Bayesian inference and Gibbs sampling in generalized true random-effects model

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Abstract

The paper investigates Bayesian approach to estimating generalized true random-effects model (GTRE) via Gibbs sampling. Simulation results show that under properly defined *priors* for transient and persistent inefficiency components the *posterior* characteristics of the GTRE model are well approximated using simple Gibbs sampling procedure. No model reparametrization is required and if such is made it leads to much lower numerical efficiency. The new model allows us to make more reasonable assumptions as regards *prior* inefficiency distribution and appears more reliable in handling especially nuisance datasets. Empirical application furthers the research into stochastic frontier analysis using GTRE by examining the relationship between inefficiency terms in GTRE, true random-effects (TRE), generalized stochastic frontier and a standard stochastic frontier model.

Keywords: generalized true random-effects model, stochastic frontier analysis, Bayesian inference, cost efficiency, firm heterogeneity, transient and persistent efficiency

JEL classification: C11, C23, C51, D24

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1. INTRODUCTION

Stochastic frontier application to panel data has led to a great deal of research into ways of modeling inefficiency variation. If inefficiency in panel data is not entirely object-specific we should reflect its variation from one period to another. This aspect seems particularly important for policymakers and managers that may be interested to know what part of overall inefficiency is due to persistent differences between companies and what part is due to changes within an organization over time. For example, transiency of inefficiency can be viewed as a short-term, within-firm part of inefficiency that resembles *gains & losses* in firm-handling over time. Such inefficiency, if determined, can be fixed relatively fast by making adjustments solely within an organization. Persistent inefficiency, however, may be viewed as beyond the reach of company management, and thus may require external interventions or even regulatory policy changes in order to “even the playing field” between competing companies. Furthermore, since we deal with panel data, we also need to worry about possible heterogeneity of the symmetric error (Baltagi, 2008). Whether or not we can treat such disturbance in the data as homogenous or heterogeneous is in fact an enquiry about the existence of firm-specific effects in the model.

A number of alternatives have been proposed within the stochastic frontier framework (see, e.g., Kumbhakar, Lien and Hardaker, 2014; or Colombi, Matini and Vittadini, 2011; for a discussion). We can summarize them in three main concepts. The first one represents an unconstraint approach to *efficiency*[1] modeling. Efficiency is both time and firm-specific effect (Koop Osiewalski and Steel 1999; Makieła, 2009, 2014). Such models can be further extended, either by adding firm-specific effects as discussed by Greene (2005a,b, 2008) or by generalizing inefficiency term (see, e.g., Kumbhakar and Heshmati, 1995; Kumbhakar and Hjalmarsson, 1995; or ‘Model 5’ in Kumbhakar, Lien and Hardaker, 2014). The second concept is usually applied to “short” panels with short time span. It treats efficiency differences as time-invariant effects (persistent). Any managerial *gains & losses* can only be captured by parametric specification of the model and thus lose their interpretation as efficiency change (see, e.g., Pitt and Lee, 1981; van den Broeck, Koop, Osiewalski and Steel, 1994; Koop, Osiewalski and Steel, 1997; Osiewalski, Wróbel-Rotter, 2008-9). The third approach tries to find some middle ground between the first two, usually by binding efficiency change over time (see, e.g., Battese and Coelli 1992; Kumbhakar and Wang, 2005; or Wang and Ho, 2012). The aim is to reduce the number of latent variables while maintaining some temporal-flexibility at the same time. This, however, is sometimes either too restrictive or simply not enough informative in terms of analyzing differences in efficiency change between firms and over time.

Colombi, Martini and Vittadini (2011) have furthered the unconstraint approach to efficiency analysis by adding firm-specific effect as well as generalizing inefficiency component. Thus the model, known as generalized true random-effects (GTRE), incorporates firm-specific (persistent), time-firm-specific (transient) inefficiency terms and a “true” firm-specific effect. It represents the most generalized form of a stochastic frontier model for panel data analysis and has caught some attention recently (see, e.g., Filippini and Greene, 2015). In a cost function framework it can be written as (Tsionas and Kumbhakar, 2014):

$$y_{it} = x_{it}'\beta + \varepsilon_{it} = x_{it}'\beta + \eta_i^+ + u_{it}^+ + \alpha_i + v_{it} \quad (1)$$

where y_{it} is the cost (in logs), x_{it}' is a k-element vector of independent variables (logs of prices, outputs etc.), β is a vector of model parameters, i (i=1,...,n) and t (t=1,...,T) are object and time indices. The composed error ε_{it} contains: i) two types of symmetric disturbances (α_i, v_{it}), one common to all

[1] *Efficiency* is a transformation of inefficiency measure; it is often used, e.g., especially in production frontier analysis due to more intuitive interpretation; traditionally: $\text{efficiency} = \exp(-\text{inefficiency})$ and $\text{inefficiency} \geq 0$; thus $\text{efficiency} \in (0,1]$. In this paper we deal with cost models, so we tend to discuss inefficiency interpretation as the “distance” to being fully cost efficient.

observations (a "standard" random disturbance v_{it}), one firm-specific (random-effect, reflecting firm heterogeneity α_i); and ii) two types of nonnegative disturbances labelled "+" (η_i^+, u_{it}^+), one common to all observations (transient inefficiency u_{it}^+), one firm-specific (persistent, firm-specific inefficiency η_i^+). Special cases (simplifications) of the composed error term ε_{it} lead to models which are already well known in the literature (see, e.g., Colombi, Martini and Vittadini, 2011; for a discussion). The stochastic components in ε_{it} are, in principle, statistically identifiable. Numerically, however, it can be virtually impossible to, e.g., obtain good estimates of α_i, if variance of v_{it} is high and the other way around. Furthermore, variances of symmetric disturbances α_i and v_{it} also impact our ability to make proper inference about inefficiency component.

The remaining part of the paper is as follows. Section 2 presents Bayesian model based on Tsionas and Kumbhakar (2014) augmented based on propositions in van den Broeck, Koop, Osiewalski and Steel (1994). Section 3 performs a series of simulations similar to the ones in Tsionas and Kumbhakar (2014) showing that new Bayesian GTRE model outperforms its predecessors. The section also discusses cases of very "noisy" datasets, where GTRE models find it difficult to yield satisfactory results and shows that in all cases considered the new model is more reliable. Section 4 presents an empirical application and Section 5 concludes with a discussion.

2. The augmented Tsionas and Kumbhakar model

Let $\theta = (\beta, \sigma_v, \sigma_u, \sigma_\eta, \sigma_\alpha, u^+, \eta^+, \alpha)$ be a vector of structural parameters $(\beta, \sigma_v, \sigma_u, \sigma_\eta, \sigma_\alpha)$ and latent variables (u^+, η^+, α). The full Bayesian model proposed by Tsionas and Kumbhakar (2014) is:

$$p(\beta)p(\sigma_v^{-2})p(\sigma_\alpha^{-2})p(\sigma_u^{-2})p(\sigma_\eta^{-2}) \times \prod_{i=1}^{n}\prod_{t=1}^{T} f_N(y_{it}|x'_{it}\beta + \alpha_i + \eta_i + u_{it}, \sigma_v^2)\, f_N(\alpha_i|0,\sigma_\alpha^2) f_N^+(\eta_i|0,\sigma_\alpha^2) f_N^+(u_{it}|0,\sigma_\alpha^2) \quad (2)$$

where $f_N(.|a, c^{-1})$ denotes density function of the Normal distribution with mean a and precision c, $f_N^+(.|a, c^{-1})$ denotes density function of the half-Normal distribution with mean a and precision c. Informative *prior* on β is $p(\beta) \propto f_N(\beta|b, C^{-1})$ with k-element vector b of *prior* mean and a k-by-k *prior* precision matrix C. Of course, a standard uninformative reference *prior* on β can be used if there is need. We focus our attention on *priors* on the variance components – $p(\sigma_v^{-2})p(\sigma_\alpha^{-2})p(\sigma_u^{-2})p(\sigma_\eta^{-2})$. In Tsionas and Kumbhakar (2014) we have that *prior* on inverse variance σ_j^{-2}, i.e. precision, is $\sigma_j^{-2}Q_j \sim \chi^2(N_j)$, and that $Q_j = 10^{-4}, N_j = 1$ for $j = v, u, \eta, \alpha$. Alternatively we can rewrite this as $p(\sigma_j^{-2}) \propto f_G(\sigma_j^{-2}|0.5 \cdot N_j, 0.5 \cdot Q_j)$, where $f_G(.|w, z)$ is the density function of the gamma distribution with mean *w/z* and variance *w/z*^{2}. This formulation, which yields a quite informative *prior* on the symmetric disturbances[2], may not be the best choice for *prior* efficiency. In fact the median of marginal *prior* density of efficiency is about 0.99, quantile 0.25 is 0.976, quantile 0.75 is 0.996, the interquartile range (IQR) is only around 0.02 and the 95% highest prior density interval is (0.878,1).[3] Clearly this very tight informative *prior* may be strongly against information in the data leading to very irregular (e.g., multimodal) *posterior*. Van den Broeck, Koop, Osiewalski and Steel (1994) discuss the problem of efficiency distribution and *prior* elicitation for model-specific parameters. The authors present their findings for several cases of stochastic frontier models with Erlang and truncated normal distribution, half-normal being its special case (simplification). That is why, following van den Broeck, Koop, Osiewalski and Steel (1994: pp. 286-7) we propose different *priors* on σ_u^{-2} and σ_η^{-2} in order to better reflect our *prior* knowledge about efficiency. The augmented Tsionas and Kumbhakar GTRE model is:

[2] The reader may find much less informative *priors* on precision parameters of the symmetric disturbances in Bayesian literature, e.g., with *prior* mean equal 1 and variance 10^{-2} or even 10^{-4}. Preliminary results have shown, however, that such *prior* can be very "unfavorable" to individual effects α in the model, especially when T is small.

