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Bouaddi, Mohammed and Belhachemi, Rachid and Douch, Mohamed

American University in Cairo, American University in Cairo, Royal Military College of Canada

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Mohammed Bouaddi,[*] Rachid Belhachemi[†] and Mohamed Douch [‡]

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Abstract

This paper explores a way to construct a new family of univariate probability distributions where the parameters of the distribution capture the dependence between the variable of interest and the continuous latent state variable (the regime). The distribution nests two well known families of distributions, namely, the skew normal family of Azzalini (1985) and a mixture of two Arnold et al. (1993) distribution. We provide a stochastic representation of the distribution which enables the user to easily simulate the data from the underlying distribution using generated uniform and normal variates. We also derive the moment generating function and the moments. The distribution comprises eight free parameters that make it very flexible. This flexibility allows the user to capture many stylized facts about the data such as the regime dependence, the asymmetry and fat tails as well as thin tails.

Keywords: Continuous Hidden threshold, Mixture Distribution, Skew-Symmetric distribution, Split Distribution.

[*]American University in Cairo, Department of Economics, Email: m.bouaddi@aucegypt.edu

[†]American University in Cairo, Department of Mathematics & Actuarial Science, Email: rbelhachemi@aucegypt.edu

[‡]Royal Military College of Canada, Department of Economics, Email: mdouch@mdouch.net

1 Introduction

Over the last three decades, extensive research papers have focused on the construction of asymmetric family of distributions which include the normal distribution as a particular case, that are flexible and able to capture a wide range of skewness and kurtosis relative to the normal distribution. For instance, skewed distributions are particularly useful in modelling empirical stock returns which are known to exhibit negative skewness and excess kurtosis.

Univariate skew-symmetric distributions have been studied by several authors. Azzalini (1985,1986) introduced the skew-normal (SN) distribution as a continuous extension of the normal distribution which accommodates asymmetry. A random variable X has a skew normal distribution with parameter λ if its density function is given by,

$$f(x|\lambda) = 2\phi(x)\Phi(\lambda x),\ x, \lambda \in \mathbf{R},$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote, respectively, the density and distribution functions of the standard normal distribution, and $\Phi(\lambda x)$ is the skewing function. The special case, where $\lambda = 0$, gives the standard normal distribution. Henze (1986) obtained a stochastic representation for the SN and used it to obtain its odd moments. A comprehensive treatment of the skew normal family of distributions can be found in Genton (2004), Arellano-Valle et al. (2004, 2005) and Gupta et al. (2002).

Another setting in which skewed-normal distributions arise is discussed in Arnold et al. (1993, 2000). Authors consider the distribution of the truncated bivariate normal random variable (X, Y) where X is observed and Y is a hidden truncation. The marginal density of X is obtained and the resulting distribution of X is skew-normal. It was also shown that their general family of distributions contains as a special case the skew-normal distribution of Azzalini (1985).

There also exists another type of general method that is used to transform symmetric distribution into a particular mixture. This class consists of truncated mixtures that are known as split distributions or two-piece distributions. This family is presented by Fernandez and Steel (1998) and generalized by Arellano-Valle et al. (2005).[1] Its generalized form is

$$f(x\,|\lambda\,) = \pi(\alpha)\frac{2g\left(\dfrac{x}{f_1(\alpha)}\right)}{f_1(\alpha)}I_{\{x\leqslant 0\}} + (1-\pi(\alpha))\frac{2g\left(\dfrac{x}{f_2(\alpha)}\right)}{f_2(\alpha)}\left(1 - I_{\{x\leqslant 0\}}\right)$$

where α is a real, $\pi(\alpha) = \dfrac{f_1(\alpha)}{f_1(\alpha) + f_2(\alpha)}$, $I_{\{x\leqslant 0\}}$ is an indicator function, $g(.)$ is a symmetric density around the origin in its standard form and $f_1(\alpha)$ and $f_2(\alpha)$ are known positive functions that govern the asymmetry and the behavior in the tails of the distribution.

The aim of this paper is to introduce a new family of univariate probability distributions capable of capturing a wide range of values of skewness and kurtosis. Our eight parameters distribution is a mixture of two asymmetric densities making it more flexible than its competitors. The most important contribution to the literature is the inclusion of a latent state variable with a continuum of states, unlike the traditional mixture distributions where the state variable is discrete with few number of states. This new class of distributions will henceforth be referred to as the hidden-threshold-skew-normal (HTSN).

This paper is outlined as follows. In section 2, the HTSN distribution is introduced. We give the stochastic representation of the proposed family of distributions which allows us to simulate data from HTSN distribution by only generating samples from the uniform and the normal distributions. Moreover, the moments generating function and formulas for centred moments are derived. In section 3, we use the HTSN distribution to model the physical distribution of US market returns and the height of Australian athletes. Our results show that the family of HTSN distributions outperforms the family of Skew-distributions introduced by Arnold et al. (1993) as well as a mixture of two normal and a class of split distributions. Section 4 concludes.

[1] Similar approach can be found in Fernandez et al. (1995), Fernandez and Steel (1998), Mudholkar and Hutson (2000), and Jones (2006).

2 The continuous hidden threshold distribution

2.1 Definition

Definition 1 *The random variable x follows a hidden threshold distribution if its probability density function is defined by*

$$f(x) = \pi \frac{\exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{x1}^2}\right\}}{(2\pi)^{\frac{1}{2}}\sigma_{x1}} \frac{\Phi\left(-\frac{\sigma_{x1}\left(1+\sigma_{x1}^{-2}\sigma_{\tau x1}\right)}{\sqrt{\sigma_{\tau 1}^2-\sigma_{x1}^{-2}\sigma_{\tau x1}^2}}\frac{x-\mu_x}{\sigma_{x1}} - \frac{\mu_\tau-\mu_x}{\sqrt{\sigma_{\tau 1}^2-\sigma_{x1}^{-2}\sigma_{\tau x1}^2}}\right)}{\Phi\left(\frac{\mu_\tau-\mu_x}{\sqrt{\sigma_{\tau 1}^2+\sigma_{x1}^2-2\sigma_{\tau x1}}}\right)}$$
$$+ (1-\pi)\frac{\exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{x2}^2}\right\}}{(2\pi)^{\frac{1}{2}}\sigma_{x2}} \frac{\Phi\left(\frac{\sigma_{x2}\left(1+\sigma_{x2}^{-2}\sigma_{\tau x2}\right)}{\sqrt{\sigma_{\tau 2}^2-\sigma_{x2}^{-2}\sigma_{\tau x2}^2}}\frac{x-\mu_x}{\sigma_{x2}} + \frac{\mu_\tau-\mu_x}{\sqrt{\sigma_{\tau 2}^2-\sigma_{x2}^{-2}\sigma_{\tau x2}^2}}\right)}{\Phi\left(-\frac{\mu_\tau-\mu_x}{\sqrt{\sigma_{\tau 2}^2+\sigma_{x2}^2-2\sigma_{\tau x2}}}\right)}. \quad (1)$$

where μ_x, μ_τ, σ_{x1}, σ_{x2}, $\sigma_{\tau 1}$, $\sigma_{\tau 2}$, $\sigma_{\tau x1}$, $\sigma_{\tau x2}$ are the parameters that govern the location, the scale and the shape of the distribution. The mixing probability is given by

$$\pi = \frac{\Phi\left(\frac{\mu_\tau-\mu_x}{\sqrt{\sigma_{\tau 1}^2+\sigma_{x1}^2-2\sigma_{\tau x1}}}\right)}{\Phi\left(\frac{\mu_\tau-\mu_x}{\sqrt{\sigma_{\tau 1}^2+\sigma_{x1}^2-2\sigma_{\tau x1}}}\right)+\Phi\left(-\frac{\mu_\tau-\mu_x}{\sqrt{\sigma_{\tau 2}^2+\sigma_{x2}^2-2\sigma_{\tau x2}}}\right)}$$

We will show below that the distribution (*1*) is the marginal of x derived from (*3*).

