The Impact of Return Nonnormality on Exchange Options

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Abstract

The Margrabe formula is used extensively by theorists and practitioners not only on exchange options, but also on executive compensation schemes, real options, weather and commodity derivatives, etc. However, the crucial assumption of a bivariate normal distribution is not fully satisfied in almost all applications. The impact of nonnormality on exchange options is studied by using a bivariate Gram-Charlier approximation. For near-the-money exchange options, skewness and coskewness induce price corrections which are linear in moneyness, while kurtosis and cokurtosis induce quadratic price corrections. The nonnormality helps to explain the implied correlation smile observed in practice.

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I. Introduction

A standard assumption often made in theoretical and empirical financial research is that the quantity under study has a normal distribution, or if several quantities are considered, then they have a jointly normal distribution. For example, the celebrated Black-Scholes (1973) formula for option pricing is derived under the assumption that the continuously compounded stock return is normally distributed. Many interest rate models, such as the Vasicek model (1977) and the Ho-Lee model (1986), are Gaussian. Cortazar and Naranjo (2006) study an N-factor jointly Gaussian model for futures prices. However, the assumption of normality is rarely fully satisfied in applications. Evidences include Fama (1976), Richardson and Smith (1993), Lo and MacKinlay (1988), Affleck-Graves and McDonald (1989), and Zhou (1993). Given the overwhelming evidence of violations of normality, there have been a plethora of theoretical and empirical studies focusing on the impact of nonnormality of security prices on various issues such as value-at-risk calculations, option pricing, cross-sectional variation of stock returns, hedging decisions, etc. See, for example, Kraus and Litzenberger (1976), Hull and White (1998), Harvey and Siddique (1999, 2000), Bakshi, Kapadia and Madan (2003), Carr and Wu (2007), and Gilbert, Jones and Morris (2006).

This paper studies the effect of nonnormality on exchange options. The most commonly used formula for the price of an exchange option is the Margrabe formula, which was independently discovered by Fischer (1978) and Margrabe (1978). Margrabe (1978) considers the price of an option to exchange one asset for another while Fischer (1978) considers the price of a call option when the exercise price is uncertain. The Margrabe formula can be regarded as an extension of the well-known Black-Scholes (1973) formula because it reduces to the latter when the other asset has fixed value in the Margrabe framework or the exercise price is constant in the Fischer framework. A crucial assumption in applying the Margrabe formula is that the returns of the two assets are jointly normal. Thus, if the asset returns deviate from the jointly normal distribution, the results obtained from the Margrabe formula should be used with care because they can give incorrect prices and price sensitivities.

Although the jointly normal assumption is often violated, during the last three decades, the Margrabe formula has been used extensively by theorists to model various financial and real options, probably because of its simplicity and the availability of a closed-form expression. Stulz (1982), Johnson (1987), Margrabe (1993), Gerber and Shiu (1996) consider extensions of the Margrabe formula. McDonald and Siegel (1985) use the Margrabe formula to study the investment and valuation of firms when there is an option to shut down. Carr (1988) extends the Margrabe formula to consider the valuation of sequential exchange opportunities. It also points out some new applications such as the pricing of variable-rate corporate debt. Hemler (1990) models the quality delivery option in treasury bond futures contracts as an exchange option. Shevlin (1991) investigates the valuation of R&D firms with R&D limited partnerships. Albizzati and Geman (1994) value the surrender option in life insurance polices by extending the Margrabe formula to a HJM stochastic interest rate framework. Grinblatt and Titman (1989), and Johnson and Tian (2000) use the Margrabe formula to study the design and effectiveness of performance-based contracts and executive stock options. Deng, Johnson and Sogomonian (2001) design a technique to value electricity generation and transmission assets based on the Margrabe formula. Davis (2001) prices weather derivatives in the spirit of the Margrabe formula.

On the practitioners' side, various types of exchange options and their extensions (such as spread options) are traded both on exchanges and over the counter. In the fixed income markets, various instruments are traded exchanging securities with different maturities (such as Treasury Notes and Bonds), with different quality levels (such as the Treasury Bill and Eurodollars), and with different issuers (such as French and German bonds, or Municipal bonds and Treasury Bonds). In the agricultural futures markets, the CBOT trades the so-called crush spread which exchanges raw soybeans with a combination of soybean oil and soybean meal. In the energy markets, crack spread options on exchanging crude oil and unleaded gasoline, and on exchanging crude oil and heating oil are both traded on NYMEX. Electricity spark spread options are also actively traded over the counter to exchange a specific fuel and electricity.

There are many possible approaches to introducing nonnormality to exchange option modeling. One approach is to use a nonnormal multivariate distribution for the returns of the two underlying assets. Kotz, Balakrishnan and Johnson (2000) and Hutchinson and Lai (1990) are two wonderful sources for this approach and contain references of many researches using this approach. Another possible approach is to use non-Gaussian processes to model asset returns. For example, one could introduce nonnormality by adding stochastic volatilities or jumps to the asset prices. In addition to these two broad approaches, Cherubini, Luciano and Vecchiato (2004) model nonnormality by using nonnormal copulas, and Adkins and Paxson (2006) consider quasi-closed-form solutions for homogeneity not equal to degree one in a real-option setup where there are multiple sources of uncertainty. The approach taken in this paper differs from the above. A multivariate Gram-Charlier approximation is used to model the joint stock distribution which explicitly takes into account the higher-order moments such as the skewness, coskewness, kurtosis, and cokurtosis. Because Hermite polynomails form a complete orthogonal function series, the Gram-Charlier approximation can be thought of as a lower-order approximation of an arbitrary density where the deviation from normality is small. The advantage of this approach is that it avoids fixing a particular multivariate distribution and allows one to look at the effects of skewness and kurtosis in an explicit way.

While researchers in natural science have been using the Gram-Charlier approximation for a long time, its application to finance was introduced by Jarrow and Rudd (1982). Since then, the univariate Gram-Charlier approximation has been applied and studied by Madan and Milne (1994), Ané (1999), Longstaff (1995), Corrado and Su (1995, 1996, 1997), Backus, Foresi and Wu (2004), Ki, Choi and Lee (2005), and many others. However, the approach here differs from the above research in that a bivariate Gram-Charlier approximation is used. This generalizes the known Gram-Charlier correction for the Black-Scholes formula to an exchange option framework. The bivariate Gram-Charlier approximation is very easy to deal with because of the availability of explicit expressions for the marginal densities, the moment generating functions of the marginal densities, and the cross-moments. It is useful to many other areas of financial modeling in which the joint distribution of multiple assets needs to be considered. Possible examples include basket options, spread options, value-at-risk calculations of portfolios, etc.

Since a closed-form formula for the option price is available, a closer look at the impact of return nonnormality on the price and the implied correlation of the exchange option is taken. The use of bivariate Gram-Charlier approximation allows us to isolate the effects of skewness, coskewness, kurtosis, and cokurtosis. For near-the-money exchange options, skewness and coskewness induce price corrections which are roughly linear functions of moneyness, while kurtosis and cokurtosis induce price corrections which are roughly quadratic functions of moneyness. Translated to the language of the greeks, for near-the-money exchange options, skewness and coskewness induce changes in the deltas but do not change the gammas much, while kurtosis and cokurtosis induce changes in the gammas but do not change the deltas much. The nonnormality in the joint distribution also helps to explain the implied correlation smile observed in practice. For near-the-money exchange options, skewness and coskewness tend to produce implied correlation skew while kurtosis and cokurtosis tend to produce implied correlation smile.

The organization of the paper is as follows. In Section II, two common methods to derive the Margrabe formula are mentioned and then a new method based on direct computation using the bivariate normal density is introduced. This method allows us to compute the price corrections in Section IV when the joint density deviates from bivariate normal. Section III introduces the bivariate Gram-Charlier approximation used in this paper. Section IV studies the impact of nonnormality on exchange option prices and derives closed-form correction terms due to higher-order moments. It also studies the impact of nonnormality on the implied correlation. Section V briefly concludes.

II. A new derivation of the Margrabe formula

Under the Margrabe formula setup, the dynamics of the two stock prices $S_1(t)$ and $S_2(t)$ under the risk-neutral measure $\mathbb{Q}$ are given by

$$\mathrm{d}S_1(t) = rS_1(t)\mathrm{d}t + \sigma_1 S_1(t)\mathrm{d}W_1(t), \tag{1}$$

$$\mathrm{d}S_2(t) = rS_2(t)\mathrm{d}t + \sigma_2 S_2(t)\mathrm{d}W_2(t), \tag{2}$$

where the two Brownian motions $W_1(t)$ and $W_2(t)$ are correlated with constant coefficient ρ. The Margrabe formula gives the time-0 price C of a European option to exchange stock 2 for stock 1 at time T as follows:

$$C = S_1(0)N(d_1) - S_2(0)N(d_2), \tag{3}$$

where

$$d_1 = \frac{\log(S_1(0)/S_2(0)) + \sigma^2 T/2}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}, \tag{4}$$

and $\sigma \equiv \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$.

