Expected utility for nonstochastic risk

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Abstract

The world of random phenomena exceeds the domain of the classical probability theory. In the general case the description of randomness requires a specific set of probability distributions (which is called statistical regularity) rather than a singe distribution. Such statistical regularity arises as a limit of relative frequencies. This approach to randomness allows to generalize the expected utility theory in order to cover the decision problems under nonstochastic random events. Applying the von Neumann − Morgenstern utility theorem, we derive the maxmin expected utility representation for statistical regularities. The derivation is based on the axiom of the preference for stochastic risk, i.e. the decision maker wishes to reduce the set of probability distributions to a single one.

Keywords: expected utility, risk, mass phenomena, statistical regularity, nonstochastic randomness, multiple prior

1. Introduction

The expected utility theory of von Neumann and Morgenstern (1947) considers the situations of objective risk relying on the frequentist notion of probability. Namely, the probability of an event is defined as its relative frequency in a large number of trials.

The problem arises when event's relative frequency do not tend to a limit (Borel (1956)). In Kolmogorov (1986) we read "Speaking of randomness in the ordinary sense of this word, we mean those phenomena in which we do not find regularities allowing us to predict their behavior. Generally speaking, there are no reasons to assume that random in this sense phenomena are subject to some probabilistic laws. Hence, it is necessary to distinguish between randomness in this broad sense and stochastic randomness (which is the subject of the probability theory)". Referring to this remark, we use the term nonstochastic

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phenomena in order to speak about random in a broad sense phenomena that are not "the subject of the probability theory".

Nowadays, the problem of revealing the regularities of nonstochastic phenomena, as well as the corresponding decision rules, becomes more and more important, in particular in relation to complex social and economic systems, e.g. financial markets (Lux (1998); Chian et al. (2006); Miller and Ratti (2009); Ivanenko and Pasichnichenko (2014)).

Some non-probabilistic mathematical formalism has been used for these purposes (see for example, Dubois and Prade (1989)). However, we rely here on the extension of the standard notions of the probability theory, given by the theorem of existence of statistical regularities of mass phenomena (Ivanenko (2010); Ivanenko and Labkovsky (2015)). Namely, every mass phenomenon (random or deterministic) possesses a statistical regularity in the form of weak* closed set of finitely additive probability distributions. The statistical regularity of a stochastic phenomenon is a singleton.

This approach to randomness makes it possible to extend the domain of the expected utility theory to cover the decision problems under nonstochastic random events. This paper proposes an axiomatic foundation of the maxmin expected utility decision rule in the statistical regularities framework.

Closed sets of probability measures have already been in use in the decision theory yet without the meaning of laws of mass random phenomena. In particular, families of a priori distributions result from the axioms of rational choice (Ivanenko and Labkovsky (1986); Gilboa and Schmeidler (1989); Chateauneuf and Faro (2009); Pasichnichenko (2016)). Jaffray (1989) studied the families of distributions that are cores of some belief function describing the situations where some true probability exists but it is known only up to a set of measures.

The next section provides a brief extract from the theory of statistical regularities. Then section 3 states the main result. Finally, section 4 provides summary and conclusions.

2. Statistical regularities

Let X be a nonempty set and Σ be an algebra of subsets of X. The simplest example of a mass phenomenon is given by an ordinary sequence.

Definition 1. Two sequences $\bar{x}^{(1)}$ and $\bar{x}^{(2)}$ of elements of the set X are called statistically equivalent (S-equivalent) if for any $m \in \mathbb{N}$ and any bounded measurable functions $\gamma_i \colon X \to \mathbb{R}$ $(i = \overline{1,m})$ the two sequences $\bar{y}^{(1)}$ and $\bar{y}^{(2)}$ defined by

$$y_n^{(k)} = \frac{1}{n}\sum_{i=1}^{n}\gamma\left(x_i^{(k)}\right),\ n \in \mathbb{N},\ k \in \{1,2\},\ \gamma = (\gamma_1, \ldots, \gamma_n)$$

have the same set of limit points in $\mathbb{R}^m$.

