# The Savings Multiplier

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**Abstract**

We develop a theory of macroeconomic development based on the novel concept of *savings multiplier*: capital accumulation changes relative prices and income shares between generations, creating further incentives to accumulate and thereby rising saving rates as the economy develops. The savings multiplier hinges on two mechanisms. First, accumulation raises wages and leads to redistribution from the consuming old to the saving young. Second, higher wages raise the price of services consumed by the old, and the anticipation of such price rise prompts the young to increase their savings. Our theory captures important aspects of China's development and suggests new channels through which the one child policy and the dismantling of cradle-to-grave social benefits have fuelled China's savings and accumulation rates.

*Keywords:* Overlapping generations, Growth, Savings.

*JEL classification:* O11, D91, E21

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## 1. Introduction

This paper presents a theory of macroeconomic development based on the novel concept of *savings multiplier*: capital accumulation sparks output growth but also induces changes in relative prices and in intergenerational income shares that create further incentives to accumulate, implying rising saving rates as the economy develops. The savings multiplier creates a feedback effect of growth on savings that magnifies the impact of exogenous shocks – such as demographic change, policy reforms, productivity shocks – on capital per capita in the long run. The scope of our results is twofold. First, the savings multiplier introduces circular causality in the savings-growth relationship and thus provides a new explanation for rising saving rates in developing countries. Second, our theory captures important aspects of China's economic performance and suggests new channels through which the one child policy and the dismantling of cradle-to-grave social benefits have fuelled China's savings and accumulation rates. We discuss each point in turn below.

Rising saving rates characterized the growth process of most developed economies. Lewis (1954) provides an early recognition of this stylized fact, stressing that

> The central problem in the theory of economic development is to understand the process by which a community which was previously saving and investing 4 or 5 per cent of its national income or less, converts itself into an economy where voluntary saving is running at about 12 to 15 per cent of national income or more. [...] We cannot explain any "industrial" revolution [...] until we can explain why saving increased relatively to national income. (Lewis, 1954: p.155).

The issue of causality in the relationship between growth and saving rates is still an open question (see Deaton, 2010). Standard growth theories tell us that saving rates drive development but empirical evidence suggests that causality may run in the opposite direction (Attanasio et al. 2000; Rodrik, 2000). The topic received attention in the growth literature of the late 1990s – mostly dedicated to the stunning performance of East Asian economies – but only a few contributions attempted at developing new theories to explain the effects of growth on saving rates. One of these contributions is the theory of Relative Consumption, where households' utility depends on current consumption relative to a benchmark level which may reflect habit formation (Carroll et al. 2000), interpersonal comparisons (Alvarez-Cuadrado et al. 2004), or international status seeking (Valente, 2009). In Relative Consumption models, economic growth raises the benchmark consumption level over time and the agents' willingness to catch-up with the benchmark prompts households to adjust savings accordingly. Our theory of the savings multiplier is different because the feedback effects of growth on saving rates hinge on the economy's demographic structure, which comprises overlapping generations, and on the allocation of labor between different production sectors.

In our model, the first channel through which growth affects saving rates is what we term the *intergenerational distribution effect*. Higher savings imply both higher capital stock and increased demand for care by the old, both fueling wage increases. The income distribution shifts in favor of the wage earners – that is, accumulation raises the income share of savers relative to the old agents – which stimulates further savings and capital accumulation. The second channel is what we term the *old-age requirement effect*. Increased savings and capital accumulation push the anticipated future wage up, making old-age care more expensive. To compensate for the increased future costs of care, young agents increase their savings relative to current income. This gives an additional channel whereby savings and capital accumulation stimulate further savings and capital accumulation. During the transition to the long-run equilibrium, savings rates increase over time, the share of employment in the manufacturing sector declines, the income distribution shifts in favor of the young, and an increasing share of private expenditures is allocated to the purchase of services.[1]

Although our contribution is theoretical, the key motivation of our analysis lies in the empirical literature on Asian economies, and on the experience of China in particular. Since 1978, real per capita GDP in China has increased tenfold, and fast output growth was accompanied by massive capital accumulation. After drastic policy changes in the late 1970s, savings and investment as a share of GDP increased sharply. Importantly, savings and investment rates continued to grow thereafter: graph (a) in Figure 1 shows that more than 40% of GDP has been invested, while more than 50% of GDP has been saved, over the last years.

[1] This mechanism clearly distinguishes our notion of savings multiplier, which operates on the supply side under full employment conditions, from the traditional concept of demand multiplier according to which income is pushed up from the side of demand when factors of production are not fully utilized. To our knowledge, neither the term 'savings multiplier' nor its underlying concept have been previously introduced in the literature.

China's saving behavior inspired a huge body of empirical literature but there is a lack of new theories that could explain the most puzzling fact, namely, that households have increased their savings rate, despite being quite poor, having fast income growth, and receiving low returns on their savings.[2] In this respect, our model provides a theory of savings that is consistent with four relevant facts that characterized China's development – most of which are direct consequences of the reforms enacted in the last forty years.

First, saving rates increased while fertility sharply declined (*Fact 1*). China's fertility rate decreased from 4.9 in 1975 to 1.7 in 2007, while life expectancy increased by ten years in the same period (Litao and Sixin, 2009). A major trigger of this acceleration in population ageing was the one-child policy implemented since 1978, which changed family composition and reduced the number of births.

Second, Chinese workers face an increased need to provide for old age with their own resources (*Fact 2*). A prominent cause is the reform of the industry sector implemented since the late 1980s, which gradually dismantled state owned enterprises and deleted cradle-to-grave social benefits for a huge fraction of workers (Ma and Yi, 2010).[3] Meanwhile, the private provision of old-age security is neither efficient nor pervasive: less than 30% of all employees are covered by pension schemes (Oksanen, 2010).

Third, a growing share of health care services is, and will increasingly need to be, purchased in the market (*Fact 3*). The share of health spending that households pay themselves increased from 16% in 1980 to 61% in 2001 (Blanchard and Giavazzi, 2006), and the growth in China's health spending is "one of the most rapid in world history" (Eggleston, 2012: p.4). The rising importance of private provision may itself be a side-effect of the one-child policy through changes in family composition.[4] But beyond its causes, the relevant consequence for our

[2]The high savings rate reported in graph (a) of Figure 1 reflects the sum of high corporate savings and high household savings. Song et al. (2011) provide a theoretical explanation for high *corporate* savings based on the existence of capital market imperfections that generate high shares of firms' retained profits. Our claim on the lack of theories refers, instead, to the analysis of *household* savings, which is the focus of our model. At present, household savings is the single largest component of total savings and according to Yang (2012), the increase in the rate of household savings from 2000 to 2008 is the most important contribution to the overall increase in the Chinese savings rate in the same period.

[3]The reform implied massive layoffs, and the enterprise-based social safety net shrank rapidly as a result (Ma and Yi, 2010). In the pre-reform system, instead, each state enterprise provided housing, medical care and old-age security to its workers and pensioneers (James 2002).

[4]The one-child policy drastically reduced the scope for family provided care during a period in which the need for such care was rapidly increasing. More and more families now consist of four grandparents, two parents and one child, making the markedet provision of care a necessity.

analysis is that the increased share of care services in private expenditures is driving structural change in production sectors. Graph (b) in Figure 1 shows that the share of employment in health and social work relative to that in manufacturing has doubled over 15 years.[5] Such sectoral change has been neglected as a possible determinant of China's saving rates whereas it plays an important role in our model.

Fourth, the income distribution is shifting in favor of young wage earners and in disfavor of the old (*Fact 4*). The share of labor income in GDP has increased (Bai and Qian, 2010) and, since 1998, real wage growth has exceeded GDP growth (Li et al., 2012). This induced a shift in the income distribution towards young workers (Song and Yang, 2010).

Our model produces equilibrium dynamics that are fully consistent with Facts 1-4: capital accumulation in the manufacturing sector raises wages and shifts labor into the care sector, boosting saving rates via both higher income for young cohorts and higher expected future cost of care services. In particular, we study exogenous shocks that plausibly capture the effects of China's past reforms – namely, a reduction in the population growth rate, an increase in the minimum level of care to be purchased – and we show that these shocks induce higher capital per capita and that saving rates increase during the transition because capital accumulation is accelerated by the savings multiplier. These results suggest that the one-child policy and the dismantling of cradle-to-grave social benefits have fuelled China's saving rates in the last decades. By the same token, the counteracting reforms that China's government recently announced – namely, the abandonment of the one-child policy as well as the intention to expand the welfare system – are predicted to reduce savings and saving rates. We analyze this mechanism quantitatively by calibrating our model on China's data to quantify the elasticity of capital accumulation to combined shocks on population growth and minimum care.

With respect to the existing literature, a specific value added of our analysis is the use of the general equilibrium framework. In our model, the economy's equilibrium path brings together Facts 1-4 and combines them with a precise causal order. The existing empirical literature – e.g., Kraay (2000), Modigliani and Cao (2004), Chamon and Prasad (2010) – provides very valuable information on each of these facts but typically focusing on one single mechanism in

[5] From 1993 to 2008, the employment share of manufacturing decreased from 37% to 29% while the employment share of health and social work increased from 2.8% to 4.7% (ILO, 2015).

isolation from the others, thus failing to deliver a complete picture.[6] Our paper is different, but complementary, to this line of research: none of the above mentioned contributions develops a general equilibrium model where capital accumulation affects subsequent saving rates, or note any of the two mechanisms behind the savings multiplier.

## 2. The Model

The key features of the model are the overlapping-generations (OLG) structure, the hypothesis of age-dependent needs, and the existence of two production sectors. The first set of firms produces the *generic good* which is partly saved as physical capital, and partly consumed by both young and old agents. The second set of firms provides services that are exclusively purchased by the old and may be interpreted as *old-age care*. The one-good OLG framework pioneered by Diamond (1965) – henceforth termed the *canonical model* – may be viewed as a special case of our model.[7]

### *2.1. Consumers*

Each agent lives two periods $(t, t+1)$. Total population, denoted $N_t$, consists of $N_t^y$ young and $N_t^o$ old agents, and grows at the exogenous net rate $n > -1$;

$$N_t = N_t^y + N_t^o, \quad N_t^y = N_t^o \cdot (1+n), \quad N_{t+1} = N_t \cdot (1+n). \tag{1}$$

Agents purchase two types of goods over their life-cycle: the generic consumption good is enjoyed in both periods of life whereas old-age care services are only purchased in the second period of life. The lifetime utility of an agent born at the beginning of period $t$ is

$$U_t \equiv u(c_t) + \beta \cdot v\left(d_{t+1}, h_{t+1} - \bar{h}\right), \tag{2}$$

where $c_t$ and $d_{t+1}$ represent consumption levels of the generic good in the first and second period of life, respectively, $h_{t+1}$ is the amount of old-age care consumed when old, $\bar{h} \geqslant 0$ is the *minimum requirement* – i.e., the minimum amount of old-age care required by old agents

[6] Kraay (2000) documents the link between the increased need to provide for old age and the dismantling of state-owned enterprises; Modigliani and Cao (2004) find a strong effect of the one-child policy on the needs to save for retirement; Blanchard and Giavazzi (2006) and Chamon and Prasad (2010) explain increased saving rates with the rising burden of expenditures such as health care and education; Song and Yang (2010) argue that the main reason for the rising saving rate is the shift in the income distribution in favor of young workers.

[7] Detailed derivations and long proofs are collected in the appendix.

– and $\beta \in (0,1)$ is the private discount factor between young and old age. The consumer problem is subject to the constraint that the minimum requirement, $h_{t+1} - \bar{h} \geqslant 0$, is at least weakly satisfied. [8] The case with zero minimum requirement, $\bar{h} = 0$, is of special interest since it will allow us to separate the two central mechanisms of the model, the 'intergenerational distribution' and the 'old-age requirement' effects (cf. Section 4.).

Young agents supply inelastically one unit of homogeneous labor and save part of their labor income. Old agents do not work and spend all their interest income in purchasing consumption goods and old-age care. The individual budget constraints read

$$\begin{aligned} c_t &= w_t - s_t, && (3) \\ s_t R_{t+1} &= d_{t+1} + p_{t+1} h_{t+1}, && (4) \end{aligned}$$

where the generic good is taken as the numeraire, $w_t$ is the wage rate, $s_t$ is savings, $R_{t+1}$ is the gross rate of return to saving, and $p_{t+1}$ is the price of old-age care. Savings consist of physical capital, which is homogeneous with the generic good. Assuming full depreciation within one period, market clearing requires that aggregate capital at the beginning of period $t+1$ equals aggregate savings of the young agents in the previous period, $K_{t+1} = N_t^y s_t$.

In order to make the analysis transparent, we consider a specific form of preferences:

$$\begin{aligned} u(c_t) &\equiv \log c_t, && (5) \\ v\left(d_{t+1}, h_{t+1} - \bar{h}\right) &\equiv \log \left[\gamma \left(d_{t+1}\right)^{\frac{\sigma-1}{\sigma}} + (1-\gamma)\left(h_{t+1} - \bar{h}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, && (6) \end{aligned}$$

where $\gamma \in [0,1]$ is a weighting parameter and $\sigma > 0$ is the elasticity of substitution between consumption goods and care services in the second period of life: $d_{t+1}$ and $h_{t+1}$ are strict complements if $\sigma < 1$, strict substitutes if $\sigma > 1$. In the limiting case $\sigma \to 1$, the term in square brackets reduces to the Cobb-Douglas form $\left(d_{t+1}\right)^{\gamma} \left(h_{t+1}\right)^{1-\gamma}$. The empirical literature shows that, when $h$ is interpreted as health care, the most plausible case is that of strict

[8] As is standard, we will focus on interior equilibria where $h_{t+1} > \bar{h}$ and verify ex-post the conditions under which this strict inequality holds. We will show that there always exists a unique equilibrium in which the allocation of labor between generic-good and health-care production is consistent with the interior solution $h_{t+1} > \bar{h}$.

complementarity with a positive requirement, $\sigma < 1$ and $\bar{h} > 0$.[9] We will nonetheless also study the case of substitutability. Preferences (5)-(6) exhibit two relevant properties. First, they allow us to treat the canonical OLG model as a special case: setting $\gamma = 1$ and $\bar{h} = 0$, old-age care services do not yield utility and, hence, are not produced in equilibrium. Second, the utility functions (5)-(6) exhibit a unit elasticity of intertemporal substitution. Therefore, setting $\gamma = 1$ yields the log-linear version of the canonical model in which the saving rate is constant over time. This implies that, in the general case $0 < \gamma < 1$, any departure from the canonical result of 'constant saving rate' must be induced by our distinctive hypothesis, namely, the fact that old agents need dedicated care services.

### *2.2. Production Sectors*

From a technological viewpoint, the different nature of generic goods – which may be interpreted as manufactured products – and old-age care services – which include health care as well as personal assistance – is captured by Baumol's (1967) hypothesis: the production of care services is strongly labor intensive because, differently from what happens in manufacturing industries, capital cannot be used as a substitute for labor. Hartwig (2008) tests this hypothesis on recent data, obtaining strong empirical support to Baumol's view and showing that health care expenditure is mainly driven by wage increases. Our model captures these aspects by assuming that care services are produced with labor as the only factor of production. The consumption good, instead, is produced by means of capital and labor as in Diamond's (1965) canonical model.[10] We denote by $\ell_t$ the fraction of workers employed in the generic sector, and by $1 - \ell_t$ the fraction employed in the care sector. Perfect labor mobility and perfectly competitive conditions in the labor market ensure wage equalization in equilibrium. The old-age care sector exhibits a simple constant returns to scale technology,

$$H_t \equiv \eta \cdot (1 - \ell_t) \cdot N_t^y, \tag{7}$$

[9] When $\bar{h} > 0$, function (6) implies that the income elasticity of old-age care falls short of unity, in line with Acemoglu et al. (2013) that estimate the income elasticity of health spending to 0.7. Finkelstein et al. (2012) estimate an elasticity of substitution between health and non-health consumption equal to $\sigma = 0.2$.