Two approaches are usually used to prove the Margrabe formula. The first one is the partial differential equation approach. In this approach, one applies Ito's lemma to $C(t, S_1, S_2)$ and derives the following PDE satisfied by C:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 C}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 C}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 C}{\partial S_1 \partial S_2} + rS_1 \frac{\partial C}{\partial S_1} + rS_2 \frac{\partial C}{\partial S_2} = rC. \tag{5}$$

The terminal boundary condition is given by $C(T, S_1, S_2) = (S_1 - S_2)^+$. The fact that $C(t, S_1, S_2)$ is homogeneous of degree 1 in S_1 and S_2 allows one to define $u(t, y) \equiv C(t, S_1, S_2)/S_2$, where $y = S_1/S_2$. With the help of Euler's theorem for homogeneous functions, the PDE can be simplified into

$$\frac{\partial u}{\partial t} + \frac{1}{2}y^2\sigma^2 \frac{\partial^2 u}{\partial y^2} = 0, \tag{6}$$

with boundary condition $u(T, y) = (y - 1)^+$, and σ as defined above. The Margrabe formula in equation (3) now follows from either a simple application of the Feynman-Kac formula (see, for example, Karatzas and Shreve 1991) or a direct mimicking of the Black-Scholes case.

The second approach to proving the Margrabe formula is through a change of numeraire. A detailed explanation of this technique applied to derivative pricing is in Geman, Karoui and Rochet (1995). Specifically, define a new measure $\widetilde{\mathbb{Q}}$ by the following Radon-Nikodym derivative

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\right|_T = \Lambda_T \equiv \frac{e^{-rT} S_2(T)}{S_2(0)}. \tag{7}$$

Under the new measure $\widetilde{\mathbb{Q}}$, traded assets deflated using asset price $S_2(t)$ are martingales. Specifically, if we let $R(t) \equiv S_1(t)/S_2(t)$, then $\mathrm{d}R = R(\sigma_1 \mathrm{d}\widetilde{W}_1 - \sigma_2 \mathrm{d}\widetilde{W}_2)$, where $\widetilde{W}_1(t) = W_1(t) - \rho\sigma_2 t$ and $\widetilde{W}_2(t) = W_2(t) - \sigma_2 t$ are two Brownian motions under $\widetilde{\mathbb{Q}}$ by Girsanov's theorem. Thus we have

$$C = \mathbb{E}^{\mathbb{Q}}[e^{-rT}(S_1(T) - S_2(T))^+] = S_2(0) \cdot \mathbb{E}^{\mathbb{Q}}[\Lambda_T \cdot (R(T) - 1)^+] \tag{8}$$

$$= S_2(0) \cdot \mathbb{E}^{\widetilde{\mathbb{Q}}}[(R(T) - 1)^+]. \tag{9}$$

The Margrabe formula now follows immediately from the Black-Scholes formula by noticing that under $\widetilde{\mathbb{Q}}$, $R(T)$ is lognormally distributed.

Unfortunately, these two methods no longer work when one steps out of the diffusion framework and specifies the joint return distribution using a bivariate Gram-Charlier approximation. A new method to derive the Margrabe formula is introduced below, which is useful in the non-normal joint distribution case. First, the "tail" joint moment generating function for a bivariate normal distribution is derived. The Margrabe formula follows from this equation immediately. Specifically, let X and Y be jointly normal with means μ_X and μ_Y, variances σ_X^2 and σ_Y^2, and correlation coefficient ρ. The quantity we are interested in is $\mathbb{E}[e^{tX+sY} \cdot 1_{X \geq Y}]$. There are many ways to compute this. The easiest way probably is to rotate X and Y and to find a value of p such that $U \equiv X - Y$ and $V \equiv X + pY$ are independent. However, a different and somewhat longer route will be taken. The benefit is that some useful results will be obtained along the way. Let $n(y; \mu, \nu^2)$ denote the density function of a normal random variable with mean μ and variance ν^2 and let $N(y)$ denote the cumulative normal distribution function. Let $n(y)$ denote $n(y; 0, 1)$, the standard normal density function.

First, two lemmas will be established. These two lemmas are crucial to prove Proposition 1 below which in turn is used to prove the Margrabe formula. They are also important later on to derive corrections terms to the Margrabe formula due to nonnormality. The proofs are in Appendix.

Lemma 1. *Let X and Y be jointly normal with means μ_X and μ_Y, variances σ_X^2 and σ_Y^2, and correlation coefficient ρ. Then*

$$\mathbb{E}[e^{tX} 1_{X \geq Y} | Y] = e^{t(\widetilde{\mu}_X + \lambda Y) + t^2 \widetilde{\sigma}_X^2/2} \cdot N\left(\frac{\widetilde{\mu}_X + t\widetilde{\sigma}_X^2 + (\lambda - 1)Y}{\widetilde{\sigma}_X}\right), \tag{10}$$

where $\lambda = \rho\sigma_X/\sigma_Y$, $\widetilde{\mu}_X = \mu_X - \lambda\mu_Y$, and $\widetilde{\sigma}_X = \sqrt{1 - \rho^2}\sigma_X$.

Lemma 2. *Let a and b be real numbers. Then we have*

$$\int_{-\infty}^{\infty} N(a + by) n(y; \mu, \nu^2) \mathrm{d}y = N\left(\frac{a + b\mu}{\sqrt{1 + b^2\nu^2}}\right), \tag{11}$$

and furthermore, for any real number s,

$$\int_{-\infty}^{\infty} N(a+by)e^{sy}n(y;\mu,\nu^2)\mathrm{d}y = e^{\mu s+\nu^2 s^2/2}N\left(\frac{a+b\mu+bs\nu^2}{\sqrt{1+b^2\nu^2}}\right). \tag{12}$$

Because the Black-Scholes formula involves two cumulative normal distributions functions, equation (12) can be useful when it is needed to integrate the Black-Scholes formula. For example, Fischer (1978) considers a situation in which one needs to integrate the Black-Scholes formula because the strike price of the option is also stochastic. Another situation where (12) is useful is a generalized Black-Scholes framework in which the interest rate r and dividend rate δ are stochastic but independent of the stock price shocks and $r-\delta$ is normally distributed.

Below is the main proposition which will be used to prove the Margrabe formula. The proof makes use of Lemma 1 and 2 and is in Appendix.

Proposition 1. *Let X and Y be jointly normal with means μ_X and μ_Y, variances σ_X^2 and σ_Y^2, and correlation coefficient ρ. For any real numbers t and s, we have*

$$\mathbb{E}[e^{tX+sY}1_{X\geq Y}] = \mathbb{E}[e^{tX+sY}]\cdot N\left(\frac{\mu_X-\mu_Y+t(\sigma_X^2-\rho\sigma_X\sigma_Y)-s(\sigma_Y^2-\rho\sigma_X\sigma_Y)}{\sqrt{\sigma_X^2+\sigma_Y^2-2\rho\sigma_X\sigma_Y}}\right), \tag{13}$$

where the joint moment generating function is given by

$$\mathbb{E}[e^{tX+sY}] = \exp\left(t\mu_X+s\mu_Y+\frac{1}{2}t^2\sigma_X^2+\frac{1}{2}s^2\sigma_Y^2+\rho st\sigma_X\sigma_Y\right). \tag{14}$$

Furthermore, let $\theta\geq 0$ and m be two real numbers, then

$$\mathbb{E}[e^{tX+sY}1_{X\geq\theta Y+m}] = \mathbb{E}[e^{tX+sY}]\cdot N\left(\frac{\mu_X-\theta\mu_Y-m+t(\sigma_X^2-\rho\theta\sigma_X\sigma_Y)-s(\theta\sigma_Y^2-\rho\sigma_X\sigma_Y)}{\sqrt{\sigma_X^2+\theta^2\sigma_Y^2-2\rho\theta\sigma_X\sigma_Y}}\right). \tag{15}$$

The Margrabe formula follows from Proposition 1 directly. Specifically, let $X\equiv\log S_1(T)$ and $Y\equiv\log S_2(T)$. Since

$$\log S_1(T) = \log S_1(0)+(r-\sigma_1^2/2)T+\sigma_1 W_1(T), \tag{16}$$

$$\log S_2(T) = \log S_2(0)+(r-\sigma_2^2/2)T+\sigma_2 W_2(T), \tag{17}$$

X and Y are jointly normally distributed with correlation coefficient ρ and

$$\mu_X = \log S_1(0)+(r-\sigma_1^2/2)T, \quad \sigma_X^2=\sigma_1 T, \tag{18}$$

$$\mu_Y = \log S_2(0)+(r-\sigma_2^2/2)T, \quad \sigma_Y^2=\sigma_2 T. \tag{19}$$

Using Proposition 1 and the fact that $\mathbb{E}^{\mathbb{Q}} e^X = S_1(0)e^{rT}$ and $\mathbb{E}^{\mathbb{Q}} e^Y = S_2(0)e^{rT}$, the Margrabe formula follows from the following calculation:

$$c = \mathbb{E}^{\mathbb{Q}}[e^{-rT}(S_1(T) - S_2(T))^+] = e^{-rT}\mathbb{E}^{\mathbb{Q}}[(e^X - e^Y)1_{X \geq Y}] \tag{20}$$

$$= S_1(0)N\left(\frac{\mu_X - \mu_Y + (\sigma_X^2 - \rho\sigma_X\sigma_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y}}\right) - S_2(0)N\left(\frac{\mu_X - \mu_Y - (\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y}}\right). \tag{21}$$