[3] The corresponding characteristics of marginal *prior* inefficiency are: median=0.01, quantile(0.25)=0.004, quantile(0.75)=0.024; 95% highest prior density interval is around $(1.59 \cdot 10^{-5}, 0.129)$. Results acquired numerically.

$$p(\beta)p(\sigma_v^{-2})p(\sigma_\alpha^{-2})f_G(\sigma_u^{-2}|5,10\ln^2(r_u^*))f_G(\sigma_\eta^{-2}|5,10\ln^2(r_\eta^*))$$
$$\times\prod_{i=1}^{n}\prod_{t=1}^{T}f_N(y_{it}|x'_{it}\beta+\alpha_i+\eta_i+u_{it},\sigma_v^2)f_N(\alpha_i|0,\sigma_\alpha^2)\,f_N^+(\eta_i|0,\sigma_\alpha^2)\,f_N^+(u_{it}|0,\sigma_\alpha^2) \tag{3}$$

The new hyperparameters of the model, r_u^* and r_η^*, are *prior* medians of transient and persistent efficiency. Since it seems intuitive to expect that a greater portion (if not all) of observed inefficiency is due to persistent differences between objects we set $r_u^*=0.85$ and $r_\eta^*=0.7$ in our simulations. This can be also interpreted that *a priori* we give more chances for persistent inefficiency to exist and treat transient inefficiency as a less likely, time-varying residual component. *Prior* elicitation leads to the following characteristics of marginal *priors* for transient and persistent efficiency distribution:

- transient efficiency: median=0.85; quantile(0.25)=0.755, quantile(0.75)=0.927; IQR=0.172; mean=0.83; std.=0.122; 95% highest *prior* density interval is (0.597,0.9997); 99%(0.476,0.9997);
- persistent efficiency: median=0.7; quantile(0.25)=0.54, quantile(0.75)=0.848; IQR=0.308; mean=0.683; std.=0.2; 95% highest *prior* density interval is (0.323,0.9993); 99%(0.196,0.9994).

It is now obvious that the proposed augmentation provides more flexible *priors,* which can also be fine-tuned to better fit the research needs. Moreover, since we control location parameter of the *prior* efficiency we can test different values of r^* as we do further in the paper.

Similarly to Tsionas and Kumbhakar (2014) conditional distributions are relatively straightforward to derive in this model and Gibbs sampling procedure can be used. We start with the conditional for a k-element vector β of the cost function parameters:

$$p(\beta|y,X,\theta_{-\beta})\propto f_N^k((C+\sigma_v^{-2}X'X)^{-1}(Cb+\sigma_v^{-2}X'\tilde{y}),(C+\sigma_v^{-2}X'X)^{-1}) \tag{4}$$

or in case of a reference *prior*:

$$p(\beta|y,X,\theta_{-\beta})\propto f_N^k((X'X)^{-1}(X'\tilde{y}),\sigma_v^2(X'X)^{-1}) \tag{5}$$

where $\tilde{y}=y-\iota_T\otimes\alpha-\iota_T\otimes\eta-u$. For precision parameters σ_v^{-2} and σ_α^{-2} the conditionals are:

$$p\left(\sigma_v^{-2}(Q_v+\tilde{v}'\tilde{v})\middle|y,X,\theta_{-\sigma_v}\right)\propto f_{\chi^2}(\sigma_v^{-2}(Q_v+\tilde{v}'\tilde{v})|nT+N_v) \tag{6}$$

$$p(\sigma_\alpha^{-2}(Q_\alpha+\alpha'\alpha)|y,X,\theta_{-\sigma_\alpha})\propto f_{\chi^2}(\sigma_\alpha^{-2}(Q_\alpha+\alpha'\alpha)|n+N_\alpha) \tag{7}$$

where $\tilde{v}=y-X\beta-\iota_T\otimes\alpha-\iota_T\otimes\eta-u$, $Q_v=Q_\alpha=10^{-4}$, $N_\alpha=N_v=1$ and "f_{χ^2}" denotes the χ^2 density function. Conditionals σ_u^{-2} and σ_η^{-2} are:

$$p(\sigma_u^{-2}|y,X,\theta_{-\sigma_u})\propto f_G(\sigma_u^{-2}|\frac{nT}{2}+5,\frac{u'u}{2}+10\ln^2(r_u^*)) \tag{8}$$

$$p(\sigma_\eta^{-2}|y,X,\theta_{-\sigma_\eta})\propto f_G(\sigma_\eta^{-2}|\frac{n}{2}+5,\frac{\eta'\eta}{2}+10\ln^2(r_\eta^*)) \tag{9}$$

Moving on to latent variables, the conditional for an nT-element vector of transient inefficiencies is:[4]

$$p(u|y,X,\theta_{-u})\propto f_N^{nT}(u|\frac{\sigma_u^2}{\sigma_v^2+\sigma_u^2}\tilde{u},\frac{\sigma_v^2\sigma_u^2}{\sigma_v^2+\sigma_u^2}I_{nT})I(u\in R_+^{nT}) \tag{10}$$

where $\tilde{u}=y-X\beta-\iota_T\otimes\alpha-\iota_T\otimes\eta$. The reader should note that I_{nT} is an nT-by-nT identity matrix and that $I(u\in R_+^{nT})$ truncates the normal distribution to only nonnegative values of u_{it}. This implicates that

[4] This is a slightly different conditional than the one reported in Tsionas and Kumbhakar (2014; p. 119). Our analytical derivations have shown, however, that this is the appropriate formula for the conditional of u in the half-normal case. Similar conditional is also reported, e.g., in van den Broeck, Koop, Osiewalski and Steel (1994; p. 281) and Makiela (2014; p. 198).

$f_N^{nT}(.|b, C^{-1})I(u \in R_+^{nT})$ is an nT-dimension truncated normal distribution function with mean vector b and diagonal precision matrix C. For n-element vector of persistent inefficiencies we have:

$$p(\eta|y, X, \theta_{-\eta}) \propto f_N^n(\eta| \frac{\sigma_\eta^2}{\frac{\sigma_v^2}{T} + \sigma_\eta^2}\tilde{\eta}, \frac{\frac{\sigma_v^2 \sigma_\eta^2}{T}}{\frac{\sigma_v^2}{T} + \sigma_\eta^2} I_n) I(\eta \in R_+^n) \quad (11)$$

where $\tilde{\eta} = \bar{\bar{y}} - \bar{\bar{X}}\beta - \alpha - \bar{\bar{u}}$ and symbol " $\bar{\bar{}}$ " denotes an n-element vector of n firm-wise averages for y, X, and u. The last but not least is the conditional for an n-element vector of firm-specific random effects α:

$$p(\alpha|y, X, \theta_{-\alpha}) \propto f_N^n(\alpha| \frac{\sigma_\alpha^2}{\frac{\sigma_v^2}{T} + \sigma_\alpha^2}\tilde{\alpha}, \frac{\frac{\sigma_v^2 \sigma_\alpha^2}{T}}{\frac{\sigma_v^2}{T} + \sigma_\alpha^2} I_n) \quad (12)$$

where this time $\tilde{\alpha} = \bar{\bar{y}} - \bar{\bar{X}}\beta - \eta - \bar{\bar{u}}$. Although the changes made may seem cosmetic they are in fact very important. Unlike in Tsionas and Kumbhakar (2014), a straightforward "naive" Gibbs sampling procedure constructed based on (4-12) has very good mixing properties. As we discuss it further in Section 3 the augmentation makes the model numerically much easier and faster to compute. It also turns out to be more reliable than the originally proposed model reparametrization discussed in Tsionas and Kumbhakar (2014).

