Index Formula of Laspeyres and the Inversion Test

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Comments on Ludwig von Auer's "Spurious Inflation: The Legacy of Laspeyres and others"

by

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Summary

The fact that the famous price index of Laspeyres is unable to pass the so called inversion test (IT) gave rise to the idea that this formula tends to measure "spurious inflation" which renders it useless and fallacious. In the IT prices and quantities of n goods, relating to two periods, the base period 0 and the current period t are interchanged in way that the sums of prices, quantities as well as values (products of prices and quantities) remain constant. It therefore appears nonsensical that Laspeyres' price index nonetheless indicates "inflation" under such conditions.

Yet this result can be explained and justified. The paper shows that violation of the IT does not prove uselessness of an index function. On the contrary a number of good reasons can be given why compliance with the IT does not at all make a formula preferable to other formulas and that the message of the IT and its relation to other tests is rather dubious.

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1. Introduction

In the paper quoted above von Auer[1] presented a really surprising result in that certain price index formulas will display a rise or decline in the price level although only some apparently insignificant and unimportant modifications of the data were made. These modifications involved in the so called "inversion test" (IT) seem to be so minuscule and innocuous that an index formula unable to satisfy this test and thus yielding counter-intuitive results, or "spurious inflation" as von Auer has put it, will in general be regarded as inappropriate[2]. Failing this test therefore seems to be a major shortcoming of the index formula of Laspeyres.

In this note we try to show that violation of the IT is not necessarily an indication of mismeasurement (or a "measurement bias", indicting inherent defects of a formula) and a poor rationale of the formula in question. "Spurious" inflation is a disadvantage only from the point of view of a certain concept of "inflation", which rests on linking expenditures (costs for a basket) to the price level. It also turned out that it is crucial to make a distinction between quantities on one hand and volumes on the other. From the point of view of the IT as a "thought experiment" the aforementioned modifications of the data are not really representing a change neither in the prices nor in the quantities. However, once we realize that volumes rather than physical quantities have to be considered the IT is loosing much of its intuitive appeal

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[1] Published in the Quarterly Review of Economic and Finance, 42 (2002), pp. 529-542.

[2] Not surprisingly von Auer thought that his inversion test would deal a blow to formulas like the one of Laspeyres which fail this test.

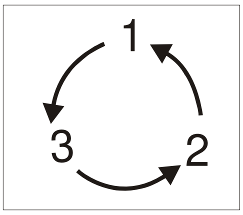
One is certainly perplexed[3] and impressed by the lucidity and compelling cogency of the argument of spurious inflation which will be depicted in detail in section 2. At first sight it seems difficult to say something against it. However in section 3 an attempt is made to show that reflecting spurious inflation is nothing an index formula should be blamed for. We therefore disagree with von Auer's conclusions. The reason is in the first place that spurious inflation (or deflation) should be viewed as tantamount to spurious decrease (or increase) in growth and growth should be measured in terms of volumes rather than quantities. Section 4 attempts at finding relationships of the inversion test to some other tests, in particular to the idea of symmetry between the two periods 0 and t, and section 5 concludes.

2. The permutation test and the inversion test as methods for detecting spurious inflation

"Spurious inflation" is referred to the situation that inflation (a price index displays $P_{0t} > 1$) is reported solely due to a measurement bias of the index formula. According to von Auer a mere permutation of price/quantity combinations as shown in table 1 where all prices and quantities and therefore also the total values (expenditures) V_{00} and V_{tt} respectively (shorthand for $V_{sk} = \sum p_s q_k$) remain unchanged, should not be regarded as "inflation". Yet price index formulas of Laspeyres (P^L) and Paasche (P^P) indicate inflation. To demonstrate the logic of the permutation test (PT) we follow von Auer's original example in the form of table 1.

Table 1 First[4] circular permutation in a three commodity scenario; the permutation test (PT)

commodity	base period		comparison period	
(i)	p_{i0}	q_{i0}	p_{it}	q_{it}
1	6	6	6	2
2	6	2	3	4
3	3	4	6	6



The price/quantity combinations (PQC) adhere to the following pattern $PQC_{1t} = PQC_{20}$, $PQC_{3t} = PQC_{10}$ and $PQC_{2t} = PQ_{30}$, that is the commodity 2 in the base period 0 takes the place of commodity 1 in the current period and what was commodity 1 in 0 now (in t) is commodity 3 etc. Of course this procedure (a circular permutation) results in $V_{tt} = V_{00} = 60$ (equality of expenditures and hence average prices (or "unit values"). Using $v_{sk}^i = p_{is} q_{ik}$ and $V_{sk} = \sum_i v_{sk}^i$ we nonetheless get[5]

$$P_{0t}^L = \frac{V_{t0}}{V_{00}} = \frac{\sum v_{t0}^i}{\sum v_{00}^i} = \frac{\sum p_t q_0}{\sum p_0 q_0} = \frac{66}{60} = 1.1, \text{ and } P_{0t}^P = \frac{V_{tt}}{V_{0t}} = \frac{\sum p_t q_t}{\sum p_0 q_t} = \frac{60}{54} = 1.111,$$

that is "inflation" in terms of both indices, the Laspeyres and the Paasche index.

[3] The present author frankly acknowledges that he also was to no small measure perplexed at first glance. On second thoughts it turned out, however, that it is not just a matter of course that spurious inflation is the ultimate proof of a nonsense - formula. In large part it is our argument that in the case of the "inversion test" where values remain constant, a price movement is unavoidably followed by a quantity movement in the opposite direction, and once we consider quantities in addition to prices spurious inflation is no longer so preposterous.

[4] There exists a second circular permutation displayed in tab. 3.

[5] Following von Auer with some modification the i-th value term (referring to the i-th commodity) is labelled v^i while $V = \Sigma v^i$ denotes the total (over all commodities) value (or total expenditure).

As von Auer said "it seems logical to assume that no change in the price level has occurred" by the simple reason that "total expenditure remains constant over time" and the commodities 1, 2, 3 in 0 simply represent different commodities, viz. 3, 1, 2 in t, and hence there "is no logical reason to claim that a change in the average price level has occurred". In what follows it will be shown that such statements, though apparently plain logic are not tenable, if not in part even simply wrong (as for example some assertions concerning the correlation between price and quantity changes).