The analysis above applies whenever $\log S_1(T)$ and $\log S_2(T)$ are jointly normally distributed. Thus it can be applied to price extensions of exchange options, such as calendar exchange options. It can also be applied to price more exotic exchange options, for example, options with final payoff $[AS_1(T)^\alpha - BS_2(T)^\beta]^+$, where $A > 0$, $B > 0$, and α, β are real numbers. Another significant application of our method is when $S_1(t)$ and $S_2(t)$ follow Gaussian processes. One example of this is when the prices follow log-OU processes, which are often used to model commodity prices with mean-reverting properties. Recently, Deng, Li and Zhou (2006) use this technique to approximate spread option prices.

Proposition 1 is useful for other things. For example, it also allows us to calculate the risk-neutral probability that the option will be exercised at maturity:

$$P^{\mathbb{Q}}[S_1(T) \geq S_2(T)] = \mathbb{E}^{\mathbb{Q}}[1_{X \geq Y}] = N\left(\frac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y}}\right) = N(d), \tag{22}$$

where

$$d = \frac{\log(S_1(0)/S_2(0)) + (\sigma_2^2 - \sigma_1^2)T/2}{\sigma\sqrt{T}}. \tag{23}$$

III. The bivariate Gram-Charlier approximation

Assume that there are two stocks whose returns are approximately bivariate normal. The current calendar time is 0. The returns of the stocks have deterministic means and variances:

$$\mu_1 = \mathbb{E}^{\mathbb{Q}} \log S_1(T), \quad \mu_2 = \mathbb{E}^{\mathbb{Q}} \log S_2(T), \tag{24}$$

and

$$\nu_1 = \sqrt{\mathrm{Var}^{\mathbb{Q}} \log S_1(T)}, \quad \nu_2 = \sqrt{\mathrm{Var}^{\mathbb{Q}} \log S_2(T)}. \tag{25}$$

Assume that the returns are correlated with constant correlation coefficient ϱ. Notice that this framework is more general than the geometric Brownian motions (GBMs) case and can incorporate all Gaussian cases, such as the popular log-OU case.

Specifically, let $W_1(t)$ and $W_2(t)$ be two Brownian motions with correlation ϱ. In the GBMs case, we have

$$\mathrm{d}S_i = (r - q_i)S_i\mathrm{d}t + \sigma_i S_i \mathrm{d}W_i, \tag{26}$$

where r is the risk-free interest rate, σ_i's are the volatilities, and q_i's are the dividend rates. A simple application of Ito's lemma yields

$$\mu_i = \log S_i(0) + (r - q_i - \sigma_i^2/2)T, \quad \nu_i = \sigma_i\sqrt{T}, \quad \rho = \varrho, \tag{27}$$

The GBMs case can be easily generalized to incorporate seasonality in parameters by allowing σ_i's, q_i's and ρ to be deterministic functions of the calendar time t. This is useful since for some spread options, their underlying assets exhibit strong seasonality in price volatilities and in their return correlations. The general framework above incorporates this generalized GBMs case too.

In the log-OU case, we have

$$\mathrm{d}S_i = -\lambda_i(\log S_i - \eta_i)S_i\mathrm{d}t + \sigma_i S_i \mathrm{d}W_i, \tag{28}$$

where λ_i's are the mean-reverting strengths and η_i's are parameters controlling the long-run means. The Brownian motions are still correlated with instantaneous correlation ϱ. The application of Ito's lemma now gives

$$\mu_i = \eta_i - \frac{\sigma_i^2}{2\lambda_i} + e^{-\lambda_i T}\left(\log S_i(0) - \eta_i + \frac{\sigma_i^2}{2\lambda_i}\right), \quad \nu_i = \sigma_i\sqrt{T}\sqrt{\frac{1 - e^{-2\lambda_i T}}{2\lambda_i}}, \tag{29}$$

$$\rho = 2\varrho\frac{\sqrt{\lambda_1\lambda_2}}{\lambda_1 + \lambda_2}\frac{1 - e^{-(\lambda_1+\lambda_2)T}}{\sqrt{1 - e^{-2\lambda_1 T}}\sqrt{1 - e^{-2\lambda_2 T}}}. \tag{30}$$

The density approximation when the joint distribution deviate from bivariate normal is introduced next. Let $\mathbf{Z}$ be a random vector. A multivariate Gram-Charlier approximation for the density of $\mathbf{Z}$ has the following general form (see, for example, McCullagh 1987):

$$f_{\mathbf{Z}}^{\mathrm{GC}}(\mathbf{z}) = f_{\mathbf{Z}}^{0}(\mathbf{z})\left(1 + \sum_i \eta_i h^i(\mathbf{z}) + \frac{1}{2!}\sum_{ij}\eta_{ij}h^{ij}(\mathbf{z}) + \frac{1}{3!}\sum_{ijk}\eta_{ijk}h^{ijk}(\mathbf{z}) + \frac{1}{4!}\sum_{ijk\ell}\eta_{ijk\ell}h^{ijk\ell}(\mathbf{z}) + \cdots\right), \tag{31}$$

where i, j, k and l all run from 1 to $\dim \mathbf{Z}$, $f_{\mathbf{Z}}^0$ is some benchmark density, η's are coefficients symmetric among their indices, and the h's are multivariate Hermite polynomials. In applications, it is often convenient to first whiten the individual components of $\mathbf{Z}$ to eliminate the first-order coefficients η^i's.

The above Gram-Charlier approximation looks quite formidable in its general form but it will be much nicer after being specialized to our application. Let the random vector $\mathbf{Z}$ be

$$\mathbf{Z} = (Z_1, Z_2)^T, \tag{32}$$

with

$$Z_1 = \frac{\log S_1(T) - \mu_1}{\nu_1}, \quad Z_2 = \frac{\log S_2(T) - \mu_2}{\nu_2}. \tag{33}$$

The benchmark density will be selected next. A popular choice among researchers is the multivariate normal density with identity correlation matrix. However, this choice is inappropriate in most applications because it implicitly assumes that the correlation coefficients are small. Instead, the benchmark density is chosen to be multivariate normal with nontrivial correlation matrix. The shortcoming of this choice of benchmark density is that it makes the correction terms in equation (31) complicated because the components z_i's are not separated. To overcome this, a modified version of the Gram-Charlier approximation is used. Let $n(\mathbf{z};\rho)$ be the bivariate normal distribution with correlation ρ. The final density approximation for $\mathbf{Z}$ is:

$$\begin{aligned} f_{\mathbf{Z}}^{\mathrm{GC}}(\mathbf{z}) =& n(\mathbf{z};\rho) + n(z_1)n(z_2)\bigg(\frac{\gamma_{3,0}}{6}h_3(z_1) + \frac{\gamma_{0,3}}{6}h_3(z_2) + \frac{\gamma_{2,1}}{2}h_2(z_1)h_1(z_2) + \frac{\gamma_{1,2}}{2}h_1(z_2)h_2(z_1) \\ &+ \frac{\kappa_{4,0}}{24}h_4(z_1) + \frac{\kappa_{0,4}}{24}h_4(z_2) + \frac{\kappa_{3,1}}{6}h_3(z_1)h_1(z_2) + \frac{\kappa_{1,3}}{6}h_1(z_1)h_3(z_2) + \frac{\kappa_{2,2}}{4}h_2(z_1)h_2(z_2)\bigg). \end{aligned} \tag{34}$$

where $\gamma_{i,j}$'s and $\kappa_{i,j}$'s are coefficients which are not functions of $\mathbf{z}$. The univariate (modified) Hermite polynomials $h_k(x)$ are defined by

$$\frac{\mathrm{d}^k n(x)}{\mathrm{d}x^k} = (-1)^k h_k(x) n(x). \tag{35}$$

Some of the properties of $f_{\mathbf{Z}}^{\mathrm{GC}}(\mathbf{z})$ are listed below. The proof is in Appendix.

