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This paper discusses a tractable approach for computing the likelihood function of non-linear Dynamic Stochastic General Equilibrium (DSGE) models that are solved using second- and third order accurate approximations. By contrast to particle filters, no stochastic simulations are needed for the method here. The method here is, hence, much faster and it is thus suitable for the estimation of medium-scale models. The method assumes that the number of exogenous innovations equals the number of observables. Given an assumed vector of initial states, the exogenous innovations can thus recursively be inferred from the observables. This easily allows to compute the likelihood function. Initial states and model parameters are estimated by maximizing the likelihood function. Numerical examples suggest that the method provides reliable estimates of model parameters and of latent state variables, even for highly non-linear economies with big shocks.

Keywords: Likelihood-based estimation of non-linear DSGE models, higher-order approximations, pruning, latent state variables.
JEL codes: C63, C68, E37

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1. Introduction

During the last three decades, Dynamic Stochastic General Equilibrium (DSGE) models have become the workhorse of modern macroeconomic research. These models have also proven to be invaluable tools for policy analysis and economic forecasting. Due to their complexity, numerical approximations are required to solve DSGE models. The bulk of DSGE-based analysis uses linear approximations. A fast growing recent literature has taken *linearized* DSGE models to the data, using likelihood-based methods (early contributions include Kim (2000), Schorfheide (2000) and Otrok (2001)).

Linearity (in state variables) greatly facilitates model estimation, as it allows to use the standard Kalman filter to infer latent variables and to compute sample likelihood functions based on prediction error decompositions. However, linear approximations are inadequate for models with big shocks, and they cannot capture the effect of risk on economic decisions and welfare. Non-linear approximations are thus, for example, needed for welfare calculations in stochastic models, or for studying asset pricing and non-linearities due to financial frictions and constraints.

Recent research has begun to estimate *non-linear* DSGE models. That work has mainly used particle filters, i.e. filters that infer latent states using Monte Carlo methods (see Fernández-Villaverde and Rubio-Ramírez (2007) and An and Schorfheide (2007) for early applications). Particle filters are slow computationally, which limits their use to small models. Other attempts at empirical estimation of non-linear DSGE models use approximate deterministic filters, essentially non-linear versions of the Kalman filter. See, e.g., Ivashchenko (2014) and Kollmann (2015a) who present 'quadratic' filters for second-order approximate DSGE models; those filters are, however, based on the assumption that the residuals of second-order equated model equations are Gaussian. Nevertheless, these filters may be more accurate than particle filters, and they are clearly much faster than particle filters.

Guerrieri and Iacoviello (2014) point out that if initial values of the state variables are (assumed) known, then one can recursively infer the value of innovations in all periods from the observable data (conditional on the initial state), if the number of observables equals the number of shocks. This makes it unnecessary to use filters, and the likelihood function can easily be computed. Guerrieri and Iacoviello (2014) apply this idea to a simple DSGE model with an occasionally binding collateral constraint (all other model equations are linear), assuming that the initial state vector equals the steady state.

The paper here uses this insight to estimate DSGE models that are solved by second- or third- order Taylor expansions of the decision rules in the neighborhood of a deterministic steady state. 'Local' higher-order approximations of the type considered here are the most widely used non-linear solution methods for DSGE models; due to their great simplicity and speed, they are also currently the only usable non-linear solution methods for medium- scale models (see survey by Kollmann, Maliar, Malin and Pichler (2011) and Kollmann, Kim and Kim (2011)).[1] For this reason, it is important to develop a tractable method that allows *estimating* higher order approximated models. This paper focuses on the estimation of third-order approximated models. The method here can also easily be used for the estimation of second-order accurate models or for models of fourth (or higher) order of accuracy.[2]

A key problem in estimating second- and third order accurate models is that the decision rules include polynomials in the innovations to exogenous variables. Given the predetermined and exogenous variables realized at date t-1, multiple date t exogenous innovations are thus consistent with the period t observables. To overcome this problem, I consider restricted third-order date t decision rules that are *linear* in the date t exogenous innovations—the coefficients of those innovations may, however, be functions of lagged state variables. I show that these restricted decision rules are observationally indistinguishable from decision rules that include higher-order powers of contemporaneous exogenous innovations. Estimating the DSGE model with the restricted decision rules is straightforward. Numerical examples show that the estimation method here is both fast and accurate, even for models with strong non-linearity and big shocks.

While Guerrieri and Iacoviello (2014) postulate that the initial state equals the steady state, I *estimate* the initial state variables (together with the structural model parameters). This allows more precise estimation of latent state variables (in the estimation sample) and of the structural model parameters. In generic DSGE models, the state variables are highly persistent. Erroneously *assuming* that the initial state equals the steady state may thus induce large and persistent estimation errors for states in subsequent periods.

