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The Exact Value for European Options on a Stock Paying a Discrete Dividend

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Abstract

In the context of a Black-Scholes economy and with a no-arbitrage argument, we derive arbitrarily accurate lower and upper bounds for the value of European options on a stock paying a discrete dividend. Setting the option price error below the smallest monetary unity, both bounds coincide, and we obtain the exact value of the option.

1 Introduction

In the seminal paper of Black and Scholes (1973), the problem of valuing a European option was solved in closed form. Among other things, their result assumes that the stochastic process associated to the underlying asset is a geometric Brownian motion, not allowing for the payment of discrete dividends. Yet the majority of stocks on which options trade do pay dividends.

Merton (1973) was the first to relax the no-dividend assumption, allowing for a deterministic dividend yield. In this case, he showed that European options can be priced in the context of a Black-Scholes economy, with either a continuous dividend yield or a discrete dividend proportional to the stock price. However, when the dividend process is discrete and does not depend on the stock level, the simplicity of the Black-Scholes model breaks down.

Let S_t denote the value of the underlying asset at time t, and let T be the maturity time of the option. When the risky asset pays a dividend D at time $\tau < T$, a jump of size D in the value process happens at that point in time. The stock price process is discontinuous at $t = \tau$ and is no more a geometric Brownian motion in the time interval $[0, T]$.

The standard approximation procedure for valuing European options written on such a risky asset, first informally suggested by Black (1975), considers a Black-Scholes

formula, where the initial price of the underlying stock S_0 is replaced by its actual value less the present value (PV) of the dividends (Div),

$$S_0 \to S_0^* = S_0 - PV(\text{Div})$$

This adjustment is made to evaluate the option at any point in time before τ. After the payment of dividends, there is no need for further adjustments. In this approximation, the input in the Black-Scholes formula is the value of the (continuous) stochastic process,

$$S_t^* = \begin{cases} S_t - De^{-r(\tau-t)}, & t < \tau \\ S_t, & t \geq \tau \end{cases}$$

where r is the risk-free rate.

For $t < \tau$, the discontinuous stock price process S_t can thus be seen as the sum of two components ($S_t = S_t^* + De^{-r(\tau-t)}$). One riskless component, $De^{-r(\tau-t)}$, corresponding to the known dividends during the life of the option, and a continuous risky component S_t^*. At any given time before τ, the riskless component is the present value of the dividend discounted at the present at the risk-free rate. For any time after τ until the time the option matures, the dividend will have been paid and the riskless component will no longer exist. We thus have $S_T = S_T^*$ and, as pointed out by Roll (1977), the usual Black-Scholes formula is correct to evaluate the option only if S_t^* follows a geometric Brownian motion. In that case, we would use in the Black-Scholes formula S_0^* for the initial value, together with the volatility of the process S_t^*, followed by the risky component of the underlying asset.

If we assume that S_t^* follows a geometric Brownian motion, a simple application of Itô Lemma shows that the original stock price process S_t does not follow a geometric Brownian motion in the time interval $[0, \tau[$. On the other hand, under the Black-Scholes assumption that S_t follows a geometric Brownian motion in $[0, \tau[$, the risky component S_t^* follows a continuous process that is not a geometric Brownian motion in $[0, \tau[$. Therefore, the standard procedure described above must be seen as an approximation to the true value of such calls under the Black-Scholes assumption. As argued by Bos and Vandermark (2002), this assumption is typically underlying the intuition of traders, but the approximation is sometimes bad. In fact, as noticed in the early papers about option pricing (Cox and Ross, 1976; Merton, 1976a; Merton, 1976b), the correct specification of the stochastic process followed by the value of the underlying stock is of prime importance in option valuation.

The deficiency of this standard procedure is reported in Beneder and Vorst (2001). Using Monte Carlo simulation methods, these authors calculate the values of call options under the Black-Scholes assumption, and compare them with the values obtained with the approach just described. Reported errors are up to 9.4%. They also find that the standard procedure above usually undervalues the options. For these reasons, Beneder and Vorst (2001) propose a different approximation, trying to improve the standard procedure by adjusting the volatility of the underlying asset. This approach consists in modifying the variance of the returns by a weighted average of an adjusted and an unadjusted variance, where the weighting depends on the time τ of the dividend payment. Performing much better than the former approximation, this method still does not allow the control of the errors committed for the given parameters of the economy. Analogously, Frishling (2002) warns on the mispricing risk due to the use of an incorrect underlying stochastic process. This discussion is followed by a series of recent papers suggesting different approximations that better match numerical results (Bos and Vandermark, 2002; Bos *et al*, 2003). More recently, Haug

et al (2003) discuss this problem. However, as these authors claim, "[i]n the case of European options, the above techniques are *ad hoc*, but the job gets done (in most cases) when the corrections are properly carried out".