In other words, S-equivalent sequences are indistinguishable with respect to their limiting averages.

Definition 2. A simple mass phenomenon is a class of S-equivalent sequences.

Let $\mathcal{P}$ be the set of all finitely additive probability measures on Σ, endowed with the weak* topology. Recall that the base is formed by the sets

$$\left\{p \in \mathcal{P} \colon \left|\int f_i(x)\,\mathrm{d}p - \int f_i(x)\,\mathrm{d}p_0\right| < \varepsilon,\ i = \overline{1,n}\right\},$$

where $f_i \colon X \to \mathbb{R}$ are bounded measurable functions, $p_0 \in \mathcal{P}$, $\varepsilon > 0$, and $n \in \mathbb{N}$.

Let $\bar{x} = \{x_n\}$ be a sequence in X. Associate to $\bar{x}$ the sequence $\{p_n\}$ of measures from $\mathcal{P}$ defined by

$$p_n(A) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_A(x_i), \quad A \in \Sigma, \tag{1}$$

where $\mathbf{1}_A$ is an indicator of a set A. Hence, p_n is the frequency distribution of the number of hits in the sets $A \in \Sigma$ of the first n terms of the sequence $\bar{x}$. Due to compactness of the space $\mathcal{P}$ this sequence has a non-empty closed set of limit points.

Definition 3. The set $P(\bar{x})$ of limit points of the sequence $\{p_n\}$ is called the statistical regularity of the sequence $\bar{x}$.

In general, $P(\bar{x})$ consists of more then one point even for finite X as was shown by Zorich et al. (2000).

Definition 4. The statistical regularity of a simple mass phenomenon is a statistical regularity of any of its sequences.

The following theorem justifies the two above definitions.

Theorem 1.

1. *For any sequence $\bar{x} \in X^{\mathbb{N}}$, any $m \in \mathbb{N}$, and any bounded measurable mappings $\gamma_i \colon X \to \mathbb{R}$ ($i = \overline{1,m}$) the set of limit points of the sequence $\bar{y}$ defined by*
$$y_n = \frac{1}{n}\sum_{i=1}^{n}\gamma(x_i),\ n \in \mathbb{N},\ \gamma = (\gamma_1, \ldots, \gamma_n)$$
can be written as
$$\left\{\int \gamma(x)\,\mathrm{d}p \colon p \in P(\bar{x})\right\}.$$
2. *The two sequences $\bar{x}^{(1)}, \bar{x}^{(2)} \in X^{\mathbb{N}}$ are S-equivalent if and only if $P\left(\bar{x}^{(1)}\right) = P\left(\bar{x}^{(2)}\right)$.*

Therefore, the statistical regularity $P(\bar{x})$ contains all the information about the limiting averages of any characteristic $\gamma(x_i)$. The proof rests on the following general lemma:

Lemma 1. *Let Y be a compact space, $f : Y \to \mathbb{R}^m$ be a continuous mapping, $\{x_n\}$ be a sequence in Y. Then*

$$\mathrm{LIM}\{f(x_n)\} = f\left(\mathrm{LIM}\{x_n\}\right),$$

where $\mathrm{LIM}\{x_n\}$ *denotes the set of limit points of the sequence* $\{x_n\}$.

Proof. For every neighborhood B of the point $f(x)$ there is a neighborhood A of a point $x \in \mathrm{LIM}\,\{x_n\}$ such that $f(A) \subseteq B$. Since the sequence $\{x_n\}$ infinitely many times hits A, the same is true for $\{f(x_n)\}$ and B. Hence, $f(x) \in \mathrm{LIM}\,\{f(x_n)\}$.

