoly: the equilibrium strategies when the payoffs of the two smallest firms are proportional to their capacities

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Abstract

The paper is the second part of a trilogy in which we extend the analysis of price competition among capacity-constrained sellers beyond duopoly to triopoly. In the first part of the trilogy we provided some general results, highlighting features of a duopolistic mixed strategy equilibrium that generalize to triopoly and provided a first partition concerning the pure strategy equilibrium regions and the mixed strategies equilibrium region and then the partition of this region in a part in which the payoffs of the two smallest firm are proportional to their capacities and another in which the smallest firm obtains a payoff proportinally higher than that of the middle sized firm. In this paper we provide a complete characterization of the set of mixed strategy equilibria in the part in which the payoffs of the two smallest firms are proportional to their capacities. This part is partitioned according to equilibrium features and in each part it is determined whether equilibria are uniquely determined or not and in the latter case it is proved that the equilibria constitute a continuum. Further we determine the circustances in which supports of an equilibrium strategy may be disconnected and show how gaps are then determined. We also prove that the union of supports is indeed connected, a property which cannot be extended to the case in which the smallest firm obtains a payoff proportinally higher than that of the middle sized firm. The third part of the trilogy will be devoted to a complete characterization of the mixed strategy equilibria when the smallest firm obtains a payoff proportinally higher than that of the middle sized firm. This will allow also to determine the payoff of the smallest firm.

1 Introduction

This paper is the second part of a trilogy concerning price competition among capacity-constrained sellers, which has attracted considerable interest since Levitan and Shubik's [16] modern reappraisal of Bertrand and Edgeworth. In the first part of the trilogy ([10]) we provided the general introduction with a survey of the literature. Here it is sufficient to remark that in the current state of the art, a complete characterization of equilibria of the price game exists only for the duopoly [15] and for special cases when the number of firms is higher than 2.[1]

In the first part of the trilogy ([10]) we have shown that several properties of a duopolistic mixed strategy equilibrium prove to generalize to triopoly: the values of the minimum and the maximum of the support of the equilibrium strategy for any firm with the highest capacity (equal to p_m and p_M, respectively, as defined here in Section 2); the equilibrium payoff of any firm with the second highest capacity. On the other hand, in a duopoly the supports of the equilibrium strategies completely overlap, which need not be the case in a triopoly.[2] In a duopoly the region of the capacity space where no pure strategy equilibrium exists can be partitioned in two subsets: one in which both firms get the same payoff per unit of capacity and one in which the smaller firm gets a higher payoff per unit of capacity. The latter subset is characterized by the fact that the capacity of the larger firm is higher than total demand at p_m. In the triopoly, on the contrary, as we have shown in [10], there are several relevant subsets of the region where no pure strategy equilibrium exists.

1. In one subset, as in the duopoly, the capacity of the largest firm is larger than or equal to demand at p_m. In this subset the other firms get the same payoff per unit of capacity, higher than that of the largest firm.

2. In another subset the sum of the capacities of the two largest firm is

[1] Vives [20] characterized the (symmetric) mixed strategy equilibrium for the case of equal capacities among all firms. In a previous paper we [9] generalized Vives result to the case in which the capacities of the largest and smallest firm are sufficiently close. Within an analysis concerning horizontal merging of firms Davidson and Deneckere [4] provided the complete analysis (apart from the fact that attention is restricted to equilibria in which strategies of equally-sized firms are symmetrical) of a Bertrand-Edgeworth game with linear demand, equally-sized small firms and one large firm with a capacity that is a multiple of the small firm's capacity.

[2] That minima of the supports of the equilibfium strategies may differ has also been recognized in [13] and [14].

smaller than or equal to demand at p_m. In this subset all firms get the same payoff per unit of capacity.

3. In another subset both the smallest firms have the same size and the capacity of the largest firm is smaller than demand at p_m. In this subset all firms get the same payoff per unit of capacity too.

4. The complement of the previous three subsets can be partitioned in two parts. In one part the smallest firm gets a higher payoff per unit of capacity than the others, that in turn get the same payoff per unit of capacity, a fact also discovered by [14]. Yet we determined the interval where the payoff of the smallest firm must be and provided examples for the exact determination of that payoff (a general rule for determining that payoff will be provided in the third paper of the trilogy). In the other part all firms get the same payoff per unit of capacity and the supports of the largest and the smallest firms have a lower bound equal to the lower bound of the overall price distribution, whereas the middle sized firm set prices only at higher levels.

In this paper we will add the following results.

- In the subset mentioned in 1 above, differently from the analogous subset in the duopoly, the equilibrium strategies of the smallest firms are constrained but not uniquely determined (there is a continuum of equilibria).

- There are other subsets (parts of the subsets mentioned in 2 and 3 above) in which the equilibrium strategies of the two smallest firms are similarly constrained and not uniquely determined, but not in the whole union of the supports of equilibrium strategies. In these subsets the largest firm can meet total demand at prices close to p_M. An example of this case was also provided in [10], here we provide the complete analysis.

- In some subsets (parts of the subsets mentioned in 2 and 3 above) the equilibrium support of strategies of some firm is necessarily disconnected and in some other subsets it may be disconnected or not (since a continuum of equilibria exists). In any case the union of supports is connected. We will also determine the gaps.[3]

[3] Osborne and Pitchik [18] clarified that in duopoly, under the set of assumptions on demand adopted here, supports of equilibrium strategies are connected, otherwise supports

- The subset mentioned in 4 above in which all firms get the same payoff per unit of capacity generally involves a continuum of equilibria (but if a special condition holds, the equilibrium is unique).

- No atom in the range $[p_m, p_M)$ may exist in any of the subsets investigated in this paper (as shown by an example in [10] this is not so in subsets investigated in the third paper of the trilogy).

One of the main aim of this paper is methodological. Indeed we will introduce a number of functions and procedures that will also be used in part three of the trilogy, where we will determine not only the complete characterization of the mixed strategy equilibria in the part in which the smallest firm obtains a payoff proportionally higher than that of the middle sized firm, but also the payoff of this firm, whereas in the first part of the trilogy we determined only a range in which this payoff must stay. However, as also the examples provided in [10] has shown, the payoff of the smallest firm cannot be determined in this subregion of the region in which a pure strategy equilibrium cannot exist without determining also the equilibrium strategies.

The paper is organized as follows. Section 2 contains definitions and the basic assumptions of the model along with a summary of the propositions proved in [10]. For the sake of simplicity we will refer only to propositions used in this paper, but we will follow the original numeration. Sections 3 and 4 introduce two sets of functions which will be used as tools. Section 5 builds the profiles of equilibrium strategies and complete the partition of the region of the capacity space where no pure strategy equilibrium exists and the payoffs of the two smallest firm are proportional to their capacities; in this section we prove that either the equilibrium is unique or there is a continuum of equilibria, and we identify the two complementary subsets of the region investigated where the former and latter hold true, respectively. We also prove that gaps do not overlap and that atoms in equilibrium strategies are absent (apart from the maximum of the support of the largest firm, which it charges with positive probability when its capacity is strictly higher than for any other firm). Section 6 provides some examples. Section 7 briefly concludes.

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need not be connected. Quite differently, we will prove that under triopoly the supports need not be connected (although their union is) even under concavity of the demand function.

2 Preliminaries

In this section we mention all the assumptions, the definitions, and the results mentioned in [10] that are relevant for this paper too. An exception is an assumption that we introduce here and we did not need to introduce in [10].

Assumption 1. There are 3 firms producing a homogeneous good at the same constant unit cost (normalized to zero), up to capacity.

Let $N = \{1, 2, 3\}$ be the set of firms and $N_{-i} = N - \{i\}$. Without loss of generality, we consider the subset of the capacity space (K_1, K_2, K_3) where

$$K_1 \geqslant K_2 \geqslant K_3 > 0 \tag{1}$$

and we define $K = K_1 + K_2 + K_3$.

Assumption 2. The market demand function is given by $D(p)$ (demand as a function of price p) and $P(x)$ (price as a function of quantity x). The function $D(p)$ is strictly positive on some bounded interval $[0, p^*)$, on which it is continuously differentiable, strictly decreasing and such that $pD(p)$ is strictly concave; it is continuous for $p \geqslant 0$ and equals 0 for $p \geqslant p^*$; $X = D(0) < \infty$. $P(x) = D^{-1}(x)$ on the bounded interval $(0, X)$; the function $P(x)$ is continuous for $x \geqslant 0$ and equals 0 for $x \geqslant X$; $p^* = P(0) < \infty$.

Assumption 3. It is assumed throughout that any rationing is according to the efficient rule.

In some propositions we will use a stronger assumption on demand, namely

Assumption 2*. Assumption 2 holds and $D''(p) \leqslant 0$.

Let p^c be the competitive price, that is

$$p^c = P(K). \tag{2}$$

Proposition 1 *Let Assumptions 1, 2, and 3 hold. (i)* $(p_1, p_2, p_3) = (p^c, p^c, p^c)$ *is an equilibrium if and only if either*

$$K - K_1 \geqslant X, \text{ if } X \leqslant K, \tag{3}$$

or

$$K_1 \leqslant -p^c \left[D'(p)\right]_{p=p^c} = -\frac{P(K)}{P'(K)}, \text{ if } X > K. \tag{4}$$

In the former case the set of equilibria includes any strategy profile such that $\Omega(0) \neq \emptyset$ *and* $\sum_{s\in\Omega(0)-\{j\}} K_s \geqslant X$ *for each* $j \in \Omega(0)$. *In the latter,* (p^c, p^c, p^c) *is the unique equilibrium.*

(ii) No pure strategy equilibrium exists if neither (3) nor (4) holds.

Proof See [10].

On the basis of Proposition 1 the region of the capacity space where no pure strategy equilibrium exists is the region where

$$K_1 > \max\left\{K - X, -\frac{P(K)}{P'(K)}\right\} \tag{5}$$

and inequalities (1) hold. In this region a strategy by firm i is denoted by $\sigma_i : (0, \infty) \to [0, 1]$, where $\sigma_i(p) = \Pr_{\sigma_i}(p_i < p)$ is the probability of firm i charging less than p under strategy σ_i. Of course, any function $\sigma_i(p)$ is non-decreasing and everywhere continuous except at p° such that $\Pr_{\sigma_i}(p_i = p^\circ) > 0$, where it is left-continuous ($\lim_{p \to p^\circ -} \sigma_i(p) = \sigma_i(p^\circ)$), but not continuous. An equilibrium is denoted by $\phi = (\phi_1, \phi_2, \phi_3)$, where $\phi_i(p) = \Pr_{\phi_i}(p_i < p)$. We denote by $\Pi_i(\sigma_i, \sigma_{-i})$ firm i's payoff (expected profit) at strategy profile (σ_i, σ_{-i}). Obviously $\Pi_i(\phi) = \Pi_i(\phi_i, \phi_{-i})$ denotes firm i's expected profit at the equilibrium strategy profile ϕ and $\Pi_i(p, \phi_{-i})$ denotes firm i's expected profit when it charges p with certainty and the rivals are playing their equilibrium profile of strategies ϕ_{-i}. Of course, $\Pi_i(\phi_i, \phi_{-i}) \geqslant \Pi_i(\sigma_i, \phi_{-i})$ for each i and and each σ_i. When no doubt can arise, and for the sake of brevity, we write Π_i^* rather than $\Pi_i(\phi_i, \phi_{-i})$ and $\Pi_i(p)$ rather than $\Pi_i(p, \phi_{-i})$. Further, we denote by $S_i(\phi_i)$ the support of ϕ_i and by $p_M^{(i)}(\phi_i)$ and $p_m^{(i)}(\phi_i)$ the maximum and minimum of $S_i(\phi_i)$, respectively. More specifically, we say that $p \in S_i(\phi_i)$ when $\phi_i(\cdot)$ is increasing at p, that is, when there is $\delta > 0$ such that $\phi_i(p + h) > \phi_i(p - h)$ for any $0 < h < \delta$, whereas $p \notin S_i(\phi_i)$ if $\phi_i(p + h) = \phi_i(p - h)$ for some $h > 0$. Obviously, $\Pi_i^* = \Pi_i(p)$ almost everywhere in $S_i(\phi_i)$. Once again, when no doubt can arise and for the sake of brevity, we write S_i, $p_M^{(i)}$, and $p_m^{(i)}$ rather than $S_i(\phi_i)$, $p_M^{(i)}(\phi_i)$, and $p_m^{(i)}(\phi_i)$, respectively. If S_i is not connected, i.e. if $\phi_i(p)$ is constant in an open interval $(\widetilde{p}, \widetilde{\widetilde{p}})$ whose endpoints are in S_i ($\widetilde{p} \in S_i$ and $\widetilde{\widetilde{p}} \in S_i$), then the interval $(\widetilde{p}, \widetilde{\widetilde{p}})$ will be referred to as a *gap* in S_i. In order to shorten notation, we denote $\lim_{p \to h+} \Pi_i(p)$ and $\lim_{p \to h-} \Pi_i(p)$ as $\Pi_i(h+)$ and $\Pi_i(h-)$, respectively, and $\lim_{p \to h+} \phi_i(p)$ as $\phi_i(h+)$.