3. Results based on simulation experiments

In order to analyze the behavior of the newly constructed Gibbs sampler based on (4-12) we generate datasets similar to the ones in Tsionas and Kumbhakar (2014: 4.2). Specifically, we set the number of observations as n=100 and number of time periods as T=10. We have a constant term and a covariate that is generated as independent standard normal and we set $\sigma_v = 0.1, \sigma_u = 0.2, \sigma_\alpha = 0.2, \sigma_\eta = 0.5$. The starting values are equal to the true parameter values.[5] We run 150,000 iterations, the first 50,000 being discarded. Following Tsionas and Kumbhakar proposition we then take every tenth draw to decrease autocorrelation in the chain and then calculate the *posterior* characteristics of model parameters and latent variables. The reader should note, however, that according to O'Hagan (1994) information about *posterior* characteristics of the model based on the full MCMC chain will always be higher than information based on any of its sub-chains. Even if autocorrelation between subsequent MCMC states is high, a new state always yields additional new information about the *posterior*. For this reason in the next section (empirical example) we use the whole MCMC chain. The last thing left to determine is the *prior* on β. Tsionas and Kumbhakar discuss both, informative as well as reference *priors* and note that they use informative *prior* in their applications (with $b = 0_{k\times1}$ and $C = 10^{-4}I_k$). Our preliminary results have shown that numerically the biggest obstacle in using "naive" Gibbs sampler for model in (2) is the *prior* on the intercept. If the *prior* is very informative (has very tight distribution around the true value) then "naive" Gibbs handles very well. This, however, is not a reasonable assumption and once we move towards less informative *prior* we run into numerical difficulties when sampling from the *posterior*. For this reason we have decided to use the reference (uninformative) *prior* on β in our simulation experiments because numerically it represents the most challenging case for Gibbs samplers to handle; we return to informative *prior* on β in the empirical example in Section 4. Also, unlike Tsionas and Kumbhakar (2014: 4.2) we do not "re-generate" datasets of the same characteristics in this section (e.g., datasets generated *M*-times using the same values of T, n, β, and σ_j's). When estimating such *M*-times generated datasets (generated using the same data generating process – DGP) we have found that for a numerically stable sampling procedure with long MCMC runs the *posterior* estimates exhibit hardly any differences, even when MCMC chain autocorrelation is high. Numerical properties of Gibbs sampler (stability, mixing speed etc.) have been monitored using cusum path plots (Yu and Mykland, 1998) and a multivariate potential scale reduction factor MPSRF (Brooks and Gelman, 1998). A more

[5] We would also initiate the sampler from the *prior* means to check if the results are dependent on the starting points (i.e., too short burn-in phase).

practical argument for not using estimates based on *M*-times generated datasets with long MCMC runs is that Gibbs sampler implementation for GTRE model in (2) based on δ –reparametrization and η –reparametrization takes much more time to compute in comparison to other implementations discussed here. This makes analyses with long chain runs especially time-consuming in this model with no practical gain to it. For the above reasons we have decided to generate several datasets of slightly different characteristics each time (slightly different DGP) and use long MCMC runs.[6] This has also allowed us to explore samplers' mixing properties under different conditions. Experiments based on datasets re-generated 100 times are provided in the Appendix (Table A.1) but are not discussed in this section. We do find particularly important, however, to check if the stochastic components $(u^{+}, \eta^{+}, \alpha, v)$ and explanatory variables (in X) that we generate are indeed independent of each other and are not "accidentally" correlated. This could have some impact and incidentally change the *posterior* characteristics of the model. Fortunately none of the datasets we generated had this problem.

Tables 1 and 2 show experiment results for Gibbs samplers constructed for 5 types of models:

1) GTRE model based on equation (3) – labeled "new GTRE",
2) GTRE model based on equation (2) and reparametrized as proposed in Tsionas and Kumbhakar (2014) – labeled "TK GTRE",
3) Bayesian stochastic frontier true random-effects model, acquired as a simplification of model in (1) so that $\varepsilon_{it} = u_{it}^{+} + \alpha_i + v_{it}$ – labeled "TRE",
4) standard Bayesian SF model, which is a simplification of model in (1) so that $\varepsilon_{it} = u_{it}^{+} + v_{it}$ (see, e.g., Koop, Osiewalski and Steel 1999; Makiela 2009, 2014) – labeled "standard SF",
5) GTRE model based on equation (2) with no reparametrization – labeled "naive GTRE".

For models in 3) and 4) we set $r^{*} = 0.7$ throughout the paper. Following propositions in Greene (2005a,b) we have reported results for true random effects model (*TRE*). This model, however, does not perform as well as a *standard SF* in identifying overall inefficiency ($\omega_{it} = \eta_i + u_{it}$) and thus we do not use it further in this section. We return to this model in empirical application where we show that *TRE* inefficiency estimates are more related to transient inefficiency from *GTRE* model.

[Table 1 here; basic results]

[Table 2 here; results for naive GTRE]

We see that Gibbs samplers for both, *new GTRE* as well as *TK GTRE* handle very well. Implementation of the new model, however, is numerically much more efficient. The time needed to acquire the results in MATLAB is nearly ten times shorter[7] and the new sampler appears to have slightly better mixing properties, as measured by the multivariate potential scale reduction factor (MPRSF=1.0235 vs. 1.0249; see Brooks and Gelman, 1998). Another method to compare samplers' performances (i.e. mixing speeds) is provided in Figure 1, which shows cusum path plot of the intercept from the two simulations. We can clearly see that cusum in *new GTRE* stabilizes more quickly, has lower excursions and a more oscillatory path (less smooth path) than its predecessor. This indicates that Gibbs sampling for the new model is indeed numerically more efficient (the sampler moves faster around parameters space).

Tsionas and Kumbhakar (2014: 4.1) report that *posterior* mean of correlation coefficient between η and $\eta^{(s)}$ is 0.856 and between and u and $u^{(s)}$ is about 0.754.[8] Exact replication of the results based on Tsionas and Kumbhakar (2014: 4.1) is provided in the Appendix (Table A.2; k=2) where the reader

[6] All datasets discussed in this section have been generate in MATLAB with restarted random number generator (zero seed), which allows their replication. Additional simulations were made using randomized datasets (random seed) to check if the simulations results are stable.

[7] In order to minimize the computation time for *TK GTRE* we used a MATLAB procedure provided by Sky Sartorius via MATLAB file exchange that allows us to fully vectorize draws for δ (no loops required). This greatly increases the computation speed of reparametrized model. When we were using only MATLAB's built-in procedures (which require loops) the computation time further increased about 7-9 times.

[8] That is the mean value of correlation coefficient between: "real values of latent variables η, v" and "each draw from the simulation $\eta^{(s)}, u^{(s)}$", where $s = 1, \ldots, S$ and S is the number of accepted draws.

can also view correlation coefficients for other cases considered in this paper (Table A.3: correlations for basic results; Table A.4 correlations for cases 1-3). We find the correlation coefficients to be on average slightly lower for both *GTRE* models. Also, even though *GTRE* models give more in-depth analysis of efficiency, *standard SF* model provides relatively good measures of overall inefficiency (ω) in the dataset. Correlation between *posterior* means of ω's and their true values is 0.78; nearly as good as in *GTRE* models. Thus, a simple SFA model is still quite useful in determining the overall efficiency ranking.

[Figure 1 here; cusum plots]

We now turn to simulation results from Gibbs sampler based on *naive GTRE* (Table 2). When we set $Q_\eta = 10^{-4}$, as in Tsionas and Kumbhakar (2014: p. 116), several marginal *posteriors* are nowhere near the values assumed in the simulation. The intercept estimate is too high, η estimate is very low and dispersion of *posterior* distribution of α is much larger than we would expect given the known DGP (*data generating process*). Considering very tight informative *prior* on η this result should not be that surprising. In fact, once we change $Q_\eta = 10^{-2}$ and double the sampling time the marginal *posterior* distributions reach much closer to values assumed in the simulation (see last column in Table 2).[9] This exercise shows that due to very tight informative *priors* on transient and persistent inefficiencies we may be dealing here with very irregular *posterior,* which is difficult to sample from (see cusum path plot in Figure 2).

[Figure 2 here; cusum 2]

In order to fully examine numerical efficiency (i.e., mixing speed) of Gibbs sampler in the *new GTRE* model let us explore other values for σ_α and σ_v in the DGP. As it has been mentioned in the introduction, practice shows that variance of α and v is crucial in acquiring proper estimates of inefficiency components. Tables 3-5 report results for model estimates once we increase σ_α, σ_v and both. For comparability we also present results for *TK GTRE* and *standard SF*.

[Table 3 here; case 1]

[Table 4 here; case 2]

[Table 5 here; case 3]

Two key findings are worth noting here. First, *new GTRE* better handles extreme cases than its predecessors. It is numerically more efficient and stable than *TK GTRE*, provides more accurate estimates of model parameters than both and, on average, its estimates have higher correlation with the true values of α, η, u, ω (especially when σ_v is high; see Table A.4 in the Appendix). Second, relatively high values of σ_v and σ_α make it extremely difficult to approximate inefficiency differences regardless of the model. For example, *new GTRE* model identifies average levels of *posterior* means for α, η, u, ω relatively well. However, correlation coefficients between simulated inefficiencies η, u, ω and their true values can be very low, especially when $\sigma_\alpha = 1$ and $\sigma_v = 0.8$, not to mention the fact that estimates from *TK GTRE* also exhibit significant numerical instability (MPSRF=1.8152). In order to help the best model (*new GTRE*) cope with low correlation in the above case one could try to fine-tune hyperparameters r_u^* and r_η^* of the *prior* transient and persistent inefficiency. As we explored this concept, however, we found that these hyperparameters have little impact on *posterior* inefficiency estimates and virtually no influence as regards relative differences in inefficiency levels between observations.