If $y \in \mathrm{LIM}\,\{f(x_n)\}$, then $f(x_{n_k}) \to y$ as $k \to \infty$ for some subsequence $\{x_{n_k}\}$. Due to compactness of X the sequence $\{x_{n_k}\}$ has a limit point $x \in \mathrm{LIM}\,\{x_n\}$. Suppose that $\|f(x) - y\| = \varepsilon > 0$. On the one hand, starting from some $k_0 \in \mathbb{N}$ we have $\|f(x_{n_k}) - y\| < \frac{\varepsilon}{2}$. On the other hand, the $\frac{\varepsilon}{2}$-neighborhood of the point $f(x)$ contains the image of some neighborhood A of x. Since there is an x_{n_k} in A after k_0, we arrive at a contradiction, that implies $f(x) = y$. □

Proof of Theorem 1. 1) Let the sequence $\{p_n\}$ correspond to $\bar{x}$ in the sense of (1) and $\pi_\gamma \colon \mathcal{P} \to \mathbb{R}^m$ be defined by

$$\pi_\gamma(p) = \int \gamma(x)\,\mathrm{d}p.$$

Since the mapping π_γ is continuous, lemma 1 implies

$$\mathrm{LIM}\,\{\pi_\gamma(p_n)\} = \pi_\gamma\left(\mathrm{LIM}\,\{p_n\}\right).$$

Rewriting the left-hand side

$$\pi_\gamma(p_n) = \int \gamma(x)\,\mathrm{d}p_n = \frac{1}{n}\sum_{i=1}^{n} \gamma\left(x_i\right) = y_n$$

and the right-hand side

$$\mathrm{LIM}\,\{p_n\} = P(\bar{x})$$

gives

$$\mathrm{LIM}\,\{y_n\} = \pi_\gamma\left(P(\bar{x})\right).$$

2) Suppose that the sequences $\bar{x}^{(1)}$ and $\bar{x}^{(2)}$ are S-equivalent and there exists $p_0 \in P\left(\bar{x}^{(1)}\right) \setminus P\left(\bar{x}^{(2)}\right)$. Since $P\left(\bar{x}^{(2)}\right)$ is closed, there is a neighborhood of the point p_0, that do not intersect with $P\left(\bar{x}^{(2)}\right)$. In other words, there are such real number $\varepsilon > 0$ and bounded measurable functions $f_i \colon X \to \mathbb{R}$ $(i = \overline{1,m})$ such that for any $p \in P\left(\bar{x}^{(2)}\right)$

$$\left|\int f_i(x)\,\mathrm{d}p - \int f_i(x)\,\mathrm{d}p_0\right| \geq \varepsilon$$

for some $i \in \overline{1,m}$. Set $\gamma = (f_1, \ldots, f_m)$. Then the vector $\int \gamma(x)\,\mathrm{d}p_0$ is not in

$$\left\{\int \gamma(x)\,\mathrm{d}p \colon p \in P\left(\bar{x}^{(2)}\right)\right\}.$$

Hence, the first part of the theorem implies that the sequences $\bar{x}^{(1)}$ and $\bar{x}^{(2)}$ are not S-equivalent, which is a contradiction.

The converse follows immediately from the first part of the theorem. □

The connection between the notions introduced above and the probabilistic notions follows directly from the strong law of large numbers (lemma 2).

Lemma 2. *Let X be a finite set, μ be a probability distribution on X, and $\{\xi_n\}$ be a sequence of independent random elements taking values in X with the distribution μ. Then with probability 1 the statistical regularity $P(\{\xi_n\})$ consists of the single element μ.*

Thus, in case of finite X the regularity of a stochastic phenomenon is a singleton.

Note that the regularity of a sequence is concentrated on the countable subset of X. The more general notion of mass phenomena may be derived using sampling nets.

Definition 5. A sampling net in X is a function φ from the directed set Λ to the sampling space $X^\infty = \bigcup_{n=1}^\infty X^n$.