[1]Computer code that allows to easily implement the approximation methods is freely available; see, e.g., Chris Sims' (2000) gensys2 code, Schmitt-Grohé and Uribe's (2004) code, and the Dynare code of Adjemian et al. (2014).
[2] Second-order accurate models have, for example, proven useful for welfare analysis (e.g., Kollmann (2002, 2004). Third-order accurate models are needed to capture endogenous fluctuations in risk-premia.

2. Model format

Standard DSGE models can be expressed as:

$$E_t M(X_{t+1}, Y_{t+1}, X_t, Y_t, \varepsilon_{t+1}) = 0,$$

where E_t is the mathematical expectation conditional on date *t* information; $M: R^{2n+m} \rightarrow R^n$ is a function, and X_t is an n_X x1 vector of exogenous variables and endogenous predetermined variables, while Y_t is an n_Y x1 vector of non-predetermined variables. ε_{t+1} is an mx1 vector of serially independent innovations to exogenous variables. In what follows, ε_t is Gaussian: $\varepsilon_t \sim N(0, \xi^2 \Sigma_\varepsilon)$, where ξ is a scalar that indexes the size of shocks. I assume that $n \equiv n_X + n_Y \geq m$.

The solution of model (1) is given by 'decision rules' $X_{t+1} = G(X_t, \varepsilon_{t+1}, \xi)$ and $Y_t = H(X_t, \xi)$ such that $E_t M(G(X_t, \varepsilon_{t+1}, \xi), H(G(X_t, \varepsilon_{t+1}, \xi), \xi), X_t, H(x_t, \xi), \varepsilon_{t+1}) = 0 \ \forall X_t$. See, e.g., Sims (2010), and Schmitt-Grohé and Uribe (2004) (who also show how generic DSGE models can be expressed in format (1)). Stacking the decision rules, we have $\Omega_{t+1} = F(X_t, \varepsilon_{t+1}, \xi)$, where Ω_{t+1} is the column vector $\Omega_{t+1} \equiv (X_{t+1}; Y_{t+1})$. This paper considers first-, second- and third-order accurate model solutions, namely first-, second- and third-order Taylor series expansions of the policy function around a deterministic stead state, i.e. around $\xi = 0$ and vectors X, Y such that $X = F(X, 0, 0)$, $Y = G(X, 0)$ and $\Omega = H(\Omega, 0, 0)$. Let $x_t \equiv X_t - X$, $y_t \equiv Y_t - Y$ and $\omega_t = (x_t; y_t)$.

First-, second- and third-order accurate model solutions have the following form:

$$\omega_{t+1} = F_1 x_t + F_2 \varepsilon_{t+1}, \tag{1}$$

$$\omega_{t+1} = F_0 \xi^2 + F_1 x_t + F_2 \varepsilon_{t+1} + F_{11} x_t \otimes x_t + F_{12} x_t \otimes \varepsilon_{t+1} + F_{22} \varepsilon_{t+1} \otimes \varepsilon_{t+1}, \tag{2}$$

and $\omega_{t+1} = F_0 \xi^2 + (F_1 + F_{1\xi} \xi^2) x_t + (F_2 + F_{2\xi} \xi^2) \varepsilon_{t+1} + F_{11} x_t \otimes x_t + F_{12} x_t \otimes \varepsilon_{t+1} + F_{22} \varepsilon_{t+1} \otimes \varepsilon_{t+1} + \ldots$

$$F_{111} x_t \otimes x_t \otimes x_t + F_{112} x_t \otimes x_t \otimes \varepsilon_{t+1} + F_{122} x_t \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1} + F_{222} \varepsilon_{t+1} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1}, \tag{3}$$

respectively. $F_0, F_1, F_{1s}, F_2, F_{2s}, F_{11}, F_{12}, F_{22}, F_{111}, F_{112}, F_{122}$ and F_{222} are matrices that are functions of the structural model parameters (i.e. parameters that describe preferences, technologies and other aspects of the economic environment). These matrices do not depend on the scale of shocks (ξ). $\otimes$ denotes the Kronecker product.

When simulating higher-order models it is common to use the 'pruning' scheme of Kim, Kim, Schaumburg and Sims (2008), under which products of state variables are replaced by products of variables approximated to lower order. Let $a_t^{(i)}$ denote a variable a_t approximated to i-th order. Under the pruning scheme, $(a_t b_t)^{(2)}$ is replaced by $a_t^{(1)} b_t^{(1)}$, $(a_t b_t)^{(3)}$ is replaced by $a_t^{(1)} b_t^{(2)} + a_t^{(2)} b_t^{(1)} - a_t^{(1)} b_t^{(1)} = a_t^{(2)} b_t^{(1)} + a_t^{(1)} (b_t^{(2)} - b_t^{(1)})$, and $a_t^{(1)} b_t^{(1)} c_t^{(1)}$ is replaced by $a_t^{(1)} b_t^{(1)} c_t^{(1)}$.