The last element that is left to explore deals with our assumptions about *prior* medians of transient (r_u^*) and persistent (r_η^*) efficiency. These are additional hyperparameters that need to be specified in the *new GTRE* model. In a standard Bayesian stochastic frontier analysis r^* should be from 0.5-0.95 interval. Values around 0.7-0.75 are usually set as reference (Osiewalski, 2000; Marzec and Osiewalski 2008), although some studies report much tighter informative *priors* with *prior* median 0.875 (Greene,

[9] $Q_\eta = 10^{-2}$ still implicates very tight informative *prior* with *prior* median efficiency about 0.9, quantile(0.25)=0.78, quantile(0.75)=0.96.

2008). In those models changing r^* only marginally impacts the level of *posterior* mean inefficiency in the sample and has virtually no influence on relative differences in efficiency levels between observations (Makieła, 2014). Although we have already mentioned that fine-tuning these hyperparameters does not help to increase accuracy of inefficiency estimates it is worth to examine what impact different values of r_u^* and r_η^* may have. Up to this point our *prior* assumption about transient and persistent efficiency distribution was that transient efficiency is higher and less likely to exist than persistent (thus $r_u^* > r_\eta^*$). Although this seems like a reasonable assumption to make, we now set both *prior* medians equal and change them between values from 0.5 to 0.9. Table 6 presents estimation results for such cases.

[Table 6 here; r* sensitivity analysis]

Simulation experiments show that the results do not change significantly for fairly reasonable values of r_u^* and r_η^* that oscillate within 0.5-0.9 interval. Once r_u^* and r_η^* reach 0.9 the *priors* on σ_v^{-2} and σ_η^{-2} become very diffused and the sampler's mixing speed may be low because high values of r^* (close to 1) give little *prior* chances that inefficiency terms exist (Koop, Osiewalski and Steel, 1995; Fernandez, Osiewalski and Steel, 1997; Ritter 1993). This also seems to be the case with *GTRE* model based on Tsionas and Kumbhakar (2014). Such strong assumption may sometimes be adequate for transient inefficiency, which existence, e.g., in "short" panels can be debatable. However, it definitely seems unreasonable to assume the same for persistent inefficiency. The overall conclusion and recommendation for r_u^* and r_η^* does not change in relation to standard Bayesian SF models. Values for r_u^* and r_η^* should be set within 0.5-0.95 interval bearing in mind that values close to 0.95 implicate considerably tight informative *prior* and may cause numerical problems if information in the data does not support this idea. Furthermore, when setting the two hyperparameters we should try to reflect our *prior* belief about the relation between levels of transient and persistent inefficiency. If we set highly unrealistic values for *prior* medians (e.g., very low *prior* median for transient and/or very high *prior* median for persistent) the results may turn out either over-optimistic or over-pessimistic with some signs of numerical instability (poor mixing properties of the sampler). This is especially important for persistent inefficiency which estimates rely only on n objects. In this example once we reposition *prior* median from 0.8 to 0.9 we notice a sharp decline in η estimate and much higher *posterior* dispersion of α. In this case information in the data seems to be not strong enough in relations to tight informative *prior* on $\eta,$ which gives little chances for persistent inefficiency to exist. Fortunately, for reasonable-enough values of r_u^* and r_η^* we find hardly any impact on the *posterior* characteristics. Furthermore, the reader should note that in *new GTRE* model we can test different values of r_u^* and r_η^* using Bayesian inference. Under equal *prior* odds we can compare competing model specifications with different *prior* median values using marginal data density. Makieła (2014) shows how marginal data density can be estimated in stochastic frontier models via harmonic mean estimator with Lenk's (2009) correction.

Tsionas and Kumbhakar (2014) also explore other values for T, n, σ_j's and shorter Gibbs runs (see Tsionas and Kumbhakar, 2014: 4.1 & 4.3). We find that both models give good results for reasonable values of T, n, and σ_j's. However, in all cases considered the new model numerically outperforms its predecessor. It takes significantly much less time to compute and it appears more reliable when simulating from the *posterior*. The latter becomes especially evident once we set T=5 and consider more regression parameters (e.g., k=3). In such datasets and comparable MCMC iterations the sampler based on *TK GTRE* significantly underscores the intercept and its implementation is numerically far less efficient in comparison to the new model (MPSRF=1.45; see Table 7).

[Table 7 here; for T=5, k=2,3]

4. Empirical application

Empirical example is based on US banking data from 1998 to 2005 as in Feng and Serletis (2009). We use translog specification with eight input variables and a time trend (3 prices and 5 products; see notes in Table 8). Although we focus here on “Group 1” from the dataset (very large banks) the findings presented in this section are consistent for other groups as well. Since the example is similar to Tsionas and Kumbhakar (2014) we do not comment extensively on the results but focus only on the main findings and differences. Economic regularity constraints are imposed at the means (always) and for the entire dataset through the support B of the *prior* density $p(\beta)$; if met, $I_B(\beta) = 1$, 0.001 otherwise. This means that subsequent state of the MCMC chain, which already meets the constraints at the means, is accepted with probability 1 if it meets the condition for the entire dataset; if not it is accepted with probability 0.001. The simulation is stopped once 100 thousand iterations are accepter – with initial 50 thousand discarded (sampler’s burn-in phase). Ideally we would set $I_B(\beta) = 0$ when regularity conditions are not met for all data points and retain only those iterations that meet the requirements. However, given information in “Group 1” it is practically impossible to impose such strict regularity conditions for the whole dataset and effectively sample from the *posterior*. A relatively straightforward way to fully address this issue in Bayesian approach would be to put much more informative *prior* on β, one that would allow us to directly satisfy theoretical regularity conditions as guided by microeconomic theory. Unfortunately, this undoubtedly may impact the *posterior* characteristics of the model, thus significantly precluding comparability with previous studies. Since it is more important for us to maintain such comparability we do not impose such strict (though more direct) regularity conditions via *prior*. Furthermore, since Tsionas and Kumbhakar (2014) find persistent inefficiency to be smaller than transient inefficiency, *a priori* we do not favor any inefficiency component and set both *prior* medians to 0.8.

[Table 8 here: empirical results]

Table 8 and Figures 3-5 compare results for four models: *GTRE, TRE, standard SF* and a *generalized SF* model, here labeled *GSF* (i.e.: $\varepsilon_{it} = u_{it}^{+} + \eta_i^{+} + v_{it}$). Similarly to Feng and Serletis (2009) we find overall annual reduction of total cost (technical progress), which is also partially in line with results from Tsionas and Kumbhakar (2014). Dependently on the model, *posterior* estimates of returns to scale are between 1.063-1.086 indicating, on average, increasing returns to scale. We also find an interesting pattern in terms of modelling inefficiency and individual effects in the analyzed models. Since the *standard SF* model does not have individual effects, *posterior* estimate of σ_v is relatively high. Symmetric individual effects (α) in the *TRE* model are quite significant, make the *posterior* estimate of σ_v much smaller (in *TRE*) and there is also less inefficiency found than in *standard SF*. *Posterior* standard deviation of symmetric individual effect α in the *GTRE* is smaller in comparison *TRE* (Figure 4), which is different to Tsionas and Kumbhakar (2014). This can be attributed to very tight *prior* on η in the previous study. Here once we “tighten” the *prior* on η in *GTRE* the *posterior* distribution of α also becomes more diffused.[10] Inefficiency components in *GSF* model are very similar to the ones from *GTRE* with only persistent inefficiency being slightly higher. This difference is likely because there are no individual effects (α) in *GSF*.

In general we find inefficiency terms to be much higher than the ones reported by Tsionas and Kumbhakar (2014). The reader should note, however, that the previous model implied very tight informative *prior* on efficiency centered around 0.99 value. Considering the tight *prior,* reasons for such low inefficiency estimates become obvious. Also, unlike in Tsionas and Kumbhakar (2014) we find that *a posteriori* persistent inefficiency distribution (η) is centered considerably higher and much more diffused than transient inefficiency (u), and thus the resulting overall inefficiency (ω) scores in *GTRE* model are also considerably higher than in in *TRE* and *standard SF* (see Figure 5). In fact, inefficiency component in *TRE* model has very similar *posterior* characteristics to transient inefficiency from *GTRE*. Their density charts from Figures 3 and 5 nearly overlap and their *posterior* inefficiency rankings are almost identical (0.998 correlation between *posterior* means of inefficiency; see Table A.5 in the Appendix). Thus, inefficiency estimates that we acquire using *TRE* model should be treated as transient rather than overall inefficiency scores. Persistent inefficiency is most likely captured via bank effects (α) in the *TRE* model. Furthermore, we find that *posterior* estimates of inefficiency scores in *standard SF*

[10] *Posterior* standard deviation of α is 0.027 if *prior* median η is 0.9, compared to 0.021 for *prior* median η 0.8.

are quite similar to overall inefficiency scores in *GTRE* model (0.895 correlation between *posterior* means of inefficiency; see Table A.5 in the Appendix).