Everything that was done using sequences can be generalized using sampling nets. First, extend the notion of S-equivalence to sampling nets and define a (nonsimple) mass phenomenon to be a class of S-equivalent sampling nets. Then, every $\lambda \in \Lambda$ associate a frequency distribution $p_\lambda \in \mathcal{P}$ of a sample φ_λ and define the statistical regularity of a sampling net φ to be a set of limit points of the net (p_λ). Theorem 1 remains true if we replace sequences with sampling nets and define

$$y_\lambda = \frac{1}{l}\sum_{i=1}^{l} \gamma(x_i),$$

when $\varphi_\lambda = (x_1, \ldots, x_l)$. Moreover, the general version of theorem 1 (Ivanenko (2010)) contains the additional statement: if P is a non-empty closed subset of the space $\mathcal{P}$, then P is a statistical regularity of some sampling net in X. The proof stems from the fact that the set of simple probability measures with rational values is dense in $\mathcal{P}$. This consideration leads to the following definition.

Definition 6. A non-empty closed subset of the space $\mathcal{P}$ is called a regularity on X.

To sum up, statistical regularities extend the method of the probability theory to the whole range of mass random phenomena. In the general case a phenomenon is described by a family of probability distributions. The approach is also appropriate to mass phenomena considered to be deterministic if we are only interested in their average characteristics.

3. Nonstochastic risk

We call nonstochastic risk a decision-making situation, in which the outcome of each decision $d \in D$ is described by the regularity P_d on the set X of consequences. We propose a decision model where the decision maker seeks to maximize the quantity

$$U(P_d) = \min_{p \in P_d} \int u(x)\,\mathrm{d}p, \quad d \in D.$$

Let $\mathcal{R}$ be the set of all regularities on X. Let's also identify a probability measure p with the regularity $\{p\}$ and thus consider $\mathcal{P}$ as a subset of $\mathcal{R}$. For $\alpha \in [0,1]$ define the convex combination of regularities $P \in \mathcal{R}$ and $q \in \mathcal{P}$ as follows:

$$\alpha P + (1-\alpha)\, q = \{\alpha p + (1-\alpha)\, q \colon p \in P\}\,. \tag{2}$$

The convex combinations in $\mathcal{P}$ are performed pointwise. The following lemma shows that the set $\mathcal{R}$ is closed under operation (2).

Lemma 3. *For any $P \in \mathcal{R}$, $q \in \mathcal{P}$ and $\alpha \in [0,1]$ the set $\alpha P + (1-\alpha)\, q$ is a regularity on X.*

Proof. The case $\alpha = 0$ is trivial. Otherwise, consider the mapping $\pi \colon \mathcal{P} \to \mathcal{P}$ defined by

$$\pi(p) = \alpha p + (1-\alpha)\, q, \quad p \in \mathcal{P},$$

and prove that it is continuous. For this we will show that for any $p, p_0 \in \mathcal{P}$, $\varepsilon > 0$, and any bounded measurable function $f \colon X \to \mathbb{R}$

$$\left|\int f(x)\,\mathrm{d}p - \int f(x)\,\mathrm{d}p_0\right| < \tfrac{\varepsilon}{\alpha}$$

implies

$$\left|\int f(x)\,\mathrm{d}\pi(p) - \int f(x)\,\mathrm{d}\pi(p_0)\right| < \varepsilon.$$

Indeed,

$$\begin{aligned}\Big|\int f(x)\,\mathrm{d}\left(\alpha p + (1-\alpha) q\right) - \int f(x)\,\mathrm{d}\left(\alpha p_0 + (1-\alpha) q\right)\Big| \\ = \alpha \left|\int f(x)\,\mathrm{d}p - \int f(x)\,\mathrm{d}p_0\right| < \varepsilon.\end{aligned}$$

Thus, for any neighborhood A of the point $\pi(p_0)$ there is a neighborhood of the point p_0 with the image in A. Therefore, the mapping π is continuous and the set $\alpha P + (1-\alpha) q$ is closed being the image of the compact set P. $\square$

Let $\mathcal{R}_0$ be a subset of $\mathcal{R}$ containing all one-point regularities and closed under convex combinations (2). Suppose there is a decision maker's preference relation $\preceq$ on $\mathcal{R}_0$.