With pruning, the second-order solution (2) is, thus replaced by:

$$\omega_{t+1}^{(2)} = F_0 \xi^2 + F_1 x_t^{(2)} + F_2 \varepsilon_{t+1} + F_{11} x_t^{(1)} \otimes x_t^{(1)} + F_{12} x_t^{(1)} \otimes \varepsilon_{t+1} + F_{22} \varepsilon_{t+1} \otimes \varepsilon_{t+1}, \text{ with } \omega_{t+1}^{(1)} = F_1 x_t^{(1)} + F_2 \varepsilon_{t+1}. \quad (4)$$

The pruned third-order solution is:

$$\omega_{t+1}^{(3)} = F_0 \xi^2 + F_1 x_t^{(3)} + F_{1\xi} \xi^2 x_t^{(1)} + (F_2 + F_{2\xi} \xi^2) \varepsilon_{t+1} + F_{11} \{ x_t^{(2)} \otimes x_t^{(1)} + x_t^{(1)} \otimes (x_t^{(2)} - x_t^{(1)}) \} + F_{12} x_t^{(2)} \otimes \varepsilon_{t+1} + F_{22} \varepsilon_{t+1} \otimes \varepsilon_{t+1} + \ldots$$
$$F_{112} x_t^{(1)} \otimes x_t^{(1)} \otimes \varepsilon_{t+1} + F_{122} x_t^{(1)} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1} + F_{222} \varepsilon_{t+1} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1}. \quad (5)$$

Unless the pruning algorithm is used, second-order approximated models often generate exploding simulated time paths. Pruning ensures that higher-order accurate model solutions are non-explosive if the first-order system (1) is stationary (i.e. when all eigenvalues of F_1 are smaller than unity in absolute value).

The motivation for pruning is that, in repeated applications of (2), third and higher-order terms of state variables appear; e.g., when ω_{t+1} is quadratic in ω_t, then ω_{t+2} is quartic in ω_t; pruning removes these higher-order terms. The unpruned systems (2) and (3) have extraneous steady states (not present in the original model)--some of these steady states mark transitions to unstable behavior. Large shocks can thus move the model into an unstable region. Pruning overcomes this problem.

3. Inferring the exogenous innovations from observables

Assume that, at date t, the econometrician knows the state vectors $x_t^{(1)}, x_t^{(2)}, x_t^{(3)}$ and that she observed 'm' of the elements of the vector $\omega_{t+1}^{(3)}$ (or 'm' linear combinations of the elements of $\omega_{t+1}^{(3)}$), i.e. a vector $z_{t+1} \equiv Q \omega_{t+1}^{(3)}$, where Q is a known matrix of dimension mxn. (Recall that 'm' is the number of exogenous innovations.) (5) implies:

$$z_{t+1} = \gamma_t + Q(F_2 + F_{2\xi} \xi^2) \varepsilon_{t+1} + Q F_{12} x_t^{(2)} \otimes \varepsilon_{t+1} + Q F_{22} \varepsilon_{t+1} \otimes \varepsilon_{t+1} + \ldots$$
$$Q F_{111} x_t^{(1)} \otimes x_t^{(1)} \otimes x_t^{(1)} + Q F_{112} x_t^{(1)} \otimes x_t^{(1)} \otimes \varepsilon_{t+1} + Q F_{122} x_t^{(1)} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1} + Q F_{222} \varepsilon_{t+1} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1}, \quad (6)$$

where $\gamma_t \equiv Q \cdot [F_0 \xi^2 + F_1 x_t^{(3)} + F_{1\xi} \xi^2 x_t^{(1)} + F_{11} \{ x_t^{(2)} \otimes x_t^{(1)} + x_t^{(1)} \otimes (x_t^{(2)} - x_t^{(1)}) \}]$ is a known quantity. As the right-hand side of (6) includes second and third powers of ε_{t+1}, one cannot uniquely solve (6) for the unknown vector of innovations ε_{t+1}. There does not appear to exist a tractable method for computing all of the vectors ε_{t+1} that solve (6) when m is larger than 2 or 3.

One approach to infer the 'true' ε_{t+1} might be to solve (6) for ε_{t+1} using a non-linear equation solve such as Chris Sims' csolve program, using $\varepsilon_{t+1}=0$ as an initial guess.[3] Experiments with a range of models suggest that when the variance of the true innovations is small, then this method detects the true ε_{t+1}.[4] However, this method is not reliable when shocks are large. Computationally, it is also relatively slow.