According to von Auer a specific type of permutation, he called "simple swap" may be more likely to occur in the real world. Assume commodity 2 in 0 takes the part of commodity 3 in t and vice versa, we then get the scenario of table 2

Table 2 Swap permutation in the three commodity scenario (inversion test IT)

commodity	base period		comparison period	
(i)	p_{i0}	q_{i0}	p_{it}	q_{it}
1	6	6	6	6
2	6	2	3	4
3	3	4	6	2

Again von Auer contends that in situations like the one depicted in table 2 a price index should indicate "no inflation", and again we have "spurious inflation" in that we get P_{0t}^{L} = 1.1 and P_{0t}^{P} =1/1.1 = 0.909. Note further that the Laspeyres price index is now equal to the result of table 1 although the scenario is quite different.

In contrast to the PT this "inversion test" (IT) applies only to one or more *pairs of* commodities and in addition to some swaps there may be some PQC's that remain constant as for example commodity 1 in this case[6]. Consider the scenario of table 2 simplified only by deleting commodity 1. With two commodities only (viz. 2 and 3) we get $P_{0t}^{L} = \frac{30}{24} = 1.25$ and $P_{0t}^{P} = \frac{24}{30} = 0.8 = \frac{1}{P_{0t}^{L}}$.

Notice that just like in the case of table 2 we get $P_{0t}^{L} P_{0t}^{P} = 1$, a simple relationship necessarily true for the IT but not of the PT (recall that there was $P_{0t}^{L} P_{0t}^{P} = \frac{66}{60} \cdot \frac{60}{54} = \frac{66}{54} = 1.222 \neq 1)$. In the IT the following holds by definition

for the pair j, k	$j \to k$: $p_{j0} = p_{kt}, q_{j0} = q_{kt}$	$k \to j$: $p_{k0} = p_{jt}, q_{k0} = q_{jt}$,

and for the unchanged good (or group of goods) m	$p_{m0} = q_{mt}, q_{m0} = q_{mt}$, hence $p_{m0} q_{m0} = p_{mt} q_{mt} = p_{m0} q_{mt} = p_{mt} q_{m0} = R$

This yields

[6] Hence the IT may be called a "restricted version" (von Auer) of the PT. In the case of only two commodities there is only one permutation left and the PT comes down to the IT.

(1) $P_{0t}^{L}=\dfrac{p_{kt}q_{k0}+p_{jt}q_{j0}+R}{p_{k0}q_{k0}+p_{j0}q_{j0}+R}=\dfrac{p_{j0}q_{k0}+p_{k0}q_{j0}+R}{p_{k0}q_{k0}+p_{j0}q_{j0}+R}$, whereas

(2) $P_{0t}^{P}=\dfrac{p_{kt}q_{kt}+p_{jt}q_{jt}+R}{p_{k0}q_{kt}+p_{jo}q_{jt}+R}=\dfrac{p_{j0}q_{j0}+p_{k0}q_{k0}+R}{p_{k0}q_{j0}+p_{j0}q_{k0}+R}=\dfrac{1}{P_{0t}^{L}}$

Obviously $V_{tt}=V_{00}$ as in the PT *and* $V_{t0}=V_{0t}$, furthermore it can be seen that[7]

(3) $p_{kt}\sqrt{q_{k0}q_{kt}}+p_{jt}\sqrt{q_{j0}q_{jt}}=p_{j0}\sqrt{q_{jt}q_{j0}}+p_{k0}\sqrt{q_{kt}q_{k0}}$, accordingly

(4) $v_{00}^{i}+v_{tt}^{i}=v_{00}^{k}+v_{tt}^{k}$,

and that arithmetic and logarithmic means of the two v-terms of j and k will be equal, and finally one price relative is the reciprocal of the other

(5) $r_j=p_{jt}/p_{j0}=p_{k0}/p_{kt}=1/r_k$.

In view of eq. 1 and 2 it comes as no surprise that Fisher's ideal index $P_{0t}^{F}=\sqrt{P_{0t}^{L}P_{0t}^{P}}$ is able to pass the IT while both indices, P_{0t}^{L} and P_{0t}^{P} fail. The question now is:

Has this result some bearing on the validity of the Fisher formula P^F as opposed to the two other formulas (Laspeyres P^L and Paasche P^P) and are these two formulas consequently rightly dismissed as having a potential of producing spurious inflation?

From eq. 1 it follows that $P_{0t}^{L}=1$ is possible only if

$$(p_{kt}-p_{k0})=(p_{j0}-p_{jt})\frac{q_{j0}}{q_{k0}} \text{ or}$$

$$(p_{j0}-p_{k0})=(p_{j0}-p_{k0})\alpha$$

where $\alpha=q_{j0}/q_{k0}$, or we should have - apart from the trivial case of no price change at all or equal prices - equality of quantities ($\alpha=1$) in the base period (and thus also in t). In other words in order to get $P_{0t}^{L}=1$ when there is a swap between j and k (j $\leftrightarrow$ k) respectively, the following condition has to be met $P_{0t}^{L}=P_{0t}^{P}=V_{0t}^{*}=\dfrac{V_{tt}}{V_{00}}=\dfrac{\sum p_t}{\sum p_0}=P_{0t}^{D}$, where V_{0t}^{*} is the "value index"[8] is and P_{0t}^{D} is an index formula known as Dutot's index. As a consequence the quantity indices of Laspeyres and Paasche respectively are $Q_{0t}^{L}=Q_{0t}^{P}=1$. Hence we now assume conditions as follows:

for the pair j, k	$j\rightarrow k$: $p_{j0}=p_{kt}$	$k\rightarrow j$: $p_{k0}=p_{jt}$	$q_{j0}=q_{kt}=q_{k0}=q_{jt}$

This result may already give a hint to the nature (and limitations) of the reasoning in terms of PT and IT. Once quantities do not matter because they do not change from 0 to t the formulas

[7] From these equations as well as $V_{t0}=V_{0t}$ it follows that (as observed already by von Auer) formulas like those of Fisher, Marshall-Edgeworth, Törnquist, Walsh (or "Walsh I") and the two Vartia formulas will satisfy the IT. Therefore they are all invariant upon a swap between any two commodities. Hence an impressive number of useful index formulas comply with the IT and to pass this test can hardly be something negative. On the other hand it may be argued that *not* complying with the IT is an indication of poor validity. This, in essence is von Auer's view with which I disagree.

[8] This index is here denoted by V* with an asterisk in order not to confound it with absolute Values V_{00}, V_{0t} etc.