So long as firm i's rivals' equilibrium strategies $\phi_{-i}(p)$ are continuous in p, $\Pi_i(p) = Z_i(p; \phi_{-i}(p))$, where

$$Z_i(p; \varphi_{-i}) := p \sum_{\psi \in \mathcal{P}(N_{-i})} q_{i,\psi}(p) \prod_{r \in \psi} \varphi_r \prod_{s \in N_{-i} - \psi} (1 - \varphi_s), \tag{6}$$

φ's are taken as independent variables (with the obvious constraints that $\varphi_j \in [0, 1]$, each j), $\mathcal{P}(N_{-i}) = \{\psi\}$ is the power set of N_{-i}, and $q_{i,\psi}(p) =$

$\max\{0, \min\{D(p) - \sum_{r\in\psi} K_r, K_i\}\}$ is firm i's output when it charges p, any firm $r \in \psi$ charges less than p and any firm $s \in N_{-i} - \psi$ charges more than p.[4] If instead $\Pr_{\phi_j}(p_j = p^\circ) > 0$ for some $j \neq i$, then $Z_i(p^\circ; \phi_{-i}(p^\circ)) \geqslant \Pi_i(p^\circ) \geqslant \lim_{p\to p^\circ+} Z_i(p; \phi_{-i}(p))$.[5] Note that since $\sum_{\psi\in\mathcal{P}(N_{-i})} \prod_{r\in\psi} \varphi_r \prod_{s\in N_{-i}-\psi}(1 - \varphi_s) = 1$, if $\varphi_i \in [0, 1]$, then the RHS of (6) is an average of the functions $pq_{i,\psi}(p)$'s. As a consequence $pq_{i,N-i}(p) \leqslant Z_i(p; \phi_{-i}) \leqslant pq_{i,\emptyset}(p)$.

Lemma 1

(ii) $Z_1(p; \varphi_2, \varphi_3)$ is concave and increasing in p throughout $[p_m, p_M]$.

(iii) $Z_i(p; \varphi_{-i})$ (each $i \neq 1$) is concave in p over any range enclosed in (p_m, p_M) in which it is differentiable; local convexity only arises at $P(K_1 + K_r) \in (p_m, p_M)$ $(r \neq 1, i)$, if $\varphi_1\varphi_r > 0$, and at $P(K_1) \in (p_m, p_M)$, if $\varphi_1(1 - \varphi_r) > 0$ $(r \neq 1, i)$.

(v) $Z_i(p; \varphi_{-i})$ is continuous and differentiable in φ_j and $\partial Z_i/\partial\varphi_j \leq 0$, each i and $j \neq i$. More precisely: if $p \in (p_m, P(K_1))$, then $\partial Z_i/\partial\varphi_j < 0$, each i and $j \neq i$; if $p \geq P(K_1)$, then $\partial Z_1/\partial\varphi_j < 0$, $\partial Z_i/\partial\varphi_1 < 0$, and $\partial Z_i/\partial\varphi_j = 0$ (each $i \neq 1$ and $j \neq 1, i$).

Proof See [10].

Let us also define

$$p_M = \arg\max_p pq_{1,N_{-1}}(p); \tag{7}$$

$$p_m = \min\left\{p : pq_{1,\emptyset}(p) = \max_p pq_{1,N_{-1}}(p)\right\}. \tag{8}$$

The definitions of p_M and p_m also make it possible to characterize the region where inequalities (5) and (1) hold by substituting inequality (5) with inequality

$$P(K) < p_m. \tag{5'}$$

Note that in the region where inequalities (5) and (1) hold we have:

$$p_M = \arg\max_p p\left[D(p) - \sum_{j\neq 1} K_j\right] \tag{9}$$

$$p_m = \max\{\widehat{p}, \widehat{\widehat{p}}\}, \tag{10}$$

[4]Note that $\prod_{r\in\psi} \varphi_r$ is the empty product, hence equal to 1, when $\psi = \emptyset$; and it is similarly $\prod_{s\in N_{-i}-\psi}(1 - \varphi_s) = 1$ when $\psi = N_{-i}$.

[5]The exact value of $\Pi_i(p^\circ)$ when $\Pr_{\phi_j}(p_j = p^\circ) > 0$ for some $j \neq i$ depends on the specific assumption made on how the residual demand is shared among firms charging the same price

where

$$\widehat{p} = \frac{\max_p p[D(p) - \sum_{j\neq 1} K_j]}{K_1} \tag{11}$$

$$\widehat{\widehat{p}} = \min\left\{p : pD(p) = \max_p p\left[D(p) - \sum_{j\neq 1} K_j\right]\right\}. \tag{12}$$

Proposition 2 *Let Assumptions 1, 2, and 3 and inequality (5′) hold. In any equilibrium $\phi_j(p_M) = 1$ for any j such that $K_j < K_1$; $p_M^{(i)} = p_M$ for some i such that $K_i = K_1$, and*

$$\Pi_i^* = \max_p p\left[D(p) - \sum_{j\neq 1} K_j\right] \tag{13}$$

for any i such that $K_i = K_1$.

Proof See [10].

Let $M = \{i \in N : p_M^{(i)} = p_M\}$ and $L = \{i \in N : p_m^{(i)} = p_m\}$.

Proposition 3 *Let Assumptions 1, 2, and 3 and inequality (5′) hold. In any equilibrium (ϕ_1, ϕ_2, ϕ_3):*
(iv) If $(p^\circ, p^{\circ\circ}) \subset S_i$, then $(p^\circ, p^{\circ\circ}) \subset \cup_{j\neq i} S_j$;
(vii) For any $i \neq 1$ such that $p_M^{(i)} \geqslant P(K_1)$, $\Pi_i^* = p_m K_i$.

Proof See [10]; Proposition 4(iv)&(vii).

We introduce the following partition of the region defined by inequalities (5) and (1).

$A = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 \geqslant K_3, D(\widehat{p}) \leqslant K_1\}$.
$B = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 > K_3, K > D(\widehat{p}) \geqslant K_1 + K_2\}$.
$C_1 = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 > K_3, K_1 + K_2 > D(\widehat{p}) > K_1 + K_3\}$.
$C_2 = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 > K_3, K_1 + K_3 \geqslant D(\widehat{p}), D(p_M) \geqslant K_1\}$
$C_3 = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 > K_3, K_1 + K_3 \geqslant D(\widehat{p}), D(p_M) < K_1 < D\left(\frac{\widehat{p}K_1}{K_1 - K_3}\right)\}$
$D = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 > K_3, K_1 + K_3 \geqslant D(\widehat{p}), D(p_M) < D\left(\frac{\widehat{p}K_1}{K_1 - K_3}\right) \leqslant K_1 < D(\widehat{p})\}$[6]
$E = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 = K_3, K_1 < D(\widehat{p})\}$.

Next we introduce and study a number of functions to be used later.

[6] If $K_1 + K_3 \geqslant D(\widehat{p})$, then $D(p_M) < D\left(\frac{\widehat{p}K_1}{K_1 - K_3}\right)$ since the latter inequality is equivalent to $p_M(K_1 - K_3) > \widehat{p}K_1 = p_M(D(p_M) - K_2 - K_3)$.

Proposition 4 *Let* $K_1 + K_j > D(p_m) > K_1$ *(some* $j \neq 1$*). (i) Denote by* $\phi^{\star}_{1j}(p) = \frac{(p-p_m)K_j}{p[K_1+K_j-D(p)]}$ *and* $\phi^{\star}_j(p) = \frac{(p-p_m)K_1}{p[K_1+K_j-D(p)]}$ *the solutions of equations* $p_m K_j = Z_j(p;\varphi_1, 0)$ *and* $p_m K_1 = Z_1(p;\varphi_j, 0)$*, respectively, over the range* $\{p_m, \min\{P(K_1), p_M\})$*. Then* $\phi^{\star}_{1j}(p)$ *and* $\phi^{\star}_j(p)$ *are increasing and lower than 1 over the range* $[p_m, \min\{\tilde{p}^{(j)}_M, P(K_1)\}]$*, where* $\tilde{p}^{(j)}_M$ *is the unique solution in* $[p_m, p_M]$ *of the equation* $K_1 p_m = [D(p) - K_j]p$*.*

(ii) Denote by $\phi^{\star\star}_{1j}(p)$ *and* $\phi^{\star\star}_j(p)$ *the solutions of equations* $p_m K_j = Z_j(p;\varphi_1, 1)$ *and* $p_m K_1 = Z_1(p;\varphi_j, 1)$*, respectively. Then:*

(ii.a) Over the range $[p_m, \min\{P(K_1 + K_3), p_M\}]$*,* $\phi^{\star\star}_{12}(p) = \frac{(p-p_m)K_2}{p[K-D(p)]}$ *and* $\phi^{\star\star}_2(p) = \frac{K_1}{K_2}\phi^{\star\star}_1(p)$*, which are both increasing.*

(ii.b) Over the range $[p_m, \min\{P(K_1 + K_3), p_M\}]$*,* $\phi^{\star\star}_{13}(p) = \frac{p-p_m}{p}$ *and* $\phi^{\star\star}_3(p) = \frac{p[D(p)-K_2]-K_1p_m}{K_3p}$*, which are both increasing.*

(ii.c) Over the range $[\max\{P(K_1 + K_3), p_m\}, p_M]$*,* $\phi^{\star\star}_{1j}(p) = \frac{p-p_m}{p}$ *and* $\phi^{\star\star}_j(p) = \frac{p[D(p)-K_i]-K_1p_m}{K_jp}$*,* $(i \neq 1, j)$ *which are both increasing.*

Proof See [10]; Proposition 7.

Next we state the parts of the main theorem proved in [10] that are relevant for the results presented here.

Theorem 1.[7] *Let the region defined by inequalities (5) and (1) be partitioned as above.* (a) *If* $(K_1, K_2, K_3) \in A$*, then in any equilibrium* $p_m = \widehat{\widehat{p}}$ *and* $\Pi^*_1 = p_m D(p_m)$*;* $\Pi^*_j = p_m K_j$ *(each* $j \neq 1$*).*

(b) *If* $(K_1, K_2, K_3) \in B$*, then in any equilibrium* $p_m = \widehat{p}$*,* $\Pi^*_i = p_m K_i$ *for all i,* $L = \{1, 2, 3\}$.

(d) *If* $(K_1, K_2, K_3) \in D$*, then in any equilibrium* $p_m = \widehat{p}$*,* $\Pi^*_i = p_m K_i$ *for all i,* $L = \{1, 3\}$ *and* $p^{(2)}_m \geqslant P(K_1)$*.*

(e) *If* $(K_1, K_2, K_3) \in E$*, then in any equilibrium* $p_m = \widehat{p}$ *and* $\Pi^*_i = p_m K_i$ *for all i.*

Proof See [10]; Theorem 1(a)-(b)&(d)-(e).

[7]Hirata discovered to a large extent that $L = \{1, 2, 3\}$ in sets $A \cup B \cup E$ ([14], Claims 3 and 6), but he was not concerned with $\Pr_{\phi_i}(p_i = p_m) = 0$. He recognized the fact that $p^{(3)}_m > p_m$ and $\Pi^*_3 > p_m K_3$ in what is here called C_1, C_2, and C_3 ([14], Claims 4 and 5), but he was not concerned with how $p^{(3)}_m$ and Π^*_3 are then determined. Hirata also recognized that $p^{(2)}_m > p_m$ and $\Pi^*_3 = p_m K_3$ in our set D ([14], Claim 5),

3 Functions $\phi_1^\circ(p)$, $\phi_2^\circ(p)$, and $\phi_3^\circ(p)$)

In this paper we will build the profiles of equilibrium strategies when $\Pi_3^* = p_m K_3$. More precisely, Theorem 2 below provides profiles of equilibrium strategies for capacity configurations in the set $A \cup B \cup D \cup E$ and Theorem 3 subsequently proves that no other equilibrium exists. In order to accomplish the first task we define a number of functions, as done in Proposition 4. This will be done in this section and in the next one.

Consider the system of equations in φ_1, φ_2, φ_3 at any $p \in [p_m, p_M]$

$$\Pi_i^* = Z_i(p; \varphi_{-i}), \qquad i \in \{1, 2, 3\} \tag{14}$$

and denote $(\phi_1^\circ(p), \phi_2^\circ(p), \phi_3^\circ(p))$ a solution of its when $\Pi_1^* = p_M[D(p_M) - K_2 - K_3]$, $\Pi_2^* = p_m K_2$, $\Pi_3^* = p_m K_3$, and $0 \leqslant \phi_1^\circ(p) \leqslant 1$.

Clearly, if $p \in S_1 \cap S_2 \cap S_3$, then $(\phi_1(p), \phi_2(p), \phi_3(p)) = (\phi_1^\circ(p), \phi_2^\circ(p), \phi_3^\circ(p))$. But in general $(\phi_1^\circ(p), \phi_2^\circ(p), \phi_3^\circ(p))$ is not a profile of strategies since $\phi_1^\circ(p)$, $\phi_2^\circ(p)$, $\phi_3^\circ(p)$ do not need to be non-decreasing and codomains of $\phi_2^\circ(p)$ and $\phi_3^\circ(p)$ do not need to be enclosed in $[0, 1]$. Next proposition will explore the properties of functions $\phi_1^\circ(p)$, $\phi_2^\circ(p)$, $\phi_3^\circ(p)$. When $p_M > P(K_1)$, $\phi_2^\circ(p)$ and $\phi_3^\circ(p)$ are not fully determined over the range $(P(K_1), p_M)$; in this case we restrict ourselves to functions $\phi_2^\circ(p)$ and $\phi_3^\circ(p)$ which are non-decreasing and such that $0 \leqslant \phi_i^\circ(p) \leqslant 1$ $(i = 2, 3)$.