To avoid these complications, I abstract from the terms in $\varepsilon_{t+1}\otimes\varepsilon_{t+1}$ and in $\varepsilon_{t+1}\otimes\varepsilon_{t+1}\otimes\varepsilon_{t+1}$ in (5), and I consider the following 'restricted' third-order decision rule:

$$\omega_{t+1}^{(3)}=F_0\xi^2+F_1x_t^{(3)}+F_{1\xi}\xi^2x_t^{(1)}+(F_2+F_{2\xi}\xi^2)\varepsilon_{t+1}+F_{11}\{x_t^{(2)}\otimes x_t^{(1)}+x_t^{(1)}\otimes(x_t^{(2)}-x_t^{(1)})\}+F_{111}x_t^{(1)}\otimes x_t^{(1)}\otimes x_t^{(1)}+\ldots$$
$$F_{12}x_t^{(2)}\otimes\varepsilon_{t+1}+F_{112}x_t^{(1)}\otimes x_t^{(1)}\otimes\varepsilon_{t+1}. \qquad (7)$$

Experiments with several models suggest that the restricted decision rule (7) is observationally almost indistinguishable from the third-order model (5), and that even for economies with strong curvature and big shocks. Simulating the decision rules (5) and (7) (using the same initial conditions and the same sequences of innovations) generates sequences of endogenous variables that are extremely highly correlated across (5) and (7) (see below).

Henceforth, I assume that the **true** data generating process is given by equations (4) and (7).

Note that when (7) is assumed, then the observation equation is given by:

$$z_{t+1}=\gamma_t+Q(F_2+F_{2\xi}\xi^2)\varepsilon_{t+1}+QF_{12}x_t^{(2)}\otimes\varepsilon_{t+1}+QF_{112}x_t^{(1)}\otimes x_t^{(1)}\otimes\varepsilon_{t+1}. \qquad (8)$$

This expression is linear in ε_{t+1}. It can be written as $z_{t+1}=\gamma_t+\lambda_t\varepsilon_{t+1}$ where λ_t is an (m *x* m) matrix. Provided that λ_t is non-singular, one can thus infer ε_{t+1} from date t+1 observables:

$$\varepsilon_{t+1}=(\lambda_t)^{-1}(z_{t+1}-\gamma_t). \qquad (9)$$

[3] I thank Matteo Iacoviello for suggesting this approach to me.

[4] I simulate various models by feeding a sequence $\{\varepsilon_{t+1}\}$ into (4),(5); I then tried to infer the innovations from the observables by solving (6) for ε_{t+1} using the csolve algorithm. When the 'true' innovations are small, the method recovers the true values.

4. Sample likelihood

Given the initial state $x_0^{(1)}, x_0^{(2)}, x_0^{(3)}$ and data $\{z_t\}_{t=1}^T$ one can recursively compute the innovations $\{\varepsilon_t\}_{t=1}^T$ and the states $\{x_t^{(i)}, y_t^{(i)}\}_{t=1}^T$ for i=1,2,3 using (4),(7) and (9). The log likelihood of the data, conditional on $x_0^{(1)}, x_0^{(2)}, x_0^{(3)}$ is:

$$\ln L(\{z_t\}_{t=1}^T | x_0^{(1)}, x_0^{(2)}, x_0^{(3)}) = -(mT/2)\ln(2\pi) - (T/2)\ln|\xi^2\Sigma_\varepsilon| - \sum_{t=1}^T \{\varepsilon_t' (\xi^2\Sigma_\varepsilon)^{-1}\varepsilon_t + \ln|\lambda_{t-1}|. \quad (10)$$

One can estimate the initial state, and the structural model parameters, by maximizing the likelihood function with respect to the initial states and parameters.

5. Application I: basic RBC model

I now illustrate the method for the basic RBC model. Assume a closed economy with a representative infinitely-lived household whose date t expected lifetime utility V_t is given by $V_t=\{\frac{1}{1-\sigma}C_t^{1-\sigma}-\frac{1}{1+1/\eta}\psi_t N_t^{1+1/\eta}\}+\lambda_t\beta E_t V_{t+1}$, where C_t and N_t are consumption and hours worked, at t, respectively. $\sigma>0$ and $\eta>0$ are the risk aversion coefficient and the (Frisch) labor supply elasticity. $0<\beta<1$ is the steady state subjective discount factor. $\psi_t>0$ and $\lambda_t>0$ are exogenous preference shocks: ψ_t is a labor supply shock, while λ_t is a shock to the subjective discount factor. ψ_t and λ_t equal unity in steady state. The household maximizes expected lifetime utility subject to the period t resource constraint

$$C_t+I_t+G_t=Y_t,$$

where Y_t and I_t are output, gross investment and exogenous government consumption, respectively. The production function is