"reduce to" the unweighted Dutot index and we may readily conclude inflation from expenditures. Things become easy if we simply disregard the columns q_{i0} and q_{it} in table 1 and 2 (or assuming $q_{i0} = q_{it} = q$ for all i) which will of course[9] yield $P_{0t}^{D} = 1$.

However, in all those cases in which quantities of different commodities will be different and subject to changes over time, there is no basis for inferring price levels from expenditure levels only. Moreover, and this is the central point to make against von Auer, when analysing the price movement, the concomitant quantity movement should be taken into account. A "spurious inflation" of the price index P should not be seen in isolation without realising that there will always be a corresponding "spurious real growth" displayed by the quantity index Q which pertains to P.

Given that the price increase is illusory (spurious), the decrease in the quantities (as a result of expenditures remaining constant) should also be deceptive, or viewed as a result of a biased measurement. We will demonstrate that the quantity is definitely *not* spurious. Hence there may exist situations in which the scenario of inversion may not easily be recognised as "inflation" (because expenditures remain constant). However the indication of a change in quantity will usually easily be justified and this should indirectly make the notion "inflation" in such a situation more acceptable.

Another interesting result is stated in the following proposition

Proposition 1

Under the conditions of the IT we get the following equations

1. $P_{0t}^{P} = \left(P_{0t}^{L}\right)^{-1}$ and consequently $P_{0t}^{L} = \left(P_{0t}^{P}\right)^{-1}$,

2. $P_{0t}^{L} = Q_{0t}^{L}$ and $P_{0t}^{P} = Q_{0t}^{P}$

3. the covariance C is given by $C = 1 - \left(Q_{0t}^{L}\right)^{2} = 1 - \left(1/Q_{0t}^{P}\right)^{2}$.

Proof

Statement 1 follows from the fact that Fisher's index satisfies the IT. The identity has also been demonstrated in the example of table 2. The second statement directly follows from

for the pair j, k	$j \to k$: $p_{j0} = p_{kt}, q_{j0} = q_{kt}$	$k \to j$: $p_{k0} = p_{jt}, q_{k0} = q_{jt}$,

We may therefore substitute prices and quantities referring to period t by prices and quantities referring to period 0 giving

(1) $$P_{0t}^{L} = \frac{p_{j0}q_{k0} + p_{k0}q_{j0} + R}{p_{k0}q_{k0} + p_{j0}q_{j0} + R}$$ and

(1a) $$Q_{0t}^{L} = \frac{q_{jt}p_{j0} + q_{kt}p_{k0} + R}{p_{k0}q_{k0} + p_{j0}q_{j0} + R} = \frac{q_{k0}p_{j0} + q_{j0}p_{k0} + R}{p_{k0}q_{k0} + p_{j0}q_{j0} + R} = P_{0t}^{L}.$$

The equality $P_{0t}^{L} = Q_{0t}^{L}$ can easily be verified in the case of table 2. In the same manner it can be shown that $P_{0t}^{P} = Q_{0t}^{P}$. In combination with the value index V* which is by definition $V_{0t}^{*} = P_{0t}^{L}Q_{0t}^{P} = P_{0t}^{P}Q_{0t}^{L} = 1$ and the theorem of L. v. Bortkiewicz we reach statement 3, which

[9] Recall 6 + 6 + 3 = 6 + 3 + 6. Interestingly von Auer also mentioned those unweighted "average prices (or unit prices)" of 15/3 = 5 in both periods.

says that whenever the covariance between price and quantity relatives is non-zero we necessarily have

- either (negative covariance C < 0) "spurious inflation" ($P_{0t}^{L} > 1$) and "spurious growth" ($Q_{0t}^{L} = P_{0t}^{L} > 1$) in terms of Laspeyres which is equivalent to "spurious" deflation and recession (negative growth) in terms of Paasche,
- or we have the other way round $P_{0t}^{L} = Q_{0t}^{L} < 1$ which is equivalent to $P_{0t}^{P} = Q_{0t}^{P} > 1$.

2. A look at quantities and individual price and quantity relatives

As already pointed out it is in large part our argument that in the case of the "inversion test" where values remain constant, a price movement is unavoidably followed by a quantity movement in the opposite direction. The notion of spurious inflation loses much of its charm once we consider the quantity dimension in conjunction with the price dimension that is once we consider quantities in addition to prices. The changes made in the IT then are not of minor importance and relevance as it might seem at first glance in view of constant values.

The explanation we attempt to provide for the violation of the inversion test (IT) in the case of the Laspeyres and Paasche index, will hopefully make clear that failing the IT is not to be viewed as "measurement bias". Price indices can often be written as average price ratios (price relatives) $r_i = p_{it}/p_{i0}$ and by analogy quantity indices (Q_{0t}) are functions of quantity ratios $m_i = q_{it}/q_{i0}$. Using weights $w_i = p_{i0}q_{i0} \big/ \sum p_{i0}q_{i0} = v_{00}^{i} \big/ V_{00}$ it turns out that the arithmetic mean of price ratios is P_{0t}^{L} and of quantity ratios Q_{0t}^{L}. Furthermore according to[10]

$$C = \sum (r_i - P_{0t}^{L})(m_i - Q_{0t}^{L}) w_i = V_{0t} - P_{0t}^{L} Q_{0t}^{L} \quad \text{where} \quad V_{0t} = V_{tt} \big/ V_{00}$$

the covariance C between price and quantity relatives is equal to the difference between the value index (value ratio) V_{0t} and the product of the two Laspeyres indices.

The scenario of table 1 gives $Q_{0t}^{L} = 54/60 = 0.9$ and $Q_{0t}^{P} = 60/66 = 0.909$.

Of course $P_{0t}^{L} Q_{0t}^{P} = P_{0t}^{P} Q_{0t}^{L} = V_{0t} = 1$. Therefore, in this case contrary to von Auer's assertion prices and quantities (or more distinct, price and quantity *relatives*) *are not uncorrelated* but rather positively correlated, in which case $P_{0t}^{L} > P_{0t}^{P}$. The covariance is $C = 1 - 1.1 \cdot 0.9 = 0.01$ and thus positive. Moreover as both quantity indices are less than unity there is definitely a decline in quantities also reflected in[11] $Q_{0t}^{F} = \sqrt{Q_{0t}^{L} Q_{0t}^{P}} = \sqrt{54/66} = 0.9045$. In general quantities can not be aggregated as such[12] but only in combination with prices, i.e. as volumes (using prices p_{i0} or p_{it} or some average of both).