Proposition 2. *For the bivariate Gram-Charlier approximation in equation (34), we have*

1. *It integrates to 1:*

$$\int_{\mathbb{R}^2} f_{\mathbf{Z}}^{\mathrm{GC}}(\mathbf{z})\mathrm{d}\mathbf{z} = 1; \tag{36}$$

2. *The marginal density of Z_1 and Z_2 are given by*

$$f_{Z_1}^{\mathrm{GC}}(z_1) = n(z_1)\left(1 + \frac{\gamma_{3,0}}{3!}h_3(z_1) + \frac{\kappa_{4,0}}{4!}h_4(z_1)\right), \tag{37}$$

$$f_{Z_2}^{\mathrm{GC}}(z_2) = n(z_2)\left(1 + \frac{\gamma_{0,3}}{3!}h_3(z_2) + \frac{\kappa_{0,4}}{4!}h_4(z_2)\right), \tag{38}$$

respectively. Both marginal densities integrate to 1. Furthermore, let t be a real number. Then the "moment generating function" for univariate Gram-Charlier approximation is given by

$$\mathbb{E}^{\mathrm{GC}} e^{tZ_1} = e^{t^2/2}\left(1 + \frac{1}{3!}t^3\gamma_{3,0} + \frac{1}{4!}t^4\kappa_{4,0}\right), \tag{39}$$

and similarly for Z_2.

3. *Let $\mathbb{E}^{\mathrm{GC}}$ denote the "expectation" under the Gram-Charlier approximation. Then*

$$\mathbb{E}^{\mathrm{GC}} z_1 = \mathbb{E}^{\mathrm{GC}} z_2 = 0, \tag{40}$$

$$\mathbb{E}^{\mathrm{GC}} z_1^2 = \mathbb{E}^{\mathrm{GC}} z_2^2 = 1, \quad \mathbb{E}^{\mathrm{GC}} z_1 z_2 = \rho, \tag{41}$$

$$\mathbb{E}^{\mathrm{GC}} z_1^3 = \gamma_{3,0}, \quad \mathbb{E}^{\mathrm{GC}} z_2^3 = \gamma_{0,3}, \tag{42}$$

$$\mathbb{E}^{\mathrm{GC}} z_1 z_2^2 = \gamma_{1,2}, \quad \mathbb{E}^{\mathrm{GC}} z_1^2 z_2 = \gamma_{2,1}, \tag{43}$$

$$\mathbb{E}^{\mathrm{GC}} z_1^4 = 3 + \kappa_{4,0}, \quad \mathbb{E}^{\mathrm{GC}} z_2^4 = 3 + \kappa_{0,4}, \tag{44}$$

$$\mathbb{E}^{\mathrm{GC}} z_1 z_2^3 = 3\rho + \kappa_{1,3}, \quad \mathbb{E}^{\mathrm{GC}} z_1^2 z_2^2 = (1 + 2\rho^2) + \kappa_{2,2}, \quad \mathbb{E}^{\mathrm{GC}} z_1^3 z_2 = 3\rho + \kappa_{3,1}. \tag{45}$$

Statement 1 shows that the density approximation integrates to 1, although it is well-known that it is possible for the Gram-Charlier approximate density to violate positivity. Statement 2 shows that the marginal densities of the bivariate Gram-Charlier approximation reduce to the usual one-dimensional Gram-Charlier approximation. Statement 3 gives meanings to the parameters γ's and κ's. It shows that $\gamma_{i,j}$'s are skewness and coskewness parameters while $\kappa_{i,j}$'s are excess kurtosis and cokurtosis parameters. This can be seen from the fact that if one sets all $\gamma_{i,j}$'s and $\kappa_{i,j}$'s to 0 in the Gram-Charlier approximation, then equations (40) to (45) give the cross-moments of the bivariate normal distribution.

While univariate Gram-Charlier approximation has been extensively used by researchers in finance, multivariate Gram-Charlier approximation has not been fully studied. The proposed multivariate Gram-Charlier approximation above can potentially be used to study portfolio returns, basket options, index options, or real options where multiple underlyings need to be considered.

IV. The impact of nonnormality on exchange option prices, greeks, and implied correlation

Let $s_1 = S_1(0)$ and $s_2 = S_2(0)$. The martingale requirement is that discounted future stock prices under the Gram-Charlier approximation are just s_1 and s_2. That is, it is required that $\mathbb{E}^{\mathrm{GC}} S_i(T) = s_i e^{rT}$ for $i = 1, 2$. In this case, the parameters μ_1, ν_1, $\gamma_{3,0}$, $\kappa_{4,0}$ are not independent. Indeed, we have

Lemma 3. *The no-arbitrage conditions* $\mathbb{E}^{\mathrm{GC}} S_i(T) = s_i e^{rT}$ *for* $i = 1, 2$ *give the following parameter restrictions:*

$$\mu_1 = \mu_1^0 - \log\Big(1 + \frac{1}{3!}\nu_1^3 \gamma_{3,0} + \frac{1}{4!}\nu_1^4 \kappa_{4,0}\Big), \tag{46}$$

$$\mu_2 = \mu_2^0 - \log\Big(1 + \frac{1}{3!}\nu_2^3 \gamma_{0,3} + \frac{1}{4!}\nu_2^4 \kappa_{0,4}\Big), \tag{47}$$

where

$$\mu_1^0 \equiv \log s_1 + (rT - \nu_1^2/2), \quad \mu_2^0 \equiv \log s_2 + (rT - \nu_2^2/2). \tag{48}$$

This martingale restriction lowers the dimension of the parameters space, as was also noticed in Corrado (2007), where he considers general one-dimensional Gram-Charlier approximation of arbitrary order. The exchange option price is now given by $C^{\mathrm{GC}} = e^{-rT}\mathbb{E}^{\mathrm{GC}}[S_1(T) - S_2(T)]^+$. It can be seen that nonzero $\gamma_{i,j}$'s and $\kappa_{i,j}$'s will now contribute to corrections to the exchange option price. The following proposition gives the correction terms in closed-form. The proof is in Appendix.

Proposition 3. *Suppose* $\log S_1(T)$ *and* $\log S_2(T)$ *after standardization follow the bivariate Gram-Charlier approximation. Then to first order in* $\gamma_{i,j}$*'s and* $\kappa_{i,j}$*'s, the spread option price is given approximately by*

$$C^{\mathrm{GC}} \approx C^{\mathrm{Margrabe}} + \sum_{i+j=3} \gamma_{i,j}\Phi_{i,j} + \sum_{i+j=4} \kappa_{i,j}\Psi_{i,j}, \tag{49}$$

where the first term is just the usual Margrabe price

$$C^{\mathrm{Margrabe}} = s_1 N\left(\frac{\log(s_1/s_2)}{\nu^M} + \frac{\nu^M}{2}\right) - s_2 N\left(\frac{\log(s_1/s_2)}{\nu^M} - \frac{\nu^M}{2}\right), \tag{50}$$

with

$$\nu^M = \sqrt{\nu_1^2 - 2\rho\nu_1\nu_2 + \nu_2^2}, \tag{51}$$

and the corrections $\Phi_{i,j}$'s and $\Psi_{i,j}$'s are given by

$$\Phi_{3,0} = \frac{\nu_1^3\omega}{6\nu^3}\left(\frac{3}{2}\nu^2 - \log\frac{s_1}{s_2}\right) + \frac{s_1\nu_1^3}{6}N\left(\frac{\log s_1/s_2}{\nu} + \frac{\nu}{2}\right), \tag{52}$$

$$\Phi_{0,3} = \frac{\nu_2^3\omega}{6\nu^3}\left(\frac{3}{2}\nu^2 + \log\frac{s_1}{s_2}\right) - \frac{s_2\nu_2^3}{6}N\left(\frac{\log s_1/s_2}{\nu} - \frac{\nu}{2}\right), \tag{53}$$

$$\Phi_{2,1} = -\frac{\nu_1^2\nu_2\omega}{2\nu^3}\left(\frac{\nu^2}{2} - \log\frac{s_1}{s_2}\right), \tag{54}$$

$$\Phi_{1,2} = -\frac{\nu_1\nu_2^2\omega}{2\nu^3}\left(\frac{\nu^2}{2} + \log\frac{s_1}{s_2}\right), \tag{55}$$

$$\Psi_{4,0} = \frac{\nu_1^4\omega}{24\nu^5}\left(-\nu^2 + \frac{7}{4}\nu^4 - 2\nu^2\log\frac{s_1}{s_2} + \left(\log\frac{s_1}{s_2}\right)^2\right) + \frac{s_1\nu_1^4}{24}N\left(\frac{\log s_1/s_2}{\nu} + \frac{\nu}{2}\right), \tag{56}$$

$$\Psi_{0,4} = \frac{\nu_2^4\omega}{24\nu^5}\left(-\nu^2 + \frac{7}{4}\nu^4 + 2\nu^2\log\frac{s_1}{s_2} + \left(\log\frac{s_1}{s_2}\right)^2\right) - \frac{s_2\nu_2^4}{24}N\left(\frac{\log s_1/s_2}{\nu} - \frac{\nu}{2}\right), \tag{57}$$

$$\Psi_{3,1} = -\frac{\nu_1^3\nu_2\omega}{6\nu^5}\left(-\nu^2 + \frac{\nu^4}{4} - \nu^2\log\frac{s_1}{s_2} + \left(\log\frac{s_1}{s_2}\right)^2\right), \tag{58}$$

$$\Psi_{1,3} = -\frac{\nu_1\nu_2^3\omega}{6\nu^5}\left(-\nu^2 + \frac{\nu^4}{4} + \nu^2\log\frac{s_1}{s_2} + \left(\log\frac{s_1}{s_2}\right)^2\right), \tag{59}$$

$$\Psi_{2,2} = \frac{\nu_1^2\nu_2^2\omega}{4\nu^5}\left(-\nu^2 - \frac{\nu^4}{4} + \left(\log\frac{s_1}{s_2}\right)^2\right), \tag{60}$$

with

$$\nu = \sqrt{\nu_1^2 + \nu_2^2}, \quad \omega = \frac{1}{\sqrt{2\pi}}\sqrt{s_1 s_2}\exp\left(-\frac{\nu^2}{8} - \frac{(\log s_1/s_2)^2}{2\nu^2}\right). \tag{61}$$