Proposition 5 *Let Assumptions 1, 2*, and 3 hold.*[8]

(i) Let $(K_1, K_2, K_3) \in B \cup E$ *and* $p \in [p_m, \min\{P(K_1 + K_2), p_M\}]$, *then there is a unique solution for* $(\phi_1^\circ(p), \phi_2^\circ(p), \phi_3^\circ(p))$; $\phi_1^\circ(p)$, $\phi_2^\circ(p)$, *and* $\phi_3^\circ(p)$ *are increasing and* $\phi_1^\circ(p)K_1 = \phi_2^\circ(p)K_2 = \phi_3^\circ(p)K_3$; *if* $p_M \leqslant P(K_1 + K_2)$, *then* $\phi_2^\circ(p_M)\phi_3^\circ(p_M) = 1$.

(ii) Let $(K_1, K_2, K_3) \in B$ *and* $p \in [P(K_1 + K_2), \min\{P(K_1 + K_3), p_M\}]$, *then there is a unique solution for* $(\phi_1^\circ(p), \phi_2^\circ(p), \phi_3^\circ(p))$; $\phi_1^\circ(p)$ *and* $\phi_2^\circ(p)$ *are increasing and* $\phi_3^\circ(p)$ *is concave;* $\phi_1^\circ(p)K_1 = \phi_2^\circ(p)K_2$; $0 \leqslant \phi_2^\circ(p) \leqslant 1$; *if* $P(K_1 + K_3) < p_M$, *then* $\frac{K_2 - K3}{K_2} < \phi_2^\circ(P(K_1 + K_3)) < 1$ *and* $0 < \phi_3^\circ(P(K_1 + K_3)) < 1$; *if* $P(K_1 + K_3) > p_M$, *then* $\phi_2^\circ(p_M) < 1$ *and* $\phi_3^\circ(p_M) > 1$; *if* $P(K_1 + K_3) = p_M$, *then* $\phi_2^\circ(p_M) = \phi_3^\circ(p_M) = 1$ *and* $\phi_3^{\circ\prime}(p_M) < 0$; $\phi_3^\circ(p)$ *is increasing in the whole range if and only if* $K_1 K_2 \geqslant \kappa^2$, *where*

$$\kappa = \left[\frac{-2D'(p)p^2}{p_m} \sqrt{\frac{p - p_m}{p}} \right]_{p = \min\{P(K_1 + K_3), p_M\}}. \tag{15}$$

[8]Note that Assumption 2* is required only in the proof of part (ii) and could be substituted by Assumption 2 in the proof of all the other parts.

(iii) Let $(K_1, K_2, K_3) \in B \cup E$ *and* $p \in [\max\{p_m, P(K_1+K_3)\}, \min\{P(K_1), p_M\}]$, *then there is a unique solution for* $(\phi_1^\circ(p), \phi_2^\circ(p), \phi_3^\circ(p))$*;* $\phi_1^\circ(p)$*,* $\phi_2^\circ(p)$*, and* $\phi_3^\circ(p)$ *are increasing and* $0 \leqslant \phi_3^\circ(p) \leqslant \phi_2^\circ(p) \leqslant 1$*; if* $p_M \leqslant P(K_1)$*, then* $\phi_2^\circ(p_M) = \phi_3^\circ(p_M) = 1$*; if* $p_M > P(K_1)$*, then* $\phi_2^\circ(p)$ *and* $\phi_3^\circ(p)$ *are not defined for* $p = P(K_1)$ *and* $\phi_i^\circ(P(K_1)-) = 1 - \frac{p_m K_1 - P(K_1)(K_1-K_2-K_3)}{2P(K_1)K_i} < 1$ *(i=2,3).*

(iv) Let $(K_1, K_2, K_3) \in A \cup B \cup D \cup E$ *and* $p \in [\max\{p_m, P(K_1)\}, p_M]$, *then* $\phi_1^\circ(p) = \frac{p-p_m}{p}$ *whereas* $\phi_2^\circ(p) \in [0,1]$ *and* $\phi_3^\circ(p) \in [0,1]$ *are any pair of functions such that*

$$pK_2\phi_2^\circ(p) + pK_3\phi_3^\circ(p) = pD(p) - \Pi_1^* \tag{16}$$

$$\frac{pD(p) - \Pi_1^*}{pK_i} - \frac{K_j}{K_i} \leqslant \phi_i^\circ(p) \leqslant \frac{pD(p) - \Pi_1^*}{pK_i}, 1 \tag{17}$$

where $i, j = 2, 3$ *and* $i \neq j$*.*

Proof (i) System (14) reads

$$\begin{cases} p_m K_1 = p[\varphi_2\varphi_3(D(p) - K_2 - K_3) + (1 - \varphi_2\varphi_3)K_1] \\ p_m K_2 = p[\varphi_1\varphi_3(D(p) - K_1 - K_3) + (1 - \varphi_1\varphi_3)K_2] \\ p_m K_3 = p[\varphi_1\varphi_2(D(p) - K_1 - K_2) + (1 - \varphi_1\varphi_2)K_3], \end{cases}$$

and the unique solution such that $0 \leqslant \phi_1^\circ(p) \leqslant 1$ is

$$\phi_1^\circ(p) = \sqrt{\frac{K_2}{K_1}\frac{(p-p_m)K_3}{p(K - D(p))}}, \phi_2^\circ(p) = \frac{K_1}{K_2}\phi_1^\circ(p), \phi_3^\circ(p) = \frac{K_1}{K_3}\phi_1^\circ(p).$$

Clearly $\phi_1^{\circ\prime}(p)$, $\phi_2^{\circ\prime}(p)$, and $\phi_3^{\circ\prime}(p)$ have the same sign. If both $\phi_2^{\circ\prime}(p)$ and $\phi_3^{\circ\prime}(p)$ are nonpositive, then a contradiction is obtained since $\frac{\partial Z_1}{\partial p} + \frac{\partial Z_1}{\partial \varphi_2}\phi_2^{\circ\prime}(p) + \frac{\partial Z_1}{\partial \varphi_3}\phi_3^{\circ\prime}(p) > 0$. This inequality holds since $\phi_2^\circ(p)\phi_3^\circ(p) = \frac{(p-p_m)K_1}{p(K-D(p))} \leqslant 1$ (which is equivalent to $p(D(p) - K_2 - K_3) \leqslant p_m K_1$). The last claim is a consequence of the fact that $p_M[K - D(p_M)] = (p_M - p_m)K_1$.

(ii) System (14) reads

$$\begin{cases} p_m K_1 = p\left[\varphi_2\varphi_3(D(p) - K_2 - K_3) + \varphi_2(1-\varphi_3)(D(p) - K_2) + (1-\varphi_2)K_1\right], \\ p_m K_2 = p[\varphi_1\varphi_3(D(p) - K_1 - K_3) + \varphi_1(1-\varphi_3)(D(p) - K_1) + (1-\varphi_1)K_2], \\ p_m K_3 = p[\varphi_1(1-\varphi_2) + (1-\varphi_1)]K_3, \end{cases}$$

and the unique solution such that $0 \leqslant \phi_1^\circ(p) \leqslant 1$ is

$$\phi_1^\circ(p) = \frac{K_2}{K_1}\phi_2^\circ(p), \phi_2^\circ(p) = \sqrt{\frac{K_1}{K_2}\frac{p-p_m}{p}}, \phi_3^\circ(p) = \frac{D(p) - K_1 - K_2}{K_3} + \frac{K_2}{K_3}\phi_2^\circ(p).$$

Functions $\phi_1^\circ(p)$ and $\phi_2^\circ(p)$ are immediately recognized as increasing and concave; $\phi_3^\circ(p)$ is the sum of two concave functions. Clearly $\phi_2^\circ(p) \leqslant 1$ since $(K_1 - K_2)p \leqslant (D(p) - K_2 - K_3)p \leqslant p_m K_1$. Both weak inequalities are satisfied as equalities if and only if $p = P(K_1 + K_3) = p_M$; hence $\phi_2^\circ(\min\{p_M, P(K_1 + K_3)\}) < 1$ if and only if $P(K_1 + K_3) \neq p_M$. If $P(K_1 + K_3) < p_M$, then $0 < \phi_3^\circ(P(K_1 + K_3)) < 1$ if and only if $\frac{K_2-K_3}{K_2} < \phi_2^\circ(P(K_1+K_3)) < 1$. Hence we just need to prove that $\phi_2^\circ(P(K_1+K_3) \geqslant \frac{K_1}{K_2}\frac{P(K_1+K_3)-p_m}{P(K_1+K_3)} > \frac{K_1}{K_2}\frac{P(K_1+K_3)-P(K_1+K_2)}{P(K_1+K_3)} = \frac{K_1}{K_2}\left[1 - \frac{P(K_1+K_2)}{P(K_1+K_3)}\right] > \frac{K_1}{K_2}\left[1 - \frac{K_1-K_2}{K_1-K_3}\right] = \frac{K_1}{K_1-K_3}\frac{K_2-K_3}{K_2} > \frac{K_2-K_3}{K_2}$. The first inequality holds since $\phi_2^\circ(P(K_1+K_3) \leqslant 1$; the second inequality holds since $p_m < P(K_1+K_2)$; the third inequality holds since $p\,[D(p) - K_2 - K_3]$ is increasing in the interval $[P(K_1 + K_2), P(K_1 + K_3)]$. If $p_M < P(K_1 + K_3)$, $\phi_2^\circ(p_M) = \sqrt{\frac{K-D(p_M)}{K_2}} > \frac{K-D(p_M)}{K_2}$ and, therefore, $\phi_3^\circ(p_M) > 1$. Differentiation of $\phi_3^\circ(p)$ yields

$$\phi_3^{\circ\prime}(p) = \frac{D'(p)}{K_3} + \frac{K_2}{K_3}\phi_2^{\circ\prime}(p) = \frac{D'(p)}{K_3} + \frac{1}{2}\left(\frac{K_1}{K_2}\frac{p-p_m}{p}\right)^{-1/2}\frac{K_1}{K_3}\frac{p_m}{p^2}$$

that equals $\frac{p_m K_1}{K_3 p_M^2}\left[-1 + \frac{1}{2}\right] < 0$ if $p = P(K_1 + K_3) = p_M$. The last claim follows from strict concavity of $\phi_3^\circ(p)$.

(iii) System (14) reads

$$\begin{cases} p_m K_1 = p\left[\varphi_2\varphi_3(D(p) - K_2 - K_3) + \varphi_2\left(1-\varphi_3\right)\left(D(p) - K_2\right)\right. \\ \qquad\qquad \left. + \left(1-\varphi_2\right)\varphi_3(D(p) - K_3) + \left(1-\varphi_2\right)\left(1-\varphi_3\right)K_1\right] \\ p_m K_2 = p\left[\varphi_1\left(1-\varphi_3\right)\left(D(p) - K_1\right) + \left(1-\varphi_1\right)K_2\right] \\ p_m K_3 = p\left[\varphi_1\left(1-\varphi_2\right)\left(D(p) - K_1\right) + \left(1-\varphi_1\right)K_3\right], \end{cases}$$

and the unique solution such that $0 \leqslant \phi_1^\circ(p) \leqslant 1$ is

$$\phi_1^\circ(p) = \sqrt{\left\{\frac{p\left(K_1 + K_2 - D\left(p\right)\right)}{\left(p-p_m\right)K_2}\frac{p\left(K_1 + K_3 - D\left(p\right)\right)}{\left(p-p_m\right)K_3} + \frac{K_1}{K_2K_3}\frac{p\left(D\left(p\right) - K_1\right)}{\left(p-p_m\right)}\right\}^{-1}},$$

$$\phi_2^\circ(p) = 1 - \frac{K_3}{D(p) - K_1}\left[1 - \frac{p-p_m}{p\phi_1^\circ(p)}\right]$$

$$\phi_3^\circ(p) = 1 - \frac{K_2}{D(p) - K_1}\left[1 - \frac{p-p_m}{p\phi_1^\circ(p)}\right]$$

It is immediately checked that $\phi_1^{\circ\prime}(p) > 0$ if and only if

$$\frac{d}{dp}\left\{\frac{p\left(K_1+K_2-D\left(p\right)\right)}{\left(p-p_m\right)K_2}\frac{p\left(K_1+K_3-D\left(p\right)\right)}{\left(p-p_m\right)K_3}+\frac{K_1}{K_2K_3}\frac{p\left(D\left(p\right)-K_1\right)}{\left(p-p_m\right)}\right\}<0$$

that is

$$\begin{aligned}z(p) := &\tfrac{\xi(p)}{(p-p_m)^2K_2}\tfrac{p(K_1+K_3-D(p))}{(p-p_m)K_3}+\tfrac{\xi(p)+p_m(K_2-K_3)}{(p-p_m)^2K_3}\tfrac{p(K_1+K_2-D(p))}{(p-p_m)K_2}+\\ &\tfrac{K_1}{K_2K_3}\tfrac{-\xi(p)-p_mK_2}{(p-p_m)^2}<0\end{aligned} \tag{18}$$

where $\xi(p)=-\left(p-p_m\right)pD'\left(p\right)-p_m\left(K_1+K_2-D\left(p\right)\right)$. It is immediately recognized that $\xi(p)$ is an increasing function for $p\geqslant p_m$.