Table 1 shows that reductions of *quantities* $q_{it} - q_{10} = 2 - 6 = -4$ in the case of commodity 1 may be offset by increases in *quantities* $q_{2t} - q_{20} = 4 - 2 = +2$ and $q_{3t} - q_{30} = 6 - 4 = +2$ in the case of commodities 2 and 3 respectively.

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[10] A famous equation first established by L. von Bortkiewicz. See v. d. Lippe, Index Theory and Price Statistics, (Peter Lang Publisher), Bern etc., 2007. p. 37f.

[11] Since the Fisher index is not subject to the spurious inflation bias (because IT is satisfied) we may put more faith in stating a decline of the quantity level as compared to a statement based on a Laspeyres or Paasche quantity index alone.

[12] This in particular applies to aggregations across very different types of commodities. Even in the case of a common unit of measurement, summation may be meaningless. One litre of wine and two litres of fuel is not simply to be regarded as a "quantity" of three litres.

A different picture will emerge, however, once the prices are taken into account such that we consider volumes instead of quantities. Remember that in practice aggregation can only be performed in terms of volumes, rather than quantities.

In the base period the price of commodity 3 is only half the price of 1 or 2, so an increase in q_3 by two units should just count as much as an increase in q_1 or q_2 by one unit only. Hence in actual fact an increase by + 1 of commodity 3 rather than + 2 has to be set against - 4 + 2 = - 2 resulting in - 1 altogether, hence in a reduction in quantity (valued at base period prices instead of measured in physical units). Taking prices p_{it} instead of p_{i0} in order to make changes in quantity commensurable, it is commodity 2 that should enter with half the weight of the other two commodities which again yields a balance of - 1. The volume - 1 is related to the quantity indices as follows

$$Q_{0t}^{L} = \frac{\sum (q_t - q_0)p_0}{\sum q_0 p_0} + 1 = \frac{(-1)\cdot 6}{60} + 1 = 0.9 \text{, and } Q_{0t}^{P} = \frac{\sum (q_t - q_0)p_t}{\sum q_0 p_t} + 1 = \frac{(-1)\cdot 6}{66} + 1 = 0.909$$

No matter which prices, p_0 or p_t are taken in order to infer volumes from quantities a Q-index on the whole indicates a downward movement. And therefore a price index rightly displays increasing prices or "inflation", which therefore should not be viewed as spurious.

Price and quantity indices should be seen as simply two sides of a coin. Given that $V_{tt} = V_{00}$ and thus $V_{0t} = 1$ it follows from Q < 1 that P should be greater than unity. Not surprisingly all price indices Laspeyres, Paasche and Fisher display a *rising* price level, by no means, however, a spurious inflation. To call a 10% or 11.1% increase in the prices "spurious" would of course imply that the growth of - 10% or - 9.09 % should be called a "spurious shrinkage".

The situation is different in the case of a swap (the inversion test) described in table 2. Applying the same kind of reasoning it will be interesting to detect the reason for the difference between the PT-type of price/quantity change as opposed to the IT-type of change. A swap in which the commodities 2 and 3 are involved entails.

(6) $q_{2t} - q_{20} = q_{30} - q_{3t} = \Delta q$.

The same absolute quantity differences may nonetheless lead to different volume changes, however, the total change valued at base period prices is amounting to

(7) $\Delta q(p_{20} - p_{30}) = \Delta V_0$ as compared to a valuation at current prices

(8) $\Delta q(p_{2t} - p_{3t}) = \Delta V_t$,

and clearly $\Delta V_0 = - \Delta V_t$[13] as $p_{20} - p_{30} = - (p_{2t} - p_{3t})$.

Hence quantity indices in which use is made of prices in a *symmetric* fashion seem to be invariant to changes of the IT type. In other words, possibly[14] all price indices which combine quantities of both periods in an even-handed (symmetrically weighted) manner will comply with the IT.

For asymmetrically weighted indices like Laspeyres and Paasche, however, we find that they can show the same direction of change. In the example of table 2 we get

$$Q_{0t}^{L} = \frac{\Delta V_0}{\sum p_0 q_0} + 1 = \frac{6}{60} + 1 = \frac{66}{60} = 1.1 > 1 \text{ (rise)}$$

[13] The example of table 2 yields $\Delta V_0 = 6$ and $\Delta V_t = - 6$.

[14] The finding just mentioned made us conjecture that the IT should be equivalent to an axiom (test) called quantity reversibility (QR). In section 4 we will see that things are not as simple as that.

$$Q_{0t}^{P} = \frac{\Delta V_t}{\sum p_t q_0} + 1 = \frac{-6}{66} + 1 = \frac{60}{66} = 0.909 < 1 \quad \text{(decline)}^{15}.$$

It may now be useful to spell out in detail the assumption of the IT in terms of price and quantity relatives and expenditure shares. As demonstrated in table 2 for the two commodities 2 and 3, both involved in a swap the following holds for price relatives $r_2 = \frac{p_{2t}}{p_{20}} = \frac{p_{30}}{P_{3t}} = \frac{1}{r_3}$ and correspondingly for quantity relatives we get $m_2 = \frac{q_{2t}}{q_{20}} = \frac{1}{m_3}$. In order to ensure compliance with the IT for each pair additionally $(r_2)^{a2}(r_3)^{a3} = 1$ or $r_2 b_2 + r_3 b_3 = 0$ should hold in the case of geometric, or arithmetic mean type price index respectively.[16]

Terms like or, as components of a will be ineffectual and thus. The condition $a_2 = a_3$ will take place in the case of symmetric means of value terms as weights. Assuming $r_j r_k = m_j m_k = 1$ we have e, that is $v_{00}^{j} = v_{tt}^{k}$ and $v_{00}^{k} = v_{tt}^{j}$ (see also eq. 4 and recall $V_{00} = V_{tt}$) such that

$\bar{v}_j = \frac{1}{2}(v_{00}^{j} + v_{tt}^{j}) = \bar{v}_k$ holds, or for the logarithmic mean

$L(v_{00}^{j}, v_{tt}^{j}) = \left(v_{tt}^{j} - v_{00}^{j}\right) / \ln\left(v_{tt}^{j} / v_{00}^{j}\right) = L(v_{00}^{k}, v_{tt}^{k})$.

This explains that the Törnquist index and both types of Vartia indices comply with the IT[17].