The development of these approximations enhance two important aspects. First, they are not exact, and it is not possible to control the error with respect to the correct value of the option. Second, there are numerical procedures to estimate the value of these options, as for example, Monte-Carlo simulation methods. However, this method is time consuming and provides a convergence of statistical nature.

The purpose of this paper is to derive a closed form for the exact value of European options on a stock paying a discrete dividend, in the context of a Black-Scholes economy. We obtain an exact result and we need not to rely on *ad hoc* assumptions.

This paper is organized as follows. In Section 2, an integral representation for the value of European options written on an asset paying a discrete dividend is obtained, and the convexity properties of the solutions of the Black-Scholes equation are derived. In section 3, we construct functional upper and lower bounds for the integral representation of the value of an option. These bounds follow from a convexity property of the solutions of the Black-Scholes equation. Theorem 3.4 is the main result of this paper and gives the algorithmic procedure to determine the price of European options on a stock paying a discrete dividend. In section 4, numerical examples are analyzed and we discuss the advantages of the proposed method. In section 5, we summarize the main conclusions of the paper.

2 Valuation of European options on a stock paying a discrete dividend

In this section, following a standard procedure to derive the Black-Scholes formula (Wilmott, 2000), we derive an integral representation for the value of a European option written on an asset paying a known discrete dividend.

We consider a European call option with maturity time T and strike price K. This call option is written on an underlying asset with value S_t, with stochastic differential equation,

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where μ and σ are the drift and volatility of the underlying asset. The quantity W_t is a continuous and normally distributed stochastic process with mean zero and variance t. Under these conditions, the underlying asset with value S_t follows a geometric Brownian motion. We also assume a risk-free asset with constant rate of return r.

In the context of the Black-Scholes economy, the value V of an option is dependent of the time t and of the price of the underlying asset S. Under the absence of arbitrage opportunities (Wilmott, 2000; Björk, 1998), it follows that $V(S,t)$ obeys the Black-Scholes equation,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{2.1}$$

The Black-Scholes equation is a quasi-linear parabolic partial differential equation, with $S \geq 0$, and $t \geq 0$. To determine the solutions of the Black-Scholes equation, we introduce the new variables,

$$\begin{cases} \theta = T - t \\ x = \log S + \left(r - \frac{\sigma^2}{2}\right)(T-t) \end{cases}$$

together with the new function $\varphi(x,\theta) = e^{r(T-t)}V(S,t)$. In the new coordinates (2.2), the Black-Scholes equation (2.1) becomes the diffusion equation,

$$\frac{\partial \varphi}{\partial \theta} = \frac{1}{2}\sigma^2 \frac{\partial^2 \varphi}{\partial x^2} \tag{2.2}$$

where $x \in \mathbf{R}$ and $\theta \geq 0$. If $\theta = 0$, by (2.2), we have $\varphi(x,0) = V(S,T)$, and $\varphi(x,T) = e^{rT}V(S,0)$. Therefore, by (2.2), the forward solution in the time θ of the diffusion equation relates with the backward solution in the time t of the Black-Scholes equation (2.1). The Black-Scholes problem for the price of a call option is to determine the option value at time $t = 0$ whose value at maturity time T is,

$$V(S,T) = \max\{0, S-K\} \tag{2.3}$$

Therefore, due to the change of coordinates (2.2), the call option solution of the Black-Scholes equation (2.1) is equivalent to an initial value problem for the diffusion equation.

Suppose now an initial data problem for the diffusion equation (2.2), $\varphi(x,\theta = 0) = f(x)$. Under these conditions, the general solution of (2.2) is (Folland, 1995),

$$\varphi(x,\theta) = \frac{1}{\sigma\sqrt{2\pi\theta}} \int_{-\infty}^{\infty} f(y) \exp\left[-\frac{(x-y)^2}{2\sigma^2\theta}\right] dy \tag{2.4}$$

and the solution of the Black-Scholes equation for a call option is,

$$V(S,0) = e^{-rT}\varphi(x,T) = \frac{e^{-rT}}{\sigma\sqrt{2\pi T}} \int_{-\infty}^{\infty} V(e^y,T) \exp\left[-\frac{(x-y)^2}{2\sigma^2 T}\right] dy \tag{2.5}$$

This integral can be easily calculated to obtain the usual Black-Scholes formula (Black and Scholes, 1973; Wilmott, 2000).

For a dividend distribution at some time $\tau \in (0,T)$, the Black-Scholes formula is no longer true, since, during the life time of the option, the value of the underlying asset does not follow a geometric Brownian motion. However, if we take the time intervals, $I_1 = [0,\tau[$ and $I_2 = [\tau,T]$, the value of the underlying asset follows a geometric Brownian motion in each interval I_1 and I_2, and, at time $t = \tau$, it has a jump equal to the dividend D.

Before considering this case, we proceed with some properties of the solutions (2.4) and (2.5) of the diffusion and of the Black-Scholes equations.