$$Y_t=\theta_t K_t^\alpha N_t^{1-\alpha}$$

where K_t is the beginning-of-period t capital stock, and $\theta_t>0$ is exogenous total factor productivity (TFP). The law of motion of the capital stock is

$$K_{t+1}=(1-\delta)K_t+I_t.$$

$0<\alpha,\delta<1$ are the capital share and the capital depreciation rate, respectively. The household's first-order conditions are:

$$\lambda_t E_t \beta (C_{t+1}/C_t)^{-\sigma}(\theta_{t+1}\alpha K_{t+1}^{\alpha-1}N_{t+1}^{1-\alpha}+1-\delta)=1, \quad C_t^{-\sigma}(1-\alpha)\theta_t K_t^\alpha N_t^{-\alpha}=\psi_t N_t^{1/\eta}.$$

The forcing variables follow independent autoregressive processes:

$\ln(\theta_t/\theta)=\rho_\theta \ln(\theta_{t-1}/\theta)+\varepsilon_{\theta,t}$, $\ln(G_t/G)=\rho_G \ln(G_{t-1}/G)+\varepsilon_{G,t}$, $\ln(\psi_t)=\rho_\psi \ln(\psi_{t-1})+\varepsilon_{\psi,t}$, $\ln(\lambda_t)=\rho_\lambda \ln(\lambda_{t-1})+\varepsilon_{\lambda,t}$, with $0<\rho_\theta,\rho_G,\rho_\Psi,\rho_\lambda<1$, where θ and G are steady state TFP and steady state government purchases. $\varepsilon_{\theta,t},\varepsilon_{G,t},\varepsilon_{\psi,t}$ and $\varepsilon_{\lambda,t}$ are normal i.i.d. white noises with standard deviations $\sigma_\theta,\sigma_G,\sigma_\psi$ and σ_λ.

The numerical simulations discussed below assume $\beta=0.99, \eta=4, \alpha=0.3, \delta=0.025$; the steady state ratio of government purchases to GDP (G/Y) is set at 0.2. The autocorrelations of all forcing variables is set at $\rho_\theta=\rho_G=\rho_\psi=\rho_\lambda=0.99$, i.e. these exogenous variables undergo persistent fluctuations. These parameter values in that range are standard in (quarterly) macro models. The risk aversion coefficient is set at a high value, $\sigma=10$, so that the model has enough curvature to allow for non-negligible differences between the second- and third-order model approximations and the and linearized model. In all model variants, I set the scalar ξ that indexes the size of shocks at $\xi=1$. One model variant, referred to as the 'small shocks' variant, assumes $\sigma_\theta=\sigma_G=\sigma_\psi=1\%$ and $\sigma_\lambda=0.025\%$. Those shock sizes (i.e. rate of time preference shocks 40-times smaller than the other shocks) ensure that each shock accounts for a non-negligible share of the variance of the endogenous variables (see Table 1). That 'small shocks' calibration is standard in the RBC literature, and it implies that the volatility of the endogenous variables in the model is roughly consistent with the empirical volatility. In the 'small shocks' variant, the behavior of endogenous variables predicted by the second- and third-order approximated model is broadly similar to that predicted by the linearized model. I thus also consider model variants with much bigger shocks—in those variants, the higher-order approximated model generates predicted behavior that differs noticeably from behavior in the first-order approximated model. In one model variant, I set the standard deviations or shocks 5 times greater than in the 'small shocks' variant $(\sigma_\theta=\sigma_G=\sigma_\psi=5\%, \sigma_\lambda=0.125\%)$; I also consider a variant in which the standard deviation of exogenous innovations is 10 time greater $(\sigma_\theta=\sigma_G=\sigma_\psi=10\%, \sigma_\lambda=0.250\%)$. I refer to these model variants as the 'big shocks' variant and the 'very big shocks' variant, respectively.

I solve the model using the Dynare toolbox (Adjemian et al. (2014)). The Taylor expansions of the model equations are taken with respect to logs of all variables.

5.1. Predicted standard deviations and mean values

Table 1 reports predicted standard deviations of GDP, consumption, investment, hours worked and the capital stock. All variables are expressed in logs. The predicted moments are shown for variables in levels, as well as for first-differenced variables. In the 'small shocks' variant, the order of approximation does not matter much for predicted behavior. For example, the predicted standard deviation of GDP is 3.00% (3.09%) [2.04%] under the first- (second-) [third-] order accurate model approximation.

By contrast, in the model variants with 'big' and with 'very big' shocks, the second- and third-order approximations generate markedly greater volatility of the endogenous variables than the linear approximation. In the 'big shocks' ['very big shocks'] variant the predicted volatility of GDP rises by one quarter [doubles] when the third-order approximation is used, instead of the linear approximation.