For the derivation see Appendix 1■

If we set $\lambda_{1i} = \frac{\sigma_{xi}\left(1+\sigma_{xi}^{-2}\sigma_{\tau xi}\right)}{\sqrt{\sigma_{\tau i}^2-\sigma_{xi}^{-2}\sigma_{\tau xi}^2}}$, $\lambda_{0i} = \frac{\mu_x-\mu_\tau}{\sqrt{\sigma_{\tau i}^2-\sigma_{xi}^{-2}\sigma_{\tau xi}^2}}$ and $\Delta_i = \frac{\mu_\tau-\mu_x}{\sqrt{\sigma_{\tau i}^2+\sigma_{xi}^2-2\sigma_{\tau xi}}} = \frac{\lambda_{0i}}{\sqrt{1+\lambda_{1i}^2}}$, $(i=1,2)$, then (*1*) becomes a mixture of two Arnold et al. (1993) distribution with mixing probability π, i.e

$$f(x) = \pi \frac{\exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{x1}^2}\right\}}{(2\pi)^{\frac{1}{2}}\sigma_{x1}} \frac{\Phi\left(-\lambda_{01} - \lambda_{11}\frac{x-\mu_x}{\sigma_{x1}}\right)}{\Phi\left(\Delta_1\right)}$$
$$+ (1-\pi) \frac{\exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{x2}^2}\right\}}{(2\pi)^{\frac{1}{2}}\sigma_{x2}} \frac{\Phi\left(\lambda_{02} + \lambda_{12}\frac{x-\mu_x}{\sigma_{x2}}\right)}{\Phi\left(-\Delta_2\right)}. \quad (2)$$

Also if $\mu_x = \mu_\tau = 0$, we get a mixture of two Azzalini's skew-normal distributions with skewness parameters $-\lambda_{11}$ and λ_{12}, scaling parameters σ_{x1} and σ_{x2} and, with mixing probability $\pi = \frac{1}{2}$. If in addition $\sigma_{x1} = \sigma_{x2} = 1$, we have a mixture of two Azzalini's standard skew-normal distributions with skewness parameters $-\frac{1+\sigma_{\tau x1}}{\sqrt{\sigma_{\tau 1}^2 - \sigma_{\tau x1}^2}}$ and $\frac{1+\sigma_{\tau x2}}{\sqrt{\sigma_{\tau 2}^2 - \sigma_{\tau x2}^2}}$. Moreover, if we set $\sigma_{\tau x1} = \sigma_{\tau x2} = 0$ we get a mixture of two Azzalini's skew-normal distributions with location parameter μ_x, scaling parameters σ_{x1} and σ_{x2} and, skewness parameters $-\frac{\sigma_{x1}}{\sigma_{\tau 1}}$ and $\frac{\sigma_{x2}}{\sigma_{\tau 1}}$with mixing probability

$$\pi = \frac{\Phi\left(\frac{\mu_\tau - \mu_x}{\sqrt{\sigma_{\tau 1}^2 + \sigma_{x1}^2}}\right)}{\Phi\left(\frac{\mu_\tau - \mu_x}{\sqrt{\sigma_{\tau 1}^2 + \sigma_{x1}^2}}\right) + \Phi\left(-\frac{\mu_\tau - \mu_x}{\sqrt{\sigma_{\tau 1}^2 + \sigma_{x1}^2}}\right)}.$$

On the other hand if we set $\sigma_{x1}^{-2}\sigma_{\tau x1} = \sigma_{x2}^{-2}\sigma_{\tau x2} = -1$ we get a mixture of two normals with mixing probability

$$\pi = \frac{\Phi\left(\frac{\mu_\tau - \mu_x}{\sqrt{\sigma_{\tau 1}^2 + 3\sigma_{x1}^2}}\right)}{\Phi\left(\frac{\mu_\tau - \mu_x}{\sqrt{\sigma_{\tau 1}^2 + 3\sigma_{x1}^2}}\right) + \Phi\left(-\frac{\mu_\tau - \mu_x}{\sqrt{\sigma_{\tau 2}^2 + 3\sigma_{x2}^2}}\right)}.$$

2.2 Derivation and stochastic representation

2.2.1 Derivation

Consider the bivariate distribution who's density is given by

$$f(Z)=c\left\{\frac{\exp\left(-\frac{(Z-\mu)'\,\Omega_1^{-1}\,(Z-\mu)}{2}\right)}{2\pi\left|\Omega_1\right|^{\frac{1}{2}}}I\right. \tag{3}$$

$$\left.+\frac{\exp\left(-\frac{(Z-\mu)'\,\Omega_2^{-1}\,(Z-\mu)}{2}\right)}{2\pi\left|\Omega_2\right|^{\frac{1}{2}}}(1-I)\right\},$$

where $Z=(x,\tau)'$, x is the observable random variable, τ is a latent random variable i.e., the hidden random threshold, $I=1$ if $x\leqslant\tau$ and $I=0$ otherwise, $\mu=(\mu_x,\mu_\tau)'$ is a vector of location parameters of the distribution, Ω_1 and Ω_2 are 2×2 symmetric and positive definite scaling matrices written as

$$\Omega_i=\begin{pmatrix}\sigma_{xi}^2 & \sigma_{\tau xi}\\ \sigma_{\tau xi} & \sigma_{\tau i}^2\end{pmatrix},\quad i=1,2,$$

and c is the normalizing parameter of the distribution as can be shown in appendix 1 to be,

$$c=\frac{1}{\Phi\left(\frac{\mu_\tau-\mu_x}{\sqrt{\sigma_{\tau 1}^2+\sigma_{x1}^2-2\sigma_{\tau x1}}}\right)+\Phi\left(-\frac{\mu_\tau-\mu_x}{\sqrt{\sigma_{\tau 2}^2+\sigma_{x2}^2-2\sigma_{\tau x2}}}\right)}. \tag{4}$$

The marginal distribution of the observable x is obtained by integrating out the latent variable τ. The following lemma gives the form of this marginal distribution.

Lemma 1 *The marginal distribution of the observable x is given by (1).*

For the Proof see Appendix 1■

From (3) it is clear that the parameters $\sigma_{\tau x1}$ and $\sigma_{\tau x2}$ in (1) capture the dependence between the observable x and the latent regime (threshold) τ in the bad state and the good state, while μ_τ is the location of the threshold. This property of the coefficient makes the distribution (1) more tractable since it departs from the traditional regime switching models in two ways. In one hand, the regime or state variable is a continuous process handling a good updating of the distribution if the regime changes. In the other hand, regimes of the distribution are also identifiable and hence the distribution doesn't suffer from the problem of label switching unlike the case of discrete mixtures.

2.2.2 Stochastic representation

In the proposition below we give a stochastic representation of the distribution (*1*). The proposed simple representation allows for easy simulation of random variables from (*1*).

Proposition 1 (Stochastic representation). *Let* $\lambda_{1i} = \frac{\sigma_{xi}\left(1+\sigma_{xi}^{-2}\sigma_{\tau xi}\right)}{\sqrt{\sigma_{\tau i}^2 - \sigma_{xi}^{-2}\sigma_{\tau xi}^2}}$, $\lambda_{0i} = \frac{\mu_\tau - \mu_x}{\sqrt{\sigma_{\tau i}^2 - \sigma_{xi}^{-2}\sigma_{\tau xi}^2}}$ and $\Delta_i = \frac{\mu_\tau - \mu_x}{\sqrt{\sigma_{\tau i}^2 + \sigma_{xi}^2 - 2\sigma_{\tau xi}}} = \frac{\lambda_{0i}}{\sqrt{1+\lambda_{1i}^2}}$, $(i = 1,2)$*,* u *and* v *be two independent standard normal variables and* η *is uniformly distributed random variable in* $[0,1]$ *independent from* u *and* v*, where* u *is truncated below at* $-\frac{\lambda_{01}}{\sqrt{1+\lambda_{11}^2}}$ *if* $\eta \leqslant \pi$ *and above at* $-\frac{\lambda_{02}}{\sqrt{1+\lambda_{12}^2}}$ *otherwise. In addition, let*

$$z = \begin{cases} \frac{\lambda_{11}}{\sqrt{1+\lambda_{11}^2}}u + \frac{1}{\sqrt{1+\lambda_{11}^2}}v, & \textit{if } \eta \leqslant \pi \\ \frac{\lambda_{12}}{\sqrt{1+\lambda_{12}^2}}u + \frac{1}{\sqrt{1+\lambda_{12}^2}}v, & \text{otherwise,} \end{cases} \tag{5}$$

and

$$x = \begin{cases} \sigma_{x1}z + \mu_x, & \textit{if } \eta \leqslant \pi \\ \sigma_{x2}z + \mu_x, & \text{otherwise,} \end{cases} \tag{6}$$

then x *has a distribution with a density function (1).*

For the Proof see Appendix 2■

2.2.3 The marginal moments of x

The moments of (1) are given in the following proposition.