Definition 2.1. *A real valued function $f(x)$, with $x \in \mathbf{R}$, is convex if, for every $x_1, x_2 \in \mathbf{R}$,*

$$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{1}{2}\left(f(x_1) + f(x_2)\right)$$

A simple property of convex functions is that, if the real-valued functions f and g are both convex, and g is increasing, then $f(g(x))$ is also convex.

Proposition 2.2. *Let $f(x)$ the initial data function of a well-posed diffusion equation problem, and suppose that $f(x)$ is non-negative and convex. Then, for fixed θ, the solution $\varphi(x,\theta)$ of the diffusion equation is also convex. Moreover, if $f(x)$ is an increasing function, then, for fixed θ, $\varphi(x,\theta)$ is also increasing.*

Proof. Suppose that the solution (2.4) of the diffusion equation (2.2) is well defined (Folland, 1995). By (2.4), with $z = y - x$, we have,

$$\varphi(x,\theta) = \frac{1}{\sigma\sqrt{2\pi\theta}} \int_{-\infty}^{\infty} f(z+x) \exp\left(-\frac{z^2}{2\sigma^2\theta}\right) dz$$

As, by hypothesis, $f(x)$ is convex, then, for every $z \in \mathbf{R}$,

$$f\left[\frac{(x_1+z)+(x_2+z)}{2}\right] = f\left(z + \frac{x_1+x_2}{2}\right) \leq \frac{1}{2}\left[f(z+x_1) + f(z+x_2)\right]$$

and, as $f(x)$ is non-negative,

$$\begin{aligned} \varphi\left(\frac{x_1+x_2}{2}, s\right) &= \frac{1}{\sigma\sqrt{2\pi\theta}} \int_{-\infty}^{\infty} f\left(z + \frac{x_1+x_2}{2}\right) \exp\left(-\frac{z^2}{2\sigma^2\theta}\right) dz \\ &\leq \frac{1}{2}\left[\varphi(x_1,\theta) + \varphi(x_2,\theta)\right] \end{aligned}$$

and so $\varphi(x,\theta)$ is also convex. Assuming now that $f(x)$ is increasing, we have that $f(x_2) \geq f(x_1)$, whenever $x_2 > x_1$. Then, for every $z \in \mathbf{R}$, we have, $f(z+x_2) \geq (z+x_1)$, and, by (2.4), the last assertion of the proposition follows. □

As (2.3) is a convex function in S, Proposition 2.2 implies that the backward solution (2.5) of the Black-Scholes equation (2.1) is also a convex function.

Suppose now that a dividend on the underlying asset is distributed at time $t = \tau$. We denote this dividend by D. According to the classical solution of the Black-Scholes equation (Wilmott, 2000), the price of the option just after the distribution of dividends at time $t = \tau$ is,

$$V(S_+,\tau) = S_+ N\left(d + \sigma\sqrt{T-\tau}\right) - Ke^{-r(T-\tau)} N(d) \tag{2.6}$$

where,

$$d = \frac{\ln S_+ - \ln K + \left(r - \frac{1}{2}\sigma^2\right)(T-\tau)}{\sigma\sqrt{T-\tau}}$$

and S_+ denotes the value of the underlying asset just after the dividend distribution. The function $N(\cdot)$ is the cumulative distribution function for the normal distribution with mean zero and unit variance. By Proposition 2.2, the function $V(S_+,\tau)$ is convex. Note that, the solution (2.6) is given by, $V(S_+,\tau) = e^{-r(T-\tau)}\phi(x, T-\tau)$, and is directly calculated from (2.5) and (2.3).

The approach taken here to value an option is equivalent (see, among others, Cox and Ross, 1976; Harrison and Krebs, 1979) to write this value at any point in time as the expected discounted payoff of the option at maturity T, under the so-called risk-neutral probability measure. Hence, knowing beforehand the amount to be distributed as dividend, the value of the option is not supposed to jump at τ. In other words, the payment of known dividends D at a known point in time τ does not affect the expectations at time τ about the final payoff of the option at maturity T, and the value of the option is continuous at τ[1] (Wilmott, 2000, pp. 129-131). Going

[1] According to Wilmott, 2000, the jump condition on the asset price is known *a priori*, implying that there is no surprise in the fall of the stock price. Therefore, in order to avoid arbitrage opportunities, the value of the option should not change across the dividend date. This is a no-arbitrage argument.

backward in time, the value of the underlying asset jumps from S_+ to $S_- = S_+ + D$, where S_- is the value of the underlying asset just before the dividend distribution. As $V(S_+, \tau) = V(S_-, \tau)$, by (2.6), the price of the option just before the distribution of dividends at time $t = \tau$ is,

$$V(S_-, \tau) = \begin{cases} (S_- - D)N(\bar{d} + \sigma\sqrt{T-\tau}) - Ke^{-r(T-\tau)}N(\bar{d}) & \text{if } S_- > D \\ 0 & \text{if } S_- \leq D \end{cases} \tag{2.7}$$

where,

$$\bar{d} = \frac{\ln(S_- - D) - \ln K + \left(r - \frac{1}{2}\sigma^2\right)(T-\tau)}{\sigma\sqrt{T-\tau}} \tag{2.8}$$

In Fig. 2.1, we plot $V(S_+, \tau)$, $V(S_-, \tau)$ and $V(S, T)$ as a function of S. The functions $V(S_+, \tau)$, $V(S_-, \tau)$ and $V(S, T)$ are convex.