In a similar fashion $r_2 b_2 = - r_3 b_3$ is requisite in order to satisfy the IT in the arithmetic case. A suitable pair b_2, b_3 would be for example

(9) $-b_3 = q_{30} q_{3t}$ and $b_2 = q_{20} q_{2t} = -b_3$ leading to the Walsh index, since

(10) $P_{0t}^{W} = \frac{\sum \sqrt{v_{t0}^{i} v_{tt}^{i}}}{\sum \sqrt{v_{00}^{i} v_{0t}^{i}}}$ or in its more familiar form

(10a) $P_{0t}^{W} = \frac{\sum p_t \sqrt{q_0 q_t}}{\sum p_0 \sqrt{q_0 q_t}}$ omitting subscripts i for convenience.

Upon substitution of eq. 9 in $-b_3 = \frac{r_2}{r_3} b_2$, and using $p_{20} = p_{3t}$ and $p_{30} = p_{2t}$ so that $-p_{3t}^{2} b_3 = p_{2t}^{2} b_2$ we get

(11) $-v_{t0}^{3} v_{tt}^{3} = v_{t0}^{2} v_{tt}^{2}$,

showing that items 2 and 3 among which a swap takes place will cause no change. If eq. 11 holds then summation over value terms $v_{00} v_{0t}$ with respect to commodities j = 2 and k = 3 will also not change the denominator because equations like $v_{tt}^{3} = v_{00}^{2}$ and $v_{0t}^{3} = v_{t0}^{2}$ etc. are resulting from the sort of interchanging characterising the IT scenario. This also shows that apart from the Walsh index arithmetic (in contrast to geometric) means of price or quantity relatives will in general not satisfy the IT.

[15] Note that the *geometric* mean (Fisher's formula) of Q^L and Q^P is unity while the *arithmetic* mean (Drobisch's formula) is not.

[16] The same applies to quantity indices if r-ratios are replaced by m-ratios.

[17] It can easily be verified that this is true also for other indices utilising symmetric means of values quoted by von Auer, s for example Walsh II, Walsh-Vartia or the formula of Theil.

It should be added that there exists a second permutation in addition to table 1 where things are in a sense reversed (one may simply reverse the arrows).

Table 3 Second circular permutation in a three commodity scenario

commodity	base period		comparison period	
(i)	p_{i0}	q_{i0}	p_{it}	q_{it}
1	6	6	3	4
2	6	2	6	6
3	3	4	6	2

resulting in $P^L_{0t} = \frac{54}{60} = 0.9$, $Q^L_{0t} = \frac{66}{60} = 1.1$, $P^P_{0t} = 1/1.11 = 0.909$, and $Q^P_{0t} = 1/0.9 = 1.11$.

Note that compared with table 1 the part of the price index is taken by the quantity index and vice versa. The covariance is therefore not surprisingly again C = 1 - 0.99 = + 0.01 and it may be interesting to look at the price and quantity relatives in this modification of table 1, that is to compare the two PT-cases of table 1 and 3 and for the sake of completeness include also table 2 (the IT).

Table 4 Price- and quantity relatives, r_i and m_i in the scenarios of table 1 and 3 (and in the IT table 2 for comparison purpose)

commodity	table 1		table 3		table 2	
i	r_i	m_i	r_i	m_i	r_i	m_i
1	1	1/3	1/2	2/3	1	1
2	1/2	2	1	3	1/2	2
3	2	3/2	2	1/2	2	1/2
covariance C*	+ 0.01		+ 0.01		$1 - (1.1)^2 = - 0.21$	

*) covariance between r and m (if positive then $P^L < P^P$ being the less likely case, however)

Looking at the price relatives instead of the expenditures (and unit values) it is far from obvious why a price index should not indicate inflation. There is even some plausibility in "inflation" in the case of table 2 provided that there are good reasons to assign a greater weight to $r_3 = 2$ than to $r_2 = 1/2$.

This result may also throw some light on thee controversy about unweighted indices. Under the regime of the two inversion-scenarios of table 1 and table 3 respectively the three well known unweighted indices of Carli (arithmetic mean), Jevons (geometric mean) and the harmonic mean will behave as shown in table 5.

Table 5 Unweighted price- and quantity indices combining three relatives, r_i and m_i in the scenarios of table 1 and 3 (the two circular permutations) and weighted Indices of Laspeyres and Paasche for comparison purposes

index	table 1 (1 → 3 → 2)		table 3 (2 → 3 → 1)	
	price index*	quantity index	price index*	quantity index
Carli (C)	1,16667	1,27778	$1{,}1667 = 0.8571^{-1}$	$1.3888 = 0.72^{-1}$
Jevons (J)	1	1	1	1
harmonic (H)	0,85714	0,72	$0{,}8571 = 1.1667^{-1}$	$0.7826 = 1.2777^{-1}$
Laspeyres	$P^L = 1.1$	$Q^L = 1/P^P = 0.9$	$P^L = 0{,}9$	$Q^L = 1.1$
Paasche	$P^P = 1.1111$	$Q^P = 1/P^L = 0{,}909$	$P^P = 0.909$	$Q^P = 1.1111$

*) in this case (and in the case of the inversion test instead of the permutation test) the harmonic mean index is simply the reciprocal arithmetic (Carli) index

To remember the formulas: Carli $P_{0t}^C = \sum r_i / n$, Jevons $P_{0t}^J = \left(\prod r_i\right)^{1/n}$ and the harmonic mean $P_{0t}^H = \dfrac{n}{\sum r_i^{-1} / n}$, and quantity indices correspondingly.

Note that the weighted indices of Laspeyres and Paasche always show in the same direction (both rising and declining respectively). This does not apply to the pair of unweighted indices, the arithmetic (Carli) may display a rise in prices or quantities while for the same data the harmonic mean index type may decline and vice versa.

To draw a conclusion: Taking quantities into account it is by no means clear that a price index ought to satisfy the inversion test (IT). In both, the permutation test (PT) as well as the inversion test (IT) the total value does not change, so that $V_{00} = V_{tt}$ and the value index therefore is given by $V_{0t}^* = 1$. Under the assumptions made in the IT (due to $V_{0t}^* = 1$) the Paasche and Laspeyres index are related as follows: $P_{0t}^L = 1/Q_{0t}^P$ and $P_{0t}^L = 1/Q_{0t}^P$. It is possible, that both price indices indicate a rise ($P_{0t}^L > 1, P_{0t}^P > 1$) in which case both quantity indices will display a decline. This will in general not be true in the case of unweighted indices. Both a price and a quantity index of Carli may show a rise.