Under the linear approximation, the unconditional means of all endogenous variables equals their values in the deterministic steady state. Under the second- and third-order approximations, the unconditional means can differ from the steady state (unconditional means implied by the second and third-order approximations are identical). In the 'small shocks' variant, the mean of capital stock and mean GDP exceeds steady state values by 0.81% and 0.25%, respectively. This is due to precautionary saving that is captured by the second-order approximation. In the 'big shocks' ['very big shocks'] model variant, the mean capital stock and mean GDP are 20.39% and 6.26% [81.56% and 25.05%] above steady state.

5.2. Comparing the 'restricted' versions of the third-order accurate model

Table 2 documents that the 'restricted' version (7) of the (pruned) third-order accurate model is observationally equivalent to the 'unrestricted' version (5). The correlation between time series generated by these variants are very close to unity, for GDP, consumption, investment, hours and the capital stock (both in levels and in first differenced), and that even when shocks are very big.

5.3. Estimating structural parameters and the initial state

I now evaluate the ability of the estimation method to estimate structural model parameters and latent state variables. For each of the three model variants, I generated 40 simulation runs of 5100 periods (each simulation run was initiated at the unconditional means of the state variables). I use the last 100 periods of each simulation run for estimation. Estimation is conducted by maximizing the sum of the likelihood function (10) and a prior log pdf of the initial state (see below). I estimate the initial states and 10 structural parameters: the risk aversion coefficient (σ), labor supply elasticity (η), as well as the autocorrelations and standard deviations of the four exogenous variables. As the model has four exogenous shocks, four observables are needed for estimation. I use first differences of log GDP, consumption, investment and hours worked as observables.

The model has 5 state variables: the capital stock, and the lagged values of each of the four exogenous variables. The likelihood depends on the first-, second- and third- order accurate initial values of these 5 state variables (see (10)). The laws of motion of the four exogenous variables are log-linear. Hence, their values are identical under (log) approximations of orders 1,2 and 3. To reduce the computational burden, I assume (in the current version of the paper) that $k_0^{(2)}=k_0^{(3)}$ and $k_0^{(1)}=k_0^{(3)}-E(k_0^{(3)}-k_0^{(1)})$; in other terms, the second-order accurate initial state capital stock is assumed to equal the third-order accurate capital stock; the first-order accurate initial capital stock is assumed to equal to the third-order accurate initial capital stock, adjusted for the difference between the mean values of these capital stocks.

The precision of the estimates of the state variables and of the model parameters is higher if prior information about the mean and variance of the initial state is used. I use a multivariate normal prior for the initial state vector $(\ln K_0^{(3)}, \ln\theta_0^{(3)}, \ln G_0^{(3)}, \ln\psi_0^{(3)}, \ln\lambda_0^{(3)})$; the prior mean and covariance are set to the unconditional means implied by the third-order accurate model (7) and the unconditional covariance implied by the second-order accurate model (4).[5]

Panel (a) of Table 3 reports the mean, median and standard deviation of the estimated model parameters across the 40 simulation runs, for the 'small shocks' model variant (Columns (1)-(3)), the 'big shocks' variant (Cols. (4)-(6)) and the 'very big shocks' model variant (Cols. (7)-(9)). For each simulation run, I compute the correlation between each estimated state variables (implied by the estimates of structural model parameters) and the true state variables. Panel (b) of Table 3 reports the mean, median and standard deviation of the correlation, across the 40 simulation runs (for each model variant).

Table 3 shows that, for all three model variants, the risk aversion coefficient, the autocorrelations of the exogenous variables and the standard deviations of exogenous

[5] I use the covariance of second-order accurate variables, as the latter can be computed using formulae in Kollmann (2015); future versions will use the unconditional variance of third-order accurate variables (formulae to be derived).

innovations are tightly estimated: the mean and median parameter estimates (across runs) are close to the true parameter values, and the standard deviations of the parameter estimates are small. The labor supply elasticity η is less tightly estimated, in the model variants with 'big shocks' and with 'very big shocks', the median estimates (across 40 runs) are close to the true value (η=4), but the standard deviation of the estimates is sizable.

The estimation method provides remarkably accurate estimates of the 5 state variables. The estimates of the capital stock, TFP, government purchases and the labor supply shock (ψ) are essentially perfectly correlated with the true values of these states, and that irrespective of the size of the shocks. The shock to the rate of time preference (λ) is somewhat less precisely estimated; the median correlations between estimates and true values of λ are 0.98-0.99 (across simulation runs).