If $(K_1,K_2,K_3)\in B$, then $\xi(p_m)\geqslant 0$ and therefore $\xi(p)>0$ for each $p\geqslant P(K_1+K_3)>p_m$. Moreover

$$z(p)<\frac{\xi(\delta)}{(p-p_m)^2K_2}\frac{\delta\left(K_1+K_3-D\left(\delta\right)\right)}{\left(\delta-p_m\right)K_3}+$$

$$\frac{\xi(\delta)+p_m(K_2-K_3)}{(p-p_m)^2K_3}\frac{\delta\left(K_1+K_2-D\left(\delta\right)\right)}{\left(\delta-p_m\right)K_2}-\frac{K_1}{K_3}\frac{p_m}{\left(p-p_m\right)^2}\leqslant\frac{p_m}{\left(p-p_m\right)^2}A<0,$$

where $\delta=\min\left\{P(K_1),p_M\right\}$ and $A=\frac{\delta(D(\delta)-K_2)-p_mK_1}{K_2}\frac{K_1+K_3-D(\delta)}{(\delta-p_m)K_3}+\frac{\delta(D(\delta)-K_3)-p_mK_1}{K_3}\times\frac{K_1+K_2-D(\delta)}{(\delta-p_m)K_2}-\frac{K_1}{K_3}$. The first inequality holds since functions $\xi(p)$, $\frac{p(K_1+K_3-D(p))}{(p-p_m)K_3}$, and $\frac{p(K_1+K_2-D(p))}{(p-p_m)K_2}$ are positive and increasing. The second inequality holds since $-\delta^2D'\left(\delta\right)\leq p_mK_1$ and therefore $\delta\xi(\delta)\leqslant p_m\left[\delta(D(\delta)-K_2)-p_mK_1\right]$. The thierd inequality holds since if $\delta=P(K_1)$, then

$$A=\frac{P(K_1)\left[K_1-K_2-K_3\right]-p_mK_1-P(K_1)\left(K_2+K_3\right)}{\left(P(K_1)-p_m\right)K_2}<0$$

whereas if $\delta=p_M$, then $A=\frac{p_M[K_1-D(p_M)-(K_2-K_3)]}{(p_M-p_m)K_2}<0$.

If $(K_1,K_2,K_3)\in E$, then either $\xi(p_m)\geqslant 0$ or $\xi(p_m)<0$. In the former case $p_m\leqslant P(K_1+K_2)$ and the same argument applies. In the latter case $p_m>P(K_1+K_2)$ and there exists $\beta\in(p_m,p_M)$ such that $\xi(\beta)=0$, since $\xi(p_M)=p_mK_3>0$. The same argument applies once again in the range (possibly empty, if $P(K_1)<p_M$) $[\beta,\delta]$. In the range $[p_m,\min\{\beta,P(K_1)\}]$ inequality (18) holds since its LHS is a sum of negative functions.

It is immediately recognized that $\phi_3^\circ(p)\leqslant\phi_2^\circ(p)\leqslant 1$ since $\phi_1^\circ(p)\geqslant\frac{p-p_m}{p}$. If $\phi_3^\circ(p)>0$, then the same argument used in the proof of part (i) proves that $\phi_2^{\circ\prime}(p)$ and $\phi_3^{\circ\prime}(p)$ are positive since they have the same sign.

If $(K_1, K_2, K_3) \in B$, then $\phi_3^\circ(P(K_1+K_3)) > 0$ because of part (i) and therefore $\phi_3^\circ(p) > 0$, $\phi_2^{\circ\prime}(p) >$, and $\phi_3^{\circ\prime}(p) > 0$ in the whole interval. If $(K_1, K_2, K_3) \in E$, then

$$\phi_2^\circ(p) = \phi_3^\circ(p) = \frac{\sqrt{(K_1+K_2-D(p))^2p^2+(p-p_m)K_1p(D(p)-K_1)}-(K_1+K_2-D(p))p}{(D(p)-K_1)p}$$

which is clearly positive in the interior of the interval and nought for $p = p_m$.

It is easily calculated that $\phi_1^\circ(p) = \frac{p-p_m}{p}$ if and only if $[D(p)-K_1]$ $[(D(p)-K_2-K_3)p-p_mK_1] = 0$. Hence if $p_M > P(K_1)$, then $\phi_2^\circ(p)$ and $\phi_3^\circ(p)$ are not defined for $p = P(K_1)$ whereas if $p_M \leqslant P(K_1)$, then $\phi_2^\circ(p_M) = \phi_3^\circ(p_M) = 1$. L'Hôpital's Rule is enough to prove the last claim.

(iv) System (14) reads

$$\begin{cases} \Pi_1^* = p\left[D(p)-\varphi_2K_2-\varphi_3K_3\right] \\ p_mK_2 = p\left(1-\varphi_1\right)K_2 \\ p_mK_3 = p\left(1-\varphi_1\right)K_3, \end{cases}$$

Hence $\phi_1^\circ(p) = \frac{p-p_m}{p}$ and equation (16) is the unique equality constraint upon $\phi_2^\circ(p)$ and $\phi_3^\circ(p)$. Inequality constraints are obvious. ■

A few remarks on Proposition 5 are appropriate. In all analyzed cases function $\phi_1^\circ(p)$ is uniquely determined, continuous, increasing, lower than 1, and such that $\phi_1^\circ(p_m) = 0$. If $p_M \leqslant P(K_1)$, functions $\phi_2^\circ(p)$ and $\phi_3^\circ(p)$ are uniquely determined and continuous; however, $\phi_2^\circ(p)$ is always increasing whereas, in a well-defined subset of the region of the capacity space under concern, $\phi_3^\circ(p)$ is decreasing on a left neighbourhood of $P(K_1+K_3)$. If $p_m < P(K_1) < p_M$, then a continuous set of pairs of functions $\phi_2^\circ(p)$ and $\phi_3^\circ(p)$ exists and only in a part (of measure 0) of this set functions $\phi_2^\circ(p)$ and $\phi_3^\circ(p)$ are continuous in $P(K_1)$. In fact $\phi_i^\circ(P(K_1)-) = 1 - \frac{p_mK_1-P(K_1)(K_1-K_2-K_3)}{2P(K_1)K_i}$ $(i = 2, 3)$ (see Proposition 5(iii)) whereas $\phi_i^\circ(P(K_1))$ may be any number satisfying constraints (17) with $D(p) = K_1$ (see Proposition 5(iv)). Moreover, apart for the possible jump downward in $P(K_1)$, $\phi_2^\circ(p)$ is always increasing whereas, in addition to the possible jump down in $P(K_1)$, in a well-defined subset of the region of the capacity space under concern, $\phi_3^\circ(p)$ is decreasing on a left neighbourhood of $P(K_1+K_3)$.

4 Functions $\Phi_2(p)$, $\Phi_2(p)$, and $\Phi_3(p)$

As mentioned above we first find an equilibrium profile (or a continuum of equilibrium profiles). In some cases, for instance if $(K_1, K_2, K_3) \in A \cup D$, the functions $(\phi_1^\circ(p), \phi_2^\circ(p), \phi_3^\circ(p))$, introduced and studied in previous section, and the functions introduced and studied in Proposition 4, are sufficient to achieve the aim; but if $(K_1, K_2, K_3) \in B \cup E$ we may need some other functions. They will be introduced and studied in this section.

A nice property of all equilibria we will find is that $S_1 = [p_m, p_M]$. This means that $\phi_2(p)$ and $\phi_3(p)$ satisfy the equation $Z_1(p, \phi_2(p), \phi_3(p)) = \Pi_1^*$, which can also be stated, with obvious meaning of the symbols, either as $\phi_2(p) = H_2(p, \phi_3(p))$ or as $\phi_3(p) = H_3(p, \phi_3(p))$.[9] Note that function $H_i(p, \varphi_j)$ is continuous, almost everywhere differentiable, such that $H_i(p_m, 0) = 0$, $H_i(p_M, 1) = 1$, $\frac{\partial H_i}{\partial p} > 0$ whenever $0 < \varphi_j \leqslant 1$ and $0 < H_i(p, \varphi_j) \leqslant 1$, and $\frac{\partial H_i}{\partial \varphi_j} < 0$ each p ($i \neq 1$ and $j \neq 1, i$). Moreover, if $\phi_i(p)$ is constant in an open interval $I \subset (p_m^{(i)}, p_M^{(i)})$, then I is (part of) a gap of S_i and, we will see, this implies that $\phi_i(p) < \phi_i^\circ(p)$ and $\phi_j(p) > \phi_j^\circ(p)$ for $p \in I$. On the contrary if $\phi_i(p)$ is increasing in an interval I, then $\phi_i(p) \geqslant \phi_i^\circ(p)$ for $p \in I$ with $\phi_i(p) = \phi_i^\circ(p)$ and $\phi_j(p) = \phi_j^\circ(p)$ if $\phi_j(p)$ is also increasing.

In this section we will introduce the functions $\Phi_1(p)$, $\Phi_2(p)$, and $\Phi_3(p)$, defined in the range $[p_m, p_M]$. The last two are transformations of functions $\phi_2^\circ(p)$ and $\phi_3^\circ(p)$; they are defined in a few steps that involve other functions as follows:[10]

$$\Phi_{31}(p) = \min_{p \leqslant y \leqslant p_M} \phi_3^\circ(y),$$

$$\Phi_{21}(p) = H_2(p, \Phi_{31}(p)),$$

$$\Phi_{22}(p) = \min_{p \leqslant y \leqslant p_M} \Phi_{21}(y),$$

$$\Phi_{32}(p) = H_3(p, \Phi_{22}(p)),$$

$$\Phi_3(p) = \min_{p \leqslant y \leqslant p_M} \max\{\phi_3^\circ(y), \Phi_{32}(y)\},$$

$$\Phi_2(p) = H_2(p, \Phi_3(p)).$$

Obviously if (K_1, K_2, K_3) is such that a continuous set of pairs of functions $\phi_2^\circ(p)$ and $\phi_3^\circ(p)$ exists, then also a continuous set of pairs of functions $\Phi_2(p)$

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[9] It is easily calculated that if $p \in [p_m, \min\{P(K_1 + K_2), p_M\}]$, then $H_2(p, \varphi_3) := \frac{(p-p_m)K_1}{p[K-D(p)]\varphi_3}$; if $p \in [P(K_1 + K_2), \min\{P(K_1 + K_3), p_M\}]$, then $H_2(p, \varphi_3) := \frac{(p-p_m)K_1}{p[K_1+K_2-D(p)+K_3\varphi_3]}$; if $p \in [\max\{p_m, P(K_1 + K_3)\}, \min\{P(K_1), p_M\}]$, then $H_2(p, \varphi_3) := \frac{(p-p_m)K_1 - p[K_1+K_3-D(p)]\varphi_3}{p[K_1+K_2-D(p)]+p[D(p)-K_1]\varphi_3}$; if $p \in [\max\{p_m, P(K_1)\}, p_M]$, then $H_2(p, \varphi_3) := \frac{pD(p)-\Pi_1^*-pK_3\varphi_3}{pK_2}$. Similarly for $H_3(p, \varphi_2)$

[10] One might equivalently obtain $\Phi_2(p)$ and $\Phi_3(p)$ by analogous transformations starting from function $\phi_2^\circ(p)$.

and $\Phi_3(p)$ exists. Note that if $\phi_i^\circ(P(K_1)) < \phi_i^\circ(P(K_1)-)$, then $\Phi_i(p)$ is constant in a left neighbourhood of $P(K_1)$. Note also that if $\phi_2^\circ(p)$ and $\phi_3^\circ(p)$ are both increasing in the range $[p_m, p_M]$, then $\phi_3^\circ(p) = \Phi_{31}(p) = \Phi_{32}(p) = \Phi_3(p)$ and $\phi_2^\circ(p) = \Phi_{21}(p) = \Phi_{22}(p) = \Phi_2(p)$ throughout $[p_m, p_M]$. Moreover, if $\phi_2^\circ(p)$ is increasing in the range $[p_m, p_M]$, but $\phi_3^\circ(p)$ is not so in part of this range, then $\Phi_{31}(p) = \Phi_{32}(p) = \Phi_3(p)$ and $\Phi_{21}(p) = \Phi_{22}(p) = \Phi_2(p)$ throughout $[p_m, p_M]$ since, as we will see, $\Phi_{21}(p)$ is increasing wherever $\Phi_{31}(p)$ is constant. Similarly, if $\phi_3^\circ(p)$ is increasing in the range $[p_m, p_M]$, but $\phi_2^\circ(p)$ is not so in part of this range (i.e $\phi_2^\circ(p)$ jumps down at $P(K_1)$), then $\phi_3^\circ(p) = \Phi_{31}(p)$, $\Phi_{32}(p) = \Phi_3(p)$, and $\Phi_{21}(p) = \Phi_{22}(p) = \Phi_2(p)$ throughout $[p_m, p_M]$ since, as we will see, $\Phi_{32}(p)$ is increasing in the range in which $\Phi_{21}(p)$ is constant. Only when both $\phi_2^\circ(p)$ and $\phi_3^\circ(p)$ are not always increasing in the range $[p_m, p_M]$ we have that in part of this range $\Phi_{32}(p) \neq \Phi_3(p)$ and $\Phi_{22}(p) \neq \Phi_2(p)$ (see Example 1 in Section 6).