4. The inversion test and its relation to other tests and properties of the Laspeyres formula

a) Chain indices and the case of zero-covariance in the inversion test

It turned out that the IT is no suitable invention for making a mockery of the Laspeyres formula. This applies also to repeated inversions. A scenario explored by von Auer is that after a change (concerning the PQCs) from 0 to 1 all prices and quantities subsequently (in period 2) return to the original situation of period 0. So we end up at the point, from where we started. For him the fact that the Laspeyres formula produces for this reverse movement again "inflation" that is $P_{01}^L = P_{12}^L = 1.1$ once more proves that the formula may lead to "a nonsensical result". Table 6 is designed to demonstrate some additional aspects (shown in the highlighted part) with three commodities and three periods 0, 1, and 2 what we had in mind so far we get:

Taking 0 as base we get $P_{01}^L = 66/60 = 1.1 = Q_{01}^L$, further $P_{01}^P = 60/66 = 0.91$ and the covariance C = - 0.21 (the negative covariance is a result of the swap between two PQCs)

Taking 1 as base we get $P_{12}^{L}=P_{01}^{L}$ and $P_{12}^{P}=P_{01}^{P}$ so that C is again – 0.21.

This finding is less of a case against the Laspeyres formula as rather an argument against chain indices (demonstrating that chain indices usually fail the identity test). For direct indices we get of course $P_{02}^{L}=P_{02}^{P}=1$ as expected since prices and quantities are the same in periods 0 and 2. Difficulties with cyclical price movements where the PQCs of 0, 2, 4, 6 etc. are identical just as the PQCs of 1, 3, 5 etc. are unavoidable with all sorts of chain indices[18]. This is not an issue related to the IT but rather to chain indices.

On the other hand it is possible to contrive examples in which the swap between two commodities, say 2 and 3 is counterbalanced by a swap between two other commodities (4 and 5 in our numerical example below) in the opposite direction.

In table 6 all sums of products are equal $\sum p_2q_0 = \sum p_2q_1 = \sum p_0q_1 = ... = 198$ with the effect that $P_{01}^{L}=P_{01}^{P}=P_{12}^{L}=P_{12}^{P}=1$.

Interestingly now the covariance between price and quantity relatives vanishes, that is C = 0 for any two adjacent periods. This shows that it is well possible to specify inversions such that they cancel out and both, the Laspeyres or Paasche index show no spurious inflation what they necessarily do, however, in the case of taking one swap in isolation.

Table 6 Inversion test with two counteracting swaps

com-modity	p_0	q_0	p_1	q_1	p_2	q_2
1	6	6	6	6	6	6
2	6	2	3	4	6	2
3	3	4	6	2	3	4
4	9	10	6	8	9	10
5	6	8	9	10	6	8

the price (r) and quantity relatives (m) are then from period t - 1 to t as follows

commodity	$0 \rightarrow 1$		$1 \rightarrow 2 = 0$	
(i)	r_i	m_i	r_i	m_i
1	1	1	1	1
2	1/2	2	2	1/2
3	2	1/2	1/2	2
4	2/3	4/5	3/2	5/4
5	3/2	5/4	2/3	4/5

So the argument, that the Laspeyres Price index will necessarily differ from unity in both phases of an IT-type way there ($0 \rightarrow 1$) and back ($1 \rightarrow 0$) is not true.

[18] These problems are one of the major defects of the chain index design and conversely it is the fact that they are absent is one of the major advantages of direct indices. See P. v. d. Lippe, Chain Indices, A Study in Index Theory, Wiesbaden 2001, pp. 121-125 for more details. I also gave a detailed account of all properties of chain indices in chapter 7 of my book P. v. d. Lippe, Index Theory and Price Statistics (Peter Lang Publisher), Bern etc., 2007.

b) Time reversal test

The usage of symmetric averages of weights in formulas that pass the IT also suggests that the IT and the time reversal test (TR) are closely related. Conspicuously all IT-compatible formulas listed above are able to pass the TR too. However, according to von Auer the tests are independent[19]. The Cobb-Douglas index for example $P_{0t}^{CD} = \prod_i \left(\frac{p_{it}}{p_{i0}}\right)^{\alpha_i} = \prod_i \left(r_{i,0t}\right)^{\alpha_i}$ does not pass the IT unless $\alpha_2 = \alpha_3$. The CD-Index satisfies the TR, as in table 4 we get $r_{3,01} = (r_{4,01})^{-1}$), and since in general $r_{i,12} = (r_{i,02})^{-1}$ (i = 3, 4). On the other hand there exists at least one index function which passes the IT but fails the TR[20] (unless very specific conditions prevail)

$$(12) \qquad P_{0t}^{A} = \frac{1}{2}\left(\frac{\sum p_t q_t}{\sum p_0 q_0} + \frac{\sum p_t q_0}{\sum p_0 q_t}\right) = \frac{1}{2}\left(\frac{V_{tt}}{V_{00}} + \frac{V_{tt}}{V_{0t}}\right) = \frac{1}{2}\left(P_{0t}^{P}\frac{V_{0t}}{V_{00}} + P_{0t}^{L}\frac{V_{00}}{V_{0t}}\right)$$

Both terms between the brackets amount to unity as by implication of the IT we have $V_{00} = V_{tt}$ and $V_{0t} = V_{t0}$.

c) Quantity reversal test (QR, symmetry of quantities)

The relation between the IT and a test that might be called quantity reversal test (QR) appears to be even closer than the relation to the TR[21].

The Quantity reversal test (QR)[22] requires that

$$(13) \qquad P(\mathbf{p}_0, \mathbf{q}_t, \mathbf{p}_t, \mathbf{q}_0) = P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t),$$

or in words that quantities of both periods must enter the index formula symmetrically so that the index remains invariant upon interchanging of quantity vectors.[23]

From the point of view of the quantity reversal test (QR) a formula may be preferred in which quantities q_0 and q_t are treated symmetrically. Except for the Törnquist index most of the so called "superlative" index formulas are built on this QR-principle.

One is led to expect that there should be some relation between this kind of symmetry (invariance upon changing the quantities $q_{i0} \leftrightarrow q_{it}$) and the invariance upon interchanging of price-quantity-combinations (PQCs) as required in the inversion test (IT).