Proposition 3 generalizes the results in Corrado and Su (1996, 1997) and Backus, Foresi and Wu (2004), where the Black-Scholes formula is adjusted for skewness and kurtosis using a one-dimensional Gram-Charlier approximation. Indeed, if one sets $s_2 = K$, $\nu_2 = 0$, and all $\Phi_{i,j}$, $\Psi_{i,j}$ to zero except $\Phi_{3,0}$ and $\Psi_{4,0}$, one will get the one-dimensional result. However, the exposition here has been more careful than previous studies. That is, the second terms in equations (52), (53), (56), and (57) have all been kept, which were often omitted by previous studies on the ground that ν is small. A careful numerical analysis show that these second terms in many cases are of comparable magnitude as the first terms in those equations. The reason is that in ω, the exponential factor can become very small, especially for away-from-the-money options. Also notice that Proposition 3 applies if the asset returns follow approximately geometric Brownian motions or approximately log-OU processes.

Figure 1 plots the price corrections $C^{\mathrm{GC}} - C^{\mathrm{Margrabe}}$ due to nonzero $\gamma_{i,j}$'s and $\kappa_{i,j}$'s with respect to moneyness, measured as $\log(s_1/s_2)$. To separate the effects of each of the higher-order excess moments, all γ's and κ's are set to 0 except for one of them in each of the nine subplots in the figure. For example, subplot 1 looks at the price correction due only to the nonzero skewness

$\gamma_{3,0}$ in the distribution of $\log S_1(T)$, that is, the term $\Phi_{3,0}$ in equation (52). In all these subplots, the only nonzero skewness and coskewness coefficients are set to be 0.25, while the only nonzero kurtosis and cokurtosis coefficients are set to be 1. We only look at positive values of $\gamma_{i,j}$'s and $\kappa_{i,j}$'s because the effects of negative values are simply the mirror images of those of positive values. We specialize Proposition 3 to the GBMs case. The parameters used are: $s_2 = 80$, $\sigma_1 = 0.2$, $\sigma_2 = 0.25$, $T = 0.5$, and $\rho = 0.6$. We allow s_1 to vary so that $\log(s_1/s_2)$ has the range of $[-0.25, 0.25]$.

Subplots 1 and 2 of Figure 1 show the effect of nonzero skewness $\gamma_{3,0}$ and $\gamma_{0,3}$ in the joint distribution of $\log S_1(T)$ and $\log S_2(T)$. As one can see, in addition to an everywhere positive correction (the correction present even when the option is at-the-money), a positive skewness in $\log S_1(T)$ increases the prices of out-of-the-money exchange options while decreases the prices of in-the-money options. On the other hand, a positive skewness in $\log S_2(T)$ decreases the prices of out-of-the-money exchange options while increases the prices of in-the-money options. Also, because in our parameter choices $\sigma_2 > \sigma_1$, the effect of $\gamma_{0,3}$ is larger than that of $\gamma_{3,0}$.

Subplots 3 and 4 of Figure 1 show the effects of nonzero coskewnesses ($\gamma_{2,1}$ and $\gamma_{1,2}$) in the distributions of $\log S_1(T)$ and $\log S_2(T)$, respectively. As one can see in Subplot 3, a positive coskewness $\gamma_{2,1}$ induces price correction somewhat similar to that of $\gamma_{0,3}$. That is, in addition to an everywhere negative background correction, a positive coskewness $\gamma_{2,1}$ increases the price of in-the-money options while decreases the price of out-of-the-money options. Subplot 4 shows that a positive $\gamma_{1,2}$ also induces an everywhere negative background correction. However, contrary to $\gamma_{2,1}$, it decreases the price of out-of-the-money options while increases the price of in-the-money options. Again, the effect of $\gamma_{1,2}$ is slightly larger than that of $\gamma_{2,1}$ because $\sigma_2 > \sigma_1$. Another thing worth noticing is that if all $\gamma_{i,j}$'s are roughly equal to each other, then the effects of the coskewnesses are usually larger than the effects of the skewnesses. Our numerical analysis (not reported here) shows that this is particularly the case if the volatilities of the two assets are high or the time-to-maturity is large.

Subplots 5 and 6 of Figure 1 show the effect of nonzero kurtosis $\kappa_{4,0}$ and $\kappa_{0,4}$ in the distributions of $\log S_1(T)$ and $\log S_2(T)$, respectively. As one can see, their effects are similar to each other. Both of them usually will decrease the prices of near-the-money options while increase the prices of away-from-the-money options. The fact that $\sigma_2 > \sigma_1$ introduces two differences. The first difference is that the effect of $\kappa_{0,4}$ is larger than that of $\kappa_{4,0}$. Another difference is that relative to $\kappa_{0,4}$, $\kappa_{4,0}$ increases the price for an out-of-the-money option slighter more than for an in-the-money option if the two options have the same absolute values of $\log(s_1/s_2)$.

Subplots 7, 8 and 9 of Figure 1 show the effect of nonzero cokurtosis $\kappa_{3,1}$, $\kappa_{1,3}$ and $\kappa_{2,2}$. There are a few things to notice. First, the effects of the cokurtosis $\kappa_{3,1}$ and $\kappa_{1,3}$ are similar to each other but opposite to those of the kurtosis $\kappa_{4,0}$ and $\kappa_{0,4}$. That is, positive cokurtosis $\kappa_{3,1}$ and $\kappa_{1,3}$ will in general increase the prices of near-the-money options while decrease the prices of away-from-the-money options. Second, the fact that $\sigma_2 > \sigma_1$ again introduces two differences. The first difference is that the effect of $\kappa_{1,3}$ is slightly larger than that of $\kappa_{3,1}$. Another difference is that relative to $\kappa_{1,3}$, $\kappa_{3,1}$ increases the price for an out-of-the-money option slighter more than for an in-the-money option if the two options have the same absolute values of $\log(s_1/s_2)$. Finally, the effect of $\kappa_{2,2}$ is opposite to those of the other two cokurtosis. It increases the prices of away-from-the-money options while decreases the prices of near-the-money options. In addition, the effect of $\kappa_{2,2}$ are usually stronger than those of the other two cokurtosis. This is especially true if the volatilities of the two underlying assets are high.

Table I presents the price effect of nonnormality in another format. The parameters used are still the same as those used in Figure 1. The asset 1 price s_1 is chosen to take one of the five values among 65, 75, 85, 95 and 105. Panel A reports the moneyness $\log(s_1/s_2)$ and the Margrabe formula prices without taking into account the nonnormality. As one can see, the Margrabe prices increase with asset 1 prices. Panel B reports the isolated effect of one of the $\gamma_{i,j}$'s and $\kappa_{i,j}$'s. As before, the nonzero $\gamma_{i,j}$'s are taken to be 0.25 and the nonzero $\kappa_{i,j}$'s are taken to be 1. Table I again shows that the price corrections due to nonnormality can be quite significant. Another feature worth pointing out is that while the general magnitudes of price corrections for in-the-money and out-of-the-money options are about the same, the percentage price corrections are usually much larger for out-of-the-money options than for in-the-money options.

Translated to the language of the greeks, for near-the-money exchange options, skewness and coskewness induce changes in the deltas but do not change the gammas much, while kurtosis and cokurtosis induce changes in the gammas but do not change the deltas much.