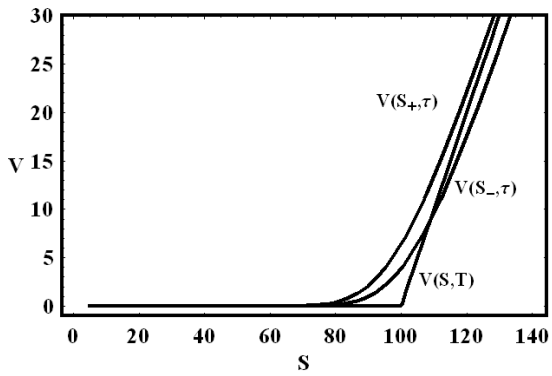

Figure 2.1: Option values $V(S_+, \tau)$, $V(S_-, \tau)$ and $V(S, T)$ as a function of the value S of the underlying asset. Parameter values are: $\mu = 0.01$, $\sigma = 0.2$, $r = 0.03$, $K = 100$, $D = 5$, $T = 1$ and $\tau = 0.5$.

To calculate the value of a call option as a function of the actual price ($t = 0$) of the underlying asset, we must introduce the change of coordinates (2.2) into (2.7) and integrate as in (2.5). By (2.5) and (2.7), it follows that the time-zero value of a European option written on an asset paying dividend D at time $t = \tau$ is given by,

$$V(S, 0) = e^{-r\tau}\varphi(x, \tau) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} V[S_-(y), \tau] \exp\left[-\frac{(x-y)^2}{2\sigma^2\tau}\right] dy \tag{2.9}$$

which has no simple representation in terms of tabulated functions. By Proposition 2.2, $V(S, 0)$ is also convex.

3 Accurate bounds for $V(S, 0)$

As it is difficult to determine a close form for the integral representation of the option's value (2.9) in terms of tabulated functions, to estimate the value $V(S, 0)$, we use the convexity property of $V(S_-, \tau)$ and its asymptotic behavior as $S_- \to \infty$.

Lemma 3.1. *If $K > 0$, then, in the limit $S_- \to \infty$, $V(S_-, \tau)$ is asymptotic to the line $V = (S_- - D) - Ke^{-r(T-\tau)}$, and $V(S_-, \tau) \geq (S_- - D) - Ke^{-r(T-\tau)}$.*

Proof. In the limit $S_- \to \infty$, $\bar{d} \to \infty$, and $N(\bar{d}) \to 1$. Hence, by (2.7), $V(S_-, \tau)$ is asymptotic to the line $V_1 = (S_- - D) - Ke^{-r(T-\tau)}$. To prove the second part of the lemma, first note that, if $V_1 = (S_- - D) - Ke^{-r(T-\tau)} \leq 0$, then $S_- \leq D + Ke^{-r(T-\tau)}$. As $V(S_-, \tau)$ is non-negative, if $S_- \leq D + Ke^{-r(T-\tau)}$, then $V(S_-, \tau) \geq V_1$.

Suppose now that $S_- > D + Ke^{-r(T-\tau)}$. By hypothesis, we assume that there exists some $S_- = \bar{S}$ such that, $V(\bar{S}, \tau) = (\bar{S} - D) - Ke^{-r(T-\tau)}$, and $V(\bar{S}, \tau) > 0$. By (2.7) and (2.8), we then have,

$$Ke^{-r(T-\tau)} = \frac{N\left[\bar{d}(\bar{S}) + \sigma\sqrt{T-\tau}\right] - 1}{N\left[\bar{d}(\bar{S})\right] - 1}(\bar{S} - D)$$

As $(S_- - D) > Ke^{-r(T-\tau)}$, from the equality above, we obtain,

$$\frac{N\left[\bar{d}(\bar{S}) + \sigma\sqrt{T-\tau}\right] - 1}{N\left[\bar{d}(\bar{S})\right] - 1}(\bar{S} - D) = Ke^{-r(T-\tau)} < (S_- - D)$$

Hence,

$$N\left[\bar{d}(\bar{S}) + \sigma\sqrt{T-\tau}\right] < N\left[\bar{d}(\bar{S})\right]$$

which contradicts the fact that $N(\cdot)$ is a monotonically increasing function of the argument. Therefore, the function $V(S_-, \tau)$ and the line $V_1 = (S_- - D) - Ke^{-r(T-\tau)}$ do not intersect for finite $\bar{S}$. As $V(S_-, \tau)$ is a continuous function of S_-, then $V(S_-, \tau) \geq V_1$ in all the range of S_-, and the lemma is proved. □

To estimate the solution (2.9) of the Black-Scholes equation, we use Proposition 2.2 and Lemma 3.1 to construct integrable upper and lower bound functions of $V(S_-, \tau)$. This constructions proceeds as follows.