It turned out, however, that surprisingly many QR-compatible index functions are not able to pass the IT. Moreover, in what follows we try to show that the two tests, IT and QR are independent.

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[19] Due to the close relation between the tests von Auer considers, however, the IT as an "appealing alternative" to the more controversial TR.

[20] See von Auer, p. 541 f.

[21] This turned out premature, however. Contrary to what we previously expected quite a few of the formulas that pass the IT will fail the QR. There exists a lot of uniqueness theorems for Fisher's "ideal index" P^F. One of these theorems is that P^F is the only index able to satisfy TR, QR and the factor reversal test.

[22] Theis test is described (and criticised) in more detail in P. v. d. Lippe, Index Theory and Price Statistics, 2007, on p.172 and p. 207f. In our view, focussing on the notion of "pure price comparison" as an indispensable property of index functions "there is prima facie not much use to be found in this property" (p. 208). Balk (1995) also derived a uniqueness theorem for Fisher's ideal index, P^F according to which P^F is the only index satisfying simultaneously the following three tests: 1. time reversal, 2. factor reversal and 3. quantity reversal test.

[23] From our results above it is apparent that weighted indices using quantities as weights in a *symmetric* fashion will more likely fulfil the IT.

To see this tt may be useful to add also the property of commensurability which can be expressed as follows

$$P(\mathbf{Lp}_0, \mathbf{L}^{-1}\mathbf{q}_0, \mathbf{Lp}_t, \mathbf{L}^{-1}\mathbf{q}_t) = P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t)$$

where $\mathbf{L}$ is a $n \times n$ diagonal matrix with elements $\lambda_1, ..., \lambda_n$, such that

$\mathbf{L} = \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{bmatrix}$ and assuming diagonal elements $\lambda_i = 1/p_{i0}$ giving

$\mathbf{Lp}_0 = \mathbf{1}$, where $\mathbf{1}' = [1\ 1\ ...\ 1]$ and

$\mathbf{Lp}_t = \mathbf{r}$, the vector of price relatives $\mathbf{r}' = [p_{1t}/p_{10}\ \ p_{2t}/p_{20}\ \ ...\ \ p_{nt}/p_{n0}]$. Furthermore

$\mathbf{L}^{-1}\mathbf{q}_0 = \mathbf{v}_0$ the vector of base period *values* $\mathbf{v}_0' =.[p_{10}\,q_{10}\ \ p_{20}\,q_{20}\ \ ...\ \ p_{n0}\,q_{n0}]$ and

$\mathbf{L}^{-1}\mathbf{q}_t = \mathbf{v}_t$ the vector of *volumes* $\mathbf{v}_t' = [p_{10}\,q_{1t}\ \ p_{20}\,q_{2t}\ \ ...\ \ p_{n0}\,q_{nt}]$.

Hence when commensurability holds the index function can be expressed in three vectors

(14) $P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t) = P(\mathbf{1}, \mathbf{v}_0, \mathbf{r}, \mathbf{v}_t)$, and in combination with eq.13 we get

(15) $P(\mathbf{v}_t, \mathbf{r}, \mathbf{v}_0) = P(\mathbf{v}_0, \mathbf{r}, \mathbf{v}_t)$.

It therefore may be useful to write some index formulas in terms of the price relatives and values v_{00} and v_{0t} respectively (see table 7).

A cursory look at the table 7 already shows that obviously not all formulas listed in this table and being in line with the IT can be conceived as symmetric in the values v_{00}^i and v_{0t}^i. This applies for example to the Törnquist formula which is symmetric in v_{00}^i and v_{tt}^i, but not in q_{i0} and q_{it} and thus in v_{00}^i and v_{0t}^i[24].

The same is true for the two Vartia index formulas and also in particular for the formula of Banerjee[25] which may be written as

(16) $$P_{0t}^{B} = \frac{V_{t0}/V_{00} + 1}{1 + V_{0t}/V_{tt}},$$

an expression clearly showing that the IT is met.

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[24] This will also be demonstrated in a simple example (see table 6).

[25] The formula discussed above has already been used by von Auer. Banerjee , however, proposed another formula (gained as by-product of his specific "economic theory approach" to index numbers) which proved to be consistent with the IT. Both formulas of Banerjee are presented in detail in von der Lippe, Index Theory and Price Statistics, 2007, p. 162 – 164.

Table 7 Some index formulas complying with the inversion test

Index formula of	expressed in terms of r, v_{00} and v_{0t} (vectors **r**, $\mathbf{v}_0$, $\mathbf{v}_t$)
Fisher $\left(P_{0t}^{F}\right)$	$\left(\frac{\sum r_i v_{00}^i}{V_{00}} \frac{\sum r_i v_{0t}^i}{V_{0t}}\right)^{1/2}$
Marshall-Edgeworth $\left(P_{0t}^{ME}\right)$	$\frac{\sum r_i\left(v_{00}^i+v_{0t}^i\right)}{V_{00}+V_{0t}}=\frac{V_{t0}+V_{tt}}{V_{00}+V_{0t}}$ where $V_{0t}=\sum_i v_{0t}^i=\sum_i p_{i0}q_{it}$, and *not* $V_{0t}=\sum p_t q_t / \sum p_0 q_0$
Walsh (= Walsh I) $\left(P_{0t}^{W}\right)$	$\frac{\sum \sqrt{v_{t0}^i v_{tt}^i}}{\sum \sqrt{v_{00}^i v_{0t}^i}}=\frac{\sum r_i \sqrt{v_{00}^i v_{0t}^i}}{\sum \sqrt{v_{00}^i v_{0t}^i}}$
Walsh II $\left(P_{0t}^{W2}\right)$	$\ln\left(P_{0t}^{W2}\right)=\sum \ln r_i \frac{\sqrt{r_i v_{00}^i v_{0t}^i}}{\sum \sqrt{r_i v_{00}^i v_{0t}^i}}$ since $v_{tt}^i=r_i v_{0t}^i$ note also that $v_{00}^i v_{tt}^i= p_{io}q_{io}p_{it}q_{it}=p_{io}q_{it}q_{it}q_{i0}=v_{0t}^i v_{t0}^i$
Walsh Vartia $\left(P_{0t}^{WV}\right)$	$\ln P_{0t}^{WV}=\sum \ln r_i \frac{\sqrt{r_i v_{00}^i v_{0t}^i}}{\sqrt{V_{00} V_{tt}}}$, note that $V_{tt}=P_{0t}^{P} V_{0t}$
Törnquist $\left(P_{0t}^{T}\right)$	$\ln P_{0t}^{T}=\sum \ln r_i \cdot \frac{1}{2}\left(\frac{v_{00}^i}{V_{00}}+\frac{r_i v_{0t}^i}{V_{tt}}\right)$