[10] For a two-sector OLG model with capital in both sectors, as well as the existence and stability properties of such models, see Galor (1992).

where $H_t$ is the aggregate output of care services, and $\eta > 0$ is a constant labor productivity parameter. In the generic good sector, aggregate sectoral output $X_t$ is given by

$$X_t = B \cdot (K_t)^{\alpha} \left(\ell_t N_t^y\right)^{1-\alpha} \tag{8}$$

where $B > 0$ is an exogenous productivity parameter, $K_t$ is aggregate capital, and $\alpha \in (0, 1)$ is an elasticity parameter.

## 3. Static Equilibrium

This section discusses the static equilibrium conditions holding in each period for a given stock of capital per worker. We first study the profit-maximizing conditions for firms, the utility-maximizing conditions for households, the labor market equilibrium, and the goods market equilibrium. We then study the joint (static) equilibrium of all the markets, the implications for the aggregate savings rate, and the implied mapping to capital accumulation.

### *3.1. Firms*

In the service sector, technology (7) implies that the wage is proportional to the market price of care services,

$$w_t = \eta p_t. \tag{9}$$

Market clearing requires that total output of old-age care services matches aggregate demand by old agents, $H_t = N_t^o h_t$. The existence of a minimum requirement, $h_t \geqslant \bar{h}$, implies that total production $H_t$ must exceed $N_t^o \bar{h}$. This imposes an upper bound on the employment share of the generic sector: using the production function (7), we obtain

$$\ell_t \leqslant \frac{\eta\,(1+n) - \bar{h}}{\eta\,(1+n)} \equiv \ell^{\max}, \tag{10}$$

where $\ell^{\max}$ is the maximum level of employment in the generic sector that is compatible with a level of old-age care output equal to the minimum requirement.[11] In the remainder of the analysis, we will work under the parameter restriction $\bar{h} \leqslant \eta\,(1+n)$, which implies $\ell^{\max} \geqslant 0$. When the minimum requirement is $\bar{h} = 0$, we have $\ell^{\max} = 1$.

[11] The level of care supply equal to the minimum requirement is $H_t^{\min} \equiv \eta\,(1 - \ell^{\max})\, N_t^y = N_t^o \bar{h}$.

In the generic good sector, factor prices equal marginal productivities,

$$w_t = B(1-\alpha)(\kappa_t/\ell_t)^{\alpha} = (1-\alpha)(x_t/\ell_t), \tag{11}$$

$$R_t = B\alpha(\ell_t/\kappa_t)^{1-\alpha} = \alpha(x_t/\kappa_t), \tag{12}$$

where $x_t \equiv X_t/N_t^y$ is sectoral output *per young*. Aggregating incomes between sectors yields

$$\frac{Y_t}{N_t^y} = w_t + R_t\kappa_t = x_t\left(\frac{1-\alpha}{\ell_t}+\alpha\right), \tag{13}$$

where $Y_t$ is aggregate income, which coincides with the total value of goods and services produced in the economy, $Y_t \equiv X_t + p_t H_t$.

### *3.2. Consumers*

Each agent maximizes (2) subject to the budget constraints (3)-(4). Using the standard notation for derivatives – i.e., $u_{c_t} \equiv \partial u/\partial c_t$ – the solution to the consumer problem yields two familiar first order conditions: the Keynes-Ramsey rule, $u_{c_t} = \beta R_{t+1} v_{d_{t+1}}$, and an efficiency condition establishing the equality between the price of care services and the marginal rate of substitution with second-period generic goods consumption, $v_{h_{t+1}}/v_{d_{t+1}} = p_{t+1}$. Under preferences (5)-(6), these conditions determine the following relationships (see appendix). Consumption and savings of young agents are given by

$$c_t = \frac{1}{1+\beta}\left(w_t - \frac{p_{t+1}}{R_{t+1}}\bar{h}\right) \text{ and } s_t = \frac{1}{1+\beta}\left(\beta w_t + \frac{p_{t+1}}{R_{t+1}}\bar{h}\right). \tag{14}$$

When $\bar{h} = 0$, these expressions are similar to those holding in the canonical model, where young agents save a constant fraction of their wage income. This similarity does not imply, however, the same accumulation dynamics: as shown in section 3.7. below, our model predicts that, even with $\bar{h} = 0$, the aggregate saving rate is not constant because the intergenerational distribution of income changes over time. In the more general case with $\bar{h} > 0$, consumption and savings are not fixed proportions of labor income: in the first period of life, consumption is lower and savings are higher the larger is $\bar{h}$. The reason is that young agents take into account the future cost of the minimum care to be purchased in the second period of life. The magnitude of this effect on savings depends on the *future* price of care in present-value terms, $p_{t+1}/R_{t+1}$, which is in turn determined by the future wage since $p_{t+1}/R_{t+1} = \eta w_{t+1}/R_{t+1}$. This mechanism, henceforth labelled the *old-age requirement effect*, establishes a precise channel through which relative factor prices affect present savings: high future wages $w_{t+1}$ and low returns on savings $R_{t+1}$ induce higher savings today in order to purchase the minimum amount of care tomorrow.

Considering generic consumption in the second period of life, each old agent purchases

$$d_t = (1+n)\left[\ell_t - (1-\alpha)\right] B\left(\kappa_t/\ell_t\right)^{\alpha}, \tag{15}$$

which is the residual (per-old) output of the generic sector after consumption and savings of young agents have been subtracted. Result (15) implies that second-period consumption is positive only if $\ell_t > 1-\alpha$, which, as we will see, always turns out to be the case in equilibrium.

The last condition for utility maximization links the old agents' expenditure shares over the two goods to their relative price:

$$\frac{p_t \cdot \left(h_t - \bar{h}\right)}{d_t} = \left(\frac{1-\gamma}{\gamma}\right)^{\sigma} p_t^{1-\sigma}. \tag{16}$$

Expression (16) shows that the expenditure share of net care services increases (decreases) with the price when the two goods are complements (substitutes). The reason is that the effect of a ceteris paribus increase in $p_t$ on the expenditure ratio $p_t\left(h_t - \bar{h}\right)/d_t$ depends on the elasticity of the relative demand for care services. Under complementarity, demand is relatively rigid and the increase in $p_t$ raises the expenditure share of net care. Under substitutability, instead, demand is elastic and the opposite happens. These substitution effects bear crucial consequences for the allocation of labor, as shown below.[12]

### *3.3. Labor Market*

The labor demand schedules of the two production sectors determine a unique equilibrium in the labor market. From (9) and (11), wage equalization between sectors implies

$$p_t = (B/\eta)(1-\alpha)\left(\kappa_t/\ell_t\right)^{\alpha} \equiv \Phi\left(\ell_t, \kappa_t\right). \tag{17}$$

Condition (17) defines $p_t$ as the level of the price of care ensuring wage equalization for given levels of sectoral employment, capital per worker, and productivity. In particular, function $p_t = \Phi\left(\ell_t, \kappa_t\right)$ is strictly decreasing in $\ell_t$. The intuition is that for a given capital per young $\kappa_t$,

[12] Substitution effects only disappear with Cobb-Douglas preferences: when $\sigma = 1$, relative expenditure shares are exclusively determined by the taste parameter $\gamma$ and do not depend on the relative price $p_t$.

higher employment in the generic sector decreases the marginal productivity of labor, implying a lower wage, and thus a lower price of care.

### 3.4. Goods Markets

We characterize the equilibrium in the goods market by solving the demand relationship (16) for the price of care, and substituting $p_t h_t / d_t$ with the market-clearing and zero-profit conditions holding for the producing firms, obtaining (see appendix)

$$p_t = \left( \frac{1-\gamma}{\gamma} \right)^{\frac{\sigma}{\sigma-1}} \left[ \frac{(1-\alpha)\left(\ell^{\max} - \ell_t\right)}{\ell_t - (1-\alpha)} \right]^{\frac{1}{1-\sigma}} \equiv \Psi\left(\ell_t\right). \tag{18}$$

Expression (18) defines $p_t$ as the price of care that ensures equilibrium in the goods market.[13] The most important insight is that the function $p_t = \Psi(\ell_t)$ is strictly decreasing when $\sigma < 1$, and strictly increasing when $\sigma > 1$. When $\sigma < 1$ the price of care is positively related to the employment share in the care sector $1 - \ell_t$. The reason is that a ceteris paribus increase in $p_t$ increases the expenditure share that old consumers devote to care services, attracting labor in the care sector. When $\sigma > 1$, in contrast, a higher price of care induces a lower expenditure share of care, and thus more labor in the generic sector.[14]

### 3.5. Employment and Capital Co-Movements

Consider now the joint equilibrium of the markets for labor and for goods. The two relevant conditions, (17) and (18), imply that the price of care and sectoral employment levels in each period $t$ depend on current capital per worker, $\kappa_t$. Formally, the employment share of the generic sector for a given level of $\kappa_t$, denoted by $\ell_t = \ell(\kappa_t)$, is the fixed point

$$\ell\left(\kappa_t\right) \equiv \arg\operatorname{solve}_{\{\ell_t \in (1-\alpha, \ell^{\max})\}} \left[ \Phi\left(\ell_t, \kappa_t\right) = \Psi\left(\ell_t\right) \right]. \tag{19}$$

The existence and uniqueness of this fixed point can be verified in graphical terms in Figure 2 (see the appendix for a formal proof). On the one hand, the function $\Phi(\ell_t, \kappa_t)$ is strictly decreasing in $\ell_t$ and exhibits positive vertical intercepts at the boundaries of the relevant

[13] Function $\Psi(\ell_t)$ does not depend on capital per worker because, with Cobb-Douglas technologies, the sector allocation of labor alone determines the sectoral output ratio $X_t / p_t H_t$.

[14] It should be noted that, in the special case of unit elasticity of substitution, $\sigma = 1$, expression (18) does not hold because price and quantity effects on the demand side balance each other. As a result, the equilibrium between demand and supply in the goods market is characterized by constant employment shares, with $\ell_t = \frac{(1-\alpha)(\gamma \ell^{\max} + 1 - \gamma)}{\gamma(1-\alpha) + 1 - \gamma}$ at each $t$.

interval $\ell_t \in (1-\alpha, \ell^{\max})$. On the other hand, the function $\Psi(\ell_t)$ is decreasing (increasing) under complementarity (substitutability), and display asymptotic properties that ensure the existence and uniqueness of the fixed point $\Psi(\ell_t) = \Phi(\ell_t, \kappa_t)$ within the relevant interval $\ell \in (1-\alpha, \ell^{\max})$.[15] The fixed point (19) simultaneously determines employment shares and the price of care. Substituting $\ell(\kappa_t)$ in $\Psi(\ell_t)$ or in $\Phi(\ell_t, \kappa_t)$ we obtain the equilibrium price of care for given capital per worker,

$$p(\kappa_t) \equiv \Psi(\ell(\kappa_t)) = \Phi(\ell(\kappa_t), \kappa_t). \tag{20}$$

Even though we have not yet specified whether and how capital grows, result (20) clarifies how capital accumulation affects the price of care and employment shares:

**Proposition 1** *An equilibrium trajectory with positive accumulation implies a rising price of care. Under complementarity the employment share in the generic sector is decreasing. Under substitutability the employment share in the generic sector is increasing;*

$$\kappa_{t+1} > \kappa_t \qquad \Longleftrightarrow \qquad p_{t+1} > p_t$$

*and*

$$\kappa_{t+1} > \kappa_t \Rightarrow \left\{ \begin{array}{c} \ell_{t+1} < \ell_t \text{ if } \sigma < 1 \\ \ell_{t+1} > \ell_t \text{ if } \sigma > 1 \end{array} \right\}$$

**Proof.** The proposition can be proved in graphical terms.[16] Since $\partial\Phi(\ell, \kappa)/\partial\kappa > 0$, an increase in $\kappa$ shifts the $\Phi(\ell, \kappa)$ curve up-rightward in Figure 2. The resulting equilibrium price $p(\kappa)$ is necessarily higher but $\ell(\kappa)$ reacts differently depending on the value of $\sigma$. The employment share $\ell(\kappa)$ increases under complementarity, decreases under substitutability:

$$\ell'_\kappa \equiv \frac{\mathrm{d}\ell(\kappa_t)}{\mathrm{d}\kappa_t} < 0 \quad \text{if } \sigma < 1; \quad > 0 \quad \text{if } \sigma > 1 \ \cdot$$

■

The intuition is that an increase in capital per young increases the equilibrium wage and thereby the price of care. Under complementarity, old agents react to the price increase by

[15] See the appendix for further details.

[16] Proposition 1 is equivalently proved by differentiating the equilibrium condition $\Psi(\ell(\kappa_t)) = \Phi(\ell(\kappa_t), \kappa_t)$. The exact relationship between $\kappa$ and $\ell$ is reported in expression (30) below, and indeed implies that $\ell'_\kappa \equiv \mathrm{d}\ell(\kappa_t)/\mathrm{d}\kappa_t$ is strictly negative (positive) under complementarity (substitutability).

raising the share of expenditure on net care, which decreases the employment share in the generic sector $\ell(\kappa)$. Under substitutability, instead, old agents reduce the expenditure share on net care and employment in the generic sector rises.

### *3.6. Static Equilibrium Comparative Statics*

For a given capital stock, the static equilibrium labor allocation depends on the parameters in the model. We investigate for later use how employment shares depend on productivity $B$, on population growth $n$, and on the level of the minimum requirement $\bar{h}$. The properties of $\ell(\kappa_t) = \ell\left(\kappa_t; B, n, \bar{h}\right)$ are summarized in the following Proposition:

**Proposition 2** *In the static equilibrium with given $\kappa_t$,*

$$\frac{d\ell\left(\kappa_t; B, n, \bar{h}\right)}{dB} \equiv \ell'_B < 0 \quad \textit{if } \sigma < 1; \quad > 0 \quad \textit{if } \sigma > 1 \ , \tag{21}$$

$$\frac{d\ell\left(\kappa_t; B, n, \bar{h}\right)}{d\bar{h}} \equiv \ell'_{\bar{h}} < 0, \tag{22}$$

$$\frac{d\ell\left(\kappa_t; B, n, \bar{h}\right)}{dn} \equiv \ell'_n > 0 \ \textit{if } \bar{h} > 0 \quad (= 0 \ \textit{if } \bar{h} = 0). \tag{23}$$

**Proof.** The proposition may be proved in graphical terms. An increase in $B$ shifts $\Phi(\ell, \kappa)$ upward in Figure 2. The employment share $\ell$ increases when $\sigma < 1$, and decreases when $\sigma > 1$. Changes in $n$ and in $\bar{h}$ operate through $\ell^{\max}$ in the expression for $\Psi(\ell)$ in equation (18). An increase in $\ell^{\max}$ shifts $\Psi(\ell)$ to the right, increasing $\ell$. Provided $\bar{h} > 0$, A higher $n$ and a lower $\bar{h}$ both imply a higher $\ell^{\max} \equiv 1 - \frac{\bar{h}}{\eta(1+n)}$. ■

A higher $B$ expands production possibilities in the generic sector and affects the labor allocation depending on the value of $\sigma$. Under complementarity, consumers wish to exploit the productivity gain to purchase more care, and such higher demand pushes labor into the care sector. Under substitutability, instead, labor is drawn into the generic sector as old agents increase their relative demand for consumption goods. The effects of changes in $\ell^{\max}$ are more clear-cut: when a larger fraction of workers is needed to satisfy the minimum care requirement, the care sector will employ more workers.