Some structural assumptions should be imposed on Σ. First, assume that Σ contains the singleton subset $\{x\}$ for each $x \in X$. Denote δ_x the one-point measure: $\delta_x(\{x\}) = 1$. A set $A \subseteq X$ is a preference interval if $x, y \in A$ implies $\{z \in X \colon \delta_x \preceq \delta_z \preceq \delta_y\} \subseteq A$. The second assumption is that Σ contains all preference intervals.

Consider the following properties.

1. *(Weak Order)* The relation $(\preceq, \mathcal{R}_0)$ is complete and transitive.
2. *(Continuity)* For any $P, Q \in \mathcal{R}_0$ and $r \in \mathcal{P}$ the sets $\{\alpha \colon \alpha P + (1-\alpha)\, r \preceq Q\}$ and $\{\alpha \colon Q \preceq \alpha P + (1-\alpha)\, r\}$ are closed.
3. *(Independence)* For any $p, q, r \in \mathcal{P}$ and $\alpha \in (0;1)$ if $p \preceq q$, then $\alpha p + (1-\alpha) r \preceq \alpha q + (1-\alpha) r$.
4. *(Dominance)* For any $p, q, r \in \mathcal{P}$ and $A \in \Sigma$

if $p(A) = 1$ and $q \preceq \delta_x$ for any $x \in A$, then $q \preceq p$;
if $p(A) = 1$ and $\delta_x \preceq r$ for any $x \in A$, then $p \preceq r$;

5. *(Monotonicity)* For any $P \in \mathcal{R}_0$ and $q \in \mathcal{P}$ if $q \preceq p$ for any $p \in P$, then $q \preceq P$.
6. *(Preference for Stochastic Risk)* $P \preceq \frac{1}{2}P + \frac{1}{2}p$ for any $P \in \mathcal{R}_0$ and $p \in P$.

Weak Order assumption is common. The key to understanding assumptions 2 and 6 is the interpretation of the convex combination (2) as a "two-step lottery" similarly to the convex combinations of measures in the expected utility theory (see the appendix). Here Continuity axiom of the expected utility theory is extended to the convex combinations of regularities, while the Independence axiom is left unchanged. Dominance axiom is used for obtaining the expected utility representation for nonsimple probability measures. Note that the latter two assumptions refer only to the preferences among measures. Monotonicity axiom links the preference relation on regularities with the one on probability measures. Assumption 6 should be understood as follows: the decision maker would not refuse a 50-50 chance to exchange the nonstochastic outcome described by a regularity P for a stochastic outcome described by a probability measure $p \in P$, i.e. to reduce nonstochastic risk to stochastic.

Theorem 2. *The preference relation $(\preceq, \mathcal{R}_0)$ satisfies assumptions 1 – 6 if and only if there exists a utility function $U \colon \mathcal{R}_0 \to \mathbb{R}$ of the form*

$$U(P) = \min_{p \in P} \int u(x)\,\mathrm{d}p, \quad P \in \mathcal{R}_0, \tag{3}$$

where $u \colon X \to \mathbb{R}$ is a bounded measurable function. Furthermore, the mapping $V : \mathcal{R}_0 \to \mathbb{R}$ is also a utility function of the form (3) if and only if there are $a, b \in \mathbb{R}$, $a > 0$, such that $V(P) = aU(P) + b$.

Proof. Due to assumptions 1, 2, and 3 the preference relation $(\preceq, \mathcal{P})$ satisfies the Herstein and Milnor (1953) conditions for the existence of linear utility function $U : \mathcal{P} \to \mathbb{R}$, which is unique up to a positive linear transformation. Assumption 4 of Fishburn (1982) implies that there is a bounded measurable function $u \colon X \to \mathbb{R}$, such that

$$U(p) = \int u(x)\,\mathrm{d}p$$

for every $p \in \mathcal{P}$.