Let us choose a fixed number $S_- = S^* > D$, and divide the interval $[D, S^*]$ into $M \geq 1$ smaller subintervals. The length of the subintervals is $\Delta S = (S^* - D)/M$, and their extreme points are denoted by,

$$S_i = D + i\,\Delta S, \qquad i = 0, \ldots, M$$

As the function $V(S_-, \tau)$ is convex, in each subinterval, the function $V(S_-, \tau)$ is bounded from above by the chord that connects the points $(S_i, V(S_i, \tau))$ and $(S_{i+1}, V(S_{i+1}, \tau))$. We define the constants,

$$\alpha_i = \frac{M}{S^* - D}\left[V(S_i, \tau) - V(S_{i-1}, \tau)\right], \qquad i = 1, \ldots, M$$

where by (2.7), $V(S_0, \tau) = 0$. Therefore, in each interval $[S_{i-1}, S_i]$, the function $V(S_-, \tau)$ is bounded from above by the function $f_i(S_-) = \alpha_i(S_- - S_{i-1}) + V(S_{i-1}, \tau)$.

Let us define the characteristic function of a set I as, $\chi_I(x) = 1$, if $x \in I$, and $\chi_I(x) = 0$, otherwise. Then, the function $V(S_-, \tau)$ in the interval $[D, S^*]$ is approached from above by the piecewise linear function,

$$V_1^+(S_-, \tau) = \sum_{i=1}^{M}\left[\alpha_i(S_- - S_{i-1}) + V(S_{i-1}, \tau)\right]\chi_{[S_{i-1}, S_i]}(S_-) \tag{3.1}$$

To extend the bound of $V(S_-, \tau)$ to $S_- > S^*$, we introduce the function,

$$V_2^+(S_-, \tau) = \left[(S_- - S^*) + V(S^*, \tau)\right]\chi_{[S^*, \infty)}(S_-) \tag{3.2}$$

By Proposition 2.2 and Lemma 3.1, for $S_- \geq S^*$, $V_2^+(S_-,\tau)$ is the chord connecting the point $(S^*, V(S^*,\tau))$ to the point at infinity. Therefore, we have proved the following:

LEMMA 3.2. *The function $V(S_-,\tau)$ has the upper bound,*

$$V(S_-,\tau) \leq V_1^+(S_-,\tau) + V_2^+(S_-,\tau), \quad \textit{if} \;\; S_- > D$$

where V_1^+ and V_2^+ are given by (3.1) and (3.2), respectively, and the function $(V_1^+ + V_2^+)$ is piecewise linear and non-negative. If $S_- \leq D$, $V(S_-,\tau) = 0$.

The construction of a lower bound for (2.7) follows the same line of reasoning.

In each subinterval $[S_{i-1}, S_i] \subset [D, S^*]$, we can construct a linear function that bounds from below the function $V(S_-,\tau)$. Due to the convexity of $V(S_-,\tau)$, we construct the lower bound through the derivative of $V(S_-,\tau)$ at the middle point of each interval $[S_{i-1}, S_i]$. We then have,

$$V_1^-(S_-,\tau) = \sum_{i=1}^{M} \left[V'\left(S_{i+\frac{1}{2}},\tau\right)\left(S_- - S_{i+\frac{1}{2}}\right) + V\left(S_{i+\frac{1}{2}},\tau\right)\right] \chi_{[S_{i-1},S_i]}(S_-) \quad (3.3)$$

where,

$$V'(S_-,\tau) = \frac{e^{-\frac{1}{2}\left(\overline{d}+\sigma\sqrt{T-\tau}\right)^2}}{\sigma\sqrt{2\pi}\sqrt{T-\tau}} - \frac{K\, e^{-r(T-\tau)}\, e^{-\frac{1}{2}\overline{d}^2}}{\sigma\sqrt{2\pi}\sqrt{T-\tau}\,(S_- - D)} + N\left(\overline{d} + \sigma\sqrt{T-\tau}\right)$$

and $\overline{d}$ is given by (2.8).