### 3.7. *Saving Rates and Accumulation*

Before studying in detail the dynamics, we describe the general relationships linking saving rates, capital accumulation and sectoral employment shares. Considering the economy's aggregate income (13) and the wage rate (11), the total labor share accruing to young agents is

$$\frac{w_t N_t^y}{Y_t} = \frac{(1-\alpha)\frac{x_t}{\ell_t}}{x_t\left(\frac{1-\alpha}{\ell_t}+\alpha\right)} = \frac{1-\alpha}{1-\alpha\,(1-\ell_t)}, \tag{24}$$

Equation (24) shows that, in static equilibrium, an increase in the generic sector employment share $\ell_t$ reduces the total income share of young agents. The intuition is that if labor moves from the care sector to generic production, the return to capital increases relative to the wage rate, and this implies a shift in the income distribution away from the young towards the old. We will refer to this result as to the *intergenerational distribution effect.*

Since only young agents save, the intergenerational distribution directly influences the economy's saving rate and, hence, capital accumulation. The savings rate is denoted by $\theta_t$ and is defined as aggregate savings relative to the total value of production. Combining the saving function in (14) with expression (24), and substituting $\ell^{\max}$ by (10), we obtain

$$\theta_t \equiv \frac{N_t^y s_t}{Y_t} = \underbrace{\frac{\beta\,(1-\alpha)}{1+\beta}}_{\text{Canonical model}} \cdot \underbrace{\frac{1}{1-\alpha\cdot(1-\ell_t)}}_{\text{Intergenerational Distribution}} \cdot \underbrace{\Gamma\left(\frac{\bar{h}}{\ell_{t+1}}\right)}_{\text{Old-age Requirement}}, \tag{25}$$

where

$$\Gamma\left(\frac{\bar{h}}{\ell_{t+1}}\right) \equiv \left[1-\frac{(1-\alpha)}{\alpha\,(1+\beta)\,\eta(1+n)}\frac{\bar{h}}{\ell_{t+1}}\right]^{-1}, \quad \Gamma'(\cdot)>0, \quad \Gamma(0)=1. \tag{26}$$

Expression (25) shows that the savings rate is negatively related to both $\ell_t$ and $\ell_{t+1}$. The *current* employment share of the generic sector, $\ell_t$, affects the saving rate through the intergenerational distribution channel described above. The anticipated *future* employment share, $\ell_{t+1}$, affects the saving rate through the function $\Gamma(\cdot)$, which captures the old-age requirement effect – i.e., extra savings induced by the existence of a minimum care requirement: being increasing in $\bar{h}$, the term $\Gamma(\cdot)$ equals unity when $\bar{h}=0$ and strictly exceeds unity when $\bar{h}>0$.[17] The comparison with the canonical model is straighforward. If we remove the care sector, the

[17] In the appendix we show that the static equilibrium conditions imply $(1-\alpha)\,\bar{h} < \alpha\,(1+\beta)\,\eta(1+n)\ell_{t+1}$, from which it follows that $\Gamma\left(\bar{h}/\ell_{t+1}\right) > 1$ for any $\bar{h}>0$.

last two terms in (25) reduce to unity, and the saving rate equals the fraction of income saved by the young, $\beta/(1+\beta)$, times the income share of the young, $1-\alpha$.

Our preliminary conclusion is twofold. First, both the intergenerational distribution and the old-age requirement effects push the saving rate *above* the level predicted by the canonical model. Second, the saving rate is, in general, *not constant* over time and in particular, it will be *increasing over time* if the economy follows an equilibrium path along which the employment share of the generic sector $\ell_t$ grows over time.

## 4. Dynamic General Equilibrium

Since the generic consumption good is produced by means of a neoclassical technology, the dynamic equilibrium path of the economy admits a long-run steady state in which capital per worker is constant, and generic production grows at the exogenous rate of population growth. This section derives the stability properties of the long-run steady state and shows that the transitional dynamics arising under complementarity match qualitatively the stylized facts that inspire our analysis (cf. Introduction). In the long run, the intergenerational distribution and the old-age requirement effects affect, through distinct channels, the steady-state level of capital per worker which is thus higher than in the canonical model.

### *4.1. Accumulation Law*

The equality between investment and savings implies that capital per worker is determined by previous savings according to

$$\kappa_{t+1} = \frac{\theta_t Y_t}{1+n}. \tag{27}$$

This market clearing condition, combined with the saving decisions of young agents, yields the dynamic law that governs capital accumulation in the economy: by substituting (25) and (13) in the right hand side of (27), we obtain

$$\kappa_{t+1} = \underbrace{\frac{B\beta\left(1-\alpha\right)}{\left(1+\beta\right)\left(1+n\right)}\kappa_t^{\alpha}}_{\text{Canonical model}} \cdot \underbrace{\ell_t^{-\alpha}}_{\text{Intergen. Distr.}} \cdot \underbrace{\Gamma\left(\frac{\bar{h}}{\ell_{t+1}}\right)}_{\text{Old-age requirement effect}}. \tag{28}$$

Expression (28) decomposes the accumulation law of capital per worker in three parts. The first term on the right hand side is the dynamic law in the canonical one-good model. The second and third terms on the right hand side of (28) directly follow from the intergenerational distribution effect and the old-age requirement effect. An increase in $\ell_t$ reduces $\kappa_{t+1}$ because a lower current wage reduces young agents' income, and thereby, current savings. An increase in $\ell_{t+1}$ reduces $\kappa_{t+1}$ because a lower future wage reduces the expected future cost of health care, and thereby, current savings.

The presence of anticipated future variables in the right hand side of (28) implies that further work is needed to characterize the equilibrium path. Recalling result (19), equilibrium employment shares are a function of the capital stock per worker in each period. By substituting $\ell_t = \ell(\kappa_t)$ and $\ell_{t+1} = \ell(\kappa_{t+1})$ into (28), we obtain the accumulation law

$$\kappa_{t+1} = \frac{B\beta(1-\alpha)}{(1+\beta)(1+n)}\kappa_t^{\alpha}\left[\ell(\kappa_t)\right]^{-\alpha}\Gamma\left(\frac{\bar{h}}{\ell(\kappa_{t+1})}\right). \tag{29}$$

Expression (29) implies that capital dynamics crucially depend on how sectoral employment shares react to variations in capital per worker. In this respect, the relevant elasticity is[18]

$$\frac{\ell'_{\kappa}(\kappa_t)\,\kappa_t}{\ell(\kappa_t)} = \frac{1}{1-\frac{1}{1-\sigma}\frac{1}{\alpha}Q_1}\left\{\begin{array}{l} <0 \text{ if } \sigma<1 \\ >0 \text{ if } \sigma>1 \end{array}\right\}, \tag{30}$$

where $Q_1 \equiv \frac{\ell_t}{\ell_t-(1-\alpha)}\cdot\frac{\ell^{\max}-(1-\alpha)}{\ell^{\max}-\ell_t} > 1$. The slope of the accumulation law can be found by taking the elasticity of (29) with respect to $\kappa_t$ and $\kappa_{t+1}$, which yields[19]

$$\frac{\mathrm{d}\kappa_{t+1}}{\mathrm{d}\kappa_t}\frac{\kappa_t}{\kappa_{t+1}} = \frac{\alpha-\alpha\frac{\ell'_{\kappa}(\kappa_t)\kappa_t}{\ell(\kappa_t)}}{1+\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\kappa_{t+1})}\frac{\ell'_{\kappa}(\kappa_{t+1})\kappa_{t+1}}{\ell(\kappa_{t+1})}}. \tag{31}$$

In the numerator of (31), the direct effect on $\kappa_{t+1}$ of an increase in $\kappa_t$ is larger under complementarity, i.e., when $\ell'_{\kappa}(\kappa_t) < 0$. When $\bar{h} > 0$, there is also an indirect effect via the increase in $\ell(\kappa_{t+1})$, captured in the denominator. We note, in passing, the possibility of (local) instability and multiple steady states which, however, turns out to be remote: non-uniqueness and instability might only occur under unreasonable parameter values (see appendix). Armed with these results, we can fully characterize the equilibrium path of the economy. The following subsections show that the intergenerational distribution and the old-age requirement effects

[18] Expression (30) is obtained by differentiating the equilibrium condition $\Psi(\ell(\kappa_t)) = \Phi(\ell(\kappa_t), \kappa_t)$ and is fully derived in appendix. The fact that $Q_1 > 1$ directly follows from the requirement $1-\alpha < \ell_t < \ell^{\max}$ and it implies the signs reported in (30). Note that (30) yields an alternative proof of Proposition 1.

[19] Totally differentiating (29) yields $\frac{\mathrm{d}\kappa_{t+1}}{\kappa_{t+1}} = \alpha\frac{\mathrm{d}\kappa_t}{\kappa_t} - \alpha\frac{\partial\ell(\kappa_t)}{\partial\kappa_t}\frac{1}{\ell(\kappa_t)}\mathrm{d}\kappa_t - \frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell_{t+1}}\frac{\partial\ell(\kappa_{t+1})}{\partial\kappa_{t+1}}\frac{1}{\ell(\kappa_{t+1})}\mathrm{d}\kappa_{t+1}$, which can be rearranged to obtain (31).

raise the long-run capital stock above the canonical level through distinct channels. In order to obtain transparent results, subsection 4.2. investigates the case without minimum care requirement, $\bar{h}=0$. Subsection 4.3. extends the analysis to the more general case with $\bar{h}>0$.

### *4.2. Dynamics without Minimum Requirement*

When there is no minimum care requirement for old agents, capital accumulation obeys a fairly simple dynamic law. This subsection assumes for simplicity that the elasticity of capital in generic production is not too high:

**Assumption 1:** $\alpha<\frac{3}{4}$.

This assumption is sufficient but not necessary for the steady state to be unique.[20] The next Proposition establishes that the steady state is globally stable under both complementarity and substitutability: the economy converges towards a long-run equilibrium in which capital per worker, the price of health care and employment shares are constant.

**Proposition 3** *In the neoclassical case with $\bar{h}=0$, capital per worker obeys*

$$\kappa_{t+1}=\frac{\beta\eta}{(1+n)(1+\beta)}p(\kappa_t), \tag{32}$$

*where $p(\kappa_t)$ is the price of health care determined by (20). Under Assumption 1 the steady state $\kappa_{ss}=\frac{\beta\eta}{(1+n)(1+\beta)}p(\kappa_{ss})$ is unique and globally stable:*

$$\lim_{t\to\infty}\kappa_t=\kappa_{ss},\quad \lim_{t\to\infty}\ell_t=\ell(\kappa_{ss}),\quad \lim_{t\to\infty}p_t=p(\kappa_{ss}).$$

*During the transition, given a positive initial stock $\kappa_0<\kappa_{ss}$, both capital per worker and the price of health care increase; under complementarity (substitutability), employment in the generic sector declines (increases) and the saving rate increases (declines):*

$$\kappa_{t+1}>\kappa_t,\quad p_{t+1}>p_t,\qquad \left\{\begin{array}{l}\ell_{t+1}<\ell_t \text{ and } \theta_{t+1}>\theta_t \text{ if } \sigma<1\\ \ell_{t+1}>\ell_t \text{ and } \theta_{t+1}<\theta_t \text{ if } \sigma>1\end{array}\right\}. \tag{33}$$

**Proof.** Expression (32) follows from setting $\bar{h}=0$ in (29) and substituting (17) and (20). Result (33) follows from Proposition 1 combined with (25) that establishes $\theta_t$ be decreasing in

[20] In Appendix B, we solve the general model for the case in which Assumption 1 is not satisfied. Moreover, under substitutability, the steady state is unique and stable independently of the parameter values.

$\ell_t$. For $\kappa_{ss}$ to be stable and unique, the elasticity (31) evaluated in $\kappa_{ss}$ must be less than unity. Inserting $\kappa_t = \kappa_{t+1} = \kappa_{ss}$ and $\Gamma = 1$ and $\Gamma' = 0$ in (31), the elasticity reduces to

$$\frac{\mathrm{d}\kappa_{t+1}}{\mathrm{d}\kappa_t} = \alpha - \alpha \frac{\ell'_\kappa(\kappa_{ss})\,\kappa_{ss}}{\ell(\kappa_{ss})},$$

where the right hand side is less than unity if and only if $m_1 < 1$, where

$$m_1(\kappa_{ss}) \equiv -\frac{\ell'_\kappa(\kappa_{ss})\,\kappa_{ss}}{\ell(\kappa_{ss})}\frac{\alpha}{1-\alpha} \tag{34}$$

In the appendix we show that Assumption 1 is a sufficient condition for $m_1 < 1$, and we also prove existence. ■

Proposition 3 suggests three remarks. First, the dynamic law (32) shows that, with no minimum care requirement, investment per young is proportional to the price of care. The reason is that, when $\bar{h} = 0$, savings only depend on current wages. Second, given that capital per worker grows monotonically, both the wage and the price of care increase over time. Employment shares, however, move in opposite directions depending on the value of $\sigma$, which determines whether the expenditure share of care services increases or decreases in response to increasing prices. The third remark is that, under complementarity, the savings rate $\theta_t$ increases during the transition because rising care prices attract labor in the care sector and the income share of young agents then grows – i.e., the intergenerational distribution effect.

The long-run consequences of the intergenerational distribution effect become evident by comparing the steady-state level of the capital stock, $\kappa_{ss}$, with that arising in the canonical model, denoted by $\kappa_{ss}^{\text{canonical}}$. From (28), imposing $\bar{h} = 0$ and $\kappa_{t+1} = \kappa_t = \kappa_{ss}$ yields

$$\kappa_{ss} = \frac{1}{\ell(\kappa_{ss})^{\frac{\alpha}{1-\alpha}}}\left[\frac{B\beta(1-\alpha)}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}} = \kappa_{ss}^{\text{canonical}} \cdot \frac{1}{\ell(\kappa_{ss})^{\frac{\alpha}{1-\alpha}}}, \tag{35}$$

where $\kappa_{ss}^{\text{canonical}}$ is obtained by setting $\ell_t = 1$ in each period, and equals

$$\kappa_{ss}^{\text{canonical}} = \left[\frac{B\beta(1-\alpha)}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}}. \tag{36}$$

It follows from (35) that $\kappa_{ss} > \kappa_{ss}^{\text{canonical}}$ always holds as long as $\ell(\kappa_{ss}) < 1$. Therefore, capital per worker in the long run is higher than in the canonical model independently of whether generic goods and care services are complements or substitutes: for any value of $\sigma$, the need for care services increases the demand for labor, pushing up the income share of young cohorts and thereby the saving rate.

### 4.3. Dynamics with Minimum Care Requirement

When the minimum old-age care requirement is strictly positive, $\bar{h} > 0$, the accumulation law (28) includes the dependency of current savings on future employment shares, i.e. the old-age requirement effect. This dynamic law determines the steady state of the system and the associated stability properties. Under substitutability, $\sigma > 1$, there always exists a unique steady state. The case of complementarity, $\sigma < 1$, can be studied more easily by assuming, again, that the production elasticity of capital is not too high:

**Assumption 2:** $\alpha < \frac{1-\alpha}{1-\sigma}$.

This assumption is sufficient but not necessary for the steady state to be unique.[21]

**Proposition 4** *Under Assumption 2, equation (29) exhibits a unique steady state $\bar{\kappa}_{ss}$ that is globally stable. The transitional dynamics of $p(\kappa_t)$ and $\ell(\kappa_t)$ comply with Proposition 1.*

**Proof.** For $\bar{\kappa}_{ss}$ to be stable and unique, the elasticity (31) evaluated in $\bar{\kappa}_{ss}$ must be less than unity. Inserting $\kappa_t = \kappa_{t+1} = \bar{\kappa}_{ss}$ in (31), the elasticity reduces to

$$\frac{\mathrm{d}\kappa_{t+1}}{\mathrm{d}\kappa_t} = \frac{\alpha - \alpha\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}}{1 + \frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}},$$

where the right hand side is less than unity if and only if

$$m_1(\bar{\kappa}_{ss}) + m_2(\bar{\kappa}_{ss}) < 1, \tag{37}$$

with

$$m_2(\bar{\kappa}_{ss}) \equiv -\frac{\ell'_\kappa(\bar{\kappa}_{ss})\,\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\frac{1}{1-\alpha}\begin{cases} < 1 \text{ if } \sigma < 1 \\ < 0 \text{ if } \sigma > 1 \end{cases}. \tag{38}$$

In the the appendix we show that Assumption 2 is a sufficient condition for (37) to be satisfied.

■

Proposition 4 establishes that, even in the general case with positive minimum care requirement, $\bar{h} > 0$, complementarity is associated with increasing savings rates during the transition.

[21] In Appendix B, we solve the model for the case in which Assumption 2 is not satisfied.