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Table 1. RBC model: predicted standard deviations (in%)

	Y	*C*	*I*	*N*	*K*
	(1)	*(2)*	*(3)*	*(4)*	*(5)*
(a) Model variant with small shocks $(\sigma_\theta=\sigma_G=\sigma_\psi=0.01, \sigma_\lambda=0.00025)$					
(a.1) Variables in levels (logs)					
1st order, all shocks	3.00	1.46	10.35	10.46	7.80
1st order, just θ shock	2.08	1.37	6.21	9.43	4.58
1st order, just G shock	1.59	0.08	1.46	1.90	1.03
1st order, just ψ shock	1.08	0.70	3.26	0.93	2.32
1st order, just λ shock	1.68	0.22	7.60	1.55	5.81
2nd order, all shocks	3.09	1.46	10.40	10.45	7.82
3rd order, all shocks	3.04	1.46	10.45	10.44	7.86
(a.2) First-differenced variables (logs)					
1st order, all shocks	0.67	0.17	2.62	1.12	0.17
2nd order, all shocks	0.67	0.17	2.62	0.12	0.17
3rd order, all shocks	0.68	0.17	2.63	0.13	0.17
(b) Model variant with big shocks $(\sigma_\theta=\sigma_G=\sigma_\psi=0.05, \sigma_\lambda=0.00125)$					
(b.1) Variables in levels (logs)					
1st order, all shocks	14.99	7.33	51.76	52.31	39.05
2nd order, all shocks	15.89	7.32	53.87	52.29	39.07
3rd order, all shocks	18.71	7.33	60.18	51.62	44.98
(b.2) First-differenced variables (logs)					
1st order, all shocks	3.35	0.85	13.09	5.63	0.86
2nd order, all shocks	3.56	0.85	13.45	5.77	0.88
3rd order, all shocks	4.00	0.83	14.63	5.94	0.92
(c) Model variant with very big shocks $(\sigma_\theta=\sigma_G=\sigma_\psi=0.10, \sigma_\lambda=0.00250)$					
(c.1) Variables in levels (logs)					
1st order, all shocks	29.99	14.66	103.52	104.62	78.01
2nd order, all shocks	35.41	14.65	115.43	105.46	86.33
3rd order, all shocks	58.77	14.87	166.39	103.71	123.94
(c.2) First-differenced variables (logs)					
1st order, all shocks	6.71	1.70	26.19	11.27	1.72
2nd order, all shocks	7.92	1.71	29.08	12.33	1.86
3rd order, all shocks	11.54	1.63	39.34	14.80	2.21

Note: Standard deviations (std.) of logged variables (listed above Cols. (1)-(5)) are shown for the RBC model. All moments are computed based on one simulation run of 5000 periods (the run is initiated at the unconditional mean of the state variables). Rows labeled '1st order', '2nd order' and '3rd order' show standard deviations predicted by the first-, second- and third-order accurate model variants, respectively. *Y*: GDP; *C*: consumption; *I*: gross investment; *N*: hours worked; *K*: capital stock.

Table 2. RBC model: correlations between variables predicted by 'full' and 'restricted' versions of third-order accurate model (see (5), (7))

	Y	C	I	N	K
	(1)	*(2)*	*(3)*	*(4)*	*(5)*
(a) Model variant with small shocks (σ_θ=σ_G=σ_ψ=0.01, σ_λ=0.00025)					
(a.1) Variables in levels (logs)					
All shocks	1.000	1.000	1.000	1.000	1.000
Just θ shock	1.000	1.000	1.000	1.000	1.000
Just G shock	1.000	1.000	1.000	1.000	1.000
Just ψ shock	1.000	1.000	1.000	1.000	1.000
Just λ shock	1.000	1.000	1.000	1.000	1.000
(a.2) First-differenced variables (logs)					
All shocks	1.000	1.000	1.000	1.000	1.000
Just θ shock	1.000	1.000	1.000	1.000	1.000
Just G shock	1.000	1.000	1.000	1.000	1.000
Just ψ shock	1.000	1.000	1.000	1.000	1.000
Just λ shock	1.000	1.000	1.000	1.000	1.000
(b) Model variant with big shocks (σ_θ=σ_G=σ_ψ=0.05, σ_λ=0.00125)					
(b.1) Variables in levels (logs)					
All shocks	1.000	1.000	1.000	1.000	1.000
Just θ shock	1.000	1.000	1.000	1.000	1.000
Just G shock	1.000	1.000	1.000	1.000	1.000
Just ψ shock	1.000	1.000	1.000	1.000	1.000
Just λ shock	1.000	1.000	1.000	1.000	1.000
(b.2) First-differenced variables (logs)					
All shocks	1.000	1.000	0.996	1.000	1.000
Just θ shock	1.000	1.000	0.999	1.000	1.000
Just G shock	0.999	0.999	1.000	0.999	1.000
Just ψ shock	1.000	1.000	1.000	1.000	1.000
Just λ shock	1.000	1.000	0.998	1.000	1.000
(c) Model variant with very big shocks (σ_θ=σ_G=σ_ψ=0.10, σ_λ=0.00250)					
(c.1) Variables in levels (logs)					
All shocks	1.000	1.000	0.999	1.000	1.000
Just θ shock	1.000	1.000	1.000	1.000	1.000
Just G shock	1.000	1.000	1.000	1.000	1.000
Just ψ shock	1.000	1.000	1.000	1.000	1.000
Just λ shock	1.000	1.000	1.000	1.000	1.000
(c.2) First-differenced variables (logs)					
All shocks	1.000	1.000	0.984	0.999	1.000
Just θ shock	1.000	1.000	0.995	1.000	1.000
Just G shock	0.998	0.998	0.998	0.998	1.000
Just ψ shock	1.000	1.000	0.998	1.000	1.000
Just λ shock	1.000	1.000	0.992	1.000	1.000