Proposition 2 Suppose x *has a distribution with a density function (1). Let* $\mu^* = \begin{pmatrix} \mu_x^* \\ \mu_\tau^* \end{pmatrix} = \begin{pmatrix} \mu_x \\ \mu_\tau - \mu_x \end{pmatrix}$,

$\Omega_i^* = \begin{pmatrix} \sigma_{xi}^{*2} & \sigma_{\tau xi}^* \\ \sigma_{\tau xi}^* & \sigma_{\tau i}^{*2} \end{pmatrix} = \begin{pmatrix} \sigma_{xi}^2 & \sigma_{\tau xi} - \sigma_{xi}^2 \\ \sigma_{\tau xi} - \sigma_{xi}^2 & \sigma_{\tau i}^2 + \sigma_{xi}^2 - 2\sigma_{\tau xi} \end{pmatrix}$ *for* $i = 1,2$, and $h_1 = -\frac{\mu_\tau^*}{\sigma_{\tau 1}^*}$ and $h_2 = -\frac{\mu_\tau^*}{\sigma_{\tau 1}^*}$*. The non-central moment of (1) of order* K *is given by,*

$$m_k^K(x) = c(I_1^K(x) + I_2^K(x)), \tag{7}$$

where

$$I_1^K(x) = \sum_{k=0}^{K} \binom{K}{k} \left[\mu_x^* - \sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^*\mu_\tau^* - E(x)\right]^{K-k} \sum_{j=0}^{k} \binom{k}{j} \left(\sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^*\right)^{k-j}$$
$$\left(\sigma_{x1}^{*2} - \sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2}\right)^{\frac{j}{2}} \left(\sigma_{\tau 1}^{*2}\right)^{\frac{k-j}{2}} I_j I_{k-j}^*, \tag{8}$$

and

$$I_2^K(x) = \sum_{k=0}^{K} \binom{K}{k} \left[\mu_x^* - \sigma_{\tau 2}^{*-2}\sigma_{\tau x2}^*\mu_\tau^* - E(x)\right]^{K-k} \sum_{j=0}^{k} \binom{k}{j} \left(\sigma_{\tau 2}^{*-2}\sigma_{\tau x2}^*\right)^{k-j}$$
$$\left(\sigma_{x2}^{*2} - \sigma_{\tau 2}^{*-2}\sigma_{\tau x2}^{*2}\right)^{\frac{j}{2}} \left(\sigma_{\tau 2}^{*2}\right)^{\frac{k-j}{2}} I_j I_{k-j}^{**}. \tag{9}$$

where

$$I_k^* = \frac{1}{\sqrt{2\pi}} 2^{\frac{k-1}{2}} \Gamma(\frac{k+1}{2}) \left\{1 + sign(h_1)(-1)^{kI_{h_1>0}}\gamma(\frac{k+1}{2}, \frac{h_1^2}{2})\right\},$$

and

$$I_k^{**} = \frac{1}{\sqrt{2\pi}} 2^{\frac{k-1}{2}} \Gamma(\frac{k+1}{2}) \left\{(-1)^k + (-1)^{(k+1)I_{h_2>0}}\gamma(\frac{k+1}{2}, \frac{h_2^2}{2})\right\}.$$

For the Proof see Appendix 3■

The four first moments are given in appendix 3.

2.3 The moment generating function and some properties

The moment generating function of (3) is given in the following theorem.

Theorem 1 *The moment generating function of (3) is*

$$M(\theta) = \pi \exp\left(\theta'\mu + \frac{\theta'\Omega_1\theta}{2}\right) \frac{\Phi\left(\frac{\mu^*_{\tau 1}}{\sqrt{\sigma^2_{\tau 1} + \sigma^2_{x1} - 2\sigma_{\tau x1}}}\right)}{\Phi\left(\frac{\mu_\tau - \mu_x}{\sqrt{\sigma^2_{\tau 1} + \sigma^2_{x1} - 2\sigma_{\tau x1}}}\right)} + (1-\pi)\exp\left(\theta'\mu + \frac{\theta'\Omega_2\theta}{2}\right) \frac{\Phi\left(-\frac{\mu^*_{\tau 2}}{\sqrt{\sigma^2_{\tau 2} + \sigma^2_{x2} - 2\sigma_{\tau x2}}}\right)}{\Phi\left(-\frac{\mu_\tau - \mu_x}{\sqrt{\sigma^2_{\tau 2} + \sigma^2_{x2} - 2\sigma_{\tau x2}}}\right)}, \tag{10}$$

where $\mu^*_{\tau i} = \mu_\tau - \mu_x + \theta_x\left(\sigma_{\tau x i} - \sigma^2_{x1}\right) + \theta_\tau\left(\sigma^2_{\tau i} - \sigma_{\tau x i}\right)$ *for* $i = 1, 2$ *and*

$$\pi = \frac{\Phi\left(\frac{\mu_\tau - \mu_x}{\sqrt{\sigma^2_{\tau 1} + \sigma^2_{x1} - 2\sigma_{\tau x1}}}\right)}{\Phi\left(\frac{\mu_\tau - \mu_x}{\sqrt{\sigma^2_{\tau 1} + \sigma^2_{x1} - 2\sigma_{\tau x1}}}\right) + \Phi\left(-\frac{\mu_\tau - \mu_x}{\sqrt{\sigma^2_{\tau 2} + \sigma^2_{x2} - 2\sigma_{\tau x2}}}\right)}. \tag{11}$$

For the Proof see Appendix 4■

The following lemma gives the moment generating function of the marginal density of x in (*1*).

Lemma 2 *The moment generating function of (1) is*

$$M(\theta_x) = \pi \exp\left(\theta_x\mu_x + \frac{\theta_x^2\sigma^2_{x1}}{2}\right) \frac{\Phi\left(\frac{\mu_\tau - \mu_x + \theta_x\left(\sigma_{\tau x1} - \sigma^2_{x1}\right)}{\sqrt{\sigma^2_{\tau 1} + \sigma^2_{x1} - 2\sigma_{\tau x1}}}\right)}{\Phi\left(\frac{\mu_\tau - \mu_x}{\sqrt{\sigma^2_{\tau 1} + \sigma^2_{x1} - 2\sigma_{\tau x1}}}\right)} + (1-\pi)\exp\left(\theta_x\mu_x + \frac{\theta_x^2\sigma^2_{x2}}{2}\right) \frac{\Phi\left(-\frac{\mu_\tau - \mu_x + \theta_x\left(\sigma_{\tau x1} - \sigma^2_{x1}\right)}{\sqrt{\sigma^2_{\tau 2} + \sigma^2_{x2} - 2\sigma_{\tau x2}}}\right)}{\Phi\left(-\frac{\mu_\tau - \mu_x}{\sqrt{\sigma^2_{\tau 2} + \sigma^2_{x2} - 2\sigma_{\tau x2}}}\right)}. \tag{12}$$

For the proof we just set $\theta_\tau = 0$ in (10).

The four first moments are given in appendix 3.

3 Applications

We apply the HTSN to model the physical distribution of two real datasets. The first dataset consists of daily market excess return of the US stock market covering the period, July 1, 1926 to June 28, 2013.[2] The second set concerns the heights (in centimeters) of 100 Australian female athletes available from the Australian Institute of Sport (AIS dataset) extensively used in the literature by Azzalini (1986) and Arellano-Valle et al. (2004).[3]

Summary statistics of the AIS data are given in Table 1 and in Table 6 for the market excess return of the US stock market. These summarizes suggest leptokurtic densities for both examples with negative skewness in all cases. Based on these two sets of data, we estimate the parameters by numerically maximizing the log-likelihood function in ((*1*), with respect to the parameter vector $\theta = (\mu_x, \mu_\tau, \sigma_{x1}, \sigma_{x2}, \sigma_{\tau 1}, \sigma_{\tau 2}, \sigma_{\tau x1}, \sigma_{\tau x2})'$.

For performance purposes, we also estimate three competing distributions namely, the hidden truncation normal, the mixture of two normals and Split-Normal distributions (densities of these mixtures are given bellow).

Hidden truncation normal distribution (HTN)

$$f(x) = \frac{\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\Phi\left(\lambda_1 + \lambda_2\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{\lambda_1}{\sqrt{1+\lambda_2^2}}\right)}$$

where $\phi(.)$ and $\Phi(.)$ are the density and the CDF of the standard normal distribution.

The mixture of two normals (MN)

$$\begin{aligned} f(x) &= \omega\frac{1}{\sqrt{2\pi}\sigma_{x1}}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu_{x1}}{\sigma_{x1}}\right)^2\right\} \\ &\quad + (1-\omega)\frac{1}{\sqrt{2\pi}\sigma_{x2}}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu_{x2}}{\sigma_{x2}}\right)^2\right\} \end{aligned}$$

[2] US stock market returns are available at: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

[3] AIS dataset is available at http://azzalini. stat.unipd.it/SN/index.html.

where

$$0 < \omega < 1$$

Split normal distribution (SN)

$$\begin{aligned} f(x) &= \frac{\sqrt{2}}{\sqrt{\pi}\left(\sigma_{x1}+\sigma_{x2}\right)}\left\{\exp\left[-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_{x1}}\right)^2\right] I_{x<\mu_x} \right. \\ &\left. + \exp\left[-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_{x2}}\right)^2\right]\left(1-I_{x<\mu_x}\right)\right\} \end{aligned}$$

where

$$I_{x<\mu_x} = \begin{cases} 1 & \text{if } x < \mu_x \\ 0 & \text{Otherwise} \end{cases} .$$

Figure 1 shows plots of the HTSN and standard normal distributions for different parameters values. We note that the HTSN distribution nests several density shapes starting from symmetric thin tails distribution to symmetric heavy tailed distributions. It also nests left and right truncated distributions as well as bimodal ones.

Tables $2-5$ show our results using AIS dataset. According to the BIC and AIC information criteria reported at the bottom of Tables $2-5$, we conclude that the HTSN model provides the best fit compared to the other distributions using this dataset. Tables $7-10$ provide results using US market excess returns and show that using the same criteria (BIC and AIC), HTSN outperforms all the three competing distributions. These results are interpreted as strong evidence in favor of the HTSN distribution.