This is the combined result of the old-age requirement and intergenerational distribution effects. By imposing $\kappa_{t+1} = \kappa_t = \bar{\kappa}_{ss}$ in (28), the steady-state level of capital per worker equals

$$\bar{\kappa}_{ss} = \kappa_{ss}^{\text{canonical}} \cdot \frac{1}{\ell\left(\bar{\kappa}_{ss}\right)^{\frac{\alpha}{1-\alpha}}} \cdot \Gamma\left(\frac{\bar{h}}{\ell\left(\bar{\kappa}_{ss}\right)}\right)^{\frac{1}{1-\alpha}}. \tag{39}$$

Since $\Gamma(\cdot)$ strictly exceeds one when $\bar{h} > 0$, result (39) establishes that $\bar{\kappa}_{ss} > \kappa_{ss} > \kappa_{ss}^{\text{canonical}}$. That is, the long-run level of capital per worker is higher when there is a positive minimum requirement of old-age care, which prompts young agents to save more during the transition in response to the continuous increase of the price of care services. Expression (39) will be exploited in the quantitative analysis of section 6. to calculate the impact of exogenous shocks on capital per worker in a calibrated version of our model.

Our main remark is that, under complementarity, $\sigma < 1$, the transitional dynamics of our model capture very well the stylized facts that inspired the analysis. During the transition to the steady state, the saving rate grows, the price of care services and the wage rate increase over time, the income distribution shifts in favor of young workers, and the employment share of the generic sector declines. Several developing countries, and in particular, China in the last two decades, experienced the same qualitative dynamics as documented in the Introduction. Since the hypothesis $\sigma < 1$ is also empirically plausible (Finkelstein et al. 2012), the remainder of the analysis will focus on the case of complementarity.

## 5. Savings Multipliers

This section introduces the concept of *savings multiplier* (subsection 5.1.) and describes its use in the analysis of three types of exogenous shocks: increased productivity (subsect. 5.2.), reduced fertility (subsect. 5.3.) and increased minimum care requirement (subsect. 5.4.). The nature of these shocks may be conceptually linked to the effects of past reforms in China, in particular, the one-child policy and the dismantling of social benefits.

### *5.1. Conceptual Definition*

The intergenerational distribution and the old-age requirement effects create *feedback mechanisms* whereby capital accumulation stimulates further savings and, hence, further accumulation. These feedback effects bear major consequences for the economy's response to exogenous shocks: following a change in the value of a parameter, the resulting change in the long-run level of capital per worker must include the cumulative impact of all the feedback effects that operate during the transition to the new steady state. Therefore, in our model with complementarity, the long-run effects of exogenous shocks are always amplified by a 'savings multiplier', which measures the impact of the feedback effects that raise savings during the transition.

### 5.2. *Productivity Shocks*

We henceforth assume $\sigma < 1$ for the reasons explained in the previous section.[22] Consider a productivity shock taking the form of an exogenous increase in $B$. In the canonical model, this shock would increase the long-run level of (log) capital per worker in (36) by

$$\frac{\mathrm{d}\log \kappa_{ss}^{\text{canonical}}}{\mathrm{d}B} = \frac{1}{B\left(1-\alpha\right)}. \tag{40}$$

In our model, the impact of the shock is magnified by both the intergenerational distribution and the old-age requirement effects. To preserve expositional clarity, we first consider the case with zero minimum requirement.

*Zero minimum requirement.* With $\bar{h} = 0$, the steady-state capital per worker is $\kappa_{ss}$ defined in (35), and the impact of the productivity shock is determined by

$$\frac{\mathrm{d}\log \kappa_{ss}}{\mathrm{d}B} = \underbrace{\frac{1}{1-m_1\left(\kappa_{ss}\right)}}_{\text{Savings Multiplier}} \left(\frac{\mathrm{d}\log \kappa_{ss}^{\text{canonical}}}{\mathrm{d}B} + m_1\left(\kappa_{ss}\right)\frac{\ell_B'\left(\kappa_{ss}\right)}{\ell_\kappa'\left(\kappa_{ss}\right)\cdot\kappa_{ss}}\right), \tag{41}$$

The crucial element in (41) is the savings multiplier, where $m_1$ is already defined in (34). Under complementarity, $m_1$ is strictly positive, and is less than unity in view of the stability of the steady state.[23] Since $0 < m_1 < 1$, the savings multiplier in (41) is strictly higher than unity. Combining this result with $\ell_\kappa' < 0$ and $\ell_B' < 0$,[24] it follows that the impact of a productivity shock on steady-state capital per worker is stronger than that predicted by the canonical model. There are two reasons for this, both related to the intergenerational distribution effect. First, the productivity increase modifies the static equilibrium of the labor market: workers move

[22] All the equations that follow are identical under substitutability, the only difference being in the strength of the effects: the saving multipliers exceed unity when $\sigma < 1$ and fall short of unity when $\sigma > 1$. Hence, shocks that are magnified with complementarity are instead dampened with substitutability.

[23] Under complementarity, $m_1$ is positive because $\ell_\kappa' < 0$ – see expression (30) – and is strictly less than unity in view of the stability condition proven in Proposition 3. Under substitutability, instead, expression (30) implies $\ell_\kappa' > 0$ and therefore $m_1 < 0$.

[24] Under complementarity, $\ell_\kappa' < 0$ follows from (30) whereas $\ell_B' < 0$ is established in Proposition 2.

out of generic production and into the care sector, increasing the wage further relative to the canonical model. This 'static reallocation effect', represented by the term $m_1 \ell_B' / (\ell_\kappa' \kappa) > 0$, increases both firms' demand for capital and current savings. Second, as the capital stock starts to grow, further labor is pushed out of generic production and into care, increasing the wage even further and thus magnifying the initial increase in savings: the cumulative impact of such 'dynamic feedback effects' is represented by the savings multiplier, $1/(1-m_1)$. The combination of these static and dynamic reallocation effects thus yields a larger overall impact of productivity shocks than in the canonical model.

*Positive minimum requirement.* With $\bar{h} > 0$, the savings multiplier is modified by the old-age requirement effect. From (39), the effects of increased productivity on long-run capital is now given by

$$\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}B} = \underbrace{\frac{1}{1 - m_1(\bar{\kappa}_{ss}) - m_2(\bar{\kappa}_{ss})}}_{\text{Savings multiplier}} \left[\frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{\mathrm{d}B} + \frac{(m_1(\bar{\kappa}_{ss}) + m_2(\bar{\kappa}_{ss}))\,\ell_B'(\bar{\kappa}_{ss})}{\ell_\kappa'(\bar{\kappa}_{ss})\,\bar{\kappa}_{ss}}\right], \quad (42)$$

where $m_2$ is defined in (38). Under complementarity, the term $m_1 + m_2$ is strictly positive, and is less than unity in view of the stability of the steady state.[25] Since $0 < m_1 + m_2 < 1$, the savings multiplier in (42) exceeds unity. Compared to the case with zero requirement – cf. expression (41) – the impact of increased productivity on steady-state capital is now strenghtened in two respects. First, the 'static reallocation effect' that raises the equilibrium wage now induces larger savings because higher wages also mean a higher anticipated cost of minimum care in the second period of life: the additional increase in savings is determined by the presence of $m_2$ inside the last term of (42). Second, the 'dynamic feedback effects' are stronger because rising wages during the transition prompt young agents to raise their savings further due, again, to the old-age requirement mechanism: this is why the savings multiplier, $1/(1-m_1-m_2)$, is larger than in the previous case with $\bar{h} = 0$.

[25] Given $\sigma < 1$, both $m_1$ and $m_2$ are positive because $\ell_\kappa' < 0$ – see expression (30) – and $m_1 + m_2$ is strictly less than unity in view of the stability condition (37) proven in Proposition 4. Under substitutability, instead, expression (30) would imply $\ell_\kappa' > 0$, $m_1 + m_2 < 0$ and, hence, a multiplier below unity.

*5.3. Reduced Fertility*

In the canonical model, a lower growth rate of population increases the steady-state level of capital per worker: from (36), we have

$$\frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{-\mathrm{d}n} = \frac{1}{(1+n)(1-\alpha)} > 0. \tag{43}$$

In contrast, from (39), the effect of reduced fertility in our model is given by

$$\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{-\mathrm{d}n} = \frac{1}{1-m_1-m_2}\left[\frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{-\mathrm{d}n} + \frac{\ell_n'}{(-\ell_\kappa')\bar{\kappa}_{ss}}(m_1+m_2) + \frac{\ell}{(1+n)(-\ell_\kappa')\bar{\kappa}_{ss}}m_2\right], \tag{44}$$

where we suppress the argument $\bar{\kappa}_{ss}$ to simplify the notation.[26] In expression (44), we can distinguish five effects that do not arise in the canonical model. The first two are included in the multiplier: as exaplined before, the term $1/(1-m_1-m_2) > 1$ represents the positive feedbacks that capital growth exherts on itself due to the intergenerational distribution and the old-age requirement effects. The second and third effects are contained in the term $\frac{\ell_n'}{(-\ell_\kappa')\bar{\kappa}_{ss}}(m_1+m_2)$, which represents the change in the static equilibrium of the labor market: the reduction in fertility increases the fraction of old agents in total population, pushing workers out of generic production and into care services; the resulting wage increase raises the savings rate through both the intergenerational distribution and the old-age requirement effects. The fifth effect is the last term appearing (44), which represents a dilution effect: lower population growth increases labor scarcity even for a fixed labor allocation. The implied rise in wages triggers further savings through the old-age requirement effect.

*5.4. Increased need for care*

In the model, a higher $\bar{h}$ represent an increased need to purchase care services through the market. Obviously, this draws resources out of generic production and into the care sector. By (39), the effect on steady-state capital is

$$\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}\bar{h}} = \frac{1}{1-m_1-m_2}\left[\frac{\ell_{\bar{h}}'}{\ell_\kappa'\cdot\bar{\kappa}_{ss}}(m_1+m_2) - \frac{\ell}{\bar{h}\cdot\ell_\kappa'\bar{\kappa}_{ss}}m_2\right]. \tag{45}$$

Besides the now familiar savings multiplier, a higher minimum requirement induces two types of static effects. First, there is a direct positive effect on the cost of care, represented by the term

[26] In (44), the terms $m_1, m_2, \ell, \ell_n', \ell_\kappa'$ are all evaluated in the steady state $\bar{\kappa}_{ss}$. Also, in deriving (44), we exploit the fact that $\frac{d\Gamma}{dn} = -\Gamma'\frac{\bar{h}}{(1+n)\ell}$ from expression (26). See the appendix for a full derivation.

$-\frac{\ell}{\bar{h}\ell'_{\kappa}\bar{\kappa}_{ss}}m_2$, which increases savings. Second, the static equilibrium of the labor market changes since higher demand for care pulls workers out of generic production and drives up the wage. This effect, represented by the term $\frac{\ell'_{\bar{h}}}{\ell'_{\kappa}\bar{\kappa}_{ss}}(m_1+m_2)$, generates higher savings through both the interegenerational distribution and the old-age requirement effects. With the additional stimulus of the savings multiplier, $1/(1-m_1-m_2)$, the increased need for market-provided care may thus have a strong positive impact on capital accumulation. This possibility is confirmed by our quantitative analysis in section 6..

## 6. Quantitative Analysis

This section presents a quantitative assessment of our theoretical results. Taking China's economy as our empirical reference, we calibrate the parameters to obtain steady-state values that match the most recent data, evaluating the effects of exogenous shocks to assess the sensitivity of steady-state capital to changes in the minimum care requirement.

### *6.1. Calibration strategy*

We consider four reference values of endogenous variables to be matched ex-post; the saving rate, the relative employment shares of manufacturing versus care services, the share of total expenditures devoted to care services (denoted by $TES$), and the capital income share (denoted by $CIS$).[27] Also, we choose ex-ante the values of $\alpha$ and $\sigma$ in line with empirical evidence:

$$\theta = 0.28, \quad \frac{1-\ell}{\ell} = 0.19, \quad TES = 0.083, \quad CIS = 0.46, \quad \alpha = 0.5, \quad \sigma = 0.2.$$

The value $\theta = 0.28$ reflects the most recent data on China's saving rate (Prasad, 2015). The value $\frac{1-\ell}{\ell} = 0.19$ corresponds to paid employment in "Health and Social Work" plus "Social and Personal Service Activities", divided by paid employment in "Manufacturing" in China (ILO, 2015: Table 2.E). The value $TES = 0.083$ is a conservative projection based on Chamon and Prasad (2010).[28] The value $CIS = 0.46$ equals one minus the long-run labor share in GDP

[27] For the mathematical definitions of TES and CIS in our model see the appendix.

[28] Chamon and Prasad (2010: Table A2, p.129) report the 1992-2004 time series of health versus non-health expenditures: the implied $TES$ goes from 2.5% in 1992 to 7.4% in 2004. More recent data on sectoral GDP shares show that, during the 2005-2014 decade, total spending in services went from from 42.9% to 48.2% of GDP (World Development Indicators, 2015). Under the conservative hypothesis that, during the 2005-2014

net of production tax calculated by Bai and Qian (2010). The value $\alpha = 0.5$ is the baseline used in most calibrated models of China (e.g., Song et al. 2011) whereas $\sigma = 0.2$ is the elasticity of substitution between consumption and health care services estimated by Finkelstein et al. (2013). We set the other parameters $(\beta, \gamma, \eta, \bar{h}, B, n)$ so as to obtain steady-state values that match the reference levels of the endogenous variables listed above.

It is worth noting that the reference value $\frac{1-\ell}{\ell} = 0.19$ implies an employment share in the generic sector $\ell = 0.84$. We are thus interpreting the 'generic good' sector of the model as a real-world sector that includes both 'manufacturing' and 'services' excluding 'care services'. The reason is that our aim is to assess to what extent the intergenerational distribution and the old-age requirement effects influence long-run capital even though the care services sector is quantitatively small in terms of both employment and expenditure shares.

### 6.2. *Steady state results*

The calibration yields the parameter values listed in the left panel of Table 1. Population growth is set to $n = 0$ and the combination of $\eta$ and $\bar{h}$ determines a threshold level for the generic sector's employment $\ell^{\max} = 0.9$. The third column shows the steady state values matching the desired levels of the four reference endogenous variables. The last column of Table 1 reports the level and the composition of capital per worker in the long run. These numbers can be interpreted straightforwardly by means of equation (39), where the steady-state capital stock $\bar{\kappa}_{ss}$ is determined by three factors: the canonical component $(\kappa_{ss}^{\text{canonical}})$, the intergenerational distribution effect (IDE), and the old-age requirement (OAR).[29] Table 1 shows that $\bar{\kappa}_{ss}$ exceeds the canonical level by 34%, that is, despite the small size of the care sector, the existence of needs for old-age care ultimately raises the capital stock by one third relative to the canonical model's baseline. The IDE and the OAR factors reported in Table 1 show that the intergenerational distribution effect alone raises capital by 18% over the canonical level, and the old-age requirement effect adds a further 13% gain. The product of IDE and OAR factors, 1.34, determines the overall distance between $\bar{\kappa}_{ss}$ and $\kappa_{ss}^{\text{canonical}}$ and is a noticeable quantitative result.

decade, health expenditures grew at the same rate as total expenditures in services, we obtain an implied $TES = 8.3\%$ for 2014.

[29] From (39), the IDE factor equals $1/\ell^{\frac{\alpha}{1-\alpha}}$ and the OAR factor is $\Gamma\left(\bar{h}/\ell_{ss}\right)^{\frac{1}{1-\alpha}}$.

*6.3. Exogenous shocks and transitional dynamics*

We now study the quantitative effects of three shocks: an increase in $n$, a decrease in $\bar{h}$, and a combination of the two. Conceptually, this exercise may be related to the important reforms recently announced by China's government, namely, the abandonment of the one-child policy and the introduction of pensions and welfare benefits.[30] A basic interpretation is that the increase in $n$ results from the dismantling of the one-child policy, the decline in $\bar{h}$ is caused by the welfare reform (e.g., covering part of the minimum care services required by the old), and the combined shock is caused by both reforms being enacted at the same time. A second possible interpretation is that the two shocks are not independent in reality because a higher $n$ enlarges family size and may thus induce a lower $\bar{h}$ by increasing family-provided care. In this view, the combined shock on $n$ and $\bar{h}$ is the overall effect of abandoning the one-child policy.