To extend the lower bound of $V(S_-,\tau)$ to $S_- > S^*$, we use Lemma 3.1 to introduce the function,

$$V_2^-(S_-,\tau) = \left[(S_- - D) - Ke^{-r(T-\tau)}\right] \chi_{[S^*,\infty)}(S_-) \quad (3.4)$$

By Lemma 3.1, $V_2^-(S_-,\tau)$ bounds from below $V(S_-,\tau)$. Therefore, we have:

LEMMA 3.3. *The function $V(S_-,\tau)$ has the lower bound,*

$$V(S_-,\tau) \geq V_1^-(S_-,\tau) + V_2^-(S_-,\tau), \quad \textit{if} \;\; S_- > D$$

where V_1^- and V_2^- are given by (3.3) and (3.4), respectively, and the function $(V_1^- + V_2^-)$ is piecewise linear and non-negative. If $S_- \leq D$, $V(S_-,\tau) = 0$.

Finally, we can state our main result:

THEOREM 3.4. *We consider the Black-Scholes equation (2.1) together with the terminal condition (2.3). We assume that $K > 0$ and a dividend $D > 0$ is payed at the time τ with $0 < \tau < T$. Let $S = S^* > D$ be a fixed constant and let $M \geq 1$ be an integer. Then, the solution of the Black-Scholes equation with terminal condition (2.3) has the following upper and lower bounds:*

$$\begin{aligned} V(S,0) \leq V^+_{S^*,M}(S,0) \;\; &= \;\; \sum_{i=1}^{M} \left\{\alpha_i A_i S + e^{-r\tau}\left[V(S_{i-1},\tau) - \alpha_i S_{i-1}\right] B_i\right\} \\ &+ \;\; SN(d^*) + e^{-r\tau}\left[V(S^*,\tau) - S^*\right] N(d^* - \sigma\sqrt{\tau}) \end{aligned}$$

and

$$
\begin{aligned}
V(S,0) \geq V^{-}_{S^*,M}(S,0) &= S\sum_{i=1}^{M} V'\left(S_{i+\frac{1}{2}},\tau\right) A_i \\
&+ e^{-r\,\tau}\sum_{i=1}^{M}\left[V\left(S_{i+\frac{1}{2}},\tau\right) - V'\left(S_{i+\frac{1}{2}},\tau\right) S_{i+\frac{1}{2}}\right] B_i \\
&+ SN(d^*) - e^{-r\tau}\left(D + Ke^{-r(T-\tau)}\right) N\left(d^* - \sigma\sqrt{\tau}\right)
\end{aligned}
$$

where,

$$
\begin{aligned}
S_i &= D + \frac{S^* - D}{M} i \\
d_i &= \frac{\log S - \log S_i + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \\
d &= \frac{\log(S-D) - \log K + (r + \frac{1}{2}\sigma^2)(T-\tau)}{\sigma\sqrt{T-\tau}} \\
d^* &= \frac{\log S - \log S^* + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \\
V(S,\tau) &= (S-D)N(d) - Ke^{-r(T-\tau)} N(d - \sigma\sqrt{T-\tau}) \\
V'(S,\tau) &= N(d) + \frac{e^{-\frac{1}{2}d^2}}{\sigma\sqrt{2\pi(T-\tau)}} - \frac{K\,e^{-r(T-\tau)}\,e^{-\frac{1}{2}\left(d-\sigma\sqrt{T-\tau}\right)^2}}{\sigma\sqrt{2\pi(T-\tau)}\,(S-D)} \\
\alpha_i &= \frac{M}{S^* - D}\left[V(S_i,\tau) - V(S_{i-1},\tau)\right] \\
A_i &= N(d_{i-1}) - N(d_i) \\
B_i &= N(d_{i-1} - \sigma\sqrt{\tau}) - N(d_i - \sigma\sqrt{\tau})
\end{aligned}
$$

and $N(\cdot)$ is the cumulative distribution function for the normal distribution with mean zero and unit variance.

Proof. By Lemmata (3.2) and (3.3),

$$
V_1^-(S_-,\tau) + V_2^-(S_-,\tau) \leq V(S_-,\tau) \leq V_1^+(S_-,\tau) + V_2^+(S_-,\tau), \quad \text{if } S_- > D
$$

Multiplying this inequality by the factors as in the integral (2.9), and integrating, we obtain the estimates of the theorem. □

Note that, for $S^* > D$ fixed, $\lim_{M\to\infty} V^-_{S^*,M}(S,0) \neq lim_{M\to\infty} V^+_{S^*,M}(S,0)$. However, if S^* is large enough, both limits can be made arbitrarily close. Technically, this is due to the way the exponential term in (2.5) contributes to the integral.

4 Calculating the price of a call option on a stock paying a discrete dividend

Theorem 3.4 is the necessary tool to determine the price of a call option when the underlying asset pays a discrete known dividend before maturity time T. In fact, Theorem 3.4 asserts that we can always find upper and a lower bound functions for $V(S,0)$, and the bounding functions approach each other as we increase M and S^*.