4 Conclusion

In this paper we propose a new family of distributions which we referred to as hidden-threshold-skew-normal (HTSN). The most important contribution to the literature is the inclusion of a latent state variable with a continuum of states unlike the traditional mixture distributions where the state variable is discrete with few number of states. The new family of distributions is regime dependent. The distribution contains eight parameters which makes it more flexible than its competitors. A wide range of shapes of HTSN are obtained. The distribution has a mixture interpretation. The information criteria shows that the HTSN distribution outperforms all the proposed competitors, including the split normal, the hidden truncation normal and the mixture of two normals with different location and scale parameters.

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Table 1: Descriptive Statistics for the AIS dataset

Sample Mean	Standard Deviation	Skewness	Kurtosis
174.594	8.242203	-0.56838	4.321205

Table 2: Parameter Estimates for the AIS dataset under HTSN

Parameter	$\hat{\mu}_x$	$\hat{\mu}_\tau$	$\hat{\sigma}_{x1}$	$\hat{\sigma}_{\tau 1}$	$\hat{\sigma}_{\tau x1}$	$\hat{\sigma}_{x2}$	$\hat{\sigma}_{\tau 2}$	$\hat{\sigma}_{\tau x2}$
Value	175.05	126.884	9.7715	3.4259	33.4762	3.7504	79946364.067	46.8781
Std Error	0.7189	0.3919	0.0206	$1.59E-007$	102771045.02	0.0009	4789.7174	0.0220
T-Stat	243.495	323.76	474.157	21592144.26	0.0000	4118.08	16691.2486	2131.909
BIC					478.3629			
AIC					477.9945			

Table 3: Parameter Estimates for the AIS dataset under GSN

Parameter	$\hat{\mu}_x$	$\hat{\sigma}_x$	$\hat{\lambda}_1$	$\hat{\lambda}_2$
Value	170.3203	9.2476	4.3800	24.1762
Std Error	0.8791	0.7225	0.4668	5.2475
T-Stat	194.0065	12.7992	9.3839	4.6072
BIC		833.1166		
AIC		833.1079		

Table 4: Parameter Estimates for the AIS dataset under MN

Parameter	$\hat{\mu}_{x1}$	$\hat{\mu}_{x2}$	$\hat{\sigma}_{x1}$	$\hat{\sigma}_{x2}$	$\hat{\omega}$
Value	174.7213	173.4616	8.6327	1.1015	0.8989
Std Error	0.9156	0.7853	0.6467	1.0793	0.0840
T-Stat	190.8349	220.8884	13.3481	1.0206	10.7053
BIC			701.2059		
AIC			700.9756		

Table 5: Parameter Estimates for the AIS dataset under SN

Parameter	$\hat{\mu}_x$	$\hat{\sigma}_{x1}$	$\hat{\sigma}_{x2}$
Value	177.0219	9.6983	6.4635
Std Error	1.5008	1.1226	0.9623
T-Stat	117.9524	8.6391	6.7166
BIC		701.8271	
AIC		701.6889	

Table 6: Descriptive Statistics for the US Market Excess Returns dataset

Sample Mean	Standard Deviation	Skewness	Kurtosis
2.7923	0.7058	−0.1333	16.6038

Table 7: Parameter Estimates for the US Market Excess Returns dataset under HTSN

Parameter	$\hat{\mu}_x$	$\hat{\mu}_\tau$	$\hat{\sigma}_{x1}$	$\hat{\sigma}_{\tau 1}$	$\hat{\sigma}_{\tau x1}$	$\hat{\sigma}_{x2}$	$\hat{\sigma}_{\tau 2}$	$\hat{\sigma}_{\tau x2}$
Value	7.6478	292.9419	175.2484	713281766.5463	13780521889.8638	51.2789	5.6377	49.0678
Std Error	0.2296	4.0687	1.3518	7995462.8944	150963852.2547	0.3902	0.0916	3.3258
T-Stat	33.3131	71.9986	129.6367	89.2108	91.2836	131.4016	61.5686	14.7535
BIC	222244.2065							
AIC	222244.2030							

Table 8: Parameter Estimates for the US Market Excess Returns dataset under HTN

Parameter	$\hat{\mu}_x$	$\hat{\sigma}_x$	$\hat{\lambda}_1$	$\hat{\lambda}_2$
Value	84.1870	134.4716	−4.2493	7.7426
Std Error	0.8194	0.7625	0.1383	0.6249
T-Stat	102.7455	176.3528	−31.5166	12.3897
BIC	307354.7683			
AIC	307354.7683			

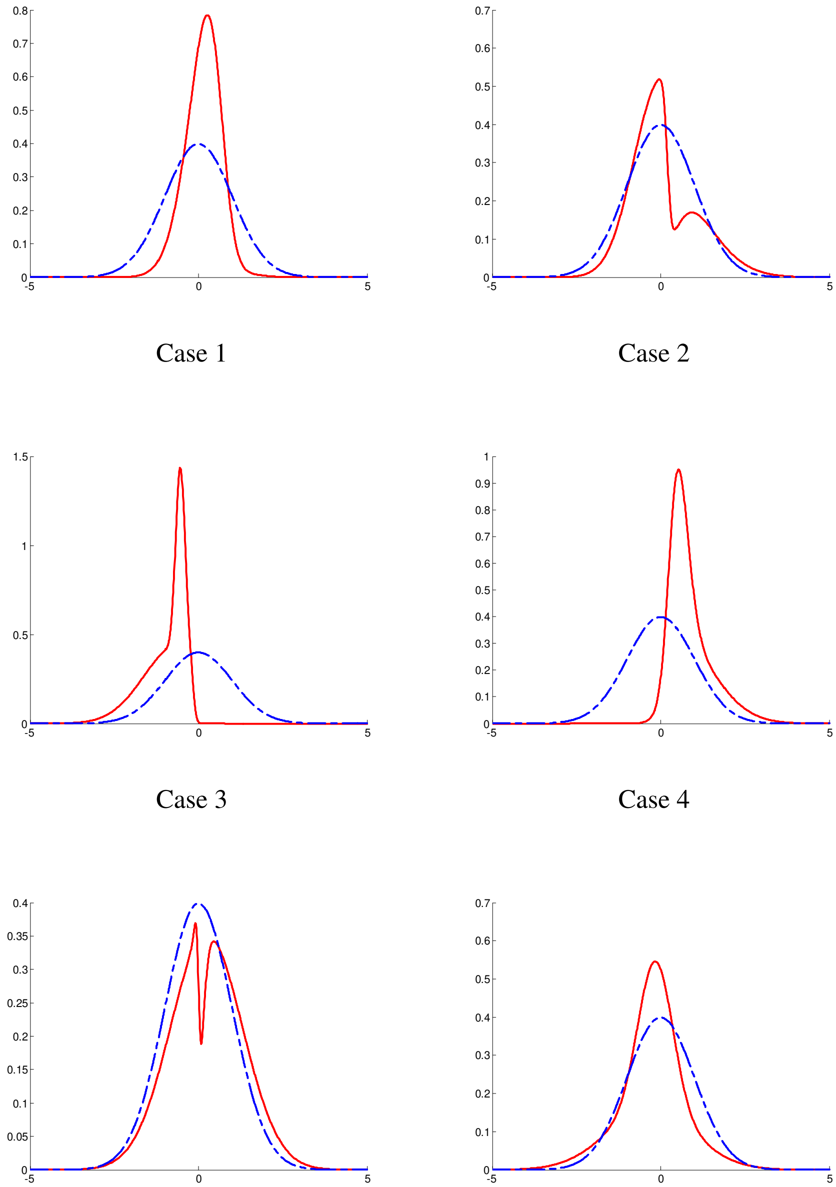
Table 9: Parameter Estimates for the US Market Excess Returns dataset under MN

Parameter	$\hat{\mu}_{x1}$	$\hat{\mu}_{x2}$	$\hat{\sigma}_{x1}$	$\hat{\sigma}_{x2}$	$\hat{\omega}$
Value	−13.3069	6.3517	214.9819	60.8539	0.1810
Std Error	3.4828	0.5519	3.6510	0.6482	0.0069
T-Stat	−3.8207	11.5093	58.8833	93.8883	26.2996
BIC	272464.6654				
AIC	272464.6632				

Table 10: Parameter Estimates for the US Market Excess Returns dataset under SN

Parameter	$\hat{\mu}_x$	$\hat{\sigma}_{x1}$	$\hat{\sigma}_{x2}$
Value	6.6882	0.6408	10.4370
Std Error	110.0249	0.3207	343.0441
T-Stat	103.9369	0.6855	151.6261
BIC	280274.0558		
AIC	280274.0545		

Figure 1: HTSN Distribution plots in particular cases



Examples of density shapes using different parameters values. Dashed line represents Standard Normal density and the solid line is for the HTSN.