If the underlying assets of the exchange option are distributed with nonzero $\gamma_{i,j}$'s and $\kappa_{i,j}$'s, the implied correlation ρ^{imp} computed using the Margrabe formula and the observed exchange option price can display various patterns observed in the actual data, such as skew, smile or frown. The implied correlation can be easily computed from a Newton-Raphson algorithm because it is a one-dimensional search. Figure 2 plots the implied correlation with respect to moneyness $\log(s1/s2)$ when one of the γ's or κ's is nonzero and positive. The effects of negative γ's or κ's are just the mirror images of the positive value cases. The parameters used in this figure are exactly the same as those in Figure 1. In particular, $\rho = 0.6$. As one can see from the first four subplots, positive $\gamma_{3,0}$ and $\gamma_{1,2}$ will produce an upward-sloping implied volatility skew, while positive $\gamma_{0,3}$ and $\gamma_{2,1}$ will produce a downward-sloping implied volatility skew.

While the γ's are mostly responsible for the slope of the implied correlation curve, the nonzero κ's are mostly responsible for the curvature of the implied correlation curve. Positive kurtosis $\kappa_{4,0}$, $\kappa_{0,4}$ or positive cokurtosis $\kappa_{2,2}$ will produce negative curvature and thus lead to the observation of implied correlation frown. On the contrary, positive $\kappa_{3,1}$ and $\kappa_{1,3}$ will generate positive curvature and implied correlation smile. Another thing worth noticing is that while the the percentage price corrections are usually much larger for out-of-the-money options than for in-the-money options, the price corrections have similar effect on implied correlations for out-of-the-money and in-the-money options.

We have studied the effect of nonnormality on exchange options using bivariate Gram-Charlier approximation. This method can also be used to study other derivatives, such as spread options, basket options and index options. The analysis also has implications beyond option pricing. For example, our result implies that if given the choice between different indexed executive stock options, company managers might choose the indexing security with the most negative skewness.

V. Conclusion

This paper studies the effect of nonnormality on exchange options. The most commonly used formula for the price of an exchange option is the Margrabe formula. A crucial assumption in applying the Margrabe formula is that the returns of the two assets are jointly normal. Thus, the results obtained from the Margrabe formula which ignores the issue of nonnormality should be used with care because they can give incorrect prices and price sensitivities.

To explicitly account for the deviation from normality, a multivariate Gram-Charlier approximation is used to model joint stock distributions. This approximation allows one to explicitly take into account the higher-order moments such as the skewness, coskewness, kurtosis, and cokurtosis. We then use a bivariate Gram-Charlier approximation to compute the corrections to the Margrabe formula due to higher-order moments in closed-form. We generalize the known Gram-Charlier correction for the Black-Scholes formula to an exchange option framework.

By utilizing the closed-form formula for the prices, a closer look at the impact of return nonnormality on the price and the implied correlation of the exchange option is taken. The use of bivariate Gram-Charlier approximation allows us to isolate the effects of skewness, coskewness, kurtosis, and cokurtosis. For near-the-money exchange options, skewness and coskewness induce price corrections which are roughly linear functions of moneyness, while kurtosis and cokurtosis induce price corrections which are roughly quadratic functions of moneyness. Translated to the language of the greeks, for near-the-money exchange options, skewness and coskewness induce changes in the deltas but do not change the gammas much, while kurtosis and cokurtosis induce changes in the gammas but do not change the deltas much. The nonnormality in the joint distribution can also help to explain the implied correlation smile observed in practice. We find that for near-the-money exchange options, skewness and coskewness tend to produce implied correlation skew while kurtosis and cokurtosis tend to produce implied correlation smile.

The bivariate Gram-Charlier approximation is very easy to deal with because of the availability of explicit expressions for the marginal densities, the moment generating functions for the marginal densities, and the cross-moments. The form of bivariate Gram-Charlier approximation should be useful to many other areas of financial modeling whenever the joint distribution of multiple assets needs to be considered. Possible examples include basket options, spread options, value-at-risk calculations of portfolios, etc.

Finally, it is well-known that the Gram-Charlier approximation only works well when the deviations from normality are mild. In situations where the deviations from normality is dramatic, the results in this section should be used with some caution. Another possible shortcoming is that sometimes the Gram-Charlier approximation can give densities that violate positivity. This in turn can give rise to non-positive option prices, especially when the option is well out of the money.

Appendix

Proof of Lemma 1: The conditional density for X given Y is $n(x; \mu_X + \lambda(Y - \mu_Y), \sigma_X^2(1-\rho^2))$. We also have the identity

$$\int_{x_0}^{\infty} e^{tx} n(x; \mu, \nu^2) \mathrm{d}x = e^{\mu t + \nu^2 t^2/2} N\left(\frac{\mu - x_0}{\nu} + \nu t\right). \tag{62}$$

Lemma 1 now follows from direct computation. ∎

Proof of Lemma 2: Define $F(\cdot)$ and $G(\cdot)$ by

$$F(a) \equiv \int_{-\infty}^{\infty} N(a+y) n(y) \mathrm{d}y; \quad \text{and} \quad G(b) \equiv \int_{-\infty}^{\infty} N(a+by) n(y) \mathrm{d}y. \tag{63}$$

Notice that

$$F(0) = \int_{-\infty}^{\infty} N(y) n(y) \mathrm{d}y = \int_{-\infty}^{\infty} N(y) \mathrm{d}N(y) = \frac{1}{2}, \tag{64}$$

and

$$F'(a) = \int_{-\infty}^{\infty} n(a+y) n(y) \mathrm{d}y = \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{a^2}{4}\right) = \frac{1}{\sqrt{2}} n\left(\frac{a}{\sqrt{2}}\right). \tag{65}$$

Thus,

$$F(a) = F(0) + \int_0^a F'(a) \mathrm{d}a = N\left(\frac{a}{\sqrt{2}}\right). \tag{66}$$

We can compute

$$G(1) = F(a) = N\left(\frac{a}{\sqrt{2}}\right), \quad \text{and} \quad G'(b) = -\frac{ab}{(1+b^2)^{3/2}} n\left(\frac{a}{\sqrt{1+b^2}}\right). \tag{67}$$

Thus

$$\int_{-\infty}^{\infty} N(a+by) n(y) \mathrm{d}y = G(b) = G(1) + \int_1^b G'(b) \mathrm{d}b = N\left(\frac{a}{\sqrt{1+b^2}}\right). \tag{68}$$

The first integral in the lemma now follows immediately:

$$\int_{-\infty}^{\infty} N(a+by) n(y; \mu, \nu^2) \mathrm{d}y = \int_{-\infty}^{\infty} N(a + b\mu + b\nu z) n(z) \mathrm{d}z = N\left(\frac{a + b\mu}{\sqrt{1 + b^2\nu^2}}\right). \tag{69}$$

Using the identity

$$e^{sy} n(y; \mu, \nu^2) = e^{\mu s + s^2\sigma^2/2} n(y; \mu + s\nu^2, \nu^2), \tag{70}$$

the second integral in the lemma now follows from the first integral. ∎

Proof of Proposition 1: By the tower property of expectation, we have

$$\mathbb{E}[e^{tX+sY}1_{X\geq Y}] = \mathbb{E}\big[\mathbb{E}[e^{tX+sY}1_{X\geq Y}|Y]\big] \tag{71}$$

$$= \mathbb{E}\left[e^{t(\widetilde{\mu}_X+\lambda Y)+t^2\widetilde{\sigma}_X^2/2+sY}\cdot N\left(\frac{\widetilde{\mu}_X+t\widetilde{\sigma}_X^2+(\lambda-1)Y}{\widetilde{\sigma}_X}\right)\right] \quad \text{(by Lemma 1)} \tag{72}$$

$$= \int_{-\infty}^{\infty} e^{t(\widetilde{\mu}_X+\lambda y)+t^2\widetilde{\sigma}_X^2/2+sy}\cdot N\left(\frac{\widetilde{\mu}_X+t\widetilde{\sigma}_X^2+(\lambda-1)y}{\widetilde{\sigma}_X}\right) n(y;\mu_Y,\sigma_Y^2)\mathrm{d}y. \tag{73}$$

The first equation in the proposition now follows from a direct computation of the above integral using Lemma 2. A useful identity in simplifying the result is $\sigma_X^2 = \widetilde{\sigma}_X^2 + \lambda^2\sigma_Y^2$.