[Table 9 here; sensitivity analysis]

Since *posterior* distribution of η is relatively diffused and centered around significantly higher values than transient inefficiency (u) it is worth exploring how *prior* median influences *posterior* characteristics of η distribution. Sensitivity analysis provided in Table 9 shows that: i) *prior* median 0.8 implicates *posterior* mean of persistent efficiency also around 0.8; ii) for *prior* median 0.7, the *posterior* mean is around 0.777; iii) if we further lower *prior* median to 0.6, which implicates a relatively diffused *prior*, the *posterior* mean is still 0.748 (0.048); and iv) if we set a relatively tight informative *prior* with *prior* median 0.9 the resulting *posterior* mean is around 0.852 (0.055). This indicates that for very high/low values of *prior* median information in the data pulls the *posterior* significantly away from the initially centered *prior*, even if the *prior* is relatively tight. More importantly, however, correlation coefficient of Bank's persistent inefficiencies between models with *prior* median 0.6 and 0.9 is 0.993 (Spearman's rank correlation is 0.997). This indicates that *prior* median level has virtually no impact on relative differences in persistent inefficiency estimates between banks.

[Figure 3 here]

[Figure 4 here]

[Figure 5 here]

5. Concluding remarks

In this paper we have proposed a revised approach to Bayesian inference in generalized true random-effects model (GTRE). As we have shown, the revised model (and its numerical implementation) significantly outperforms its predecessors. Artificial examples have shown that both models handle well in favorable conditions; that is: i) if the dataset is large-enough, ii) symmetric disturbances are relatively small in respect to inefficiencies, and iii) we do not have that many regression parameters in the model. However, in more nuisance datasets advantages of the new model are evident, no doubt due less strict and better-tuned *priors* on efficiency terms. The new model is not only easier and faster to compute but it also allows for more robust analysis. By controlling our *prior* beliefs about η and u we can learn how much information in the data alters the *posterior* in relation to the *prior*. This becomes especially important in case of firm-specific effects (η, α), which *posterior* characteristics in the GTRE model are quite diffused and may dependent on η *prior*.

In empirical application we show that the GTRE specification is interconnected with other models already known in the literature. This seems especially interesting because we can acquire these models by reducing selected stochastic components of the GTRE and it may impact the remaining components of the simplified model. By using GTRE model we can have full view of how each component is relevant in describing the given data and we can make more informed decision as to which stochastic frontier model should be chosen.

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FIGURES

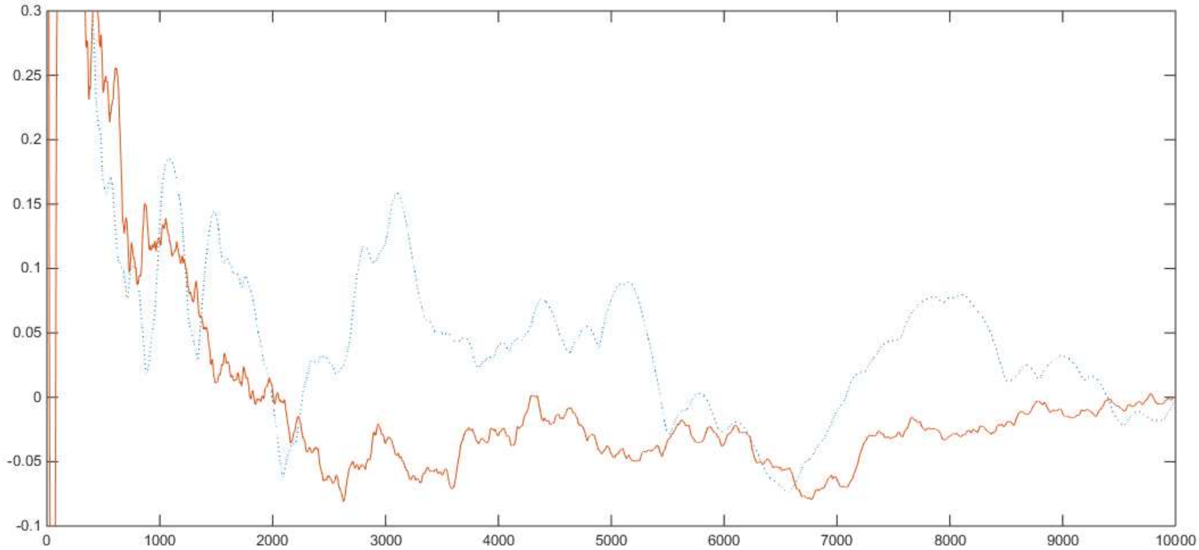

Figure 1. CUSUM path plots for new GTRE model (solid line) and TK GTRE (dotted line)

Source: author's calculations.

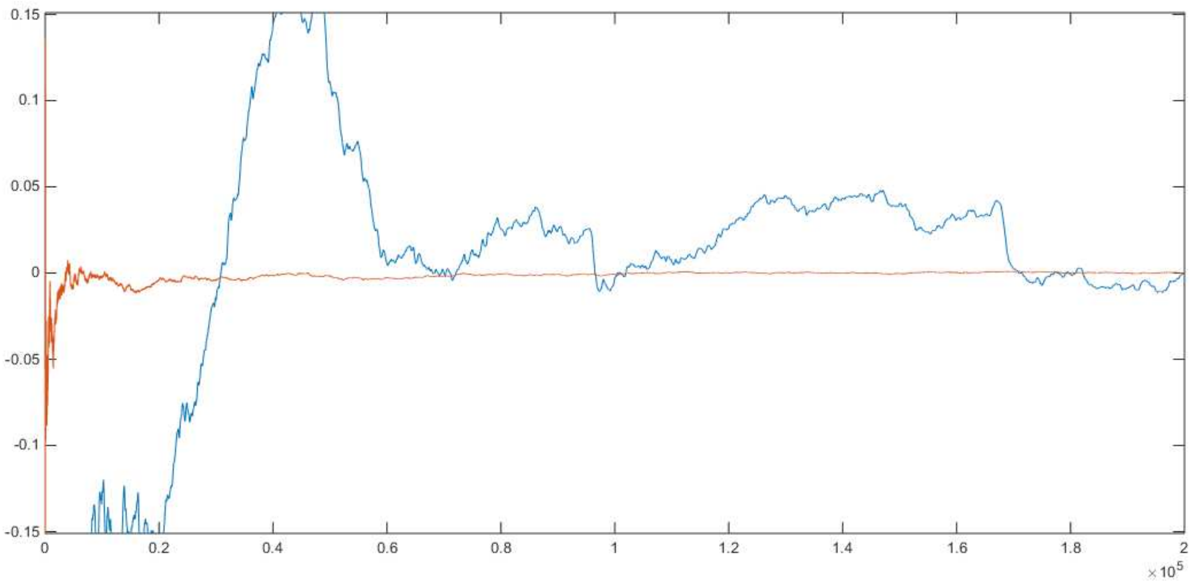

Figure 2. CUSUM path plot for naive GTRE model ($Q_\eta = 10^{-2}$)

Note: CUSUM path plot is for the intercept. The other (almost flat) line is a benchmark path based on independent sampler with the same mean and standard deviation. *Source:* author's calculations.

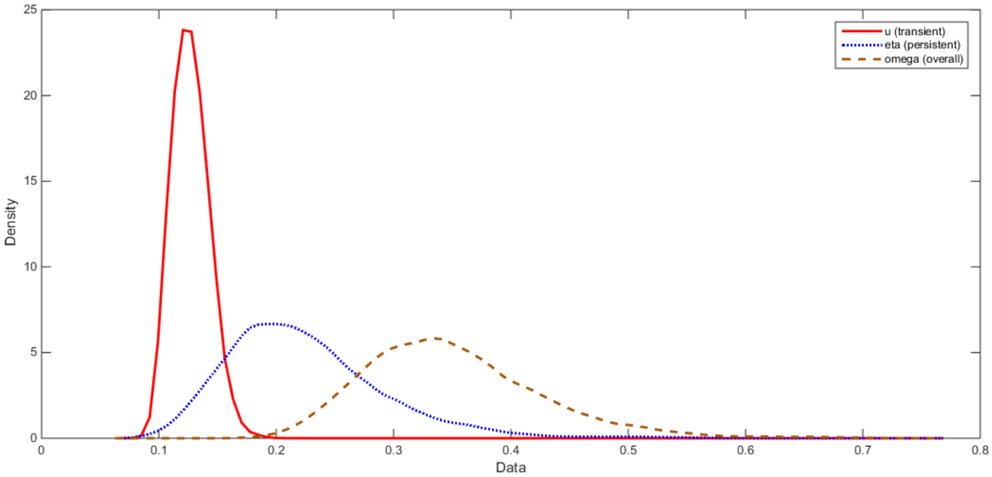

Figure 3. ***Posterior*** **distributions of inefficiency components in the GTRE model**

Source: author's calculations.