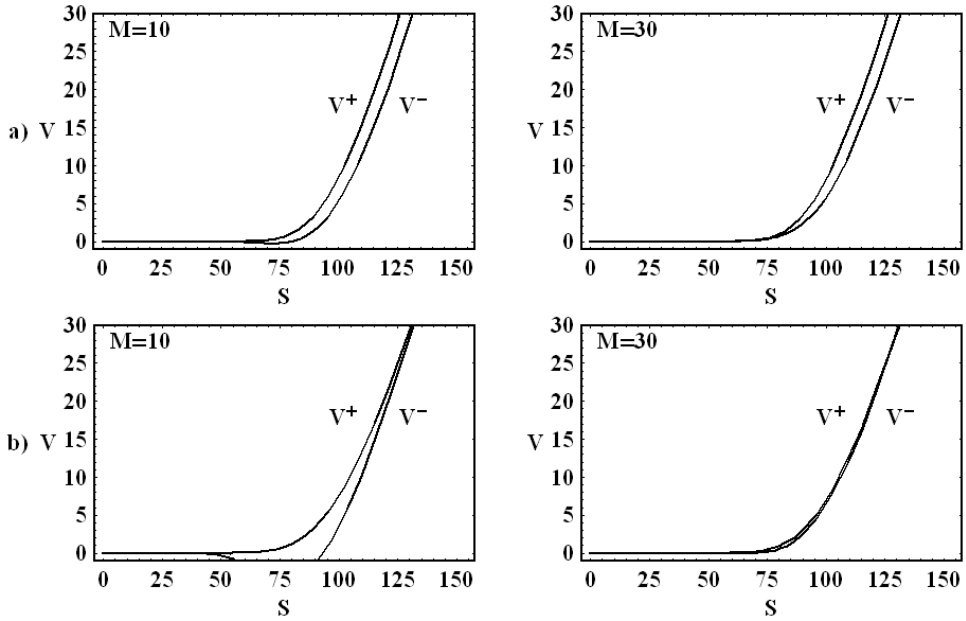

Figure 4.1: Bounds $V^+_{S^*,M}(S,0)$ and $V^-_{S^*,M}(S,0)$ for $V(S,0)$, calculated from Theorem 3.4, for several values of S^* and M. In a) we have chosen $S^* = D+Ke^{-r(T-\tau)} = 103.5$. In b), $S^* = 2(D + Ke^{-r(T-\tau)}) = 207.0$. Parameter values are: $\mu = 0.01$, $\sigma = 0.2$, $r = 0.03$, $K = 100$, $D = 5$, $T = 1$ and $\tau = 0.5$.

Table 1: Bounds $V^+_{S^*,M}(S,0)$ and $V^-_{S^*,M}(S,0)$ for $V(S,0)$, calculated from Theorem 3.4, for several values of S^* and M, and $S = 110$. The exact value $V(S,0)$ has been obtained by the numerical integration of (2.9). The interval error ε is given by (2.9). Parameter values are the same as in Fig. 4.1, and we have chosen $S^* = D + Ke^{-r(T-\tau)} = 103.5$, $S^* = 1.5(D+Ke^{-r(T-\tau)}) = 155.3$ and $S^* = 2(D+Ke^{-r(T-\tau)}) = 207.0$.

S	S^*	M	$V^-_{S^*,M}(S,0)$	$V(S,0)$	$V^+_{S^*,M}(S,0)$	ε
110	103.5	10	11.24	12.87	15.41	4.166
110	103.5	50	11.61	12.87	15.35	3.739
110	103.5	400	11.63	12.87	15.35	3.721
110	155.3	10	11.39	12.87	13.20	1.807
110	155.3	50	12.79	12.87	12.88	0.096
110	155.3	200	12.87	12.87	12.87	0.006
110	155.3	400	12.87	12.87	12.87	0.002
110	207.0	10	10.64	12.87	13.45	2.813
110	207.0	50	12.72	12.87	12.89	0.170
110	207.0	200	12.86	12.87	12.87	0.011
110	207.0	400	12.87	12.87	12.87	0.003

To determine the price of the option, we first choose fixed values for the approximation parameters S^* and M. If $V^+_{S^*,M}(S,0)$ and $V^-_{S^*,M}(S,0)$ differ too much within some fixed precision, we then increase S^* and M.