Let p_0 be an element of $P \in \mathcal{R}_0$ satisfying

$$U(p_0) = \min_{p \in P} U(p).$$

Such p_0 exists, since the mapping U is continuous on the compact set P. Assumption 5 implies $p_0 \preceq P$. On the other hand, assumption 6 gives $P \preceq \frac{1}{2}P + \frac{1}{2}p_0$. Since $p_0 \in \frac{1}{2}P + \frac{1}{2}p_0$, the repeated use of assumption 6 gives

$\frac{1}{2}P + \frac{1}{2}p_0 \preceq \frac{1}{4}P + \frac{3}{4}p_0$. Following the same pattern we obtain the sequence of regularities

$$P \preceq \frac{1}{2^n}P + \left(1 - \frac{1}{2^n}\right)p_0.$$

Since $\frac{1}{2^n} \to 0$ as $n \to \infty$, assumption 2 implies $P \preceq p_0$. Extend U to $\mathcal{R}_0$ by letting $U(P) = U(p_0)$. Obviously, U is a utility function of the form (3).

The necessity of assumptions 2 and 6 follows from the linearity of U, i.e.

$$U(\alpha P + (1-\alpha)q) = \alpha U(P) + (1-\alpha)U(q)$$

for any $P \in \mathcal{R}_0$, $q \in \mathcal{P}$, and $\alpha \in [0, 1]$. □

4. Conclusion

Theorem 2 provides an axiomatic foundation of the maxmin expected utility rule for the decision problems under nonstochastic risk. In such problems the choice has to be made among the weak* closed sets of probability measures. This reflects the fact that in the general case the random phenomenon is described by a family of probability distributions (Theorem 1). In case of stochastic phenomenon with finite set of outcomes this family is a singleton. Correspondingly, when $\mathcal{R}_0 = \mathcal{P}$ Theorem 2 degenerates into the expected utility theorem. The key assumption we use is that the decision maker wishes to reduce the set of probability distributions to a single one.

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Appendix

Let the statistical regularities of the phenomena $\mathfrak{A}$ and $\mathfrak{B}$ be $P \in \mathcal{R}$ and $q \in \mathcal{P}$ respectively. The phenomenon $\mathfrak{C}$ is represented by the following sampling net $\varphi : \Lambda \to X^\infty$: for each $\lambda \in \Lambda$ before each observation there is an α-chance of observing $\mathfrak{A}$ and a complementary chance of observing $\mathfrak{B}$. Denote r_λ the frequency distribution of a sample φ_λ. If the sample is big enough, then approximately α percentage of the observations belongs to $\mathfrak{A}$. They alone constitute the sample from $\mathfrak{A}$ with some distribution p_λ. Similarly, denote the distribution of the observations belonging to $\mathfrak{B}$ as q_λ. Then, the following equalities hold (the first holds approximately):

$$r_\lambda = \alpha p_\lambda + (1-\alpha)q_\lambda, \quad P = \mathrm{LIM}(p_\lambda), \quad q = \mathrm{LIM}(q_\lambda).$$

The following lemma implies that the statistical regularity $\mathrm{LIM}(r_\lambda)$ of the phenomenon $\mathfrak{C}$ coincides with $\alpha P + (1-\alpha)q$.

Lemma 4. *If Λ is a directed set, $p_\lambda, q_\lambda \in \mathcal{P}$ for every $\lambda \in \Lambda$, $\alpha \in [0,1]$, and* $\mathrm{LIM}\,(q_\lambda)$ *is a singleton, then*

$$\mathrm{LIM}\,(\alpha p_\lambda + (1-\alpha)\,q_\lambda) = \alpha \mathrm{LIM}\,(p_\lambda) + (1-\alpha)\,\mathrm{LIM}\,(q_\lambda)\,.$$