Appendix 1

Consider $f(x)$ as the probability density function of a random variable x, we have that

$$\int_{-\infty}^{+\infty} f(x)dx = c\left(I_1 + I_2\right), \tag{13}$$

where

$$\begin{cases} I_1 = \displaystyle\int_{x}^{+\infty} \frac{1}{2\pi\left|\Omega_1\right|^{\frac{1}{2}}} \exp\left\{-\frac{(Z-\mu)'\,\Omega_1^{-1}\,(Z-\mu)}{2}\right\} d\tau \\ I_2 = \displaystyle\int_{-\infty}^{x} \frac{1}{2\pi\left|\Omega_2\right|^{\frac{1}{2}}} \exp\left\{-\frac{(Z-\mu)'\,\Omega_2^{-1}\,(Z-\mu)}{2}\right\} d\tau \end{cases}, \tag{14}$$

and

$$c = \frac{1}{\Phi\left(\frac{\mu_\tau - \mu_x}{\sqrt{\sigma_{\tau 1}^2 + \sigma_{x1}^2 - 2\sigma_{\tau x1}}}\right) + \Phi\left(-\frac{\mu_\tau - \mu_x}{\sqrt{\sigma_{\tau 2}^2 + \sigma_{x2}^2 - 2\sigma_{\tau x2}}}\right)}.$$

Let $Y = Z - \mu$ and

$$\Omega_1 = \begin{pmatrix} \sigma_{\tau 1}^2 & \sigma_{\tau x1} \\ \sigma_{\tau x1} & \sigma_{x1}^2 \end{pmatrix} \tag{15}$$

then,

$$Y\Omega_1^{-1}Y = \omega_{11}y_\tau^2 + 2\omega_{12}y_x y_\tau + \omega_{22}y_x. \tag{16}$$

Noting that

$$\Omega_1^{-1}\Omega_1 = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{pmatrix} \begin{pmatrix} \sigma_{\tau 1}^2 & \sigma_{\tau x1} \\ \sigma_{\tau x1} & \sigma_{x1}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{17}$$

then

$$\omega_{11}\sigma_{\tau 1}^2 + \omega_{12}\sigma_{\tau x1} = 1, \tag{18}$$

$$\omega_{11}\sigma_{\tau x1} + \omega_{12}\sigma_{x1}^2 = 0, \tag{19}$$

$$\omega_{12}\sigma_{\tau 1}^2 + \omega_{22}\sigma_{\tau x1} = 0, \tag{20}$$

and

$$\omega_{12}\sigma_{\tau x1} + \omega_{22}\sigma_{x1}^2 = 1, \tag{21}$$

from (18-21) we deduce

$$\omega_{12} = -\sigma_{x1}^{-2}\omega_{11}\sigma_{\tau x1}, \tag{22}$$

$$\omega_{22} = \sigma_{x1}^{-2}\left(1 - \omega_{12}\sigma_{\tau x1}\right), \tag{23}$$

and

$$\begin{aligned}\omega_{11} &= \sigma_{\tau 1}^{-2}\left(1 - \omega_{12}\sigma_{\tau x1}\right) \\ &= \sigma_{\tau 1}^{-2}\left(1 + \sigma_{x1}^{-2}\omega_{11}\sigma_{\tau x1}^{2}\right) \\ &= \left(\sigma_{\tau 1}^{2} - \sigma_{x1}^{-2}\sigma_{\tau x1}^{2}\right)^{-1}.\end{aligned} \tag{24}$$

Thus

$$Y\Omega_1^{-1}Y = \omega_{11}y_\tau^2 + 2\omega_{12}y_x y_\tau + \omega_{22}y_x. \tag{25}$$

$$\begin{aligned}Y'\Omega_1^{-1}Y &= \omega_{11}y_\tau^2 + 2\omega_{12}y_x y_\tau + \frac{y_x^2}{\sigma_{x1}^2} \\ &+ \omega_{12}^2\omega_{11}^{-1}y_x^2 \\ &= \frac{y_x^2}{\sigma_{x1}^2} + \omega_{11}\left(y_\tau + \varpi\right)^2 \\ &= \frac{y_x^2}{\sigma_{x1}^2} + \frac{\left(y_\tau + \varpi\right)^2}{\left(\sigma_{\tau 1}^2 - \sigma_{x1}^{-2}\sigma_{\tau x1}^2\right)},\end{aligned} \tag{26}$$

where $\varpi = (\omega_{12}y_x)/\omega_{11} = -\sigma_{x1}^{-2}\sigma_{\tau x1}y_x = -\sigma_{x1}^{-2}\sigma_{\tau x1}\left(x - \mu_x\right)$. This implies that,

$$\begin{aligned}I_1 &= \int_x^{+\infty}\frac{1}{2\pi\left|\Omega_1\right|^{\frac{1}{2}}}\exp\left\{-\frac{\left(Z-\mu\right)'\Omega_1^{-1}\left(Z-\mu\right)}{2}\right\}d\tau \\ &= \frac{1}{(2\pi)^{\frac{1}{2}}\left|\Omega_1\right|^{\frac{1}{2}}}\exp\left\{-\frac{\left(x-\mu_x\right)^2}{2\sigma_{x1}^2}\right\}\int_x^{+\infty}\frac{1}{(2\pi)^{\frac{1}{2}}}\exp\left\{-\frac{\left(\tau-\mu_\tau+\varpi\right)^2}{2\left(\sigma_{\tau 1}^2 - \sigma_{x1}^{-2}\sigma_{\tau x1}^2\right)}\right\}d\tau,\end{aligned}$$

where $u=\dfrac{\tau-\mu_\tau+\varpi}{\left(\sigma_{\tau 1}^2-\sigma_{x1}^{-2}\sigma_{\tau x1}^2\right)^{\frac{1}{2}}}$. It follows that

$$I_1=\frac{\left(\sigma_{\tau 1}^2-\sigma_{x1}^{-2}\sigma_{\tau x1}^2\right)^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}\left|\Omega_1\right|^{\frac{1}{2}}}\exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{x1}^2}\right\}\int_{\frac{x-\mu_\tau+\varpi}{\sqrt{\sigma_{\tau 1}^2-\sigma_{x1}^{-2}\sigma_{\tau x1}^2}}}^{+\infty}\frac{1}{(2\pi)^{\frac{1}{2}}}\exp\left\{-\frac{u^2}{2}\right\}du \tag{27}$$

$$=\frac{\exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{x1}^2}\right\}}{\sqrt{2\pi}\sigma_{x1}}\Phi\left(-\frac{x-\mu_\tau+\varpi}{\left(\sigma_{\tau 1}^2-\sigma_{x1}^{-2}\sigma_{\tau x1}^2\right)^{\frac{1}{2}}}\right) \tag{28}$$

$$=\frac{\exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{x1}^2}\right\}}{(2\pi)^{\frac{1}{2}}\sigma_{x1}}\Phi\left(-\lambda_{01}-\lambda_{11}\frac{x-\mu_x}{\sigma_{x1}}\right). \tag{29}$$

Similarly, we have

$$I_2=\int_{-\infty}^{x}\frac{1}{2\pi\left|\Omega_2\right|^{\frac{1}{2}}}\exp\left\{-\frac{(Z-\mu)'\,\Omega_2^{-1}\,(Z-\mu)}{2}\right\}d\tau \tag{30}$$

$$=\frac{\exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{x2}^2}\right\}}{(2\pi)^{\frac{1}{2}}\sigma_{x2}}\Phi\left(\lambda_{02}+\lambda_{12}\frac{x-\mu_x}{\sigma_{x2}}\right), \tag{31}$$

where $\lambda_{1i}=\dfrac{\sigma_{xi}\left(1+\sigma_{xi}^{-2}\sigma_{\tau xi}\right)}{\sqrt{\sigma_{\tau i}^2-\sigma_{xi}^{-2}\sigma_{\tau xi}^2}}$, $\lambda_{0i}=\dfrac{\mu_x-\mu_\tau}{\sqrt{\sigma_{\tau i}^2-\sigma_{xi}^{-2}\sigma_{\tau xi}^2}}$ and the result follows.

End of the proof■

Appendix 2 Proof of proposition 1

Let $\lambda_{1i}=\dfrac{\sigma_{xi}\left(1+\sigma_{xi}^{-2}\sigma_{\tau xi}\right)}{\sqrt{\sigma_{\tau i}^2-\sigma_{xi}^{-2}\sigma_{\tau xi}^2}}$, $\lambda_{0i}=\dfrac{\mu_x-\mu_\tau}{\sqrt{\sigma_{\tau i}^2-\sigma_{xi}^{-2}\sigma_{\tau xi}^2}}$ and $\Delta_i=\dfrac{\mu_\tau-\mu_x}{\sqrt{\sigma_{\tau i}^2+\sigma_{xi}^2-2\sigma_{\tau xi}}}$ $(i=1,2)$, u and v be two independent standard normal variables and let η be a uniformly distributed random variable in $[0,1]$ independent from u and v, where u is truncated below at $\frac{\lambda_{01}}{\sqrt{1+\lambda_{11}^2}}$ if $\eta\leqslant\pi$ and above at $\frac{\lambda_{02}}{\sqrt{1+\lambda_{12}^2}}$ otherwise, in addition let

$$z=-\frac{\lambda_{1i}}{\sqrt{1+\lambda_{1i}^2}}u+\frac{1}{\sqrt{1+\lambda_{1i}^2}}v$$

then the joint density of u and v given that $\eta\leqslant\pi$ is,

$$f\left(u,v\,|\eta\leqslant\pi\right)=\frac{1}{2\pi}\frac{\exp\left(-\frac{v^2}{2}-\frac{u^2}{2}\right)}{\Phi\left(\Delta_1\right)}I_{u\geqslant\frac{\lambda_{01}}{\sqrt{1+\lambda_{11}^2}}}$$