In general, an increase in $n$ and a decrease in $\bar{h}$ modify the allocation of labor in the same direction: the employment share of the generic good's sector increases. The difference is that the positive shock on $n$ raises $\ell$ by increasing *total* labor supply whereas the negative shock on $\bar{h}$ raises $\ell$ by increasing *sectoral* labor supply: a lower minimum care requirement increases $\ell^{\max}$, the maximum share of labor that can be devoted to the production of the generic good.[31] Building on this observation, we specify the first shock as 1% increase in the population growth rate, and the second shock as a decline in the value of $\bar{h}$ that corresponds to a 1% increase in $\ell^{\max}$. Starting from the equilibrium values reported in Table 1, we thus have a first scenario where $n$ goes from 0 to 0.01; a second scenario where $\bar{h}$ falls from 1 to 0.9, implying that $\ell^{\max}$ increases from 0.90 to 0.91; and a third scenario in which both shocks arise simultaneously.

Table 2 reports the impact of each shock on the steady-state levels of the relevant variables. In the first scenario, higher population growth reduces capital per worker by 2.3%. In the second scenario, a lower minimum care requirement implies a comparable and even larger drop in capital per worker: $\bar{\kappa}_{ss}$ falls by 2.5% and this decline is entirely determined by non-canonical mechanisms. The third scenario shows that the combined shock reduces capital per worker by 4.7%, of which only 2% is due to canonical mechanisms. According to these numbers,

[30] See, e.g., The Economist (September 8th, 2012) and The Wall Street Journal (October 30th, 2015).

[31] See the definition of $\ell^{\max}$ in expression (10). The intuition is that a reduction in $\bar{h}$ reduces the minimum share of labor that *must* be devoted to care services (for supply to meet the requirement) and therefore increases the maximum share of labor that *can* be devoted to the production of generic goods.

reducing by one tenth the minimum care purchased by old agents, $\bar{h}$, yields larger effects on capital than a 1% increase in the population growth rate, $n$. From a policy perspective, the sensitivity of long-run capital to changes in $\bar{h}$ suggests that welfare reforms may induce strong 'non-canonical' effects on savings and income per capita in the long run.

Considering the transitional dynamics, Figure 3 draws, for each scenario, the equilibrium path followed by the economy for ten periods after the shocks. The time paths of capital per worker induced by the $n$-shock and by the $\bar{h}$-shock look similar but the associated time paths of $\ell_t$ are rather different because the shock on the minimum care requirement involves stronger labor reallocation. This is confirmed in Table 2, which shows that the $\bar{h}$-shock induces more drastic changes in the spending share of care services, $TES_{ss}$, as well as in sectoral employment, $\ell_{ss}$, in the long run.

## 7. Conclusion

This paper introduced the concept of savings multiplier, a general equilibrium mechanism that induces rising saving rates over time and that magnifies the impact of exogenous shocks on capital per capita in the long run. In our theory, capital accumulation yields positive feedbacks on saving rates via two channels. First, real wages increase as the capital stock grows at the same time as workers move from the manufacturing sector to the labor-intensive service sector, implying a shift of the income distribution in favor of young savers (*intergenerational distribution effect*). Second, growth in real wages raises the anticipated cost of providing for the old age, prompting the currently young to save a higher fraction current income (*old-age requirement effect*). Both these mechanisms provide a novel explanation for rising saving rates in developing countries and, more specifically, are consistent with the stylized facts that characterize China's economic performance.

Our analysis of exogenous shocks suggests that China's past reforms – in particular, the one-child policy and the dismantling of cradle-to-grave social benefits – have fuelled China's saving rates in the past decades. Our quantitative analysis shows that capital in the long run may be quite sensitive to changes in the minimum care services required by old agents even though the care sector is small relative to manufacturing and other services. This suggests that the recently announced policy reforms, i.e., the abandonement of the one-child policy and the introduction of welfare benefits, may reduce savings and long-run capital to a much larger extent than what the traditional neoclassical model would predict.

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## Tables

| Parameters | | Parameters | | Steady state values | | Steady state values | |
|---|---|---|---|---|---|---|---|
| $\alpha$ | 0.50 | $\bar{h}$ | 1.00 | $\theta_{ss}$ | 0.281 | $\bar{\kappa}_{ss}$ | 0.210 |
| $\beta$ | 0.95 | $n$ | 0.00 | $\ell_{ss}$ | 0.846 | $\kappa_{ss}^{\text{canonical}}$ | 0.157 |
| $\gamma$ | 0.50 | $B$ | 1.628 | $TES_{ss}$ | 0.083 | (gap) | (34%) |
| $\sigma$ | 0.20 | $\ell^{\max}$ | 0.90 | $CIS_{ss}$ | 0.458 | IDE factor | 1.181 |
| $\eta$ | 10.0 | | | | | OAR factor | 1.133 |

Table 1. Calibration and simulation results (see section 6. for details)

| | Before shock | Shock on $n$ | | Shock on $\bar{h}$ | | Combined shocks | |
|---|---|---|---|---|---|---|---|
| | $(\bar{h} = 1.0)$ | $(\bar{h} = 1.0)$ | change | $(\bar{h} = 0.9)$ | change | $(\bar{h} = 0.9)$ | change |
| $\ell^{\max}$ | 0.90 | 0.90 | | 0.91 | | 0.91 | |
| $n$ | 0.00 | 0.01 | | 0.00 | | 0.01 | |
| $\ell_{ss}$ | 0.847 | 0.848 | 0.2% | 0.856 | 1.1% | 0.857 | 1.2% |
| $TES_{ss}$ | 0.083 | 0.083 | −0.9% | 0.078 | −6.8% | 0.077 | −7.6% |
| $\theta_{ss}$ | 0.281 | 0.280 | −0.1% | 0.277 | −1.2% | 0.277 | −1.3% |
| $\bar{\kappa}_{ss}$ | 0.210 | 0.206 | −2.3% | 0.205 | −2.5% | 0.201 | −4.7% |
| $\kappa_{ss}^{\text{canonical}}$ | 0.157 | 0.154 | −2.0% | 0.157 | 0 | 0.154 | −2.0% |

Table 2. Impact of exogenous shocks (see section 6. for details)

## Figures

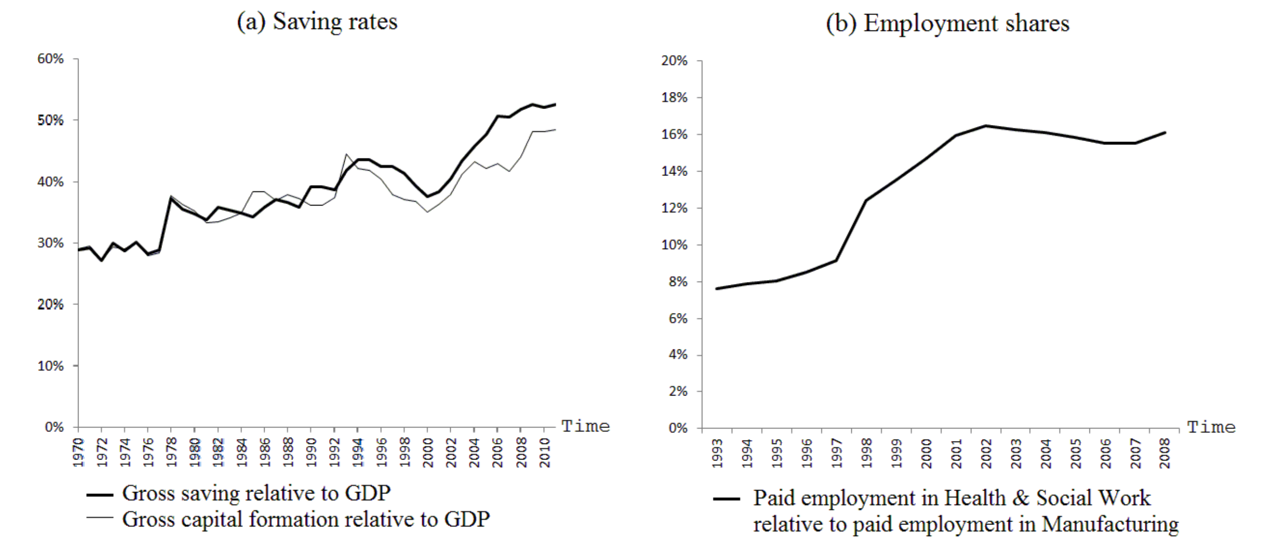


Figure 1: Graph (a): saving and investment shares of GDP in China 1970-2010 (source: World Bank). Graph (b): paid employment in Health and Social Work relative to paid employment in Manufacturing in China 1993-2008 (source: authors calculations on LABORSTA Table 2E, International Labor Organization).

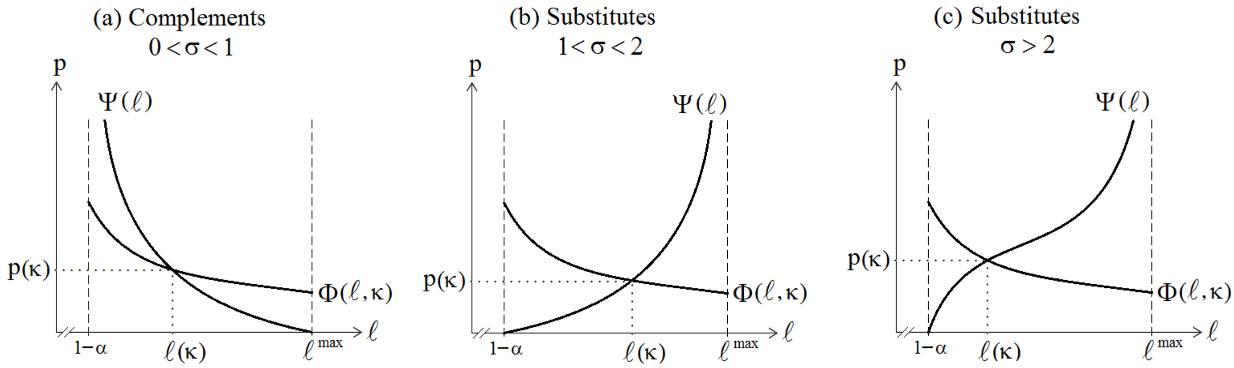


Figure 2: Static equilibrium: determination of $\ell_t$ and $p_t$ for given $\kappa_t$. The case of strong substitution ($\sigma > 2$) implies local concavity of $\Psi(\ell)$ for low $\ell$ without altering existence, uniqueness, and comparative-statics properties.

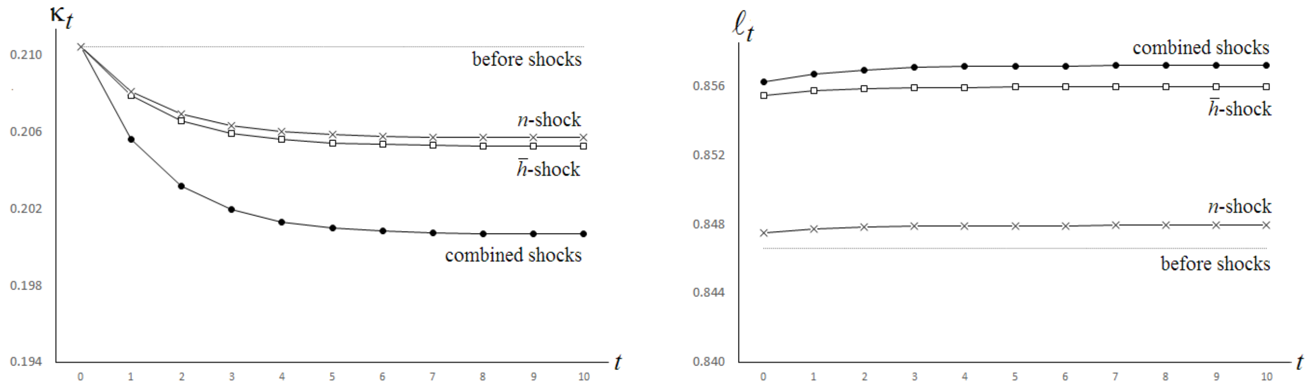


Figure 3: Transitional dynamics induced by exogenous shocks (see section 6. for details).

## A Appendix (main proofs)

**Consumption levels: derivation of (14)-(15).** The household maximizes (2) subject to (3)-(4). The Lagrangean at time $t$ reads

$$\mathcal{L} \equiv u\left(c_t\right) + \beta v\left(d_{t+1}, h_{t+1} - \bar{h}\right) + \lambda_1\left(w_t - s_t - c_t\right) + \lambda_2\left(s_t R_{t+1} - d_{t+1} - p^h_{t+1} h_{t+1}\right)$$

where $\lambda_1$ and $\lambda_2$ are the multipliers. The first-order conditions with respect to $(c_t, d_{t+1}, h_{t+1}, s_t)$ are

$$u_{c_t}\left(c_t\right) = \lambda_1, \tag{A.1}$$

$$\beta v_{d_{t+1}}\left(d_{t+1}, h_{t+1} - \bar{h}\right) = \lambda_2, \tag{A.2}$$

$$\beta v_{h_{t+1}}\left(d_{t+1}, h_{t+1} - \bar{h}\right) = \lambda_2 p_{t+1}, \tag{A.3}$$

$$\lambda_1 = \lambda_2 R_{t+1}. \tag{A.4}$$

Combining (A.1) with (A.2) and (A.2) with (A.3), we respectively obtain

$$u_{c_t}\left(c_t\right) = \beta R_{t+1} v_{d_{t+1}}\left(d_{t+1}, h_{t+1} - \bar{h}\right), \tag{A.5}$$

$$v_{h_{t+1}}\left(d_{t+1}, h_{t+1} - \bar{h}\right) = p_{t+1} v_{d_{t+1}}\left(d_{t+1}, h_{t+1} - \bar{h}\right), \tag{A.6}$$

where (A.5) is the Keynes-Ramsey rule for the generic good, and (A.6) equates the relative price of health care services to the marginal rate of substitution with second-period consumption. Exploiting the assumed utility functions (5)-(6), conditions (A.5)-(A.6) respectively read

$$d_{t+1} = c_t \beta R_{t+1} - \left(h_{t+1} - \bar{h}\right) \frac{1-\gamma}{\gamma} \left(\frac{d_{t+1}}{h_{t+1} - \bar{h}}\right)^{\frac{1}{\sigma}}, \tag{A.7}$$

$$p_{t+1} = \frac{1-\gamma}{\gamma} \left(\frac{d_{t+1}}{h_{t+1} - \bar{h}}\right)^{\frac{1}{\sigma}}. \tag{A.8}$$

Substituting (A.8) in (A.7) gives

$$d_{t+1} + p_{t+1}\left(h_{t+1} - \bar{h}\right) = c_t \beta R_{t+1}. \tag{A.9}$$

Substituting (A.9) in the second-period budget constraint (4) and then using the first-period budget constraint (3) to eliminate savings, we obtain expression (14) in the text. Next, substitute the market clearing condition $K_{t+1} = N_t^y s_t$ into (4) to obtain

$$R_{t+1}K_{t+1} = N_t^y \left(d_{t+1} + p_{t+1}h_{t+1}\right). \tag{A.10}$$

Given the market clearing condition $N_t^o h_t = H_t$, the zero-profit condition for the health care sector reads

$$p_t N_t^o h_t = w_t \left(1 - \ell_t\right) N_t^y. \tag{A.11}$$

Substituting (A.11) for period $t+1$ into (A.10), we obtain

$$N_{t+1}^o d_{t+1} = R_{t+1}K_{t+1} - w_{t+1}\left(1 - \ell_{t+1}\right) N_{t+1}^y. \tag{A.12}$$

From the profit maximizing conditions (12) and (11), respectively, we have

$$R_t K_t = B\alpha \left[N_t^y \left(\kappa_t\right)^\alpha \left(\ell_t\right)^{1-\alpha}\right], \tag{A.13}$$

$$w_t \left(1 - \ell_t\right) N_t^y = B\left(1 - \alpha\right) \frac{1 - \ell_t}{\ell_t} \left[N_t^y \left(\kappa_t\right)^\alpha \left(\ell_t\right)^{1-\alpha}\right]. \tag{A.14}$$

Setting (A.12) at period $t$ and substituting (A.13) and (A.14), we obtain equation (15) in the text. Note that result (15) implies a restriction: second-period generic consumption is positive if and only if $\ell_t > 1 - \alpha$.