Finally $\Phi_1(p)$ is defined as the solution in φ_1 of the equations

$$[\max\{\Phi_2'(p-), \Phi_2'(p+)\}][p_m K_2 - Z_2(p, \varphi_1, \Phi_3(p))] = 0$$

$$[\max\{\Phi_3'(p-), \Phi_3'(p+)\}][p_m K_3 - Z_3(p, \varphi_1, \Phi_2(p))] = 0$$

Note that if $\max\{\Phi_2'(p-), \Phi_2'(p+)\}$ and $\max\{\Phi_3'(p-), \Phi_3'(p+)\}$ are both positive, then $\Phi_2(p) = \phi_2^\circ(p)$ and $\Phi_3(p) = \phi_3^\circ(p)$, the two equations are equivalent, and $\Phi_1(p) = \phi_1^\circ(p)$. The following proposition explores the properties of $\Phi_1(p)$, $\Phi_2(p)$, $\Phi_3(p)$ over a well-specified subset of the region of the capacity space under concern in this paper.

Proposition 6 *Let $(K_1, K_2, K_3) \in B \cup E$ and $K_1 + K_3 > D(p_M)$. Then functions $\Phi_2(p)$ and $\Phi_3(p)$ are (i) continuous, (ii) almost everywhere differentiable, and (iii) non decreasing; (iv) $\Phi_2(p_m) = \Phi_3(p_m) = 0$, (v) $\Phi_2(p_M) = \Phi_3(p_M) = 1$; (vi) $\Phi_i(p)$ is increasing over any range in which $\Phi_j(p)$ is constant ($i, j \in \{2, 3\}$ and $j \neq i$); (vii) if $\Phi_i'(p') = 0$, then $\Phi_i(p') \leqslant \phi_i^\circ(p')$ and $\Phi_j(p') \geqslant \phi_j^\circ(p')$, whereas (viii) if $\max\{\Phi_i'(p'-), \Phi_i'(p'+)\} > 0$, then $\Phi_i(p') \geqslant \phi_i^\circ(p')$; (ix) $\Phi_1(p) \geqslant \phi_1^\circ(p)$ is everywhere increasing.*

Proof $\Phi_{31}(p)$ is nondecreasing and never jumps down (by definition); it is almost everywhere differentiable; $\Phi_{31}(p_m) = 0$ and $\Phi_{31}(p_M) = 1$ since $K_1 + K_3 > D(p_M)$; hence $0 \leqslant \Phi_{31}(p) \leqslant 1$ for $p \in [p_m, p_M]$. Because of the properties of $H_2(\cdot, \cdot)$ $\Phi_{21}(p)$ is larger than $\phi_2^\circ(p)$ and increasing wherever $\Phi_{31}'(p) = 0$ and is equal to $\phi_2^\circ(p)$ elsewhere; it never jumps up, since $\Phi_{31}(p)$ never jumps down, and is almost everywhere differentiable; $\Phi_{21}(p_m) = 0$ and $\Phi_{21}(p_M) = 1$. $\Phi_{22}(p)$ is nondecreasing and continuous, by definition and since $\Phi_{21}(p)$ never jumps up; it is almost everywhere differentiable; $\Phi_{22}(p_m) = 0$ and $\Phi_{22}(p_M) = 1$;hence $0 \leqslant \Phi_{21}(p) \leqslant 1$ for $p \in [p_m, p_M]$. Because of the properties of $H_3(\cdot,\cdot)$ $\Phi_{32}(p)$ is larger than $\Phi_{31}(p)$ and increasing wherever $\Phi_{22}(p)$ is constant and is equal to $\Phi_{31}(p)$ elsewhere; it is continuous, almost everywhere differentiable, and nondecreasing; $\Phi_{32}(p_m) = 0$ and $\Phi_{32}(p_M) = 1$; hence $0 \leqslant \Phi_{32}(p) \leqslant 1$. Thus the properties (i)-(vi) claimed for $\Phi_2(p)$ and $\Phi_3(p)$ hold for functions $\Phi_{22}(p)$ and $\Phi_{32}(p)$. Moreover function $\Phi_{32}(p)$ satisfies property (vii) claimed for function $\Phi_3(p)$ and function $\Phi_{22}(p)$ satisfies property (viii) claimed for function $\Phi_2(p)$. Indeed function $\Phi_{31}(p)$ satisfies property (vii) and $\Phi'_{32}(p') = 0$ only if $\Phi_{32}(p') = \Phi_{31}(p')$ and $\Phi'_{31}(p') = 0$; $\Phi_{21}(p) \geqslant \phi^\circ_2(p)$ each p and $\Phi_{22}(p) = \Phi_{21}(p)$ wherever $\max\{\Phi'_{22}(p'-), \Phi'_{22}(p'+)\} > 0$. It is immediately checked that $\Phi_2(p)$ and $\Phi_3(p)$ satisfy properties (i)-(vi). Next we prove that function $\Phi_3(p)$ satisfies both properties (vii)-(viii). This is certainly so in any interval in which $\Phi_3(p) = \Phi_{32}(p)$: property (vii) was proved to hold in these intervals and property (viii) is a consequence of the fact that $\phi^\circ_3(p) \leqslant \Phi_{32}(p)$ in any of these intervals. In any interval in which $\Phi_3(p) \neq \Phi_{32}(p)$, and hence, $\Phi_3(p) > \Phi_{32}(p)$, $\phi^\circ_3(p) > \Phi_{32}(p)$ and therefore if $\max\{\Phi'_3(p'-), \Phi'_3(p'+)\} > 0$, then $\Phi_3(p) = \phi^\circ_3(p)$ and if $\Phi'_3(p) = 0$, $\Phi_3(p) < \phi^\circ_3(p)$. Similarly for function $\Phi_2(p)$. In any interval in which $\Phi_3(p) = \Phi_{32}(p)$, and therefore $\Phi_2(p) = \Phi_{22}(p)$, property (viii) was proved to hold and property (vii) is a consequence of the fact that $\phi^\circ_3(p) \leqslant \Phi_{32}(p)$, and therefore $\phi^\circ_2(p) \geqslant \Phi_{22}(p)$. In any interval in which $\Phi_3(p) > \Phi_{32}(p)$, and therefore $\Phi_2(p) < \Phi_{22}(p)$, if $\max\{\Phi'_3(p'-), \Phi'_3(p'+)\} > 0$, then $\Phi_3(p) = \phi^\circ_3(p)$ and therefore $\Phi_2(p) = \phi^\circ_2(p)$ whereas if $\Phi'_3(p) = 0$, $\Phi_2(p) > \phi^\circ_2(p)$. Claim (ix) is easily obtained from Proposition 5 and the following facts.

- If $p \in [p_m, P(K_1 + K_2)]$, then $\Phi_1(p) = \frac{K_i}{K_1}\Phi_i(p)$ whenever $\Phi_j(p)$ is constant ($\{i, j\} = N_{-1}$);
- if $p \in [P(K_1 + K_2), P(K_1 + K_3)]$, then $\Phi_1(p) = \frac{K_2}{K_1}\Phi_2(p)$ whenever $\Phi_3(p)$ is constant and $\Phi_1(p) = \frac{p - p_m}{p\Phi_2(p)}$ whenever $\Phi_2(p)$ is constant.
- If $p \in [\max\{p_m, P(K_1 + K_3)\}, P(K_1)]$ and $P(K_1) < p_M$,[11] it is easily calculated that in this range $\frac{d\Phi_i}{dp} = \frac{K_1}{K_i}\frac{d\Phi_1}{dp} + D'(p)\Phi_j(p)\frac{K_i - K_j(1-\Phi_j(p))}{[K_1 + K_i - D(p) + \Phi_j(p)(D(p) - K_1)]^2}$, whenever $\Phi_j(p)$ is constant ($i, j = 2, 3$ and $i \neq j$). Since $\Phi_i(p)$ is increasing in this range, then the required property holds if $K_i - K_j\,(1 - \Phi_j(p)) \geqslant 0$. This inequality obviously holds when $(K_1, K_2, K_3) \in$

[11]From Proposition 5(iii) we know that if $p_M \leqslant P(K_1)$, then $\phi^\circ_2(p)$ and $\phi^\circ_3(p)$ are both increasing in this range and so are also $\Phi_2(p)$ and $\Phi_3(p)$.

E and when $(K_1, K_2, K_3) \in B$ and $i = 2$. Then we analyze the case in which $(K_1, K_2, K_3) \in B$ and $i = 3$. Since $\Phi_2(p) \geqslant \min\{\phi_2^\circ(P(K_1 + K_3)), \phi_2^\circ(P(K_1))\}$ it is enough to prove that $\phi_2^\circ(P(K_1)) > \frac{K_2 - K_3}{K_2}$, that is $(P(K_1) - p_m)K_1 > P(K_1)K_2$ since $\phi_2^\circ(P(K_1)) \geqslant \frac{(P(K_1)-p_m)K_1 - P(K_1)K_3}{P(K_1)K_2}$ because of Proposition 5(iv). Indeed $\frac{p_m K_1}{K_1 - K_2} \leqslant \frac{P(K_1+K_2)K_1}{K_1-K_2} < \frac{K_1}{K_1-K_2}\frac{K_1-K_2-K_3}{K_1-K_3}P(K_1) < P(K_1)$. The first inequality holds since $p_m \leqslant P(K_1 + K_2)$ in set B; the second inequality is a consequence of the fact that $p_M > P(K_1)$ and therefore function $p[D(p) - K_2 - K_3]$ is increasing over $[P(K_1 + K_2), P(K_1)]$.

■

From the properties highlighted by Proposition 6, it should already be apparent that for $(K_1, K_2, K_3) \in B \cup E$ and $K_1 + K_3 > D(p_M)$ any triple $(\Phi_1(p), \Phi_2(p), \Phi_3(p))$ is actually an equilibrium of the price game.

5 Profiles of equilibrium strategies when $\Pi_3^* = p_m K_3$

We can now characterize equilibria when $\Pi_3^* = p_m K_3$. In order to accomplish this task we introduce the following partitions of sets B and E.

$B_1 = \{(K_1, K_2, K_3) \in B : D(p_M) \geqslant K_1 + K_2\}$
$B_2 = \{(K_1, K_2, K_3) \in B : K_1 + K_2 > D(p_M) \geqslant K_1 + K_3\}$
$B_3 = \{(K_1, K_2, K_3) \in B : K_1 + K_3 > D(p_M) \geqslant K_1, K_1K_2 \geqslant \kappa^2\}$
$B_4 = \{(K_1, K_2, K_3) \in B : K_1 + K_3 > D(p_M) \geqslant K_1, K_1K_2 < \kappa^2\}$
$B_5 = \{(K_1, K_2, K_3) \in B : K_1 > D(p_M), K_1K_2 \geqslant \kappa^2\}$
$B_6 = \{(K_1, K_2, K_3) \in B : K_1 > D(p_M), K_1K_2 < \kappa^2\}$
$E_1 = \{(K_1, K_2, K_3) \in E : D(p_M) \geqslant K_1\}$
$E_2 = \{(K_1, K_2, K_3) \in E : D(p_M) < K_1 < D\left(\frac{\widehat{p}K_1}{K_1-K_3}\right)\}$
$E_3 = \{(K_1, K_2, K_3) \in E : D(p_M) < D\left(\frac{\widehat{p}K_1}{K_1-K_3}\right) \leqslant K_1 < D(\widehat{p})\}$
where κ is defined by equation (15).

We will see that a unique equilibrium exists for capacity configurations in the set $B_1 \cup B_2 \cup B_3 \cup B_4 \cup E_1$. In the set $B_1 \cup B_2 \cup B_3 \cup E_1$, the supports of equilibrium strategies are connected whereas in the set B_4 $\phi_3'(p) = 0$ throughout a well-defined range $(\widetilde{p}, P(K_1 + K_3))$. A continuum of equilibria exists for any capacity configuration in the set $B_5 \cup B_6 \cup E_2 \cup E_3$. Actually, the degree of freedom in the determination of $\phi_2(p)$ and $\phi_3(p)$ for $p \geqslant P(K_1)$ in set $B_5 \cup B_6 \cup E_2 \cup E_3$ allows for equilibria in which either $\phi_2(p)$ or $\phi_3(p)$ is constant on a left neighbourhood of $P(K_1)$. In the set B_6, equilibria with such a gap also entail the gap referred above in connection with set B_4 (see Example 1 in next section).

Theorem 2 Let Assumptions 1, 2*, and 3 hold.[12]

(i) Let $(K_1, K_2, K_3) \in A$. Let $\phi_3^{\circ\circ}(p)$ be any non-decreasing function such that $\phi_2^{\circ\circ}(p) = \frac{pD(p)-\Pi_1^*-pK_3\phi_3^{\circ\circ}(p)}{pK_2}$ is non-decreasing,[13] $\phi_3^{\circ\circ}(p_m) = 0$, $\phi_3^{\circ\circ}(p_M) = 1$. Then $(\phi_1^{\circ}(p), \phi_2^{\circ\circ}(p), \phi_3^{\circ\circ}(p))$ is an equilibrium strategy profile; in such equilibrium $S_1 = [p_m, p_M] = S_2 \cup S_3$ and $\Pr(p_i = p) = 0$ for $p < p_M$ (each i).

(ii) Let $(K_1, K_2, K_3) \in B_1 \cup B_2$. There is an equilibrium in which $S_1 = S_2 = [p_m, p_M]$, $S_3 = [p_m, p_M^{(3)}]$ and $p_M^{(3)} < p_M$: hence $(\phi_1(p), \phi_2(p), \phi_3(p)) = (\phi_1^{\circ}(p), \phi_2^{\circ}(p), \phi_3^{\circ}(p))$ throughout S_3 and $(\phi_1(p), \phi_2(p), \phi_3(p)) = (\phi_{12}^{\star\star}(p), \phi_2^{\star\star}(p), 1)$ throughout $S_1 - S_3$.