and

$$v = \sqrt{1+\lambda_{11}^2} z + \lambda_{11} u$$

$$\begin{aligned} f\left(z, u \left| \eta \leqslant \pi \right.\right) &= \frac{\sqrt{1+\lambda_{11}^2}}{2\pi\Phi\left(\Delta_1\right)} \exp\left(-\frac{\left(\sqrt{1+\lambda_{11}^2} z + \lambda_{11} u\right)^2}{2} - \frac{u^2}{2} \right) I_{u \geqslant \frac{\lambda_{01}}{\sqrt{1+\lambda_{11}^2}}} \\ &= \frac{\sqrt{1+\lambda_{11}^2}}{2\pi\Phi\left(\Delta_1\right)} \exp\left(-\frac{z^2}{2}\right) \exp\left(-\frac{\left(1+\lambda_{11}^2\right)\left(u + \frac{\lambda_{11}}{\sqrt{1+\lambda_{11}^2}} z\right)^2}{2} \right) I_{u \geqslant \frac{\lambda_{01}}{\sqrt{1+\lambda_{11}^2}}} \end{aligned}$$

the marginal density of z is

$$f\left(z \left| \eta \leqslant \pi \right.\right) = \frac{\sqrt{1+\lambda_{11}^2}}{2\pi\Phi\left(\Delta_1\right)} \exp\left(-\frac{z^2}{2}\right) \int_{\frac{\lambda_{01}}{\sqrt{1+\lambda_{11}^2}}}^{+\infty} \exp\left(-\frac{\left(1+\lambda_{11}^2\right)\left(u + \frac{\lambda_{11}}{\sqrt{1+\lambda_{11}^2}} z\right)^2}{2} \right) du.$$

Let $h = \sqrt{1+\lambda_{11}^2}\left(u + \frac{\lambda_{11}}{\sqrt{1+\lambda_{11}^2}} z\right)$ then,

$$\begin{aligned} f\left(z \left| \eta \leqslant \pi \right.\right) &= \frac{1}{\sqrt{2\pi}\Phi\left(\Delta_1\right)} \exp\left(-\frac{z^2}{2}\right) \int_{\lambda_{01}+\lambda_{11} z}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{h^2}{2}\right) dh \\ &= \frac{1}{\sqrt{2\pi}\Phi\left(\Delta_1\right)} \exp\left(-\frac{z^2}{2}\right) \Phi\left(-\lambda_{01} - \lambda_{11} z\right) \end{aligned}$$

Let $x = \sigma_{x1} z + \mu_x$ then

$$f\left(x \left| \eta \leqslant \pi \right.\right) = \frac{1}{\sqrt{2\pi}\sigma_{x1}} \exp\left(-\frac{\left(x-\mu_x\right)^2}{2\sigma_{x1}^2}\right) \frac{\Phi\left(-\lambda_{01} - \lambda_{11}\frac{x-\mu_x}{\sigma_{x1}}\right)}{\Phi\left(\Delta_1\right)}$$

where

$$\Phi\left(-\lambda_{01} - \lambda_{11} z\right) = \int_{\lambda_{01}+\lambda_{11} z}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{h^2}{2}\right) dh.$$

We can similarly show that,

$$f\left(x \left| \eta > \pi \right.\right) = \frac{1}{\sqrt{2\pi}\sigma_{x2}} \exp\left(-\frac{\left(x-\mu_x\right)^2}{2\sigma_{x2}^2}\right) \frac{\Phi\left(\lambda_{02} + \lambda_{12}\frac{x-\mu_x}{\sigma_{x2}}\right)}{\Phi\left(-\Delta_2\right)}$$

then it follows that

$$
\begin{aligned}
f(x) &= \pi f\left(x\left|\eta \leqslant \pi\right.\right)+(1-\pi) f\left(x\left|\eta>\pi\right.\right) \\
&= \pi \frac{1}{\sqrt{2\pi}\sigma_{x1}} \exp\left(-\frac{(x-\mu_x)^2}{2\sigma_{x1}^2}\right) \frac{\Phi\left(-\lambda_{01}-\lambda_{11}\dfrac{x-\mu_x}{\sigma_{x1}}\right)}{\Phi\left(\Delta_1\right)} \\
&\quad +(1-\pi)\frac{1}{\sqrt{2\pi}\sigma_{x2}} \exp\left(-\frac{(x-\mu_x)^2}{2\sigma_{x2}^2}\right) \frac{\Phi\left(\lambda_{02}+\lambda_{12}\dfrac{x-\mu_x}{\sigma_{x2}}\right)}{\Phi\left(-\Delta_2\right)}
\end{aligned}
$$

which is just the density function given by (1).

End of the proof■

Appendix 3

The central moment of order K of (1)

$$
\begin{aligned}
m^K(x) &= \int [x-E(x)]^K f(Z|_{\mu,\Omega_1,\Omega_2,})dZ, \\
&= c\int_{-\infty}^{+\infty}\int_{x}^{+\infty} [x-E(x)]^K \frac{\exp\left\{-\dfrac{(Z-\mu)'\,\Omega_1^{-1}\,(Z-\mu)}{2}\right\}}{2\pi\left|\Omega_1\right|^{\frac{1}{2}}} d\tau dx \\
&+ c\int_{-\infty}^{+\infty}\int_{-\infty}^{x} [x-E(x)]^K \frac{\exp\left\{-\dfrac{(Z-\mu)'\,\Omega_2^{-1}\,(Z-\mu)}{2}\right\}}{2\pi\left|\Omega_2\right|^{\frac{1}{2}}} d\tau dx, \\
&= c(I_1^K(x)+I_2^K(x)),
\end{aligned} \tag{32}
$$

$$
I_1^K(x) = \int_{-\infty}^{+\infty}\int_{x}^{+\infty} [x-E(x)]^K \frac{\exp\left\{-\dfrac{(Z-\mu)'\,\Omega_1^{-1}\,(Z-\mu)}{2}\right\}}{2\pi\left|\Omega_1\right|^{\frac{1}{2}}} d\tau dx, \tag{33}
$$

$$
Y = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} Z. \tag{34}
$$

$$
Z = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} Y. \tag{35}
$$

$$I_1^K(x) = \int_{-\infty}^{+\infty} \int_0^{+\infty} [y_x - E(x)]^K \frac{\exp\left\{-\frac{(\Upsilon Y - \mu)' \Omega_1^{-1} (\Upsilon Y - \mu)}{2}\right\}}{2\pi |\Omega_1|^{\frac{1}{2}}} dy_\tau dy_x \tag{36}$$

$$= \int_{-\infty}^{+\infty} \int_0^{+\infty} [y_x - E(x)]^K \frac{\exp\left\{-\frac{(Y - \mu^*)' \Omega_1^{*-1} (\Upsilon Y - \mu^*)}{2}\right\}}{2\pi |\Omega_1|^{\frac{1}{2}}} dy_\tau dy_x, \tag{37}$$

where

$$\Upsilon = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \tag{38}$$

$$Z = \Upsilon Y, \tag{39}$$

$$\mu^* = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mu = \begin{pmatrix} \mu_x \\ \mu_\tau - \mu_x \end{pmatrix}, \tag{40}$$

$$\Omega_1^* = \begin{pmatrix} \sigma_{xi}^{*2} & \sigma_{\tau xi}^* \\ \sigma_{\tau xi}^* & \sigma_{\tau i}^{*2} \end{pmatrix}$$

$$= \left(\Upsilon' \Omega_1^{-1} \Upsilon\right)^{-1} \tag{41}$$

$$= \Upsilon^{-1} \Omega_1 \Upsilon'^{-1}$$

$$= \begin{pmatrix} \sigma_{x1}^2 & \sigma_{\tau x1} - \sigma_{x1}^2 \\ \sigma_{\tau x1} - \sigma_{x1}^2 & \sigma_{\tau 1}^2 + \sigma_{x1}^2 - 2\sigma_{\tau x1} \end{pmatrix}, \tag{42}$$

$$(\Upsilon Y - \mu)' \Omega_1^{-1} (\Upsilon Y - \mu) = (Y - \mu^*)' \Omega_1^{*-1} (Y - \mu^*)$$

$$= \frac{(y_\tau - \mu_\tau^*)^2}{\sigma_{\tau 1}^{*2}} + \frac{(y_x - \mu_x^* + \overline{\omega})^2}{\sigma_{x1}^{*2} - \sigma_{\tau 1}^{*-2} \sigma_{\tau x1}^{*2}}, \tag{43}$$

where $\overline{\omega} = -\sigma_{\tau 1}^{*-2} \sigma_{\tau x1}^* y_\tau$. It follows that

$$I_1^K(x) = \int_{-\infty}^{+\infty} \int_0^{+\infty} [y_x - E(x)]^K \frac{\exp\left\{-\frac{(y_\tau - \mu_\tau^*)^2}{2\sigma_{\tau 1}^{*2}}\right\} \exp\left\{-\frac{(y_x - \mu_x^* + \overline{\omega})^2}{2\left(\sigma_{x1}^{*2} - \sigma_{\tau 1}^{*-2} \sigma_{\tau x1}^{*2}\right)}\right\}}{2\pi |\Omega_1|^{\frac{1}{2}}} dy_\tau dy_x. \tag{44}$$