**Consumer problem: derivation of (16)**. Setting expression (A.8) at time $t$, raising both sides to the power of $\sigma$, and dividing both sides by $p_t$, we obtain expression (16) in the text.

**Goods Market: derivation of (18)**. Starting from (16), multiply both $d_t$ and $\left(h_t - \bar{h}\right)$ by old population $N_t^o$, and substitute the old agents' constraint $N_t^o d_t = N_t^o s_{t-1} R_t - p_t H_t$, to obtain

$$p_t^{\sigma-1} = \left(\frac{1-\gamma}{\gamma}\right)^\sigma \frac{N_t^o s_{t-1} R_t - p_t H_t}{p_t H_t - p_t N_t^o \bar{h}}. \tag{A.15}$$

Substituting capital income with the profit-maximizing condition $N_t^o s_{t-1} R_t = \alpha X_t$, we get

$$p_t^{\sigma-1} = \left(\frac{1-\gamma}{\gamma}\right)^\sigma \frac{\alpha X_t - p_t H_t}{p_t H_t - p_t N_t^o \bar{h}}. \tag{A.16}$$

Recalling that constant returns to scale in both production sectors imply the zero profit conditions

$$X_t = N_t^y \left( w_t \ell_t + R_t \kappa_t \right),$$
$$p_t H_t = N_t^y w_t \left(1 - \ell_t\right),$$
$$\text{and } \tfrac{R_t \kappa_t}{w_t} = \tfrac{\alpha}{1-\alpha} \ell_t,$$

expression (A.16) reduces to

$$p_t^{\sigma-1} = \left( \frac{1-\gamma}{\gamma} \right)^{\sigma} \frac{1}{1-\alpha} \frac{N_t^y w_t}{p_t} \frac{\ell_t - (1-\alpha)}{H_t - N_t^o \bar{h}}. \tag{A.17}$$

From (7) and the definition of $\ell^{\max}$, we have $H_t - N_t^o \bar{h} = \eta \left( \ell^{\max} - \ell_t \right) N_t^y$. Plugging this result into (A.17), and substituting $\frac{w_t}{\eta p_t} = 1$ from (9), we obtain expression (18) in the text.

**Existence and uniqueness of the fixed point (19).** The function $\Phi \left( \ell_t, \kappa_t \right)$ defined in (17) exhibits the following properties:

$$\begin{aligned} &\lim_{\ell_t \to 1-\alpha} \Phi \left( \ell_t, \kappa_t \right) = (B/\eta) \left(1-\alpha\right)^{1-\alpha} \left( \kappa_t \right)^{\alpha} \\ &\lim_{\ell_t \to \ell^{\max}} \Phi \left( \ell_t, \kappa_t \right) = (B/\eta) \left(1-\alpha\right) \left( \kappa_t / \ell^{\max} \right)^{\alpha}, \end{aligned} \tag{A.18}$$

with derivatives

$$\Phi_{\ell_t} \equiv \tfrac{\partial \Phi(\ell_t, \kappa_t)}{\partial \ell_t} = -\alpha \tfrac{\Phi(\ell_t, \kappa_t)}{\ell_t} < 0 \quad \text{and} \quad \Phi_{\ell_t \ell_t} \equiv \tfrac{\partial^2 \Phi(\ell_t, \kappa_t)}{\partial \ell_t^2} > 0. \tag{A.19}$$

The elasticity of $\Phi \left( \ell_t, \kappa_t \right)$ is

$$\frac{\Phi_{\ell_t} \ell_t}{\Phi} = -\alpha \tag{A.20}$$

The function defined in (18), instead, exhibits

$$\begin{aligned} &\lim_{\ell_t \to 1-\alpha} \Psi \left( \ell_t \right) = \left\{ \infty \quad \text{if } \sigma < 1; \quad 0 \quad \text{if } \sigma > 1 \right\}, \\ &\lim_{\ell_t \to \ell^{\max}} \Psi \left( \ell_t \right) = \left\{ 0 \quad \text{if } \sigma < 1; \quad \infty \quad \text{if } \sigma > 1 \right\}, \end{aligned} \tag{A.21}$$

with

$$\Psi' \left( \ell_t \right) \equiv \frac{\partial \Psi \left( \ell_t \right)}{\partial \ell_t} = \frac{\Psi \left( \ell_t \right)}{\sigma - 1} \frac{\ell^{\max} - (1-\alpha)}{\left( \ell^{\max} - \ell_t \right) \left[ \ell_t - (1-\alpha) \right]} \left\{ \begin{array}{ll} < 0 & \text{if } \sigma < 1 \\ > 0 & \text{if } \sigma > 1 \end{array} \right\}. \tag{A.22}$$

The elasticity is therefore

$$\frac{\Psi'(\ell_t)\,\ell_t}{\Psi(\ell_t)} = -\frac{1}{1-\sigma}\underbrace{\frac{\ell^{\max}-(1-\alpha)}{\ell^{\max}-\ell_t}}_{>1}\underbrace{\frac{\ell_t}{\ell_t-(1-\alpha)}}_{>1}. \tag{A.23}$$

Under substitutability, existence and uniqueness of the fixed point (19) are guaranteed by the derivatives (A.19)-(A.22) along with the limits (A.18) and (A.21). Under complementarity, expression (A.20) implies $\Phi_{\ell_t}\ell_t/\Phi > -1$ whereas expression (A.23) implies $\Psi'(\ell_t)\,\ell_t/\Psi(\ell_t) < -1$. These values of elasticities imply existence and uniqueness of the fixed point (19) even with $\sigma < 1$ despite the fact that both $\Phi(\ell_t, \kappa_t)$ and $\Psi(\ell_t)$ are strictly decreasing. For future reference, note that the limiting properties of $\Phi(\ell_t, \kappa_t)$ and $\Psi(\ell_t)$ described in (A.18) and (A.21) imply

$$\lim_{\kappa\to 0}\ell(\kappa) = \left\{\begin{array}{ll} \ell^{\max} & \text{if } \sigma < 1 \\ 1-\alpha & \text{if } \sigma > 1 \end{array}\right\} \text{ and } \lim_{\kappa\to\infty}\ell(\kappa) = \left\{\begin{array}{ll} 1-\alpha & \text{if } \sigma < 1 \\ \ell^{\max} & \text{if } \sigma > 1 \end{array}\right\}. \tag{A.24}$$

**Neoclassical growth: elasticity of $\ell(\kappa)$, derivation of (30)**. Totally differentiating the fixed-point condition $\Psi(\ell(\kappa)) = \Phi(\ell(\kappa), \kappa)$, we obtain

$$\ell'_\kappa(\kappa) = \frac{\Phi_\kappa(\ell(\kappa), \kappa)}{\Psi'(\ell(\kappa)) - \Phi_\ell(\ell(\kappa), \kappa)}. \tag{A.25}$$

The function $\Phi(\ell(\kappa), \kappa)$ exhibits the partial derivatives

$$\Phi_\kappa(\ell, \kappa) = \alpha\Phi(\ell, \kappa)/\kappa \text{ and } \Phi_\ell(\ell, \kappa) = -\alpha\Phi(\ell, \kappa)/\ell. \tag{A.26}$$

Substituting (A.26) and the equilibrium condition $\Psi(\ell(\kappa)) = \Phi(\ell(\kappa), \kappa)$ in (A.25), we obtain

$$\frac{\ell'_\kappa(\kappa)\,\kappa}{\ell(\kappa)} = \frac{\alpha}{\alpha + \frac{\Psi'(\ell(\kappa))\ell(\kappa)}{\Psi(\ell(\kappa))}}. \tag{A.27}$$

Result (A.27) establishes a clear link between the elasticity of the generic-sector employment share $\ell(\kappa)$ to the capital stock $\kappa$ and the elasticity of the price of health care $\Psi(\ell(\kappa))$ to the generic-sector employment share $\ell(\kappa)$. In particular, substituting (A.23) in (A.27), we have

$$\frac{\ell'_\kappa(\kappa)\,\kappa}{\ell(\kappa)} = \frac{1}{1 - \frac{1}{1-\sigma}\frac{1}{\alpha}\left\{\frac{\ell^{\max}-(1-\alpha)}{\ell^{\max}-\ell(\kappa)}\frac{\ell(\kappa)}{\ell(\kappa)-(1-\alpha)}\right\}}, \tag{A.28}$$

where the term in curly brackets equals $Q_1 \equiv \frac{\ell_t}{\ell_t-(1-\alpha)}\frac{\ell^{\max}-(1-\alpha)}{\ell^{\max}-\ell_t}$ in expression (30). The fact that $Q_1 > 1$ directly follows from the equilibrium requirement $1-\alpha < \ell_t < \ell^{\max}$, and it implies that

$$\begin{array}{ll} \frac{1}{1-\sigma}\frac{1}{\alpha}Q_1 > 1 & \text{if } 0 < \sigma < 1, \\ \frac{1}{1-\sigma}\frac{1}{\alpha}Q_1 < 0 & \text{if } \sigma > 1. \end{array} \tag{A.29}$$

Results (A.29) imply the signs reported in expression (30) in the text.

**Proposition 3: Existence, Uniqueness and Stability.** To prove existence, consider equation (32) and substitute $p(\kappa_t) = \Psi(\ell(\kappa_t))$ from (20), obtaining

$$\kappa_{t+1} = \frac{\eta\beta}{(1+n)(1+\beta)}\Psi(\ell(\kappa_t)). \tag{A.30}$$

The right-hand side of (A.30) is strictly increasing in $\kappa_t$: differentiation with respect to $\kappa_t$ yields

$$\frac{\mathrm{d}\kappa_{t+1}}{\mathrm{d}\kappa_t} = \frac{\eta\beta}{(1+n)(1+\beta)}\Psi'(\ell(\kappa_t))\,\ell'_\kappa(\kappa_t) > 0, \tag{A.31}$$

where the positive sign derives from the fact that both $\Psi'(\ell_t)$ and $\ell'_\kappa(\kappa_t)$ are negative (positive) under complementarity (substitutability). From (A.22), (A.21) and (A.24), we have

$$\lim_{\kappa\to 0}\Psi'(\ell(\kappa)) = \infty \text{ and } \lim_{\kappa\to\infty}\Psi'(\ell(\kappa)) = 0 \tag{A.32}$$

under both complementarity and substitutability. Results (A.31) and (A.32) imply existence of at least one steady state $\kappa_{ss} = \frac{\eta\beta}{(1+n)(1+\beta)}\Psi(\ell(\kappa_{ss}))$. Moreover, if the elasticity condition (34) is valid for any $\kappa$, i.e.

$$-\frac{\ell'_\kappa(\kappa)\,\kappa}{\ell(\kappa)}\frac{\alpha}{1-\alpha} < 1, \tag{A.33}$$

then the steady state $\kappa = \kappa_{ss}$ is unique and globally stable. Under substitutability, inequality (A.33) is necessarily satisfied: when $\sigma > 1$, expression (30) implies $\ell'_\kappa(\kappa) > 0$ and therefore a strictly negative left hand side in (A.33). To study the case of complementarity, substitute the elasticity $\frac{\ell'_\kappa(\kappa)\kappa}{\ell(\kappa)}$ by means of expression (A.28), and rearrange terms, to rewrite condition (A.33) as

$$1 - \frac{(1-\sigma)(1-\ell(\kappa))(\ell(\kappa)-(1-\alpha))}{\ell(\kappa)} > \alpha. \tag{A.34}$$

Condition (A.34) implies a more restrictive requirement on parameters the lower is the left hand side. In this respect, the left hand side of (A.34) is strictly increasing in $\sigma$ so that, all else equal, it reaches its smallest value when $\sigma = 0$. Letting $\sigma = 0$, the stability condition becomes

$$\left(\ell(\kappa) - \frac{1}{2}\right)^2 > \alpha - \frac{3}{4}, \tag{A.35}$$

which is surely satisfied when $\alpha < 3/4$. Therefore, a generously sufficient, not necessary condition for stability and uniqueness under complementarity is $\alpha < 3/4$. Given uniqueness and stability, as $\kappa_t$ converges to $\kappa_{ss}$ in the long run, both the price of health care $p_t = p(\kappa_t)$ and the employment share $\ell_t = \ell(\kappa_t)$ converge to constant levels. Results (A.31) and (A.32) guarantee that the transitional dynamics of $\kappa_t$ are monotonic. The transitional dynamics of $p(\kappa_t)$ and $\ell_t = \ell(\kappa_t)$ then follow directly from Proposition 1. ■

**Proposition 4: Uniqueness and Stability**. The existence of the steady state $\bar{\kappa}_{ss}$ is proved in Appendix B along with the discussion of possible multiple steady states. Given existence, the steady state $\kappa = \bar{\kappa}_{ss}$ is unique and globally stable if the elasticity condition (37) is valid for any $\kappa$, i.e.

$$m_1(\kappa) + m_2(\kappa) < 1, \tag{A.36}$$

where

$$m_1(\kappa) \equiv -\frac{\ell'_\kappa(\kappa)\,\kappa}{\ell(\kappa)}\frac{\alpha}{1-\alpha}, \tag{A.37}$$

$$m_2(\kappa) \equiv -\frac{\ell'_\kappa(\kappa)\,\kappa}{\ell(\kappa)}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\kappa)}\frac{1}{1-\alpha}. \tag{A.38}$$

Expression (A.37) follows from generalizing the definition of $m_1$ given in (34) whereas (A.38) follows from generalizing the definition of $m_2$ given in (38). The inequalities appearing in (38) are part of the following proof. First, consider substitutability. When $\sigma > 1$, both $m_1(\kappa)$ and $m_2(\kappa)$ are strictly negative because expression (30) implies $\ell'_\kappa(\kappa) > 0$ and expression (26) implies $\Gamma' > 0$. Therefore, condition (A.36) is necessarily satisfied when $\sigma > 1$. To study the case of complementarity, note that (26) implies

$$\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\kappa)} = \frac{\frac{(1-\alpha)}{\alpha(1+\beta)\eta(1+n)}\frac{\bar{h}}{\ell(\kappa)}}{1-\frac{(1-\alpha)}{\alpha(1+\beta)\eta(1+n)}\frac{\bar{h}}{\ell(\kappa)}} = \frac{\frac{(1-\alpha)(1-\ell^{\max})}{\alpha(1+\beta)}}{\ell(\kappa)-\frac{(1-\alpha)(1-\ell^{\max})}{\alpha(1+\beta)}}, \tag{A.39}$$

where the last term follows from substituting $\bar{h} = \eta(1+n)(1-\ell^{\max})$ by definition (10). Defining the convenient parameter

$$q_1 \equiv \frac{(1-\alpha)(1-\ell^{\max})}{\alpha(1+\beta)} > 0, \tag{A.40}$$

we can substitute (A.39) in (A.38) and rewrite the stability condition (A.36) as

$$-\frac{\ell'(\kappa)\,\kappa}{\ell(\kappa)}\cdot\frac{\alpha}{1-\alpha} - \frac{\ell'(\kappa)\,\kappa}{\ell(\kappa)}\cdot\frac{q_1}{\ell(\kappa)-q_1}\cdot\frac{1}{1-\alpha} < 1. \tag{A.41}$$

Fro future reference, note that parameter $q_1$ is always strictly less than $1-\alpha$.[32] This implies that $\ell(\kappa) > q_1$ holds in any interior equilibrium:

$$q_1 < 1-\alpha < \ell(\kappa)\,. \tag{A.42}$$

Going back to (A.41), substituting $\frac{\ell'_\kappa(\kappa)\kappa}{\ell(\kappa)}$ by means of (A.28), the stability condition reduces to

$$\frac{1-\alpha}{\alpha(1-\sigma)}\left[\frac{\ell(\kappa)-q_1}{\ell(\kappa)-(1-\alpha)}\frac{\ell^{\max}-(1-\alpha)}{\ell^{\max}-\ell(\kappa)}\right] > 1. \tag{A.43}$$

From result (A.42), the term in square brackets in (A.43) is strictly greater than unity.[33] Therefore, a sufficient but not necessary condition for satisfying (A.43) is that $\frac{1-\alpha}{\alpha(1-\sigma)} > 1$, which is equivalent to Assumption 2 in the text. The conclusion is that, when $\sigma < 1$, satisfying Assumption 2 is sufficient to guarantee stability and uniqueness of the steady state. Also note that the stability condition under complementarity guarantees $m_2(\kappa) < 1$, as reported in expression (38).[34]

**Derivation of (40)-(41)** . Expression (40) directly follows from log-differentiating (36) with respect to $B$. In (35), instead, log-differentiation with respect to $B$ yields

$$\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}B} = -\frac{\alpha}{1-\alpha}\left[\frac{\ell'_\kappa(\kappa_{ss})\,\kappa_{ss}}{\ell(\kappa_{ss})}\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}B} + \frac{\ell'_B(\kappa_{ss})}{\ell(\kappa_{ss})}\right] + \frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{\mathrm{d}B}, \tag{A.44}$$

where the term in square brackets is the chain derivative $\frac{\mathrm{d}\log\ell(\kappa_{ss})}{\mathrm{d}B}$, with $\ell'_B(\kappa_{ss})$ representing the static derivative $\frac{\mathrm{d}\ell(\kappa;B)}{\mathrm{d}B}$ defined in Proposition 2, evaluated in the steady state $\kappa_{ss}$. Substituting the definition of $m_1 \equiv -\frac{\alpha}{1-\alpha}\frac{\ell'_\kappa\cdot\kappa}{\ell}$ from (34) into (A.44), and rearranging terms, we obtain (41).