(iii) Let $(K_1, K_2, K_3) \in B_3 \cup E_1$. There is an equilibrium in which $S_1 = S_2 = S_3 = [p_m, p_M]$: hence $(\phi_1(p), \phi_2(p), \phi_3(p)) = (\phi_1^{\circ}(p), \phi_2^{\circ}(p), \phi_3^{\circ}(p)) = (\Phi_1(p), \Phi_2(p), \Phi_3(p))$ throughout $[p_m, p_M]$. In E_1, $\phi_2(p) = \phi_3(p)$.

(iv) Let $(K_1, K_2, K_3) \in B_4$. There is an equilibrium in which $(\phi_1(p), \phi_2(p), \phi_3(p)) = (\Phi_1(p), \Phi_2(p), \Phi_3(p))$. Note that $S_1 = S_2 = [p_m, p_M]$, $S_3 = [p_m, \widetilde{p}] \cup [P(K_1 + K_3), p_M]$, and $(\phi_1(p), \phi_2(p), \phi_3(p)) = (\phi_1^{\circ}(p), \phi_2^{\circ}(p), \phi_3^{\circ}(p))$ throughout S_3.

(v) Let $(K_1, K_2, K_3) \in B_5 \cup B_6 \cup E_2 \cup E_3$. Let $\phi_3^{\circ\circ}(p)$ be any non-decreasing function defined in the interval $[P(K_1), p_M]$ such that $\phi_2^{\circ\circ}(p) = \frac{pD(p)-\Pi_1^*-pK_3\phi_3^{\circ\circ}(p)}{pK_2}$ is non-decreasing, $0 \leqslant \phi_3^{\circ\circ}(P(K_1)) \leqslant \frac{P(K_1)-p_m}{P(K_1)}\frac{K_1}{K_3}$, and $\phi_3^{\circ\circ}(p_M) = 1$. There is an equilibrium in which $(\phi_1(p), \phi_2(p), \phi_3(p)) = (\Phi_1(p), \Phi_2(p), \Phi_3(p))$, where $\Phi_2(p)$ and $\Phi_3(p)$ are built using $\phi_2^{\circ\circ}(p)$ and $\phi_3^{\circ\circ}(p)$ as $\phi_2^{\circ}(p)$ and $\phi_3^{\circ}(p)$ in the range $[P(K_1), p_M]$. Note that $S_1 = [p_m, p_M] = S_2 \cup S_3$ and $(\phi_1(p), \phi_2(p), \phi_3(p)) = (\phi_1^{\circ}(p), \phi_2^{\circ\circ}(p), \phi_3^{\circ\circ}(p))$ in $[P(K_1), p_M]$.[14]

(vi) Let $(K_1, K_2, K_3) \in D$. (vi.a) Let $\phi_3^{\circ\circ}(p)$ be any non-decreasing function defined in interval $[P(K_1), p_M]$ such that $\phi_2^{\circ\circ}(p) = \frac{pD(p)-\Pi_1^*-pK_3\phi_3^{\circ\circ}(p)}{pK_2}$ is non-decreasing, $\phi_3^{\circ\circ}(P(K_1)) = \phi_3^{\star}(P(K_1))$, and $\phi_3^{\circ\circ}(p_M) = 1$. Further let $\phi_1^{\circ\circ}(p) = \phi_{13}^{\star}(p)$, $\phi_2^{\circ\circ}(p) = 0$, and $\phi_3^{\circ\circ}(p) = \phi_3^{\star}(p)$ over $[p_m, P(K_1)]$ and $\phi_1^{\circ\circ}(p) = \phi_1^{\circ}(p) = \frac{p-p_m}{p}$ over $[P(K_1), p_M]$. Then $(\phi_1^{\circ\circ}(p), \phi_2^{\circ\circ}(p), \phi_3^{\circ\circ}(p))$ is

[12] Note that Assumption 2* is required only when Proposition 5(ii) is involved and could be substituted by Assumption 2 otherwise.

[13] This condition is equivalent to $0 \leqslant \phi_3^{\circ\circ\prime}(p) \leqslant \frac{\Pi_1^*+D'(p)p^2}{K_3p^2}$ wherever $\phi_3^{\circ\circ}(p)$ is differentiable

[14] Note that in $B_5 \cup E_2 \cup E_3$ if $\phi_i^{\circ\circ}(P(K_1)) < \phi_i^{\circ}(P(K_1)-)$ (some $i \notin 1$), there is $\widetilde{p} \in [p_m, P(K_1)]$ such that $[p_m, P(K_1)] \cap S_i = [p_m, \widetilde{p}]$ and $[p_m, P(K_1)] \subseteq S_j \cap S_1$. Clearly, in $E_2 \cup E_3$, $\phi_2(p) = \phi_3(p)$ over $[p_m, \widetilde{p}]$. Obviously $\widetilde{p} = p_m$ if $\phi_i^{\circ\circ}(P(K_1)) = 0$ (see Example 2 in next section). Note also that in B_6 there are further varieties of equilibria (see Example 1 in next section).

an equilibrium strategy profile: $S_1 = [p_m, p_M] = S_2 \cup S_3$, $[p_m, P(K_1)] \subseteq S_3$, $[p_m, P(K_1)) \cap S_2 = \emptyset$. (vi.b) In the special case in which $D\left(\frac{\widehat{p}K_1}{K_1-K_3}\right) = K_1$ function $\phi_3^{\circ\circ}(p)$ is uniquely determined, $S_3 = [p_m, P(K_1)]$ and $S_2 = [P(K_1), p_M]$.

Proof

(i) It is immediately calculated that $\phi_2^{\circ\circ}(p_m) = 0$ and $\phi_2^{\circ\circ}(p_M) = 1$. Hence $(\phi_1^{\circ}(p), \phi_2^{\circ\circ}(p), \phi_3^{\circ\circ}(p))$ is a strategy profile and it is an equilibrium strategy profile since it satisfies system (14) because of Proposition 5(iv). If $\phi_3^{\circ\circ}(p)$ jumps upward, then $\phi_2^{\circ\circ}(p)$ jumps downward and, therefore, is not non-decreasing. Hence functions $\phi_2^{\circ\circ}(p)$ and $\phi_3^{\circ\circ}(p)$ are continuous throughout $[p_m, p_M]$.

(ii) Let $(K_1, K_2, K_3) \in B_1$. Because of Proposition 5(i) $\phi_1^{\circ}(p)$, $\phi_2^{\circ}(p)$, $\phi_3^{\circ}(p)$ are increasing all over $[p_m, p_M]$ and there is a single $p' < p_M$ such that $\phi_3^{\circ}(p') = 1$; note that $\phi_1^{\circ}(p') = \frac{K_3}{K_1} < 1$, $\phi_2^{\circ}(p') = \frac{K_3}{K_2} < 1$. Hence there is an equilibrium in which $p_M^{(3)} = p'$ and $\phi_1(p) = \phi_{12}^{\star\star}(p)$ and $\phi_2(p) = \phi_2^{\star\star}(p)$ in the interval $\left[p_M^{(3)}, p_M\right]$.

Let $(K_1, K_2, K_3) \in B_2$. Because of Proposition 5(i) $\phi_1^{\circ}(p)$, $\phi_2^{\circ}(p)$, $\phi_3^{\circ}(p)$ are increasing all over $[p_m, P(K_1+K_2)]$. If $\phi_3^{\circ}(P(K_1+K_2)) = \frac{K_1}{K_3}\sqrt{\frac{K_2}{K_1}\frac{P(K_1+K_2)-p_m}{P(K_1+K_2)}} > 1$,[15] then there exists a single $p' \in (p_m, P(K_1 + K))$ such that $\phi_3^{\circ}(p') = 1$. If $\phi_3^{\circ}(P(K_1 + K_2)) \leqslant 1$, then because of Proposition 5(ii) there exists a single $p' \in [P(K_1 + K_3), p_M)$ such that $\phi_3^{\circ}(p') = 1$ and $\phi_3^{\circ\prime}(p') > 0$. Note that $\phi_1^{\circ}(p') = \frac{K-D(p')}{K_1} < \frac{K_2}{K_1}$ and $\phi_2^{\circ}(p') = \frac{K-D(p')}{K_2} < \frac{K_2}{K_2}$. Thus, in either case, there exists an equilibrium in which $S_3 = [p_m, p_M^{(3)}]$ with $p_M^{(3)} = p'$, and in which $S_1 = S_2 = [p_m, p_M]$, with $(\phi_1(p), \phi_2(p)) = (\phi_{12}^{\star\star}(p), \phi_2^{\star\star}(p))$ in the interval $\left[p_M^{(3)}, p_M\right]$.

(iii) From Proposition 5(i)-(iii).

(iv)-(v) We have just to prove that wherever $\Phi_{-j}(p') > \phi_{-j}^{\circ}(p')$ it does not pay for firm j to charge price p'. Indeed $Z_j(p', \Phi_{-j}(p')) < Z_j(p', \phi_{-j}^{\circ}(p')) = \Pi_j^*$ because of Lemma 1(v).[16]

(vi.a) Clearly $(\phi_1^{\circ\circ}(p), \phi_2^{\circ\circ}(p), \phi_3^{\circ\circ}(p))$ is a strategy profile either by con-

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[15]Note that $\phi_1^{\circ}(P(K_1 + K_2)) = \sqrt{\frac{K_2}{K_1}\frac{P(K_1+K_2)-p_m}{P(K_1+K_2)}} < \phi_2^{\circ}(P(K_1 + K_2)) = \sqrt{\frac{K_1}{K_2}\frac{P(K_1+K_2)-p_m}{P(K_1+K_2)}} < 1$. To prove the last inequality it is enough to remark that $(K_1 - K_2)P(K_1 + K_2) < (K_1 - K_3)P(K_1 + K_2) < p_m K_1$.

[16]Note that if $(K_1, K_2, K_3) \in B_5 \cup B_6 \cup E_2$, then $\phi_3^{\circ\circ}(P(K_1)) > 0$. Indeed $\phi_2^{\circ\circ}(P(K_1)) \leqslant \phi_2^{\circ\circ}(p_M) = 1$ and therefore $\phi_3^{\circ\circ}(P(K_1))P(K_1)K_3 \geqslant P(K_1)(K_1 - K_2) - \Pi_1^*$. Hence we are done since $P(K_1) > \frac{\Pi_1^*}{K_1-K_2}$. This inequality holds in E_2 by definition and was proved to hold in $B_5 \cup B_6$ in the proof of Proposition 6(ix). Conversely, if $(K_1, K_2, K_3) \in E_3$,

struction ($p \geqslant P(K_1)$) or by Proposition 4(i). It is an equilibrium, actually, since $\Pi_i(p) = Z_i(p; \phi^{\circ\circ}_{-i}(p)) = \Pi^*_i$ at any $p \in S_i$ (each i) and $\Pi_2(p) < \Pi^*_2$ over the range $(p_m, P(K_1))$ since over this range $\Pi_2(p) = p\phi^\star_{13}(p)(1 - \phi^\star_3(p))(D(p) - K_1) + p(1 - \phi^\star_{13}(p))K_2 < (K_2/K_3)[p\phi^\star_{13}(p)(D(p) - K_1) + p(1 - \phi^\star_{13}(p))K_3] = (K_2/K_3)Z_3(p; \phi_{-3}(p)) = p_m K_2$. The inequality holds since $K_3(1 - \phi^\star_3(p)) - K_2 < 0$.

(vi.b) Since $\phi^\star_3(P(K_1)) = 1$ only the constant function $\phi^{\circ\circ}_3(p) = 1$ meets the requirements. ■

Theorem 3 Let Assumptions 1, 2*, and 3 hold. Apart from the continuum of equilibria due to the arbitrary choice of function $\phi^{\circ\circ}_3(p)$ in interval $[P(K_1), p_M]$ in parts (i), (v), and (vi) of Theorem 2, no other equilibrium exists.

Before proving Theorem 3 we will prove that no atom may exist in the range $[p_m, p_M)$ in the subsets of the capacity space studied in this paper.

Proposition 7 *Let Assumptions 1, 2, and 3 and inequality* ($6'$) *hold and let* (ϕ_1, ϕ_2, ϕ_3) *be an equilibrium in which* $\phi_j(\widetilde{p}) < \phi_j(\widetilde{p}+)$ *for some* j *and some* $\widetilde{p} \in (p_m, p_M)$*. Then*

(i) $\Pi^*_j = \Pi_j(\widetilde{p}) = Z_j(\widetilde{p}; \phi_{-j}(\widetilde{p}))$*;*

(ii) there is $p^\circ > \widetilde{p}$ *such that* $(S_1 \cup S_2 \cup S_3) \cap (\widetilde{p}, p^\circ) = \emptyset$ *and* $p^\circ \in S_1 \cup S_2 \cup S_3$*;*

(iii) $\lim_{p \to \widetilde{p}+} \partial Z_j(p, \phi_{-j}(\widetilde{p}))/\partial p \leq 0$*;*

(iv) $K_j < K_1$*;*

(v) $\widetilde{p} < P(K_1)$*;*

(vi) $\widetilde{p} \geqslant P(K_1 + K_3)$*;*

(vii) if $p^{(j)}_M \leqslant P(K_1)$*, then* $\widetilde{p} = p^{(j)}_M$*;*

(viii) there is no other equilibrium $(\widehat{\phi}_1, \widehat{\phi}_2, \widehat{\phi}_3)$ *such that* $\widehat{\Pi}^*_j = \Pi^*_j$*,* $(p^\circ, p^{\circ\circ}) \subset \widehat{S}_j$*, with* $p^\circ < \widetilde{p} < p^{\circ\circ}$*, and* $\widehat{\phi}_i(p) = \phi_i(p)$ *(each* i*) for* $p \in (p^\circ, \widetilde{p})$*;*

(ix) $K_j < K_2$*;*

(x) $(K_1, K_2, K_3) \in C_1 \cup C_2 \cup C_3$*.*

Proof

(i)-(v) See [10]; Lemma 2.