To compute $\mathbb{E}[e^{tX+sY}1_{X\geq \theta Y+m}]$, consider the pair X and $Z \equiv \theta Y + m$ and use the first equation in the proposition. ∎

Proof of Proposition 2: The $h_i(x)$'s are orthogonal to each other:

$$\int_{-\infty}^{\infty} h_i(x)h_j(x)n(x)\mathrm{d}x = \delta_{ij}\cdot i!. \tag{74}$$

A very useful shorthand for equation (34) is that

$$f_{\mathbf{Z}}^{\mathrm{GC}}(\mathbf{z}) = n(\mathbf{z};\rho) + n(z_1)n(z_2)\sum_{i+j=3,4}\frac{1}{i!\,j!}\,\theta_{i,j}h_i(z_1)h_j(z_2), \tag{75}$$

where $\theta_{i,j} = \gamma_{i,j}$ if $i+j=3$ and $\theta_{i,j} = \kappa_{i,j}$ if $i+j=4$. All the statements in the proposition can be shown using repeated integration by parts and the orthogonality of the Hermite polynomials. For statement 1, notice that

$$\int_{\mathbb{R}^2} f_{\mathbf{Z}}^{\mathrm{GC}}(\mathbf{z})\mathrm{d}\mathbf{z} = 1 + \sum_{i+j=3,4}\frac{1}{i!\,j!}\,\theta_{i,j}\delta_{i,0}\delta_{j,0} = 1. \tag{76}$$

For the "moment generating function", repeated integration by parts gives

$$\int_{-\infty}^{\infty} e^{tz}n(z)h_3(z)\mathrm{d}z = -\int_{-\infty}^{\infty} e^{tz}\mathrm{d}n''(z) = \int_{-\infty}^{\infty} te^{tz}n''(z)\mathrm{d}z = \cdots = t^3e^{t^2/2}. \tag{77}$$

Finally, notice that polynomials in x can be expressed in terms of the Hermite polynomials as follows:

$$x = h_1(x), \tag{78}$$

$$x^2 = h_2(x) + h_0(x), \tag{79}$$

$$x^3 = h_3(x) + 3h_1(x), \tag{80}$$

$$x^4 = h_4(x) + 6h_2(x) + 3. \tag{81}$$

Statement 3 can be easily proven using the above identities. ∎

Proof of Lemma 3: Proof essentially follows from equation (39):

$$s_i e^{rT} = \mathbb{E}^{\mathrm{GC}} S_i(T) = \mathbb{E}^{\mathrm{GC}} e^{\mu_1+\nu_1 Z_1} = e^{\mu_1+\nu_1^2/2}\Big(1+\frac{1}{3!}\nu_1^3\gamma_{3,0}+\frac{1}{4!}\nu_1^4\kappa_{4,0}\Big). \tag{82}$$

Now take the logarithm and rearrange and we get Lemma 3. ∎

Proof of Proposition 3: Let $\mathbb{E}^0$ denote the expectation under the benchmark density $n(\mathbf{z};\rho)$. Then by Proposition 1,

$$e^{-rT}\mathbb{E}^0[S_1(T)-S_2(T)]^+ = s_1 N\left(\frac{\mu_1-\mu_2+(\nu_1^2-\rho\nu_1\nu_2)}{\sqrt{\nu_1^2+\nu_2^2-2\rho\nu_1\nu_2}}\right) - s_2 N\left(\frac{\mu_1-\mu_2-(\nu_2^2-\rho\nu_1\nu_2)}{\sqrt{\nu_1^2+\nu_2^2-2\rho\nu_1\nu_2}}\right). \tag{83}$$

Because of the symmetry

$$s_1 n\left(\frac{\mu_1-\mu_2+(\nu_1^2-\rho\nu_1\nu_2)}{\sqrt{\nu_1^2+\nu_2^2-2\rho\nu_1\nu_2}}\right) = s_2 n\left(\frac{\mu_1-\mu_2-(\nu_2^2-\rho\nu_1\nu_2)}{\sqrt{\nu_1^2+\nu_2^2-2\rho\nu_1\nu_2}}\right), \tag{84}$$

a Taylor series expansion gives us

$$\begin{aligned} &e^{-rT}\mathbb{E}^0[S_1(T)-S_2(T)]^+ \\ &\quad= s_1 N\left(\frac{\mu_1^0-\mu_2^0+(\nu_1^2-\rho\nu_1\nu_2)}{\sqrt{\nu_1^2+\nu_2^2-2\rho\nu_1\nu_2}}\right) - s_2 N\left(\frac{\mu_1^0-\mu_2^0-(\nu_2^2-\rho\nu_1\nu_2)}{\sqrt{\nu_1^2+\nu_2^2-2\rho\nu_1\nu_2}}\right) + \mathcal{O}(\gamma^2,\kappa^2) \\ &\quad= C^{\mathrm{Margrabe}} + \mathcal{O}(\gamma^2,\kappa^2), \end{aligned} \tag{85}$$

where $\mathcal{O}(\gamma^2,\kappa^2)$ denote a term that is at least second order in $\gamma_{i,j}$'s or $\kappa_{i,j}$'s.

Next let us consider the $\Phi_{i,j}$'s and $\Psi_{i,j}$'s. Their calculations are very similar to each other and make repeated uses of Lemma 2, equation (35) and integration by parts. Below we only show the calculation for $\Phi_{3,0}$. Let

$$\theta = \nu_2/\nu_1, \quad m = (\mu_2-\mu_1)/\nu_1. \tag{86}$$

From the Gram-Charlier approximation and equation (35), we have

$$\Phi_{3,0} = -\frac{e^{-rT}}{6}\int_{-\infty}^{\infty} n(z_2)\mathrm{d}z_2 \int_{\theta z_2+m}^{\infty} n'''(z_1)\left[e^{\nu_1 z_1+\mu_1} - e^{\nu_2 z_2+\mu_2}\right]\mathrm{d}z_1. \tag{87}$$

By repeated use of integration by parts, we have

$$\Phi_{3,0} = -\frac{e^{-rT}}{6}(\mathbf{J}_1+\mathbf{J}_2+\mathbf{J}_3), \tag{88}$$

where

$$\mathbf{J}_1 = \int_{-\infty}^{\infty} n(z_2)n(\theta z_2 + m)e^{\nu_2 z_2 + \mu_2}\nu_2 z_2 \mathrm{d}z_2, \tag{89}$$

$$\mathbf{J}_2 = \int_{-\infty}^{\infty} n(z_2)n(\theta z_2 + m)e^{\nu_2 z_2 + \mu_2}\big(\mu_2 - \mu_1 + \nu_1^2\big)\mathrm{d}z_2, \tag{90}$$

$$\mathbf{J}_3 = \int_{-\infty}^{\infty} n(z_2)e^{\mu_1 + \nu_1^2/2}\nu_1^3\ N\big(\nu_1 - m - \theta z_2\big)\mathrm{d}z_2. \tag{91}$$

The integrals for $\mathbf{J}_1$ and $\mathbf{J}_2$ are simple integration involving the normal densities. They can be computed by completing the squares. By Lemma 2,

$$\mathbf{J}_3 = e^{\mu_1 + \nu_1^2/2}\nu_1^3 N\bigg(\frac{\nu_1 - m}{\sqrt{1+\theta^2}}\bigg). \tag{92}$$

Finally, using Lemma 3, we Taylor expand $\Phi_{3,0}$ around $\gamma_{i,j} = 0$ and $\kappa_{i,j} = 0$ and keep only the lowest order (constant) term. ∎

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Table I
The effect of nonnormality on the price of an exchange option

Panel A of this table reports the Margrabe formula prices of the exchange options for various moneyness values. Panel B reports the price corrections due to the nonzero skewness, coskewness, kurtosis and cokurtosis. Each row reports the isolated effect of one of the $\gamma_{i,j}$'s or $\kappa_{i,j}$'s. In each row, the single nonzero $\gamma_{i,j}$ and the single nonzero $\kappa_{i,j}$ are taken to be 0.25 and 1, respectively. For each of the nine cases, the price corrections in absolute dollar terms are reported. The price corrections in terms of percentages of the approximate actual prices C^{GC} are reported in parenthesis. In both panels, the parameters used are: $s_2 = 80$, $\sigma_1 = 0.2$, $\sigma_2 = 0.25$, $T = 0.5$, $\rho = 0.6$.

Panel A: Margrabe formula prices

	s_1				
	65	75	85	95	105
$\log(s_1/s_2)$	−0.2076	−0.0645	0.0606	0.1719	0.2719
C^{Margrabe}	0.3645	2.4363	7.7014	15.7431	25.1612

Panel B: Price correction due to nonnormality

	s_1				
	65	75	85	95	105
$\gamma_{3,0} = 0.25$	0.0559	0.0462	0.0117	−0.0159	−0.0238
	(13.29%)	(1.86%)	(0.15%)	(−0.10%)	(−0.09%)
$\gamma_{0,3} = 0.25$	−0.0515	0.0009	0.0757	0.1142	0.1063
	(−16.45%)	(0.04%)	(0.97%)	(0.72%)	(0.42%)
$\gamma_{2,1} = 0.25$	−0.1668	−0.1013	0.0420	0.1443	0.1657
	(−84.34%)	(−4.34%)	(0.54%)	(0.91%)	(0.65%)
$\gamma_{1,2} = 0.25$	0.1627	0.0546	−0.1295	−0.2436	−0.2502
	(30.86%)	(2.19%)	(−1.71%)	(−1.57%)	(−1.00%)
$\kappa_{4,0} = 1$	0.0096	−0.0291	−0.0426	−0.0239	0.0013
	(2.56%)	(−1.21%)	(−0.56%)	(−0.15%)	(0.01%)
$\kappa_{0,4} = 1$	−0.0324	−0.1002	−0.0812	−0.0015	0.0639
	(−9.75%)	(−4.29%)	(−1.07%)	(−0.01%)	(0.25%)
$\kappa_{3,1} = 1$	−0.0769	0.1447	0.2547	0.1868	0.0612
	(−26.73%)	(5.61%)	(3.20%)	(1.17%)	(0.24%)
$\kappa_{1,3} = 1$	0.0745	0.3211	0.3025	0.0695	−0.1442
	(16.97%)	(11.64%)	(3.78%)	(0.44%)	(−0.58%)
$\kappa_{2,2} = 1$	−0.0434	−0.3699	−0.3997	−0.1523	0.1023
	(−13.50%)	(−17.90%)	(−5.47%)	(−0.98%)	(0.40%)