**Derivation of (42).** From definition (26), we have

$$\frac{\mathrm{d}}{\mathrm{d}B}\log\Gamma\left(\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\right) = -\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}B},$$

[32] Because any interior equilibrium satisfies $(1-\alpha) < \ell(\kappa_t) < \ell^{\max}$, it is necessarily true that $\ell^{\max} > 1-\alpha-\alpha\beta$. Consequently, the factor $\frac{(1-\ell^{\max})}{\alpha(1+\beta)}$ is strictly less than unity and this, in turn, implies that $q_1 \equiv (1-\alpha)\frac{(1-\ell^{\max})}{\alpha(1+\beta)}$ is strictly less than $(1-\alpha)$.

[33] The fact that $q_1 < 1-\alpha$ implies $\frac{\ell(\kappa)-q_1}{\ell(\kappa)-(1-\alpha)} > 1$. Also, the equilibrium restriction $\ell(\kappa) > (1-\alpha)$ implies $\frac{\ell^{\max}-(1-\alpha)}{\ell^{\max}-\ell(\kappa)} > 1$.

[34] Since condition (A.43) is equivalent to (A.36), satisfying (A.43) implies $m_2(\kappa) < 1-m_1(\kappa)$, where $m_1(\kappa) > 0$ under complementarity. Therefore, $m_2(\kappa) < 1$.

where both $\Gamma$ and $\Gamma'$ in the right hand side are evaluated in $\left(\bar{h}/\ell\left(\bar{\kappa}_{ss}\right)\right)$. Therefore, log-differentiating (39) with respect to $B$ yields

$$\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}B}=-\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell\left(\bar{\kappa}_{ss}\right)}\frac{\mathrm{d}\log\ell\left(\bar{\kappa}_{ss}\right)}{\mathrm{d}B}-\frac{\alpha}{1-\alpha}\frac{\mathrm{d}\log\ell\left(\bar{\kappa}_{ss}\right)}{\mathrm{d}B}+\frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{\mathrm{d}B},$$

where we can substitute $\frac{\mathrm{d}\log\ell(\kappa)}{\mathrm{d}B}=\frac{\ell'_\kappa\kappa}{\ell}\frac{\mathrm{d}\log\kappa}{\mathrm{d}B}+\frac{\ell'_B}{\ell}$ to obtain

$$\begin{aligned}&\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}B}=-\frac{\alpha}{1-\alpha}\left[\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}B}+\frac{\ell'_B(\kappa_{ss})}{\ell(\bar{\kappa}_{ss})}\right]+\\&-\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\left[\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}B}+\frac{\ell'_B(\kappa_{ss})}{\ell(\bar{\kappa}_{ss})}\right]+\frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{\mathrm{d}B},\end{aligned}\tag{A.45}$$

where $\ell'_B\left(\bar{\kappa}_{ss}\right)$ is the static derivative $\frac{\mathrm{d}\ell(\kappa;B)}{\mathrm{d}B}$ defined in Proposition 2, evaluated in the steady state $\bar{\kappa}_{ss}$. Recalling (34) and (34), the definitions $m_1\equiv-\frac{\alpha}{1-\alpha}\frac{\ell'_\kappa\kappa}{\ell}$ and $m_2\equiv-\frac{\ell'_\kappa\kappa}{\ell}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell}\frac{1}{1-\alpha}$ imply that (A.45) reduces to equation (42) in the text.

**Derivation of (44).** From definition (26), we have

$$\frac{\mathrm{d}}{\mathrm{d}n}\log\Gamma\left(\frac{\bar{h}}{\ell\left(\bar{\kappa}_{ss}\right)}\right)=-\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell\left(\bar{\kappa}_{ss}\right)}\frac{\mathrm{d}\log\left[(1+n)\ell\left(\bar{\kappa}_{ss}\right)\right]}{\mathrm{d}n},$$

where both $\Gamma$ and $\Gamma'$ in the right hand side are evaluated in $\left(\bar{h}/\ell\left(\bar{\kappa}_{ss}\right)\right)$. Therefore, log-differentiating (39) with respect to $n$ yields

$$\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n}=-\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell\left(\bar{\kappa}_{ss}\right)}\frac{\mathrm{d}\log\left[(1+n)\ell\left(\bar{\kappa}_{ss}\right)\right]}{\mathrm{d}n}-\frac{\alpha}{1-\alpha}\frac{\mathrm{d}\log\ell\left(\bar{\kappa}_{ss}\right)}{\mathrm{d}n}+\frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{\mathrm{d}n}.$$

Substituting in the above expression the chain derivatives

$$\begin{aligned}\frac{\mathrm{d}\log[(1+n)\ell(\bar{\kappa}_{ss})]}{\mathrm{d}n}&=\frac{1}{1+n}+\frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}n},\\ \frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}n}&=\frac{\ell'_n(\bar{\kappa}_{ss})}{\ell(\bar{\kappa}_{ss})}+\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n},\end{aligned}$$

we obtain

$$\begin{aligned}&\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n}=-\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\left[\frac{\ell'_n(\bar{\kappa}_{ss})}{\ell(\bar{\kappa}_{ss})}\right]-\frac{\alpha}{1-\alpha}\left[\frac{\ell'_n(\bar{\kappa}_{ss})}{\ell(\bar{\kappa}_{ss})}\right]+\\&-\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\left[\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n}\right]-\frac{\alpha}{1-\alpha}\left[\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n}\right]+\\&-\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\left[\frac{1}{1+n}\right]+\frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{\mathrm{d}n}.\end{aligned}\tag{A.46}$$

Recalling (34) and (34), the definitions $m_1 \equiv -\frac{\alpha}{1-\alpha}\frac{\ell'_\kappa \kappa}{\ell}$ and $m_2 \equiv -\frac{\ell'_\kappa \kappa}{\ell}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell}\frac{1}{1-\alpha}$ imply that (A.46) reduces to

$$\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n} = \frac{1}{1-(m_1+m_2)}\left[(m_1+m_2)\frac{\ell'_n}{\ell'_\kappa \kappa} + \frac{m_2}{1+n}\frac{\ell}{\ell'_\kappa \kappa} + \frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{\mathrm{d}n}\right],$$

where we can invert the sign of the variation $\mathrm{d}n$ to $-\mathrm{d}n$, and rearrange terms, to obtain equation (44) in the text.

**Derivation of (45)**. From definition (26), we have

$$\frac{\mathrm{d}}{\mathrm{d}\bar{h}}\log\Gamma\left(\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\right) = \frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\left[\bar{h}/\ell(\bar{\kappa}_{ss})\right]}{\mathrm{d}\bar{h}},$$

where both $\Gamma$ and $\Gamma'$ in the right hand side are evaluated in $\left(\bar{h}/\ell(\bar{\kappa}_{ss})\right)$. Therefore, log-differentiating (39) with respect to $\bar{h}$ (recalling that $\mathrm{d}\kappa_{ss}^{\text{canonical}}/\mathrm{d}\bar{h} = 0$) yields

$$\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}\bar{h}} = \frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\left[\bar{h}/\ell(\bar{\kappa}_{ss})\right]}{\mathrm{d}\bar{h}} - \frac{\alpha}{1-\alpha}\frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}\bar{h}}.$$

Substituting in the above expression the chain derivatives

$$\begin{aligned}\frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}\bar{h}} &= \frac{\ell'_{\bar{h}}(\bar{\kappa}_{ss})}{\ell(\bar{\kappa}_{ss})} + \frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}\bar{h}},\\ \frac{\mathrm{d}\log\left[\bar{h}/\ell(\bar{\kappa}_{ss})\right]}{\mathrm{d}\bar{h}} &= \frac{1}{\bar{h}} - \frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}\bar{h}},\end{aligned}$$

we obtain

$$\begin{aligned}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}\bar{h}} &= \frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\frac{1}{\bar{h}} - \frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\left[\frac{\ell'_{\bar{h}}(\bar{\kappa}_{ss})}{\ell(\bar{\kappa}_{ss})} + \frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}\bar{h}}\right] +\\ &-\frac{\alpha}{1-\alpha}\left[\frac{\ell'_{\bar{h}}(\bar{\kappa}_{ss})}{\ell(\bar{\kappa}_{ss})} + \frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n}\right].\end{aligned} \tag{A.47}$$

Recalling (34) and (34), the definitions $m_1 \equiv -\frac{\alpha}{1-\alpha}\frac{\ell'_\kappa \kappa}{\ell}$ and $m_2 \equiv -\frac{\ell'_\kappa \kappa}{\ell}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell}\frac{1}{1-\alpha}$ imply that (A.47) reduces to (45).

**Quantitative analysis: Total Expenditure Share (TES)**

The definition of TES is

$$TES \equiv \frac{N^o_{t+1}p_{t+1}h_{t+1}}{N^o_{t+1}d_{t+1} + N^o_{t+1}p_{t+1}h_{t+1} + N^y_{t+1}c_{t+1}} = \frac{p_{t+1}h_{t+1}}{d_{t+1} + p_{t+1}h_{t+1} + (1+n)\cdot c_{t+1}}$$

Substituting in the above expression the zero profit conditions for the care sector $ph = w(1-\ell)(1+n)$ and for the generic sector, $\frac{R\kappa}{w} = \frac{\alpha}{1-\alpha}\ell$, we have

$$TES = \frac{w_{t+1}(1-\ell_{t+1})}{\kappa_{t+1}R_{t+1} + c_{t+1}}$$

where we can substitute the first-order condition for consumption $c_{t+1} = \frac{1}{1+\beta}\left(w_{t+1} - \frac{p_{t+2}}{R_{t+2}}\bar{h}\right)$ and the marginal product of labor in care $p_{t+2} = w_{t+2}/\eta$ to get so as to get

$$TES = \frac{w_{t+1}(1-\ell_{t+1})}{\kappa_{t+1}R_{t+1} + \frac{1}{1+\beta}\left(w_{t+1} - \frac{w_{t+2}}{R_{t+2}} \cdot \frac{\bar{h}}{\eta}\right)}.$$

Imposing the steady state and substituting again the profit-maximization condition $\frac{R\kappa}{w} = \frac{\alpha}{1-\alpha}\ell$, we have

$$TES_{ss} \equiv \frac{(1-\ell_{ss})}{\frac{\alpha}{1-\alpha}\ell_{ss} + \frac{1}{1+\beta}\left(2 - \frac{1}{R_{ss}} \cdot \frac{\bar{h}}{\eta}\right)}. \tag{A.48}$$

This is the relveant equation for TES that we use in the calibration.

**Quantitative analysis: Capital Income Share (CIS)**

In our model, the capital income share directly follows from the profit-maximization conditions in the generic sector,

$$CIS \equiv 1 - \frac{w_t N_t^y}{Y_t} = 1 - \frac{1-\alpha}{1-\alpha(1-\ell_t)} = \frac{\alpha \cdot \ell_t}{1-\alpha \cdot (1-\ell_t)}. \tag{A.49}$$

Imposing the steady state, we obtain the relevant equation for CIS that we use in the calibration.

## B Appendix (further details)

**Proposition 4: Further Details on Existence and Uniqueness of the steady state.** To prove existence, we transform equation (29) into an equivalent dynamic law that maps $\ell(\kappa_t)$ into $\ell(\kappa_{t+1})$. Starting from expression (29), substitute $p_t = \Phi(\ell_t, \kappa_t) = (B/\eta)(1-\alpha)(\kappa_t/\ell_t)^\alpha$ from (17) to write

$$\frac{\kappa_{t+1}}{\ell(\kappa_{t+1})}\left[\ell(\kappa_{t+1}) - \frac{(1-\alpha)(1-\ell^{\max})}{\alpha(1+\beta)}\right] = \frac{\eta\beta}{(1+\beta)(1+n)}p_t. \tag{B.1}$$

Imposing the equilibrium condition $p_t = p(\kappa_t) \equiv \Psi(\ell(\kappa_t))$ from (20), we have

$$\frac{\kappa_{t+1}}{\ell(\kappa_{t+1})}\left[\ell(\kappa_{t+1}) - \frac{(1-\alpha)(1-\ell^{\max})}{\alpha(1+\beta)}\right] = \frac{\eta\beta}{(1+\beta)(1+n)}\Psi(\ell(\kappa_t)). \tag{B.2}$$

Also notice that, setting (17) at time $t+1$ and solving for the input ratio, we have $\frac{\kappa_{t+1}}{\ell_{t+1}} = \left[\frac{p_{t+1}}{(B/\eta)(1-\alpha)}\right]^{\frac{1}{\alpha}}$. Plugging this result in (B.2), and imposing the static equilibrium condition $p_{t+1} = p(\kappa_{t+1}) \equiv \Psi(\ell(\kappa_{t+1}))$ from (20), we obtain the *dynamic law*

$$\Psi(\ell(\kappa_{t+1}))(\ell(\kappa_{t+1}) - q_1)^{\alpha} = q_2\Psi(\ell(\kappa_t))^{\alpha}, \tag{B.3}$$

where we again have used that the definition of $q_1$ and also have defined the convenient variable $q_2$:

$$q_1 \equiv \tfrac{(1-\alpha)(1-\ell^{\max})}{\alpha(1+\beta)} > 0 \quad \text{and} \quad q_2 \equiv \tfrac{B}{\eta}(1-\alpha)\left[\tfrac{\eta\beta}{(1+\beta)(1+n)}\right]^{\alpha} > 0. \tag{B.4}$$

Expression (B.3) fully characterizes the dynamics of capital per worker. Dynamics are well defined only if both sides are strictly positive, which requires $\ell(\kappa_{t+1}) > q_1$ in each period $t+1$: this inequality is always satisfied as shown in (A.42). In (B.3), the steady state condition $\kappa_{t+1} = \kappa_t = \bar{\kappa}_{ss}$ is satisfied when

$$\Psi(\ell(\bar{\kappa}_{ss})) = \underbrace{\left[q_2(\ell(\bar{\kappa}_{ss}) - q_1)^{-\alpha}\right]^{\frac{1}{1-\alpha}}}_{\Omega(\ell(\bar{\kappa}_{ss}))}. \tag{B.5}$$

For future reference, we define the elasticities of $\Psi(.)$ and $\Omega(.)$ with respect to $\bar{\kappa}_{ss}$ as

$$\epsilon_1 \equiv \frac{d\Psi(\ell(\bar{\kappa}_{ss}))}{d\bar{\kappa}_{ss}}\frac{\bar{\kappa}_{ss}}{\Psi(\ell(\bar{\kappa}_{ss}))} = \frac{\Psi'(\ell(\bar{\kappa}_{ss}))}{\Psi(\ell(\bar{\kappa}_{ss}))}\ell'_{\kappa}(\bar{\kappa}_{ss})\bar{\kappa}_{ss}, \tag{B.6}$$

$$\epsilon_2 \equiv \frac{d\Omega(\ell(\bar{\kappa}_{ss}))}{d\bar{\kappa}_{ss}}\frac{\bar{\kappa}_{ss}}{\Omega(\ell(\bar{\kappa}_{ss}))} = -\frac{\frac{\alpha}{1-\alpha}}{\ell(\bar{\kappa}_{ss}) - q_1}\ell'_{\kappa}(\bar{\kappa}_{ss})\bar{\kappa}_{ss}. \tag{B.7}$$

The remainder of the proof studies separately the two cases of substitutability and complementarity.