(vi) Assume first that $\widetilde{p} < P(K_1 + K_2)$, then $\Pi^*_i = p_m K_i$ (each i) because of Theorem 1(b). Hence $p_m K_1 \geqslant \widetilde{p}[\phi_i(\widetilde{p})\phi_j(\widetilde{p})(D(\widetilde{p}) - K) + K_1]$ and $p_m K_j = \widetilde{p}[\phi_1(\widetilde{p})\phi_i(\widetilde{p})(D(\widetilde{p}) - K) + K_j]$ $(i \neq 1, j)$; the equality being a consequence of parts (i) and (iv). As a consequence, $\phi_1(\widetilde{p})K_1 \leqslant \phi_j(\widetilde{p})K_j$. On the other hand, taking account of Lemma 1(ii) and part (iii) we obtain $\phi_1(\widetilde{p})K_1 >$

$\phi^{\circ\circ}_2(P(K_1)) \leqslant 1$ and $\phi^{\circ\circ}_3(P(K_1)) \leqslant 1$ since now $P(K_1) \leqslant \frac{\Pi^*_1}{K_1 - K_2}$ by definition.

$\phi_j(\widetilde{p})K_j$. The contradiction implies that $\widetilde{p} \geqslant P(K_1+K_2)$. Assume now that $P(K_1+K_2) \leqslant \widetilde{p} < P(K_1+K_3)$, then the same argument implies that $j \neq 2$. But $j \neq 3$ too since in this range $Z_3(p,\varphi_{-3}) = p[\varphi_1(1-\varphi_2)+(1-\varphi_1)]K_3$.

(vii) It is enough to remark that $\Pi_j^* = Z_j(\widetilde{p},\phi_{-j}(\widetilde{p})) > Z_j(p,\phi_{-j}(\widetilde{p})) \geqslant Z_j(p,\phi_{-j}(p))$, each $p \in (\widetilde{p}, P(K_1)]$. The equality is established in part (i), the strict inequality is a consequence of parts (iii) and (vi) and Lemma 1(iii), the weak inequality is a consequence of Lemma 1(v). The same argument proves also that if $p_M^{(j)} > P(K_1)$, then there is $p^\circ > P(K_1)$ such that $(\widetilde{p},p^\circ) \cap S_j = \emptyset$ and $P(K_1) \notin S_1 \cup S_2 \cup S_3$.

(viii) Otherwise the following contradiction is obtained (see the proof of part (vii)): $\Pi_j^* = Z_j(\widetilde{p},\phi_{-j}(\widetilde{p})) = Z_j(\widetilde{p},\widehat{\phi}_{-j}(\widetilde{p})) > Z_j(p,\widehat{\phi}_{-j}(\widetilde{p})) \geqslant Z_j(p,\widehat{\phi}_{-j}(p)) = \widehat{\Pi}_j^*$ for $p \in (\widetilde{p}, \min\{p^{\circ\circ}, P(K_1)\}$.

(ix) We will find a contradiction when $\Pi_j^* = p_m K_j$ and $K_j - (1-\phi_i(\widetilde{p}))K_i > 0$: both conditions hold when $K_j = K_2$. Taking into account parts (v) and (vi) and arguing as in the proof of part (vi) we obtain that $(\phi_1(\widetilde{p})K_1 - \phi_i(\widetilde{p})K_i - \phi_j(\widetilde{p})K_j)$ $[(1-\phi_i(\widetilde{p}))(D(\widetilde{p})-K_1)-K_j] \geqslant \phi_i(\widetilde{p})[K_j-(1-\phi_i(\widetilde{p}))K_i](D(\widetilde{p})-K_1)$ since $Z_j(p,\phi_{-j}(\widetilde{p})) = p_m K_j$ and $p_m K_1 \geqslant Z_1(p,\phi_{-1}(\widetilde{p}))$. Hence $\phi_1(\widetilde{p})K_1 - \phi_3(\widetilde{p})K_3 - \phi_2(\widetilde{p})K_2 < 0$. On the other hand, taking account of Lemma 1(ii) and part (iii) we obtain $(1-\phi_i(\widetilde{p}))[\phi_1(\widetilde{p})K_1 - \phi_j(\widetilde{p})K_j] > \phi_i(\widetilde{p})K_j - \phi_1(\widetilde{p})\phi_i(\widetilde{p})[K_j-(1-\phi_i(\widetilde{p}))K_i]$, and therefore $(1-\phi_i)[\phi_1 K_1 - \phi_j K_j - \phi_i K_i] > \phi_i(1-\varphi_1)[K_j-(1-\phi_i)K_i]$.

(x) If $(K_1,K_2,K_3) \in A \cup B_1 \cup B_2$, then no atom exists because of parts (v) and (vi). If $(K_1,K_2,K_3) \in D$, then no atom exists because of parts (v) and (viii) and Theorems 1(d) and 2(vi). If $(K_1,K_2,K_3) \in E$, then no atom exists because of part (ix). If $(K_1,K_2,K_3) \in B_3 \cup B_4$, then no atom exists because of parts (v), (vi), and (vii) since otherwise $Z_j(p,\phi_{-j}(p)) = p[\phi_1(p)(1-\phi_i(p))(D(p)-K_1)+(1-\phi_1(p))K_j] > \frac{K_j}{K_i}p(1-\phi_1(p))K_i = \frac{K_j}{K_i}Z_i(p,\phi_{-i}(p)) = p_m K_j$ in interval $[\widetilde{\widetilde{p}}, p_M]$, where $\widetilde{\widetilde{p}} = \min\{p > \widetilde{p}, p \in S_1 \cup S_i\}$. If $(K_1,K_2,K_3) \in B_5 \cup B_6$, we can follow the proof of part (ix) since $\Pi_3^* = p_m K_3$ and $K_3 - (1-\phi_2(\widetilde{p}))K_2 > 0$. The last inequality holds in any of the equilibria found in Theorem 2 (it is enough to explore the proof of Proposition 6(ix)). In order to prove that such inequality holds in any equilibrium we first remind that if $\phi_2(\widetilde{p}) < \phi_2(\widetilde{p}+)$ for some $\widetilde{p} \in [P(K_1+K_3), P(K_1))$, then $P(K_1) \notin S_1 \cup S_2 \cup S_3$ (see the proof of part vii). Then we remark that if $K_3 - (1-\phi_2(\widetilde{p}))K_2 \leqslant 0$, the following contradiction is obtained: $Z_1(P(K_1),\phi_{-1}(P(K_1))) = P(K_1)[K_1 - \phi_2(P(K_1))K_2 - \phi_3(P(K_1))K_3 \geqslant P(K_1)[K_1 - K_2 + (1-\phi_3)K_3] > P(K_1)[K_1-K_2] > p_m K_1$. The last inequality holds since $(P(K_1)-p_m)K_1 > P(K_1)K_2$ (see the proof of Proposition 6(ix)).

■

Proof of Theorem 3

Let $(\phi_1(p), \phi_2(p), \phi_3(p))$ be any equilibrium profile characterized by Theorem 2 and $(\widehat{\phi}_1(p), \widehat{\phi}_2(p), \widehat{\phi}_3(p))$ be any equilibrium profile not characterized by Theorem 2. Further, let

$$\rho = \inf\{p : (\phi_1(p), \phi_2(p), \phi_3(p)) \neq (\widehat{\phi}_1(p), \widehat{\phi}_2(p), \widehat{\phi}_3(p))\}. \tag{19}$$

we will prove that it cannot be that

1. $\rho \geqslant P(K_1)$,
2. $\rho < P(K_1)$ is internal to $S_2 \cap S_3$,
3. $\rho < P(K_1)$ is internal to $[p_m, p_M] - (S_2 \cap S_3)$,
4. $\rho = \widetilde{p}$, where $(\widetilde{p}, \widetilde{\widetilde{p}})$ is a gap of S_3 and $\widetilde{\widetilde{p}} \leqslant P(K_1 + K_3)$.
5. $\rho = \widetilde{p}$, where $(\widetilde{p}, \widetilde{\widetilde{p}})$ is a gap of S_j and $\widetilde{\widetilde{p}} \geqslant P(K_1)$.

If $\rho \geqslant P(K_1)$ and $(\widehat{\phi}_1(p), \widehat{\phi}_2(p), \widehat{\phi}_3(p))$ is not any of the equilibria characterized by Theorem 2(i)&(v)&(vi), then either $\widehat{\phi}_1(p^\circ) \neq \frac{p^\circ - p_m}{p^\circ}$ or $\widehat{\phi}_2(p^\circ) \neq \frac{p^\circ D(p^\circ) - \Pi_1^* - p^\circ K_3 \widehat{\phi}_3(p^\circ)}{p^\circ K_2}$ (some $p^\circ \in [\rho, p_M]$). In the former case there is a neighbourhood I of p° such that $I \cap (S_2 \cup S_3) = \emptyset$ and therefore either Proposition 7(x) or Proposition 3(iv) is violated. In the latter $p^\circ \notin S_1$, then $\Pi_i(p)$ is increasing (each $i \neq 1$) in a neighbourhood I of p° and once again $I \cap (S_2 \cup S_3) = \emptyset$.

If $\rho < P(K_1)$ is internal to $S_2 \cap S_3$, then there is an open interval (ρ, ρ') such that $(\rho, \rho') \subset S_j$ and $(\rho, \rho') \cap \widehat{S}_j = \emptyset$ (some $j \in \{1, 2, 3\}$). Because of Propositions 3(iv) and 7(x) $(\rho, \rho') \subset \cap_{i \neq j} \widehat{S}_i$. Let $\rho^\circ = \min\{p \geqslant \rho' : p \in \widehat{S}_j\}$. Then $\widehat{\phi}_j(\rho^\circ) = \widehat{\phi}_j(\rho) = \phi_j^\circ(\rho) < \phi_j^\circ(\rho^\circ)$ and there is $i \neq j$ such that $\widehat{\phi}_i(\rho^\circ) \leqslant \phi_i^\circ(\rho^\circ)$ since $Z_j(\rho^\circ, \widehat{\phi}_{-j}(\rho^\circ)) = Z_j(\rho^\circ, \phi_{-j}^\circ(\rho^\circ))$. As a consequence of Lemma 1(v) $Z_h(\rho^\circ, \widehat{\phi}_{-h}(\rho^\circ)) > Z_h(\rho^\circ, \phi_{-h}^\circ(\rho^\circ)) = \Pi_h^*$ $(h \neq j, i)$: an obvious contradiction.

If $\rho < P(K_1)$ is internal to $[p_m, p_M] - (S_2 \cap S_3)$, then either Proposition 3(iv) or Proposition 7(x) is contradicted.

If $\rho = \widetilde{p}$, where $(\widetilde{p}, \widetilde{\widetilde{p}})$ is a gap of S_3 and $\widetilde{\widetilde{p}} \leqslant P(K_1 + K_3)$, then $\widehat{\phi}_3(p) > \phi_3(p)$ in the range $(\widetilde{p}, \widetilde{\widetilde{p}})$ and therefore $\widehat{\phi}_3(\widetilde{\widetilde{p}}) > \phi_3^\circ(\widetilde{\widetilde{p}}) = \phi_3^\circ(\widetilde{p})$. Clearly $\widetilde{\widetilde{p}} \notin \widehat{S}_1 \cap \widehat{S}_2 \cap \widehat{S}_3$, otherwise $\widehat{\phi}_3(\widetilde{\widetilde{p}}) = \phi_3^\circ(\widetilde{\widetilde{p}})$. But $\widetilde{\widetilde{p}} \in \widehat{S}_1 \cup \widehat{S}_2 \cup \widehat{S}_3$ otherwise Proposition 7(x) is contradicted. Then $\widetilde{\widetilde{p}} \in \widehat{S}_3$ otherwise $Z_s(\widetilde{\widetilde{p}}, \widehat{\phi}_{-s}(\widetilde{\widetilde{p}})) = Z_s(\widetilde{\widetilde{p}}, \phi_{-s}^\circ(\widetilde{\widetilde{p}})) = p_m K_s$ (each $s \in N_{-3}$) and, because of Lemma 1(v), $\widehat{\phi}_r(\widetilde{\widetilde{p}}) <$

$\phi_r^\circ(\widetilde{\widetilde{p}})$ (each $r \neq s, 3$), implying $Z_3(\widetilde{\widetilde{p}}, \widehat{\phi}_{-3}(\widetilde{\widetilde{p}})) > p_m K_3$, an obvious contradiction. As a consequence $\widetilde{\widetilde{p}} \notin \widehat{S}_j$ (some $j \neq 3$) and $\widehat{\phi}_j(\widetilde{\widetilde{p}}) < \phi_j^\circ(\widetilde{\widetilde{p}})$. Then, arguing as for the case in which $\rho < P(K_1)$ is internal to $S_2 \cap S_3$, we obtain that $Z_h(\rho'', \widehat{\phi}_{-h}(\rho'')) > \Pi_h^*$, where $\rho'' = \min\{p \geqslant P(K_1 + K_3)) : p \in \widehat{S}_j\}$, some $h \neq j$.