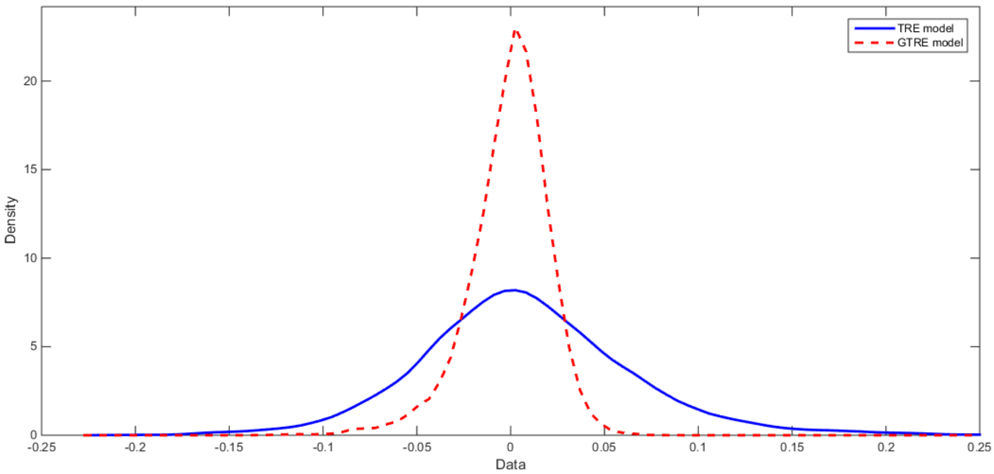

Figure 4. ***Posterior*** **distribution of bank effects**

Source: author's calculations.

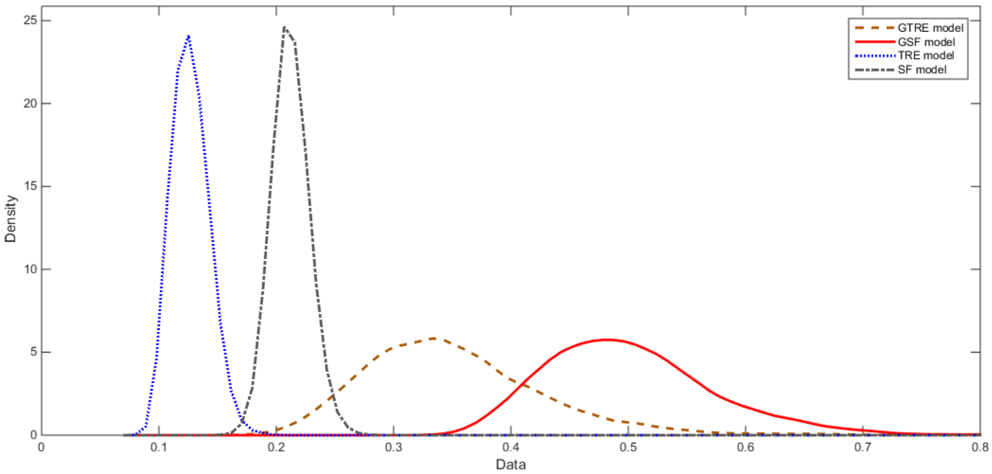

Figure 5. ***Posterior*** **distribution of overall inefficiency ω in GTRE, TRE, GSF and standard SF**

Source: author's calculations.

TABLES

Table 1. Basic results for new GTRE, TK GTRE, TRE and standard SF

	True values		new GTRE		TK GTRE		TRE		standard SF	
	Value	Std	$E(m)$	$D(m)$	$E(m)$	$D(m)$	$E(m)$	$D(m)$	$E(m)$	$D(m)$
β_0	1		1,031	0,051	0,950	0,047	1,401	0,034	1,244	0,035
β_1	1		1,004	0,005	1,008	0,005	1,004	0,005	1,004	0,012
σ_α	0,2		0,187	0,037	0,170	0,030	0,333	0,024		
σ_η	0,5		0,494	0,056	0,512	0,057				
σ_v	0,1		0,109	0,008	0,070	0,009	0,100	0,007	0,271	0,021
σ_u	0,2		0,189	0,014	0,237	0,014	0,212	0,011	0,406	0,042
α	0,000	0,200	0,000	0,153	0,008	0,146	0,000	0,058		
η	0,408	0,274	0,387	0,159	0,419	0,151				
u	0,160	0,120	0,151	0,079	0,192	0,067	0,167	0,080	0,325	0,176
ω	0,569	0,297	0,538	0,175	0,611	0,162	0,167	0,080	0,325	0,176
MPSRF			1,0235		1,0249		1,0143		1,0019	
Time			155		1354		103		85	

Note: $\sigma_v = 0.1, \sigma_u = 0.2, \sigma_\alpha = 0.2, \sigma_\eta = 0.5, \beta_0$ is intercept; β_1 is slope parameter; Std is the standard deviation calculated based on true values; E(m) is posterior mean of m; D(m) is posterior standard deviation of m; for parameters α, η, u and ω we report average posterior mean and standard deviation of posterior means; MPSRF is multivariate potential scale reduction factor; time is simulation duration given in seconds. *Source*: author's calculations.

Table 2. Results for naive GTRE under $Q_\eta = 10^{-4}$ and $Q_\eta = 10^{-2}$

			$Q_\eta = 10^{-4}$ 150 000 draws		$Q_\eta = 10^{-4}$ 300 000 draws		$Q_\eta = 10^{-2}$ 150 000 draws		$Q_\eta = 10^{-2}$ 300 000 draws	
	True values									
	Value	Std	$E(m)$	$D(m)$	$E(m)$	$D(m)$	$E(m)$	$D(m)$	$E(m)$	$D(m)$
β_0	1		**1,382**	0,098	**1,405**	0,048	**1,224**	0,134	**1,047**	0,089
β_1	1		1,003	0,005	1,003	0,005	1,003	0,005	1,003	0,005
σ_α	0,2		**0,322**	0,040	**0,331**	0,025	**0,275**	0,063	**0,209**	0,057
σ_η	0,5		**0,061**	0,113	**0,028**	0,037	**0,254**	0,160	**0,435**	0,109
σ_v	0,1		0,115	0,011	0,115	0,010	0,115	0,011	0,110	0,009
σ_u	0,2		0,175	0,022	0,176	0,021	0,175	0,022	0,181	0,018
α	0,000	0,200	-0,001	**0,091**	-0,002	**0,065**	-0,001	**0,151**	0,001	**0,162**
η	0,408	0,274	0,049	0,106	0,022	0,039	0,203	0,180	0,348	0,177
u	0,160	0,120	0,140	0,079	0,141	0,079	0,140	0,079	0,144	0,078
ω	0,569	0,297	0,188	0,136	0,163	0,089	0,343	0,198	0,493	0,192
MPSRF			1,063		1,005		1,083		1,002	
Time			183		322		163		326	

Note: For 150 000 draws we discard first 50 thousand, for 300 thousand we discard first 100 thousand; see notes in Table 1 for notation. *Source:* author's calculations.

Table 3. Extreme case 1: estimations results when $\sigma_\alpha = 1$

	True values		new GTRE		TK GTRE		standard SF	
	Value	Std	$E(m)$	$D(m)$	$E(m)$	$D(m)$	$E(m)$	$D(m)$
β_0	1		0,940	0,131	**0,386**	**0,071**	1,186	0,109
β_1	1		1,004	0,005	1,005	0,005	1,017	0,033
$\boldsymbol{\sigma_\alpha}$	**1,0**		**0,957**	**0,085**	**0,742**	**0,100**		
σ_η	0,5		0,615	0,126	1,226	0,148		
σ_v	0,1		0,109	0,008	0,092	0,008	0,991	0,037
σ_u	0,2		0,190	0,014	0,208	0,015	0,479	0,131
α	0,000	1,000	-0,010	0,356	0,028	0,507		
η	0,408	0,274	0,488	0,355	0,988	0,509		
u	0,160	0,120	0,151	0,079	0,167	0,076	0,382	0,299
ω	0,569	0,297	0,639	0,364	1,155	0,514	0,382	0,299
MPSRF			1,0171		1,0540		1,0243	
Time			220		1386		66	

Note: See notes for Table 1. *Source:* author's calculations.

Table 4. Extreme case 2: estimations results when $\sigma_v = 0.8$

	True values		new GTRE		TK GTRE		standard SF	
	Value	Std	$E(m)$	$D(m)$	$E(m)$	$D(m)$	$E(m)$	$D(m)$
β_0	1		0,915	0,076	**1,548**	**0,041**	1,202	0,091
β_1	1		1,022	0,026	1,021	0,028	1,022	0,028
σ_α	0,2		0,031	0,041	0,022	0,021		
σ_η	0,5		0,567	0,064	0,019	0,017		
$\boldsymbol{\sigma_v}$	**0,8**		**0,795**	**0,021**	**0,872**	**0,020**	**0,825**	**0,032**
σ_u	0,2		0,252	0,064	0,025	0,038	0,459	0,109
α	0,000	0,200	0,000	0,050	-0,015	0,026		
η	0,408	0,274	0,453	0,201	0,015	0,020		
u	0,160	0,120	0,201	0,161	0,020	0,041	0,366	0,277
ω	0,569	0,297	0,653	0,254	0,035	0,045	0,366	0,277
MPSRF			1,0026		1,1076		1,013	
Time			206		1323		64	

Note: See notes for Table 1. *Source:* author's calculations.