Let $h_x = y_x - \mu_x^*$ and $h_\tau = y_\tau - \mu_\tau^*$ then $\overline{\omega} = -\sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^* \left(h_\tau + \mu_\tau^*\right)$ and

$$
\begin{aligned}
I_1^K(x) &= \int_{-\infty}^{+\infty}\int_{-\mu_\tau^*}^{+\infty} \left[h_x + \mu_x^* - E(x)\right]^K \frac{\exp\left\{-\dfrac{h_\tau^2}{2\sigma_{\tau 1}^{*2}}\right\}\exp\left\{-\dfrac{\left(h_x+\overline{\omega}\right)^2}{2\left(\sigma_{x1}^{*2} - \sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2}\right)}\right\}}{2\pi\left|\Omega_1\right|^{\frac{1}{2}}} dh_\tau dh_x \\
&= \sum_{k=0}^{K}\binom{K}{k}\left[\mu_x^* - E(x)\right]^{K-k}\int_{-\infty}^{+\infty}\int_{-\mu_\tau^*}^{+\infty} h_x^k \frac{\exp\left\{-\dfrac{h_\tau^2}{2\sigma_{\tau 1}^{*2}}\right\}\exp\left\{-\dfrac{\left(h_x+\overline{\omega}\right)^2}{2\left(\sigma_{x1}^{*2} - \sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2}\right)}\right\}}{2\pi\left|\Omega_1\right|^{\frac{1}{2}}} dh_\tau dh_x,
\end{aligned}
\tag{45}
$$

Now let $u_x = h_x + \overline{\omega}$ and $u_\tau = h_\tau$ then

$$
\begin{aligned}
I_1^K(x) = \frac{1}{\left|\Omega_1\right|^{\frac{1}{2}}}&\sum_{k=0}^{K}\binom{K}{k}\left[\mu_x^* - E(x)\right]^{K-k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\left(\sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^*\right)^{k-j} \\
&\int_{-\infty}^{+\infty} u_x^j \frac{\exp\left\{-\dfrac{u_x^2}{2\left(\sigma_{x1}^{*2} - \sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2}\right)}\right\}}{\sqrt{2\pi}} du_x \int_{-\mu_\tau^*}^{+\infty} u_\tau^{k-j}\frac{\exp\left\{-\dfrac{u_\tau^2}{2\sigma_{\tau 1}^{*2}}\right\}}{\sqrt{2\pi}} du_\tau,
\end{aligned}
\tag{46}
$$

Let $v_x = \dfrac{u_x}{\sqrt{\sigma_{x1}^{*2} - \sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2}}}$ and $v_\tau = \dfrac{u_\tau}{\sigma_{\tau 1}^*}$ then

$$
\begin{aligned}
I_1^K(x) &= \frac{1}{\left|\Omega_1\right|^{\frac{1}{2}}}\sum_{k=0}^{K}\binom{K}{k}\left[\mu_x^* - E(x)\right]^{K-k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\left(\sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^*\right)^{k-j} \\
&\left(\sigma_{x1}^{*2} - \sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2}\right)^{\frac{j+1}{2}}\left(\sigma_{\tau 1}^{*2}\right)^{\frac{k-j+1}{2}}\int_{-\infty}^{+\infty} v_x^j\frac{\exp\left\{-\dfrac{v_x^2}{2}\right\}}{\sqrt{2\pi}} dv_x \int_{-\frac{\mu_\tau^*}{\sigma_{\tau 1}^*}}^{+\infty} v_\tau^{k-j}\frac{\exp\left\{-\dfrac{v_\tau^2}{2}\right\}}{\sqrt{2\pi}} dv_\tau \\
&= \frac{\left(\sigma_{x1}^{*2} - \sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2}\right)^{\frac{1}{2}}\left(\sigma_{\tau 1}^{*2}\right)^{\frac{1}{2}}}{\left|\Omega_1\right|^{\frac{1}{2}}}\sum_{k=0}^{K}\binom{K}{k}\left[\mu_x^* - E(x)\right]^{K-k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\left(\sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^*\right)^{k-j} \\
&\left(\sigma_{x1}^{*2} - \sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2}\right)^{\frac{j}{2}}\left(\sigma_{\tau 1}^{*2}\right)^{\frac{k-j}{2}} I_j I_{k-j}^*,
\end{aligned}
\tag{47}
$$

where

$$
I_j = \int_{-\infty}^{+\infty} v_x^j \frac{\exp\left\{-\dfrac{v_x^2}{2}\right\}}{\sqrt{2\pi}} dv_x.
\tag{48}
$$

and

$$I_k^* = \int_{-\frac{\mu_\tau^*}{\sigma_{\tau 1}^*}}^{+\infty} v^k \frac{\exp\left\{-\frac{v^2}{2}\right\}}{\sqrt{2\pi}} dv \tag{49}$$

$$= \int_0^{+\infty} v^k \frac{\exp\left\{-\frac{v^2}{2}\right\}}{\sqrt{2\pi}} dv + \int_{-\frac{\mu_\tau^*}{\sigma_{\tau 1}^*}}^{0} v_\tau^k \frac{\exp\left\{-\frac{v^2}{2}\right\}}{\sqrt{2\pi}} dv \tag{50}$$

From the standard normal properties, we have

$$I_j = \begin{cases} 0, & \text{if } j \text{ is odd} \\ \frac{j!}{\left(\frac{j}{2}\right)! 2^{\frac{j}{2}}}, & \text{if } j \text{ is even} \end{cases} . \tag{51}$$

Let $y = \frac{v^2}{2}$ then from the first part of (49) we have,

$$\int_0^{+\infty} v^k \frac{\exp\left\{-\frac{v^2}{2}\right\}}{\sqrt{2\pi}} dv = \frac{2^{\frac{k-1}{2}}}{\sqrt{2\pi}} \int_0^{+\infty} y^{\frac{k-1}{2}} \exp(-y) dy$$

$$= \frac{2^{\frac{k-1}{2}}}{\sqrt{2\pi}} \Gamma\left(\frac{k+1}{2}\right)$$

where

$$\Gamma(a) = \int_0^{+\infty} y^{a-1} \exp(-y) dy.$$

The second part of (49) where we make use of $y = \frac{v^2}{2}$, to get

$$\int_{-\frac{\mu_\tau^*}{\sigma_{\tau 1}^*}}^{0} v_\tau^k \frac{\exp\left\{-\frac{v^2}{2}\right\}}{\sqrt{2\pi}} dv = \frac{2^{\frac{k-1}{2}}}{\sqrt{2\pi}} \begin{cases} -\int_0^{\frac{h_1^2}{2}} y^{\frac{k-1}{2}} \exp(-y) dy & \text{if } h_1 < 0 \\ (-1)^k \int_0^{\frac{h_1^2}{2}} y^{\frac{k-1}{2}} \exp(-y) dy & \text{otherwise} \end{cases}$$

$$= \frac{2^{\frac{k-1}{2}}}{\sqrt{2\pi}} \Gamma\left(\frac{k+1}{2}\right) sign(h_1) (-1)^{k I_{h_1 > 0}} \gamma\left(\frac{k+1}{2}, \frac{h_1^2}{2}\right)$$

where $h_1 = -\frac{\mu_\tau^*}{\sigma_{\tau 1}^*}$ and

$$\gamma(a,x)=\frac{\int_0^x y^{a-1}\exp(-y)dy}{\Gamma(a)}.$$

Therefore,

$$I_k^*=\frac{1}{\sqrt{2\pi}}2^{\frac{k-1}{2}}\Gamma(\frac{k+1}{2})\left\{1+sign(h_1)(-1)^{kI_{h_1>0}}\gamma(\frac{k+1}{2},\frac{h_1^2}{2})\right\}.$$

It can be shown, using the same procedure that,

$$I_k^{**}=\frac{1}{\sqrt{2\pi}}2^{\frac{k-1}{2}}\Gamma(\frac{k+1}{2})\left\{(-1)^k+(-1)^{(k+1)I_{h_2>0}}\gamma(\frac{k+1}{2},\frac{h_2^2}{2})\right\}.$$

Using (48), (**??**) and the fact that $|\Omega_1|=|\Omega_1^*|=\left(\sigma_{x1}^{*2}-\sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2}\right)^{\frac{1}{2}}\left(\sigma_{\tau 1}^{*2}\right)^{\frac{1}{2}}$, we get

$$I_1^K(x)=\sum_{k=0}^{K}\binom{K}{k}\left[\mu_x^*-E(x)\right]^{K-k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\left(\sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*}\right)^{k-j}$$
$$\left(\sigma_{x1}^{*2}-\sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2}\right)^{\frac{j}{2}}\left(\sigma_{\tau 1}^{*2}\right)^{\frac{k-j}{2}}I_j I_{k-j}^*. \quad (52)$$