Proof. Fix $p \in \mathrm{LIM}\,(p_\lambda)$, $q \in \mathrm{LIM}\,(q_\lambda)$, $\lambda_0 \in \Lambda$, and show that $\alpha p_\lambda + (1-\alpha)\,q_\lambda$ is in the $(f_1, \ldots, f_n, \varepsilon)$-neighborhood of $\alpha p + (1-\alpha)\,q$ for some $\lambda \geq \lambda_0$. Since $\mathcal{P}$ is compact, q is a limit of the net (q_λ) and there is $\lambda_1 \in \Lambda$ such that for any $\lambda \geq \lambda_1$ q_λ is in the $(f_1, \ldots, f_n, \varepsilon)$-neighborhood of q. On the other hand, there is $\lambda_2 \in \Lambda$, such that $\lambda_2 \geq \lambda_0$, $\lambda_2 \geq \lambda_1$, and p_{λ_2} is in the $(f_1, \ldots, f_n, \varepsilon)$-neighborhood of p. Then

$$\begin{aligned} &\left|\int f_i(x)\,\mathrm{d}\,(\alpha p + (1-\alpha)\,q) - \int f_i(x)\,\mathrm{d}\,(\alpha p_{\lambda_2} + (1-\alpha)\,q_{\lambda_2})\right| \\ &\qquad \leq \alpha \left|\int f_i(x)\,\mathrm{d}p - \int f_i(x)\,\mathrm{d}p_{\lambda_2}\right| \\ &\qquad + (1-\alpha)\left|\int f_i(x)\,\mathrm{d}q - \int f_i(x)\,\mathrm{d}q_{\lambda_2}\right| < \varepsilon \end{aligned}$$

for each $i = \overline{1,n}$.

To prove the converse, suppose $r \in \mathrm{LIM}\,(\alpha p_\lambda + (1-\alpha)\,q_\lambda)$. Let M be the directed set of pairs (λ, A), such that $\lambda \in \Lambda$, A is a neighborhood of r, $\alpha p_\lambda + (1-\alpha)\,q_\lambda \in A$, and $(\lambda_1, A_1) \geq (\lambda_0, A_0)$ if and only if $\lambda_1 \geq \lambda_0$ and $A_1 \subseteq A_0$. For each $\mu \in M$ define

$$r_\mu = \alpha p_\lambda + (1-\alpha)\,q_\lambda, \quad p_\mu = p_\lambda, \quad q_\mu = q_\lambda,$$

when $\mu = (\lambda, A)$. Clearly, (r_μ), (p_μ), and (q_μ) are the subnets of $(\alpha p_\lambda + (1-\alpha)\,q_\lambda)$, (p_λ), and (q_λ) respectively. Moreover, $\lim\,(r_\mu) = r$. Since $\mathcal{P}$ is compact, (p_μ) has a limit point $p \in \mathrm{LIM}\,(p_\lambda)$. We will show that $r = \alpha p + (1-\alpha)\,q$.

For any $\mu \geq \mu_1$ r_μ is in the (f, ε)-neighborhood of r and q_μ is in the (f, ε)-neighborhood of q. On the other hand, there is $\mu_2 \geq \mu_1$ such that p_{μ_2} is in the (f, ε)-neighborhood of p. Then

$$\begin{aligned} &\left|\int f(x)\,\mathrm{d}r - \int f(x)\,\mathrm{d}\,(\alpha p + (1-\alpha)\,q)\right| \\ &\qquad \leq \left|\int f(x)\,\mathrm{d}r - \int f(x)\,\mathrm{d}\,(r_{\mu_2})\right| \\ &+ \left|\int f(x)\,\mathrm{d}\,(\alpha p_{\mu_2} + (1-\alpha)\,q_{\mu_2}) - \int f(x)\,\mathrm{d}\,(\alpha p + (1-\alpha)\,q)\right| \\ &\qquad < \varepsilon + \alpha\varepsilon + (1-\alpha)\,\varepsilon = 2\varepsilon. \end{aligned}$$

Since f and ε are arbitrary, $r = \alpha p + (1-\alpha)\,q$. $\square$

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