Table 5. Extreme case 3: estimations results when $\sigma_\alpha = 1$ and $\sigma_v = 0.8$

	True values		new GTRE		TK GTRE		standard SF	
	Value	Std	$E(m)$	$D(m)$	$E(m)$	$D(m)$	$E(m)$	$D(m)$
β_0	1		0,840	0,172	1,127	0,521	1,201	0,108
β_1	1		1,022	0,027	1,025	0,026	1,036	0,041
$\boldsymbol{\sigma_\alpha}$	**1**		**0,933**	**0,095**	**0,905**	**0,099**		
σ_η	0,5		0,664	0,160	**0,171**	**0,273**		
$\boldsymbol{\sigma_v}$	**0,8**		**0,793**	**0,021**	0,731	0,084	**1,264**	**0,037**
σ_u	0,2		0,251	0,063	**0,323**	**0,367**	0,461	0,125
α	0,0	1,000	-0,001	0,434	0,047	0,284		
η	0,408	0,274	0,530	0,388	**0,136**	**0,273**		
u	0,160	0,120	0,201	0,160	**0,259**	**0,374**	0,368	0,295
ω	0,569	0,297	0,731	0,421	0,394	0,556	0,368	0,295
MPSRF			1,0031		1,8152		1,0249	
Time			173		1180		67	

Note: See notes for Table 1. *Source:* author's calculations.

Table 6. Simulation results for different values of r_u^* and r_η^* in new GTRE model

	True values		$r_u^* = r_\eta^* = 0.5$			$r_u^* = r_\eta^* = 0.6$			$r_u^* = r_\eta^* = 0.7$			$r_u^* = r_\eta^* = 0.8$			$r_u^* = r_\eta^* = 0.9$		
	Value	Std	$E(m)$	$D(m)$	$\rho_{\hat{m}}$	$E(m)$	$D(m)$	$\rho_{\hat{m}}$	$E(m)$	$D(m)$	$\rho_{\hat{m}}$	$E(m)$	$D(m)$	$\rho_{\hat{m}}$	$E(m)$	$D(m)$	$\rho_{\hat{m}}$
β_0	1		0,914	0,047		0,969	0,049		1,029	0,055		**1,118**	0,070		**1,296**	0,050	
β_1	1		1,007	0,006		1,006	0,005		1,005	0,005		1,004	0,005		1,003	0,005	
σ_α	0,200		0,161	0,033		0,178	0,035		0,201	0,040		0,242	0,047		0,313	0,028	
σ_η	0,500		0,600	0,055		0,528	0,055		0,458	0,060		0,363	0,076		0,169	0,044	
σ_v	0,100		0,078	0,007		0,086	0,007		0,095	0,007		0,104	0,008		0,116	0,009	
σ_u	0,200		0,269	0,010		0,245	0,010		0,223	0,011		0,201	0,013		0,174	0,019	
α	0,000	0,200	0,000	0,144	0,521	0,000	0,151	0,537	0,000	0,159	0,552	-0,001	0,162	0,561	-0,001	0,112	0,558
η	0,408	0,274	0,449	0,153	0,806	0,409	0,158	0,803	0,364	0,164	0,799	**0,292**	0,167	0,796	**0,135**	0,104	0,797
u	0,160	0,120	0,206	0,077	0,747	0,191	0,079	0,750	0,176	0,080	0,751	0,159	0,080	0,752	0,139	0,078	0,752
ω	0,569	0,297	0,655	0,164	0,796	0,600	0,171	0,794	0,540	0,179	0,791	0,452	0,185	0,783	0,274	0,131	0,606
MPSRF			1,0467			1,0354			1,0346			1,0343			1,0409		
Time			226			214			246			244			222		

Note: See notes for Table 1. *Source:* author's calculations.

Table 7. Comparison between GTRE models for T=5 and different number of regression parameters (k=2,3)

	True values		new GTRE		TK GTRE	
	n=100 T=5, k=2					
β_0	1		0,969	0,048	**0,328**	**0,070**
β_1	1		1,002	0,008	0,997	0,008
σ_α	0,2		0,198	0,039	0,142	0,071
σ_η	0,5		0,557	0,057	**1,095**	**0,103**
σ_v	0,1		0,090	0,015	**0,024**	**0,011**
σ_u	0,2		0,208	0,021	0,263	0,013
α	0,00	0,2	0,001	0,164	0,020	0,156
η	0,418	0,319	0,437	0,171	**1,012**	**0,176**
u	0,155	0,119	0,166	0,082	0,216	0,056
ω	0,573	0,349	0,603	0,182	**1,228**	**0,172**
MPSRF			1,0541		1,0388	
Time			13,5		108,1	
	n=100 T=5, k=3					
β_0	1		0,996	0,066	**0,216**	**0,068**
β_1	1		1,011	0,008	1,009	0,007
β_2	1		0,994	0,008	1,001	0,008
σ_α	0,2		0,206	0,041	0,148	0,082
σ_η	0,5		0,519	0,066	**1,185**	**0,105**
σ_v	0,1		0,093	0,014	**0,024**	**0,013**
σ_u	0,2		0,220	0,019	0,282	0,014
α	0,00	0,2	0,000	0,169	0,023	0,166
η	0,411	0,301	0,411	0,177	**1,114**	**0,187**
u	0,170	0,126	0,175	0,085	0,228	0,061
ω	0,582	0,322	0,586	0,190	**1,342**	**0,182**
MPSRF			1,0107		**1,4507**	
Time			13,7		105,9	

Note: Based on 15 thousand draws with initial 5 thousand discarded; example based on Tsionas and Kumbhakar (2014: p. 120). *Source:* author's calculations.

Table 8. Empirical results for the four models

	standard SF		TRE		GTRE		GSF	
	$E(m)$	$D(m)$	$E(m)$	$D(m)$	$E(m)$	$D(m)$	$E(m)$	$D(m)$
σ_v	0,161	0,009	0,085	0,008	0,096	0,009	0,101	0,009
σ_u	0,271	0,020	0,190	0,017	0,162	0,019	0,159	0,018
σ_α			0,213	0,021	0,146	0,027		
σ_η					0,280	0,070	0,427	0,064
α			0,008	0,058	-0,001	0,021		
u	0,212	0,016	0,147	0,015	0,127	0,016	0,124	0,015
η					0,224	0,067	0,377	0,065
ω	0,212	0,016	0,147	0,015	0,351	0,076	0,502	0,073
$El(p_1)$	0,549	0,019	0,543	0,020	0,546	0,020	0,538	0,020
$El(p_2)$	0,401	0,012	0,378	0,012	0,376	0,011	0,384	0,012
$El(p_3)$	0,050	0,017	0,079	0,018	0,078	0,018	0,078	0,018
$El(y_1)$	0,108	0,007	0,086	0,010	0,085	0,010	0,087	0,010
$El(y_2)$	0,416	0,023	0,472	0,024	0,480	0,024	0,492	0,023
$El(y_3)$	0,216	0,018	0,213	0,020	0,213	0,020	0,198	0,021
$El(y_4)$	0,082	0,029	0,082	0,030	0,079	0,029	0,074	0,030
$El(y_5)$	0,098	0,011	0,083	0,014	0,080	0,014	0,090	0,014
TC	-0,047	0,004	-0,049	0,004	-0,049	0,004	-0,048	0,003
$intercept$	-0,904	1,040	0,097	0,995	-0,353	0,964	-0,096	0,972
RTS	1,086	0,008	1,068	0,013	1,069	0,013	1,063	0,012
$MPSRF$	1,002		1,014		1,043		1,007	

Note: $El(m)$ denotes cost elasticity of m; the table only provides average levels of elasticities due to space constrains; $E(m)$ and $D(m)$ are *posterior* mean and *posterior* standard deviation respectively; *TC* is technical change ($\partial \ln C / \partial t$); RTS are returns to scale; p_1 is wage rate for labor; p_2 is interest rate for borrowed funds; p_3 is price of capital; $y_1$are consumer loans; y_2 are non-consumer loans; y_3 are securities; y_4 is financial equity capital; y_5 are non-traditional banking activities; see Feng and Serletis (2009) for more details. *Source:* author's calculations.