Note: Correlations between variables predicted by the 'full' and 'restricted' third-order models are reported. 'All shocks': simulations with all 4 shocks. 'Just θ shocks', 'Just G shocks' etc. pertain to simulations in which just one type of shock is fed into the model; the other exogenous variables are set at steady state values (model is solved assuming 4 shocks). Reported statistics are based on one simulation run of 5000 periods (the run is initiated at the unconditional mean of the state variables). Y: GDP; C: consumption; I: gross investment; N: hours worked; K: capital stock. Correlations greater than 0.9995 are reported as 1.000.

Table 3. RBC model: estimates of structural parameters and of state variables, 40 simulation runs (100 periods)

	Model variant with 'small shocks'			Model variant with 'big shocks'			Model variant with 'very big shocks'		
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*
(a) Parameter estimates									
	Mean	**Median**	**Std**	**Mean**	**Median**	**Std**	**Mean**	**Median**	**Std**
σ	9.97	9.96	0.55	10.06	9.97	0.79	10.44	10.31	1.27
η	4.47	4.04	1.38	7.05	4.24	7.88	4.74	3.59	5.54
ρ_θ	0.99	0.99	0.002	0.99	0.99	0.002	0.99	0.99	0.01
ρ_G	0.98	0.99	0.013	0.98	0.99	0.018	0.98	0.99	0.01
ρ_ψ	0.99	0.99	0.003	0.99	0.99	0.005	0.98	0.99	0.03
ρ_λ	0.99	0.99	0.006	0.99	0.99	0.009	0.99	0.99	0.01
σ_θ (%)	1.01	0.99	0.06	5.07	5.02	0.32	9.98	9.96	0.79
σ_G (%)	1.00	0.99	0.08	4.91	4.77	0.73	12.42	10.88	4.73
σ_ψ (%)	0.99	0.99	0.05	4.97	4.79	0.65	10.69	9.93	2.49
σ_λ (%)	0.025	0.025	0.000	0.13	0.12	0.01	0.25	0.25	0.02
(b) Correlation between estimated & true states									
	Mean	**Median**	**Std**	**Mean**	**Median**	**Std**	**Mean**	**Median**	**Std**
$\ln K$	1.00	1.00	0.00	1.00	1.00	0.00	1.00	1.00	0.00
$\ln \theta$	1.00	1.00	0.00	1.00	1.00	0.00	1.00	1.00	0.00
$\ln G$	1.00	1.00	1.00	1.00	1.00	0.00	1.00	1.00	0.00
$\ln \psi$	1.00	1.00	0.00	1.00	1.00	0.00	1.00	1.00	0.00
$\ln \lambda$	0.88	0.98	0.28	0.95	0.99	0.20	0.94	0.99	0.16

Note: The Table summarizes estimation results across 40 simulation runs of 100 periods each. Panel (a) reports the mean, median and standard deviation of the estimated model parameters across the 40 runs, for the 'small shocks' model variant (Columns (1)-(3)), the 'big shocks' variant (Cols. (4)-(6)) and the 'very big shocks' variant (Cols. (7)-(9)). For each simulation run, the correlation between the estimated state variables and the true state variables was computed. Panel (b) reports the mean, median and standard deviation of that correlation, across the 40 simulation runs (for each model variant). Mean/median correlations above 0.9995 are reported as 1.00.

The ***true*** values of the estimated parameters are: σ=10, η=4, $\rho_\theta=\rho_G=\rho_\psi=\rho_\lambda$=0.99. In the 'small shocks' model variant, the true standard deviations of exogenous innovations are: $\sigma_\theta=\sigma_G=\sigma_\psi$=1%, σ_λ=0.025%. 'Big shocks' model variant: $\sigma_\theta=\sigma_G=\sigma_\psi$=5%, σ_λ=0.125%. 'Very big shocks' model variant: $\sigma_\theta=\sigma_G=\sigma_\psi$=10%, σ_λ=0.250%.