To analyze the convergence of the functional bounds V^+ and V^- to the true price of a call option, we take, as an example, the parameters: $\mu = 0.01$ (drift), $\sigma = 0.2$ (volatility), $r = 0.03$ (interest rate), $K = 100$ (strike price), $D = 5$ (dividend), $T = 1$ (expiration time) and $\tau = 0.5$ (time of dividend paying). In Fig. 4.1, we show $V^+_{S^*,M}(S,0)$ and $V^-_{S^*,M}(S,0)$, for several values of S^* and M, and calculated from Theorem 3.4. Increasing M and S^*, the upper and lower bounds $V^+_{S^*,M}(S,0)$ and $V^-_{S^*,M}(S,0)$ approach each other, increasing the accuracy to which the functionals bounds approach the option price. To quantify this approximation to the value of the option, we define the interval error as,

$$\varepsilon = |V^+_{S^*,M}(S,0) - V^-_{S^*,M}(S,0)| \tag{4.1}$$

In Table 1, we compare the values of the upper and lower bounds $V^+_{S^*,M}(S,0)$ and $V^-_{S^*,M}(S,0)$, calculated from Theorem 3.4, with the exact value of $V(S,0)$, obtained by the numerical integration of (2.9). We show also the interval error ε associated to both bounds. Assuming an interval error below the smallest unit of the monetary currency, for example, $\varepsilon < 10^{-2}$, we obtain the true value of the option. Therefore, for a choice of S^* and M such that $\varepsilon < 10^{-2}$, the difference between $V^+_{S^*,M}(S,0)$ and $V^-_{S^*,M}(S,0)$, is below the smallest unit of the monetary currency, and the rounded values of $V^+_{S^*,M}(S,0)$ and $V^-_{S^*,M}(S,0)$ coincide. This rounded value is the option value within the chosen monetary accuracy

To analyze the global convergence behavior of $V^+_{S^*,M}(S,0)$ and $V^-_{S^*,M}(S,0)$, we chose a fixed value of S, and we change the approximation parameters S^* and M. In Fig. 4.2, we show $V^+_{S^*,M}(S,0)$ and $V^-_{S^*,M}(S,0)$ as a function of S^*, for several values of M. Increasing M, the upper and lower bounds of $V(S,0)$ become close in a region of the S^* axis. A choice of S^* in this region, gives better bounds to the value of the option, for lower values of M (Table 1 and Fig. 4.2).

For all the examples we have analyzed, a good compromise to determine the value of the call option is to choose $S^* = 2(D + Ke^{-r(T-\tau)})$. Then, increasing M, the interval error decreases. Due to the fast computational convergence of the expressions in Theorem 3.4, bounds with interval error below the smallest unit of the monetary currency are straightforwardly obtained.

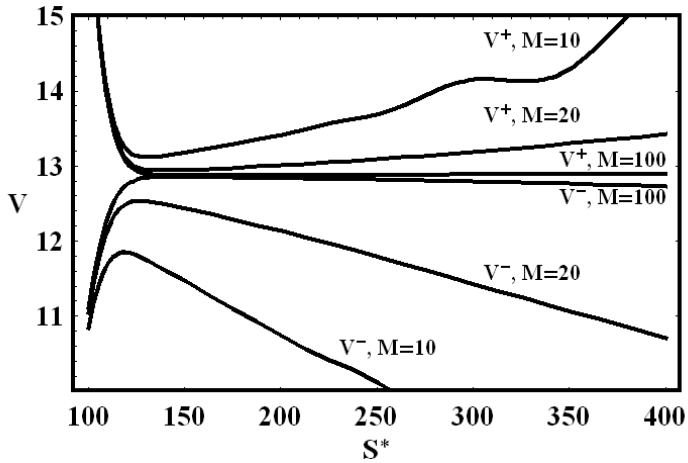

Figure 4.2: Bounds $V^+_{S^*,M}(S,0)$ and $V^-_{S^*,M}(S,0)$ as a function of S^*, for $S = 110$ and several values of M. The parameter values are the same as in Fig. 4.1 and Table 1.

5 Concluding remarks

We have obtained an upper and a lower bound for the exact value of a call option on a stock paying a known discrete dividend at a known future time. We have assumed the context of a Black-Scholes economy, where, away from the dividend time paying, the underlying asset price follows a geometric Brownian motion type stochastic process. The upper and lower bounds both approach the exact value of the option when two parameters are varied. In practical terms, one of these parameters (S^*) can be fixed to the value, $S^* = 2\left(D + Ke^{-r(T-\tau)}\right)$, where K is the strike, D is the dividend, τ is the time of paying the discrete dividend, and T is the length of the contract. Increasing the second parameter M, we obtain bounds for the option value with increasing accuracy. If this accuracy is below the smallest unit of the monetary currency, both bounds coincide, and we obtain the exact value of the option.

The technique used to construct these bounds relies on the convexity properties of the option value at maturity, and on a property of the Black-Scholes and diffusion equations that preserves the convexity of propagated initial conditions. Under this framework, a similar methodology can be used to determine the value of a put option on a stock paying a known discrete dividend at a known future time.

From the numerical point of view, the technique developed here reduces to the sum of a few Black-Scholes type terms, whereas numerical Monte Carlo methods rely on the poor convergence properties determined by the classical central limit theorem. In our numerical tests for the determination of the exact price of a call option, the computing time of our technique (using the Mathematica programming language) is several orders of magnitude faster than the computing time of finite diferences integration algorithms and of Monte Carlo methods.

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