Another equivalent expression is, however

(16a) $$P_{0t}^{B}=P_{0t}^{P}\frac{P_{0t}^{L}+1}{P_{0t}^{P}+1}=\frac{\sum r_i v_{0t}^i\left(\sum r_i v_{00}^i+V_{00}\right)}{V_{00}\left(\sum r_i v_{0t}^i+V_{0t}\right)},$$

which shows that P^B fails the QR-test. Invariance upon interchanging of quantities would, however, require that P_{0t}^{P} could be replaced by P_{0t}^{L} and vice versa. And this in turn would imply $Q_{0t}^{F}(P_{0t}^{L}+1)=V_{0t}+Q_{0t}^{L}$ which is true only if very special conditions are met (as for example in the case of table 2 where $Q_{0t}^{L}=P_{0t}^{L}=1.1$ and $V_{0t}=Q_{0t}^{F}=1)$. Accordingly interchanging v_{00} and v_{0t} in the second formula of eq. 16a does not leave the index unchanged.

We may now express the result in terms of a proposition.

Proposition 2

The quantity reversal test (QR) and the inversion test (IT) are independent.

Proof

It is sufficient to show that there exist a number of formulas that will

1. satisfy the IT but violate the QR as for example the indices of Törnquist, Banerjee, Walsh-Vartia, Vartia I and II and Theil, and

2. vice versa, i.e. the QR is satisfied while the IT is not.

An examole for s . On the other hand it is obvious that the quadratic mean type price index

(17) $$P_{0t}^{QM} = \sqrt{\sum \left(\frac{p_{it}}{p_{i0}}\right)^2 \frac{v_{00}^{i} + v_{0t}^{i}}{V_{00} + V_{0t}}}$$

is able to pass the QR-test (as $p_{i0}\, q_{i0} = p_{i0} q_{it}^{*}$ etc. in table 8 – see next page -)[26]. This can easily be seen as follows: Assume a swap takes place among j and k, then due to $v_{00}^{j} = v_{tt}^{k} = r_k v_{0t}^{k}$ and $v_{0t}^{j} = r_j^{-1} v_{00}^{k} = r_k v_{00}^{k}$ (hence $r_k = 1/r_j$) we may conclude that $r_j^2 (v_{00}^{j} + v_{0t}^{j}) = r_j^2 r_k (v_{0t}^{k} + v_{00}^{k})$ will not offset the term $r_k^2 (v_{0t}^{k} + v_{00}^{k})$ in a summation as required in eq. 17. On the other hand P^{QM} is unable to satisfy the IT which can be shown with the help of various examples. Reviewing the numerical example portrayed in table 2 we see that

$$P_{0t}^{QM} = \sqrt{1^2 \cdot \frac{72}{126} + \frac{1}{4} \cdot \frac{36}{126} + 2^2 \cdot \frac{18}{126}} = 1.10195 \neq 1,$$

showing that the QM-index in fact fails the inversion test. This may also be verified in the case of table 4, because

$$P_{0t}^{QM} = \left[\left(72 + \frac{1}{4} \cdot 36 + 4 \cdot 18 + \left(\frac{2}{3}\right)^2 162 + \left(\frac{3}{2}\right)^2 108\right) \Big/ 396\right]^{1/2} = 1.08711$$

which again differs from unity whereas for example the Walsh I – type index is unity in both examples (table 2 and table 4) because this index satisfies the IT.

To pull things together we can give examples for all four situations in the following table 7 which is sufficient in order to show that the QT and the IT are in fact independent.

Table 7 Independence of inversion test (IT) and quantity reversal test (QR)

	quantity reversal test (QR)	
inversion test (IT)	satisfied	violated
satisfied	Fisher, Marshall-Edgeworth Walsh I, Walsh II, P^{A} index*	Törnquist, Walsh-Vartia, Theil, Vartia I + II, Stuvel, Banerjee
violated	quadratic mean (QM) index	Laspeyres, Paasche

* irrelevant formula, devised for the sole purpose of finding a formula which passes the inversion test but fails time reversibility.

The following table presents a very simple numerical example. I may be useful in order to check whether a formula complies with the QR or fails this test.

[26] It is easy to verify that in the case of the example of table 8 we get for P^{QM} 1.18577 (the arithmetic mean with the same weights yields 1.0909) in both situations (q- as well as q*-quantities).

Table 8 A two commodity example to demonstrate the quantity reversal test (QR)

i	p_{i0}	q_{i0}	p_{it}	q_{it}	q_{i0}^*	q_{it}^*
1	6	2	4	4	4	2
2	5	4	8	2	2	4

To verify that the QR is violated calculate

formula	with q_0, q_t	with q_{i0}^*, q_{it}^*
Törnquist	1.09088	1.07623
Walsh-Vartia a)	1.09108	1.07857
Vartia I b)	1.09101	1.08420
Vartia II	1.09152	1.08620
Stuvel	1.09813	1.08465 c)
Banerjee	1.09091	1.07843
Theil	1.09152	1.09314 d)

On the other hand we get for example indices satisfying QR

QM-index (eq. 17)	1.18577	1.18577
Walsh I	1.09091 e)	1. 09091 e)
Walsh II	1.09184	1.09184
P^A index (eq. 12)	1.08824	1.08824

a) summation of the weights yields 0.99203 and 0.8608 respectively.
b) weights sum up to 0.99468 or 0.90680 respectively.
c) in this case we get $P^L = Q^L = 32/34$.
e) = 11/12.

5. Final Remark

The fact that so many quite different index formulas, differing enormously with respect to the rationale on which they are built or the (economic) interpretation they provide are equally well suited to fulfil the criteria of the inversion test (IT) does not really lend support to this test. The message of the IT is far from clear and it may be useful to try to find out more interesting implications of the IT, hoping that they may give more insight in the inversion scenario. Another interesting exercise should be a systematic and more thorough study of the relationship between the IT and other tests. We have shown that the IT and two other tests are independent, viz. the time reversal and the quantity reversal test. But this is only a rather rudimentary axiomatic approach to the idea of inversion. We also should give more thought to differences between weighted and unweighted price indices as far as the IT is concerned.

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