*Substitutability.* When $\sigma > 1$, results (A.22) imply that

$$\frac{d\Psi(\ell(\bar{\kappa}_{ss}))}{d\bar{\kappa}_{ss}} = \underbrace{\Psi'(\ell(\bar{\kappa}_{ss}))}_{\text{positive}}\underbrace{\ell'_{\kappa}(\bar{\kappa}_{ss})}_{\text{positive}} > 0 \text{ and } \ell'(\bar{\kappa}_{ss}) > 0. \tag{B.8}$$

Result (B.8) implies that, given the definitions in (B.5), function $\Psi(\ell(\bar{\kappa}_{ss}))$ is strictly increasing in $\bar{\kappa}_{ss}$ whereas $\Omega(\ell(\bar{\kappa}_{ss}))$ is strictly decreasing in $\bar{\kappa}_{ss}$. Using the limiting properties (A.21) and (A.24), we also have

$$\begin{aligned} &\lim_{\bar{\kappa}_{ss}\to 0}\Psi(\ell(\bar{\kappa}_{ss})) = 0, && \lim_{\bar{\kappa}_{ss}\to 0}\Omega(\ell(\bar{\kappa}_{ss})) = \left[q_2/(1-\alpha-q_1)^{\alpha}\right]^{\frac{1}{1-\alpha}} > 0, \\ &\lim_{\bar{\kappa}_{ss}\to\infty}\Psi(\ell(\bar{\kappa}_{ss})) = \infty, && \lim_{\bar{\kappa}_{ss}\to\infty}\Omega(\ell(\bar{\kappa}_{ss})) = \left[q_2/(\ell^{\max}-q_1)^{\alpha}\right]^{\frac{1}{1-\alpha}} > 0. \end{aligned} \tag{B.9}$$

These results imply that, under substitutability, there exists a unique steady state $\bar{\kappa}_{ss}$ satisfying condition (B.5).

*Complementarity.* When $\sigma < 1$, results (A.22) imply that

$$\frac{\mathrm{d}\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right)}{\mathrm{d}\bar{\kappa}_{ss}} = \underbrace{\Psi'\left(\ell\left(\bar{\kappa}_{ss}\right)\right)}_{\text{negative}}\underbrace{\ell'_{\kappa}\left(\bar{\kappa}_{ss}\right)}_{\text{negative}} > 0 \text{ and } \ell'_{\kappa}\left(\bar{\kappa}_{ss}\right) < 0. \tag{B.10}$$

Result (B.10) implies that, given the definitions in (B.5), both $\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right)$ and $\Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right)$ in (B.5) are strictly increasing in $\bar{\kappa}_{ss}$. Using the limiting properties (A.21) and (A.24), we also have

$$\begin{array}{ll} \lim_{\bar{\kappa}_{ss}\to 0}\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = 0, & \lim_{\bar{\kappa}_{ss}\to 0}\Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = \left[q_2/\left(\ell^{\max}-q_1\right)^{\alpha}\right]^{\frac{1}{1-\alpha}} > 0, \\ \lim_{\bar{\kappa}_{ss}\to\infty}\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = \infty, & \lim_{\bar{\kappa}_{ss}\to\infty}\Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = \left[q_2/\left(1-\alpha-q_1\right)^{\alpha}\right]^{\frac{1}{1-\alpha}} > 0. \end{array} \tag{B.11}$$

The limits in (B.11) imply that *there always exists at least one steady state $\bar{\kappa}_{ss(1)}$ satisfying condition (B.5) in which $\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right)$ cuts $\Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right)$ from below*: this steady state therefore satisfies

$$\frac{\mathrm{d}\Psi\left(\ell\left(\bar{\kappa}_{ss(1)}\right)\right)}{\mathrm{d}\bar{\kappa}_{ss(1)}} > \frac{\mathrm{d}\Omega\left(\ell\left(\bar{\kappa}_{ss(1)}\right)\right)}{\mathrm{d}\bar{\kappa}_{ss(1)}}. \tag{B.12}$$

Considering stability, equation (B.3) implies that any steady state $\bar{\kappa}_{ss(i)}$ is stable if

$$\frac{\Psi'\left(\ell\left(\bar{\kappa}_{ss(i)}\right)\right)}{\Psi\left(\ell\left(\bar{\kappa}_{ss(i)}\right)\right)} < -\frac{\alpha}{1-\alpha}\frac{1}{\ell\left(\bar{\kappa}_{ss(i)}\right)-q_1}. \tag{B.13}$$

It is easily shown that (B.12) implies that the steady state $\bar{\kappa}_{ss(1)}$ satisfies the stability condition (B.13). Hence, under complementarity, there always exist a stable steady state $\bar{\kappa}_{ss(1)}$. In order to assess the uniqueness of the steady state, re-write the steady-state condition (B.5) in explicit form by substituting $\Psi\left(\cdot\right)$ with (18), obtaining

$$\ell\left(\bar{\kappa}_{ss}\right) - q_1 = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\sigma(1-\alpha)}{\alpha(1-\sigma)}} q_2^{\frac{1}{\alpha}} \left[\frac{\ell\left(\bar{\kappa}_{ss}\right)-(1-\alpha)}{(1-\alpha)\left(\ell^{\max}-\ell\left(\bar{\kappa}_{ss}\right)\right)}\right]^{\frac{1-\alpha}{\alpha(1-\sigma)}}. \tag{B.14}$$

Hence, defining $q_3 \equiv \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\sigma(1-\alpha)}{\alpha(1-\sigma)}} q_2^{\frac{1}{\alpha}}$, the steady-state condition reads

$$\ell\left(\bar{\kappa}_{ss}\right) = F\left(\ell\left(\bar{\kappa}_{ss}\right)\right) \quad \text{where} \quad F\left(\ell\left(\bar{\kappa}_{ss}\right)\right) \equiv q_1 + q_3\left[\frac{\ell\left(\bar{\kappa}_{ss}\right)-(1-\alpha)}{(1-\alpha)\left(\ell^{\max}-\ell\left(\bar{\kappa}_{ss}\right)\right)}\right]^{\frac{1-\alpha}{\alpha(1-\sigma)}}. \tag{B.15}$$

In general, the function $F\left(\ell\right)$ is strictly increasing and exhibits the following properties

$$\lim_{\ell\to 1-\alpha} F\left(\ell\right) = q_1 < \ell \text{ and } \lim_{\ell\to\ell^{\max}} F\left(\ell\right) = \infty, \tag{B.16}$$

$$F'\left(\ell\right) = \frac{q_3}{\alpha(1-\sigma)}\left[\frac{\ell-(1-\alpha)}{(1-\alpha)\left(\ell^{\max}-\ell\right)}\right]^{\frac{1-\alpha}{\alpha(1-\sigma)}-1}\frac{\ell^{\max}-(1-\alpha)}{\left(\ell^{\max}-\ell\right)^2} > 0. \tag{B.17}$$

Evaluating $\digamma'(\ell)$ in a steady state $\ell(\bar{\kappa}_{ss}) = \digamma(\ell(\bar{\kappa}_{ss}))$, we have

$$\digamma'(\ell(\bar{\kappa}_{ss})) = \frac{1-\alpha}{\alpha(1-\sigma)} \underbrace{\frac{\ell(\bar{\kappa}_{ss}) - q_1}{\ell(\bar{\kappa}_{ss}) - (1-\alpha)}}_{>1} \underbrace{\frac{\ell^{\max} - (1-\alpha)}{\ell^{\max} - \ell(\bar{\kappa}_{ss})}}_{>1}. \tag{B.18}$$

Comparing (B.18) with (A.43), it is evident that the stability condition (A.43) is equivalent to $\digamma'(\ell(\bar{\kappa}_{ss})) > 1$. In graphical terms, this means that a stable steady state is a an intersection $\ell = \digamma(\ell)$ in which the function $\digamma(\ell)$ cuts the 45-degree line $\ell = \ell$ from below – e.g., like the steady state shown in Figure B.1, graph (a). Properties (B.16)-(B.17) thus confirm the existence of at least one stable steady state. Concerning the uniqueness of the steady state, we must consider two sub-cases, depending on whether the parameter values imply $\frac{1-\alpha}{\alpha(1-\sigma)} > 1$ or $\frac{1-\alpha}{\alpha(1-\sigma)} < 1$.

**Subcase I**. When $\frac{1-\alpha}{\alpha(1-\sigma)} > 1$, expression (B.17) implies $\digamma''(\ell) > 0$ for all $\ell \in (1-\alpha, \ell^{\max})$, so that $\digamma(\ell)$ is strictly increasing and strictly convex for all $\ell \in (1-\alpha, \ell^{\max})$. This means that there is a unique steady state $\ell(\bar{\kappa}_{ss}) = \digamma(\ell(\bar{\kappa}_{ss}))$, as shown in Figure B.1, graph (a). Moreover, $\ell(\bar{\kappa}_{ss})$ is stable, as is immediately evident from (B.18): when $\frac{1-\alpha}{\alpha(1-\sigma)} > 1$, all the three terms at the right hand side are strictly greater than unity, implying $\digamma'(\ell(\bar{\kappa}_{ss})) > 1$. Recalling that $\ell'_{\kappa}(\kappa) < 0$ under complementarity, an initial condition $\kappa(0) < \bar{\kappa}_{ss}$ implies positive accumulation and declining employment in generic production: the economy starts from an initial level $\ell_0 = \ell(\kappa(0))$ and then declines towards $\ell_{ss} = \ell(\bar{\kappa}_{ss})$ as shown in Figure B.1, graph (a).

**Subcase II**. When $\frac{1-\alpha}{\alpha(1-\sigma)} < 1$, we have $\lim_{\ell \to 1-\alpha} \digamma'(\ell) = \infty$ and $\lim_{\ell \to \ell^{\max}} \digamma'(\ell) = \infty$. Expression (B.17) implies that $\digamma(\ell)$ is initially concave and then convex: from

$$\frac{\digamma''(\ell)}{\digamma'(\ell)} = \frac{1}{\ell^{\max} - \ell} \left\{ 2 - \left(1 - \frac{1-\alpha}{\alpha(1-\sigma)}\right) \frac{\ell^{\max} - (1-\alpha)}{\ell - (1-\alpha)} \right\}, \tag{B.19}$$

there exists a point of inflection

$$\tilde{\ell} \equiv (1-\alpha) + \frac{1}{2}\left(1 - \frac{1-\alpha}{\alpha(1-\sigma)}\right)\left[\ell^{\max} - (1-\alpha)\right]$$

such that $\digamma''\left(\tilde{\ell}\right) = 0$, with $\digamma''(\ell)$ is negative for $\ell < \tilde{\ell}$ and positive $\tilde{\ell} > \ell$ . This implies that, in the subcase $\frac{1-\alpha}{\alpha(1-\sigma)} < 1$, we may have in principle two possible outcomes: a unique stable steady state or three steady states, as shown in Figure B.1, graphs (b) and (c). When there are three steady states, $\bar{\kappa}_{ss(1)} < \bar{\kappa}_{ss(2)} < \bar{\kappa}_{ss(3)}$, the middle steady state $\bar{\kappa}_{ss(2)}$ is unstable because $F'\left(\ell\left(\bar{\kappa}_{ss(2)}\right)\right) < 1$, whereas $\bar{\kappa}_{ss(1)}$ and $\bar{\kappa}_{ss(3)}$ are both stable. This scenario is thus characterized by

$$F\left(\ell_{ss3}\right) < F\left(\ell_{ss2}\right) < F\left(\ell_{ss1}\right), \tag{B.20}$$

$$F'\left(\ell_{ss3}\right) < 1, \quad F'\left(\ell_{ss2}\right) > 1, \quad F\left(\ell_{ss1}\right) < 1, \tag{B.21}$$

where $\ell^i_{ss} \equiv \ell\left(\bar{\kappa}_{ss(i)}\right)$. Recalling that $\ell'_{\kappa}(\kappa) < 0$ under complementarity, an initial condition $\kappa(0) < \bar{\kappa}_{ss(1)}$ implies positive accumulation and declining employment in generic production: the economy starts from an initial level $\ell_0 = \ell(\kappa(0))$ and then declines towards $\ell^1_{ss} = \ell\left(\bar{\kappa}_{ss(1)}\right)$ as shown in Figure B.1, graph (d).

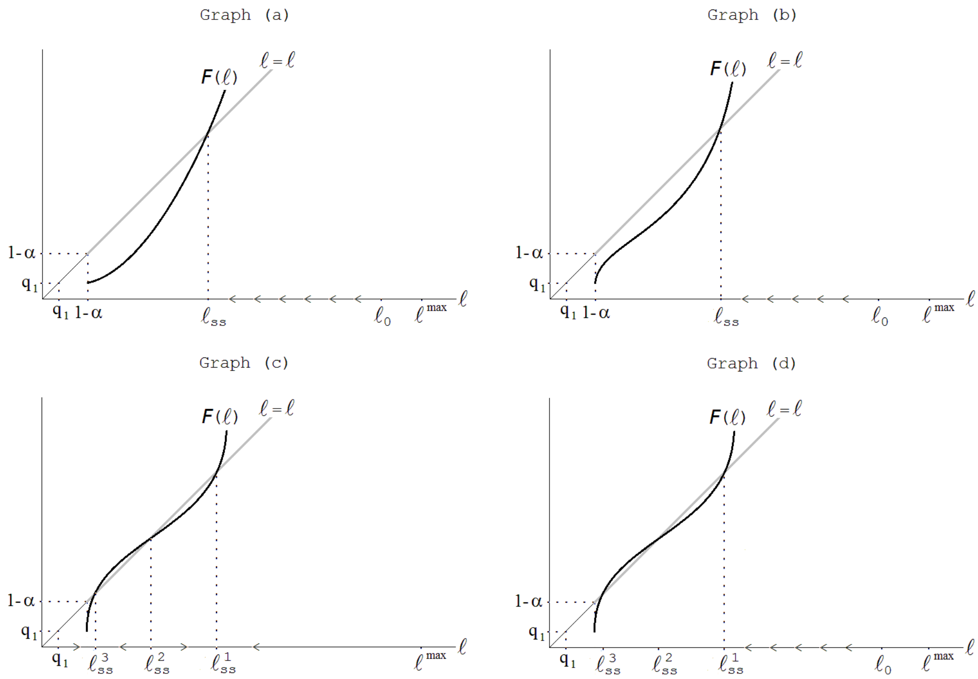


**Figure B.1** Existence and uniqueness of steady states under complementarity. Graph (a): the subcase $\frac{1-\alpha}{\alpha(1-\sigma)} > 1$ features a unique stable steady state. Graph (b): the subcase $\frac{1-\alpha}{\alpha(1-\sigma)} < 1$ when the steady state is unique. Graphs (c)-(d): the subcase $\frac{1-\alpha}{\alpha(1-\sigma)} < 1$ when the steady states are three – the middle one being unstable.