If $\rho = \widetilde{p}$, where $(\widetilde{p}, \widetilde{\widetilde{p}})$ is a gap of S_j and $\widetilde{\widetilde{p}} \geqslant P(K_1)$, then $\widehat{\phi}_j(p) > \phi_j(p)$ in the range $(\widetilde{p}, \widetilde{\widetilde{p}})$ and therefore $\widehat{\phi}_j(P(K_1)) > \phi_j(P(K_1)) = \phi_j(\widetilde{p})$. In order not to be $(\widehat{\phi}_1(p), \widehat{\phi}_2(p), \widehat{\phi}_3(p))$ any of the equilibria characterized by Theorem 2(v), either $\widehat{\phi}_1(P(K_1)) \neq \frac{P(K_1)-p_m}{P(K_1)}$ or $\widehat{\phi}_2(P(K_1)) \neq \frac{P(K_1)K_1 - \Pi_1^* - P(K_1)K_3\widehat{\phi}_3(P(K_1))}{P(K_1)K_2}$: then the argument developed above applies. ■

Corollary 1 Let Assumptions 1, 2*, and 3 hold and let $(K_1, K_2, K_3) \in A \cup B \cup D \cup E$. Then $S_1 = S_2 \cup S_3 = [p_m, p_M]$.

6 Examples

We provide two numerical examples in order to illustrate Theorem 2(v). Example 1 characterizes the several varieties of equilibria for a point in the capacity space belonging to subset B_6. In one of these varieties $\Phi_3(p) > \Phi_{32}(p)$ and $\Phi_2(p) < \Phi_{22}(p)$ over a range. Example 2 characterizes equilibria for a point in the capacity space belonging to subset E_3; apart from the symmetric equilibrium there are basically two varieties of equilibria: equilibria in which $L = \{1, 2, 3\}$ and there is a gap in S_i (some $i \neq 1$) and equilibria in which $p_m^{(i)} \geq P(K_1)$ (some $i \neq 1$) and there is no gap.

Example 1. Let $D(p) = 14 - p$, and $(K_1, K_2, K_3) = (7.5, 0.2, 0.1)$. Then, $p_M = \frac{137}{20}$, $\Pi_1^* = \frac{18769}{400}$, $p_m = \frac{18769}{3000}$, $\Pi_2^* = \frac{18769}{15000}$, and $\Pi_3^* = \frac{18769}{30000}$. Note that $p_m = 6.256\bar{3} < 6.3 = P(K_1 + K_2)$ and $P(K_1) = 6.5 < p_M = 6.85$. According to Proposition 8 $\phi_1^\circ(p)$ and $\phi_2^\circ(p)$ are uniquely determined and increasing in the range $[p_m, P(K_1))$ whereas $\phi_3^\circ(p)$ is also uniquely determined in the range $[p_m, P(K_1))$, but is increasing for $p_m \leqslant p < p^\circ \approx 6.314627855$ and $P(K_1 + K_3) < p < P(K_1)$ and is decreasing for $p^\circ < p < P(K_1 + K_3)$. Moreover $\phi_3^\circ(p^\circ) \approx 1.030475263$ and $\phi_3^\circ(P(K_1 + K_3)) = \sqrt{\frac{431}{128}} - 1$. Let real number $h \in [\frac{211}{260}, 1]$ and function $\phi_3^{\circ\circ}(p)$ be any non-decreasing function defined in the range $[P(K_1), p_M]$ whose derivative is not higher than $2\frac{p_m K_1 - p^2}{p^2 K_2}$, $\phi_3^{\circ\circ}(P(K_1)) = h$, and $\phi_3^{\circ\circ}(p_M) = 1$; then the following profile of strategies is an equilibrium.[17]

[17]Note that $\phi_2^{\circ\circ}(P(K_1)) = 1$ for $h = \frac{211}{260}$, $\sqrt{\frac{431}{128}} - 1 = \phi_3^\circ(P(K_1 + K_3))$, $\frac{471}{520} =$

- if $h \in \left[\frac{211}{260}, \sqrt{\frac{431}{128}} - 1\right]$, then $\phi_i(p) = \phi_i^\circ(p)$ (each i) in the range $[p_m, \widetilde{p}]$ where $\widetilde{p} < P(K_1 + K_3)$ is such that $\phi_3^\circ(\widetilde{p}) = h$ whereas $\phi_3(p) = h$ and $\phi_1(p)$ and $\phi_2(p)$ are calculated accordingly (they are the solutions to the equations in $\varphi_j \ \Pi_i^* = Z_i(p; \varphi_j, h) \quad (\{i, j\} = \{1, 2\})$), in the range $[\widetilde{p}, P(K_1)]$.

- if $h \in \left(\sqrt{\frac{431}{128}} - 1, \frac{471}{520}\right)$, then $\phi_i(p) = \phi_i^\circ(p)$ (each i) in the ranges $[p_m, \widetilde{p}]$ and $[P(K_1+K_3), \widetilde{\widetilde{p}}]$ where $\widetilde{p} < P(K_1+K_3)$ is such that $\phi_3^\circ(\widetilde{p}) = \phi_3^\circ(P(K_1 + K_3))$ and $\widetilde{\widetilde{p}} > P(K_1 + K_3)$ is such that $\phi_3^\circ(\widetilde{\widetilde{p}}) = h$; $\phi_3(p) = \phi_3^\circ(P(K_1 + K_3))$ in the range $[\widetilde{p}, P(K_1 + K_3)]$ and $\phi_3(p) = h$ in the range $[\widetilde{\widetilde{p}}, P(K_1)]$; in both these ranges $\phi_1(p)$ and $\phi_2(p)$ are calculated accordingly.

- if $h \in \left[\frac{471}{520}, \frac{731}{260} - \sqrt{\frac{431}{128}}\right]$, then $\phi_i(p) = \phi_i^\circ(p)$ (each i) in the ranges $[p_m, \widetilde{p}]$ and $[P(K_1+K_3), \widetilde{\widetilde{p}}]$ where $\widetilde{p} < P(K_1+K_3)$ is such that $\phi_3^\circ(\widetilde{p}) = \phi_3^\circ(P(K_1 + K_3))$ and $\widetilde{\widetilde{p}} > P(K_1 + K_3)$ is such that $\phi_2^\circ(\widetilde{\widetilde{p}}) = \frac{731}{520} - \frac{1}{2}h$; $\phi_3(p) = \phi_3^\circ(P(K_1+K_3))$ in the range $[\widetilde{p}, P(K_1+K_3)]$ and $\phi_2(p) = \frac{731}{520} - \frac{1}{2}h$ in the range $[\widetilde{\widetilde{p}}, P(K_1)]$; $\phi_1(p)$ and $\phi_2(p)$ (in the range $[\widetilde{p}, P(K_1 + K_3)]$) and $\phi_1(p)$ and $\phi_3(p)$ (in the range $[\widetilde{\widetilde{p}}, P(K_1)]$) are calculated accordingly. (Note that $\widetilde{\widetilde{p}} = P(K_1)$ if $h = \frac{471}{520}$.)

- if $h \in \left(\frac{731}{260} - \sqrt{\frac{431}{128}}, 1\right]$, then $\phi_i(p) = \phi_i^\circ(p)$ (each i) in the range $[p_m, \widetilde{p}]$ where $\widetilde{p}$ is such that $\phi_3^\circ(\widetilde{p}) = \phi_3^\circ(\widetilde{\widetilde{p}})$ and $\widetilde{\widetilde{p}}$ is such that $\phi_2^\circ(\widetilde{\widetilde{p}}) = \frac{731}{520} - \frac{1}{2}h$; $\phi_3(p) = \phi_3^\circ(\widetilde{p})$ in the range $[\widetilde{p}, \widetilde{\widetilde{p}}]$ and $\phi_2(p) = \frac{731}{520} - \frac{1}{2}h$ in the range $[\widetilde{\widetilde{p}}, P(K_1)]$; $\phi_1(p)$ and $\phi_2(p)$ (in the range $[\widetilde{p}, \widetilde{\widetilde{p}}]$) and $\phi_1(p)$ and $\phi_3(p)$ (in the range $[\widetilde{\widetilde{p}}, P(K_1)]$) are calculated accordingly.

- whatever is h, for $p \in [P(K_1), p_M]$, $\phi_1(p) = \frac{p - p_m}{p}$, $\phi_2(p) = \frac{pD(p) - p_m K_1 - p\phi_3^{\circ\circ}(p) K_3}{pK_2}$, and $\phi_3(p) = \phi_3^{\circ\circ}(p)$.

Note that if $h = \frac{211}{260}$, then $\phi_2(P(K_1)) = 1$ and if $h = 1$, then $\phi_3(P(K_1)) = 1$. In both these cases, and in no other, functions $\phi_2(p)$ and $\phi_3(p)$ are uniquely determined in the range $[P(K_1)), p_M]$. Note also that if $h \in \left[\frac{211}{260}, \frac{471}{520}\right]$, then $\Phi_3(p) = \Phi_{31}(p)$ and $\Phi_2(p) = \Phi_{21}(p)$ throughout $[p_m, p_M]$; if $h \in \left(\frac{471}{520}, \frac{731}{260} - \sqrt{\frac{431}{128}}\right]$, then $\Phi_2(p) = \Phi_{22}(p) < \Phi_{21}(p)$ and $\Phi_3(p) = \Phi_{32}(p) >$

$\lim_{p \to P(K_1)-} \phi_3^\circ(P(K_1))$, $\phi_2^{\circ\circ}(P(K_1)) = \phi_2^\circ(P(K_1 + K_3)$ for $h = \frac{731}{260} - \sqrt{\frac{431}{128}}$.

$\Phi_{31}(p)$ over the range $(\widetilde{\widetilde{p}}, P(K_1))$; finally, if $h \in \left[\frac{731}{260} - \sqrt{\frac{431}{128}}, 1\right]$, then $\Phi_3(p) > \Phi_{32}(p)$ and $\Phi_2(p) < \Phi_{22}(p)$ over the range $(p^\circ, \widetilde{\widetilde{p}})$, where p° is such that $\phi_3^\circ(p^\circ) = \phi_3^\circ(P(K_1 + K_3))$.

Example 2. Let $D(p) = 1 - p$, and $(K_1, K_2, K_3) = (0.8, 0.125, 0.125)$. Then, $p_M = \frac{3}{8}$, $\Pi_1^* = \frac{9}{64}$, $p_m = \frac{45}{256}$, $\Pi_2^* = \Pi_3^* = \frac{45}{2048}$. Note that $p_M = 0.375 > \frac{p_m K_1}{K1 - K3} = 0.208\bar{3} > P(K_1) = 0.2 > p_m = 0.17578125$. Because of Proposition 8 $\phi_1^\circ(p)$, $\phi_2^\circ(p)$, and $\phi_3^\circ(p)$ are uniquely determined and increasing in the range $[p_m, P(K_1))$. Let real number $h \in [0, \frac{31}{40}]$ and function $\phi_3^{\circ\circ}(p)$ be any non-decreasing function defined in the range $[P(K_1), p_M]$ whose derivative is not higher than $\frac{p_m K_1 - p^2}{p^2 K_2}$, $\phi_3^{\circ\circ}(P(K_1)) = h$, and $\phi_3^{\circ\circ}(p_M) = 1$; then the following profile of strategies is an equilibrium.[18]

- if $h \in \left[0, \frac{31}{80}\right]$, then $\phi_i(p) = \phi_i^\circ(p)$ (each i) in the range $[p_m, \widetilde{p}]$ where $\widetilde{p}$ is such that $\phi_3^\circ(\widetilde{p}) = h$ whereas $\phi_3(p) = h$ and $\phi_1(p)$ and $\phi_2(p)$ are calculated accordingly in the range $[\widetilde{p}, P(K_1)]$. Note that if $h = 0$, then the range $(p_m, \widetilde{p})$ is empty.
- if $h \in \left[\frac{31}{80}, \frac{31}{40}\right]$, then $\phi_i(p) = \phi_i^\circ(p)$ (each i) in the range $[p_m, \widetilde{p}]$ where $\widetilde{p}$ is such that $\phi_2^\circ(\widetilde{p}) = \frac{31}{40} - h$ whereas $\phi_2(p) = \frac{31}{40} - h$ and $\phi_1(p)$ and $\phi_3(p)$ are calculated accordingly in the range $[\widetilde{p}, P(K_1)]$. Note that if $h = \frac{31}{40}$, then $\widetilde{p} = p_m$ and the range $(p_m, \widetilde{p})$ is empty.
- whatever is $h \in \left[0, \frac{31}{40}\right]$ for $p \in [P(K_1), p_M]$, $\phi_1(p) = \frac{p - p_m}{p}$, $\phi_2(p) = \frac{p(D(p) - p) - p_m K_1}{p K_2} - \phi_3^{\circ\circ}(p)$, and $\phi_3(p) = \phi_3^{\circ\circ}(p)$.

Note that $S_3 \subseteq [P(K_1), p_M]$ for $h = 0$, $S_2 \subseteq [P(K_1), p_M]$ for $h = \frac{31}{40}$, whereas the symmetric solution (for the equal-capacity firms) arises for $h = \frac{31}{80}$ and $\phi_3^{\circ\circ}(p) = \frac{pD(p) - p_m K_1}{2pK_2}$. For any other value of h, there is a gap in S_j (some $j \neq 1$) which includes $P(K_1)$.

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