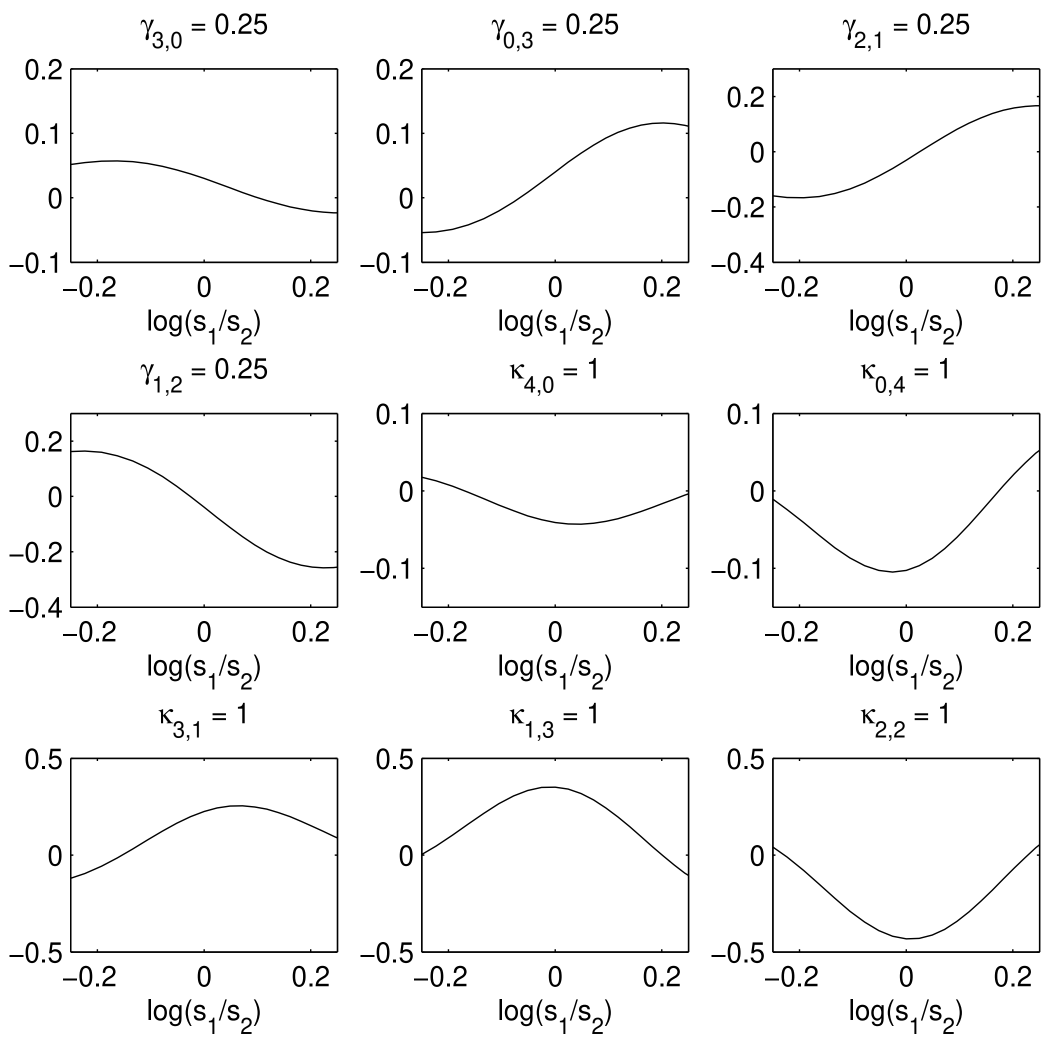

Figure 1. The effect of nonzero skewness, coskewness, kurtosis and cokurtosis on the price of an exchange option. In each of the nine subplots, the price correction in absolute dollar terms due to a nonzero $\gamma_{i,j}$ or $\kappa_{i,j}$ is plotted against the moneyness $\log(s_1/s_2)$. The parameters used are: $s_2 = 80$, $\sigma_1 = 0.2$, $\sigma_2 = 0.25$, $T = 0.5$, $\rho = 0.6$. We allow s_1 to vary so that $\log(s_1/s_2)$ has the range of $[-0.25, 0.25]$.

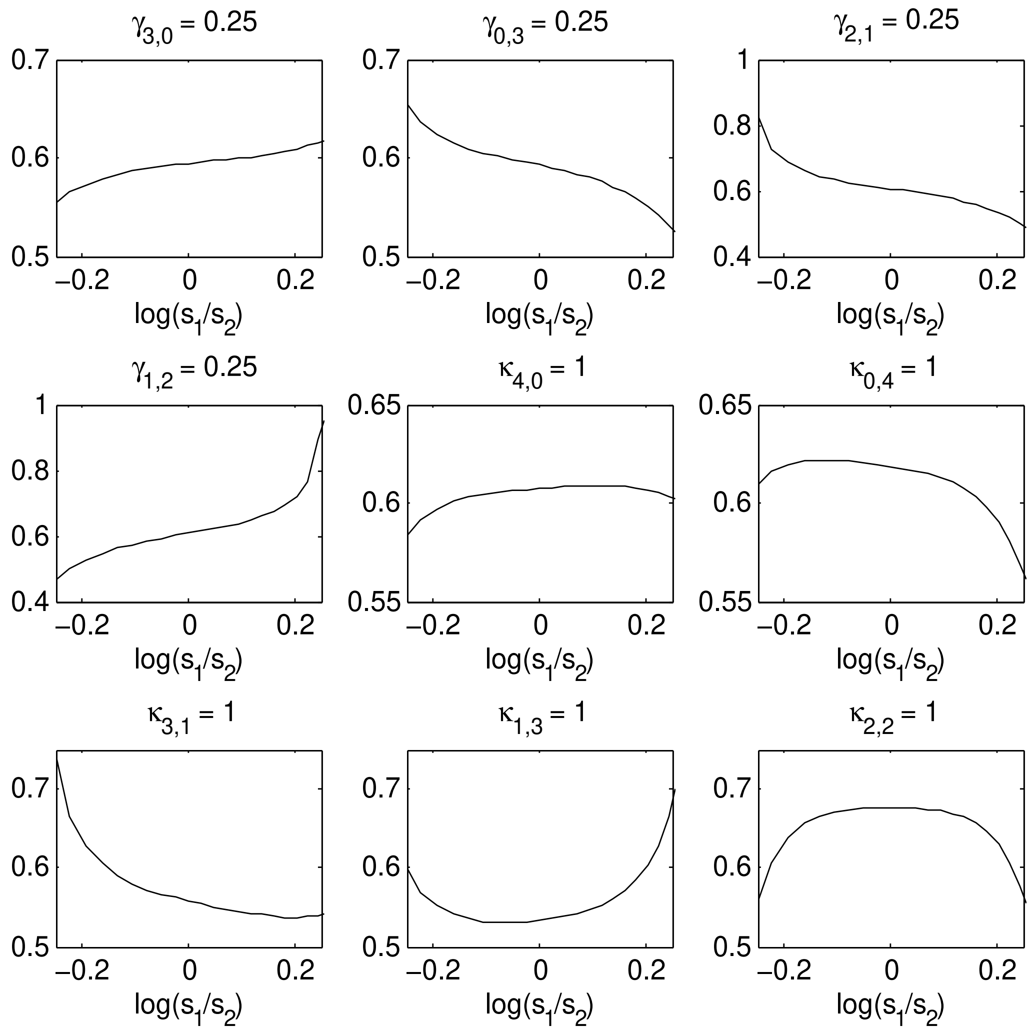

Figure 2. The effect of nonzero skewness, coskewness, kurtosis and cokurtosis on the implied correlation of an exchange option. In each of the nine subplots, the implied correlation with a nonzero $\gamma_{i,j}$ or $\kappa_{i,j}$ is plotted against the moneyness $\log(s_1/s_2)$. The parameters used are: $s_2 = 80$, $\sigma_1 = 0.2$, $\sigma_2 = 0.25$, $T = 0.5$, $\rho = 0.6$. We allow s_1 to vary so that $\log(s_1/s_2)$ has the range of $[-0.25, 0.25]$.