Table 9. Prior and posterior distribution of η under different *prior* median values

prior median	0,6		0,7		0,8		0,875		0,9	
	Efficiency distribution characteristics ($\exp(-\eta)$)									
	prior	*posterior*	*prior*	*posterior*	*prior*	*posterior*	*prior*	*posterior*	*prior*	*posterior*
mean	0,595	0,748	0,683	0,777	0,778	0,801	0,856	0,838	0,884	0,852
st.dev.	0,235	0,048	0,200	0,047	0,152	0,052	0,105	0,051	0,087	0,055
median	0,6	0,753	0,7	0,782	0,800	0,807	0,875	0,842328	0,900	0,850
	Inefficiency distribution characteristics (η)									
mean	0,625	0,292	0,436	0,254	0,273	0,224	0,163	0,179	0,129	0,173
st.dev.	0,513	0,065	0,358	0,061	0,224	0,067	0,134	0,063	0,106	0,066
median	0,506	0,283	0,352	0,247	0,221	0,214	0,132	0,172	0,104	0,161

APPENDIX

Table A.1. Sampling behavior of Bayes estimator in the new GTRE model

	α			η			u			ω		
	mean	median	std	mean	median	std	mean	median	std	mean	median	std
n=50, T=5												
True	0,000222	0,000201	0,003129	0,403	0,400	0,046	0,159	0,158	0,007	0,562	0,561	0,048
Est.	-0,000267	0,000025	0,003167	0,490	0,479	0,069	0,179	0,176	0,018	0,669	0,561	0,072
n=100, T=5												
True	-0,000071	-0,000010	0,002941	0,399	0,394	0,030	0,158	0,158	0,005	0,557	0,554	0,030
Est.	-0,000072	0,000103	0,002162	0,448	0,446	0,049	0,169	0,169	0,014	0,617	0,554	0,050
n=100, T=5												
True	0,000083	0,000239	0,002886	0,405	0,403	0,028	0,399	0,398	0,014	0,804	0,805	0,031
Est.	-0,000330	-0,000060	0,002270	0,433	0,432	0,035	0,396	0,399	0,034	0,829	0,805	0,051
n=100, T=10												
True	-0,000264	-0,000604	0,002990	0,396	0,394	0,025	0,400	0,399	0,009	0,796	0,794	0,026
Est.	-0,000011	-0,000002	0,000594	0,429	0,430	0,042	0,398	0,397	0,014	0,826	0,794	0,042
	σ_α			σ_η			σ_v			σ_u		
True		0,2			0,5			0,1			0,2	
n=50, T=5												
Est.	0,147	0,154	0,083	0,546	0,546	0,052	0,081	0,092	0,029	0,224	0,223	0,028
n=100, T=5												
Est.	0,154	0,164	0,058	0,560	0,561	0,051	0,091	0,092	0,014	0,212	0,211	0,017
True		0,1			0,5			0,1			0,5	
n=100, T=5												
Est.	0,046	0,027	0,040	0,546	0,544	0,038	0,085	0,085	0,049	0,493	0,494	0,041
n=100, T=10												
Est.	0,048	0,037	0,034	0,539	0,541	0,046	0,102	0,103	0,014	0,497	0,497	0,018

Note: "Est." is *posterior* estimate; results are mean estimates calculated based on 100 datasets of the same characteristics (re-generated 100 times); simulation results based on 5000 burn-in and 5000 accepted draws; example similar to Tsionas and Kumbhakar (2014: p. 124). *Source:* author's calculations.

Table A.2. Correlations between *posterior* means and true values of latent variables; *posterior* means and standard deviations of the correlation coefficient

	new GTRE			TK GTRE		
m	$\rho_{\hat{m}}$	$E(\rho_m)$	$D(\rho_m)$	$\rho_{\hat{m}}$	$E(\rho_m)$	$D(\rho_m)$
	n=100, T=5, k=2					
α	0,514	0,286	0,089	0,559	0,039	0,101
η	0,862	0,738	0,054	0,824	0,760	0,055
u	0,712	0,535	0,049	0,656	0,609	0,017
ω	0,862	0,744	0,047	0,836	0,782	0,049
	n=100, T=5, k=3					
α	0,560	0,328	0,091	0,441	0,026	0,103
η	0,830	0,670	0,077	0,810	0,730	0,068
u	0,732	0,559	0,049	0,721	0,671	0,016
ω	0,826	0,678	0,067	0,818	0,754	0,063

Note: Based on 15 thousand draws with initial 5 thousand discarded; results for two and three regression parameters (k=2,3); $\rho_{\hat{m}}$ is correlation coefficient between *posterior* mean of "m" ($\hat{m}$) and true value, e.g., for $m \coloneqq \alpha, \rho_{\hat{\alpha}} = \rho_\alpha(\hat{\alpha}, \alpha_{true})$; $E(\rho_m)$ is the *posterior* mean of a correlation coefficient, e.g., $E(\rho_\alpha) = E(\rho_\alpha|data)$; $D(\rho_m)$ is the *posterior* standard deviation of a correlation coefficient; example based on Tsionas and Kumbhakar (2014: p. 120). *Source:* author's calculations.

Table A.3. Basic results; correlations between *posterior* means and true values of latent variables; *posterior* means and standard deviations of the correlation coefficient

	new GTRE			TK GTRE			naive GTRE			TRE			standard SF		
	$\rho_{\hat{m}}$	$E(\rho_m)$	$D(\rho_m)$	$\rho_{\hat{m}}$	$E(\rho_m)$	$D(\rho_m)$	$\rho_{\hat{m}}$	$E(\rho_m)$	$D(\rho_m)$	$\rho_{\hat{m}}$	$E(\rho_m)$	$D(\rho_m)$	$\rho_{\hat{m}}$	$E(\rho_m)$	$D(\rho_m)$
α	0,544	0,313	0,089	0,530	0,270	0,091	0,558	0,535	0,056	0,555	0,550	0,012			
η	0,800	0,658	0,071	0,804	0,692	0,053	0,796	0,088	0,190						
u	0,752	0,528	0,040	0,744	0,651	0,021	0,752	0,497	0,061	0,752	0,569	0,030	0,283	0,194	0,029
ω	0,792	0,647	0,061	0,794	0,693	0,045	0,487	0,252	0,113	0,320	0,242	0,030	0,781	0,534	0,052

Note: $\rho_{\hat{m}}$ is correlation coefficient between *posterior* mean of "m" ($\hat{m}$) and true value, e.g., for $m \coloneqq \alpha, \rho_{\hat{m}} = \rho_m(\hat{\alpha}, \alpha_{true})$; $E(\rho_m)$ is the *posterior* mean of a correlation coefficient, e.g., $E(\rho_\alpha) = E(\rho_\alpha|data)$; $D(\rho_m)$ is the *posterior* standard deviation of a correlation coefficient. *Source:* author's calculations.

Table A.4. Extreme cases 1-3: correlations between *posterior* means and true values of latent variables; *posterior* means and standard deviations of the correlation coefficient

	new GTRE			TK GTRE			standard SF		
	$\rho_{\hat{m}}$	$E(\rho_m)$	$D(\rho_m)$	$\rho_{\hat{m}}$	$E(\rho_m)$	$D(\rho_m)$	$\rho_{\hat{m}}$	$E(\rho_m)$	$D(\rho_m)$
	when $\sigma_\alpha = 1$								
α	0,962	0,895	0,034	0,940	0,679	0,071			
η	0,239	0,091	0,095	0,242	0,170	0,073			
u	0,753	0,531	0,038	0,749	0,591	0,031	0,084	0,024	0,032
ω	0,328	0,141	0,083	0,266	0,190	0,065	0,252	0,072	0,036
	when $\sigma_v = 0.8$								
α	0,396	0,030	0,108	0,421	0,042	0,105			
η	0,660	0,523	0,055	0,653	0,040	0,108			
u	0,132	0,024	0,031	0,116	0,002	0,032	0,108	0,0131	0,0313
ω	0,605	0,445	0,050	0,553	0,033	0,077	0,323	0,0447	0,0333
	when $\sigma_\alpha = 1$ and $\sigma_v = 0.8$								
α	0,930	0,826	0,049	0,931	0,896	0,053			
η	0,248	0,097	0,095	0,246	0,030	0,109			
u	0,140	0,025	0,032	0,119	0,030	0,044	0,060	0,013	0,031
ω	0,226	0,085	0,082	0,147	0,034	0,070	0,203	0,045	0,033

Note: See notes for Table A.3. *Source:* author's calculations.

Table A.5. Correlations between inefficiencies from four models

Correlation between overall inefficiency (ω_{it})				
	standard SF	TRE	GTRE	GSF
standard SF	1	0,328 (0,039)	0,466 (0,058)	0,568 (0,040)
TRE	*0,606*	1	0,314 (0,051)	0,260 (0,044)
GTRE	***0,895***	*0,574*	1	0,638 (0,084)
GSF	***0,845***	*0,375*	***0,961***	1
Correlation between transient inefficiency (u_{it})				
	standard SF	TRE	GTRE	GSF
standard SF	1	0,328 (0,039)	0,290 (0,042)	0,286 (0,040)
TRE	*0,606*	1	0,502 (0,058)	0,478 (0,059)
GTRE	*0,595*	***0,998***	1	0,432 (0,062)
GSF	*0,609*	***0,987***	***0,992***	1

Note: Lower triangles in the cross-tables (in *italic*) contain correlation coefficients between *posterior* means of inefficiencies in different models; upper triangles are *posterior* means and standard deviations (in brackets) of correlation coefficients between inefficiencies in different models; for models *standard SF* and *TRE* overall inefficiency is equal to transient. *Source:* author's calculations.