Similarly we can show that

$$I_2^K(x)=\sum_{k=0}^{K}\binom{K}{k}\left[\mu_x^*-E(x)\right]^{K-k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\left(\sigma_{\tau 2}^{*-2}\sigma_{\tau x2}^{*}\right)^{k-j}$$
$$\left(\sigma_{x2}^{*2}-\sigma_{\tau 2}^{*-2}\sigma_{\tau x2}^{*2}\right)^{\frac{j}{2}}\left(\sigma_{\tau 2}^{*2}\right)^{\frac{k-j}{2}}I_j I_{k-j}^{**}. \quad (53)$$

It follows that the four moments are given by,

$$E(x)=\mu=\mu_x^*-c(\sigma_{\tau 1}^{*-1}\sigma_{\tau x1}^{*}I_1^*+\sigma_{\tau 2}^{*-1}\sigma_{\tau x2}^{*}I_1^{**})$$

$$\begin{aligned}
var(x) &= m^2(x)=E\left((x-\mu)^2\right)\\
&= \left(\mu_x^*-\mu\right)^2-c[2\left(\mu_x^*-\mu\right)\left(\sigma_{\tau 1}^{*-1}\sigma_{\tau x1}^{*}I_1^*+\sigma_{\tau 2}^{*-1}\sigma_{\tau x2}^{*}I_1^{**}\right)-(\sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2}I_2^*+\sigma_{\tau 2}^{*-2}\sigma_{\tau x2}^{*2}I_2^{**})\\
&\quad -(\sigma_{x1}^{*2}-\sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2})I_2I_0^*-\left(\sigma_{x2}^{*2}-\sigma_{\tau 2}^{*-2}\sigma_{\tau x2}^{*2}\right)I_2I_0^{**}]
\end{aligned}$$

$$
\begin{aligned}
m^3(x) &= E\left((x-\mu)^3\right) \\
&= (\mu_x^*-\mu)^3 - c[3(\mu_x^*-\mu)^2(\sigma_{\tau 1}^{*-1}\sigma_{\tau x1}^* I_1^* + \sigma_{\tau 2}^{*-1}\sigma_{\tau x2}^* I_1^{**}) \\
&\quad -3(\mu_x^*-\mu)(\sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2} I_2^* + \sigma_{\tau 2}^{*-2}\sigma_{\tau x2}^{*2} I_2^{**}) - (\sigma_{x1}^{*2} - \sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2}) I_2 I_0^* - \left(\sigma_{x2}^{*2} - \sigma_{\tau 2}^{*-2}\sigma_{\tau x2}^{*2}\right) I_2 I_0^{**} \\
&\quad +(\sigma_{\tau 1}^{*-3}\sigma_{\tau x1}^{*3} I_3^* + \sigma_{\tau 2}^{*-3}\sigma_{\tau x2}^{*3} I_3^{**}) + \sigma_{\tau 1}^{*-1}\sigma_{\tau x1}^*(\sigma_{x1}^{*2} - \sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2}) I_2 I_1^* + \sigma_{\tau 2}^{*-1}\sigma_{\tau x2}^*\left(\sigma_{x2}^{*2} - \sigma_{\tau 2}^{*-2}\sigma_{\tau x2}^{*2}\right) I_2 I_1^{**}]
\end{aligned}
$$

$$
\begin{aligned}
m^4(x) &= E\left((x-\mu)^4\right) \\
&= (\mu_x-\mu)^4 - c[4(\mu_x^*-\mu)^3(\sigma_{\tau 1}^{*-1}\sigma_{\tau x1}^* I_1^* + \sigma_{\tau 2}^{*-1}\sigma_{\tau x2}^* I_1^{**}) \\
&\quad -6(\mu_x^*-\mu)^2[\sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2} I_2^* + \sigma_{\tau 2}^{*-2}\sigma_{\tau x2}^{*2} I_2^{**} + (\sigma_{x1}^{*2} - \sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2}) I_2 I_0^* + \left(\sigma_{x2}^{*2} - \sigma_{\tau 2}^{*-2}\sigma_{\tau x2}^{*2}\right) I_2 I_0^{**}] \\
&\quad +4(\mu_x^*-\mu)[\sigma_{\tau 1}^{*-3}\sigma_{\tau x1}^{*3} I_3^* + \sigma_{\tau 2}^{*-3}\sigma_{\tau x2}^{*3} I_3^{**} + 3\sigma_{\tau 1}^{*-1}\sigma_{\tau x1}^*(\sigma_{x1}^{*2} - \sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2}) I_2 I_1^* + 3\sigma_{\tau 2}^{*-1}\sigma_{\tau x2}^*\left(\sigma_{x2}^{*2} - \sigma_{\tau 2}^{*-2}\sigma_{\tau x2}^{*2}\right) \\
&\quad I_2 I_1^{**}] - \sigma_{\tau 1}^{*-4}\sigma_{\tau x1}^{*4} I_4^* - \sigma_{\tau 2}^{*-4}\sigma_{\tau x2}^{*4} I_4^{**} - 6\sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2}(\sigma_{x1}^{*2} - \sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2}) I_2 I_2^* - 6\sigma_{\tau 2}^{*-2}\sigma_{\tau x2}^{*2}\left(\sigma_{x2}^{*2} - \sigma_{\tau 2}^{*-2}\sigma_{\tau x2}^{*2}\right) I_2 I_2^{**} \\
&\quad -(\sigma_{x1}^{*2} - \sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2})^2 I_4 I_0^* - \left(\sigma_{x2}^{*2} - \sigma_{\tau 2}^{*-2}\sigma_{\tau x2}^{*2}\right)^2 I_4 I_0^{**}
\end{aligned}
$$

where,

$$
\begin{aligned}
I_0^* &= \frac{1}{2}\left\{1 + sign(h_1)\gamma(\frac{1}{2}, \frac{h_1^2}{2})\right\}, \\
I_1^* &= \frac{1}{\sqrt{2\pi}}\left\{1 + sign(h_1)(-1)^{I_{h_1>0}}\gamma(1, \frac{h_1^2}{2})\right\}, \\
I_2^* &= \frac{1}{2}\left\{1 + sign(h_1)\gamma(\frac{3}{2}, \frac{h_1^2}{2})\right\}, && (54) \\
I_3^* &= \frac{2}{\sqrt{2\pi}}\left\{1 + sign(h_1)(-1)^{3I_{h_1>0}}\gamma(2, \frac{h_1^2}{2})\right\} && (55) \\
I_4^* &= \frac{3}{2}\left\{1 + sign(h_1)\gamma(\frac{5}{2}, \frac{h_1^2}{2})\right\} && (56)
\end{aligned}
$$

and

$$I_0^{**} = \frac{1}{2}\left\{1 + (-1)^{I_{h_1>0}}\gamma(\frac{1}{2}, \frac{h_2^2}{2})\right\},$$

$$I_1^{**} = \frac{1}{\sqrt{2\pi}}\left\{-1 + \gamma(1, \frac{h_2^2}{2})\right\},$$

$$I_2^{**} = \frac{1}{2}\left\{1 + (-1)^{3I_{h_1>0}}\gamma(\frac{3}{2}, \frac{h_2^2}{2})\right\}, \tag{57}$$

$$I_3^{**} = \frac{2}{\sqrt{2\pi}}\left\{-1 + \gamma(2, \frac{h_2^2}{2})\right\}, \tag{58}$$

$$I_4^{**} = \frac{3}{2}\left\{1 + (-1)^{5I_{h_1>0}}\gamma(\frac{5}{2}, \frac{h_2^2}{2})\right\}. \tag{59}$$

End of the proof■

Appendix 4

The moment generating function of (3) is given by

$$\begin{aligned} M(\theta) &= \int \exp(\theta' Z) f(Z|_{\mu,\Omega_1,\Omega_2,}) dZ, \\ &= c\int_{-\infty}^{+\infty}\int_{x}^{+\infty} \exp(\theta' Z)\frac{\exp\left\{-\frac{(Z-\mu)'\,\Omega_1^{-1}\,(Z-\mu)}{2}\right\}}{2\pi\,|\Omega_1|^{\frac{1}{2}}} d\tau dx \\ &+ c\int_{-\infty}^{+\infty}\int_{-\infty}^{x} \exp(\theta' Z)\frac{\exp\left\{-\frac{(Z-\mu)'\,\Omega_2^{-1}\,(Z-\mu)}{2}\right\}}{2\pi\,|\Omega_2|^{\frac{1}{2}}} d\tau dx, \\ &= c(I_1(\theta) + I_{,2}(\theta)), \end{aligned} \tag{60}$$

where

$$\left\{ \begin{array}{l} I_1(\theta) = \int_{-\infty}^{+\infty}\int_{x}^{+\infty} \exp(\theta' Z)\frac{\exp\left\{-\frac{(Z-\mu)'\,\Omega_1^{-1}\,(Z-\mu)}{2}\right\}}{2\pi\,|\Omega_1|^{\frac{1}{2}}} d\tau dx \\ I_2(\theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{x} \exp(\theta' Z)\frac{\exp\left\{-\frac{(Z-\mu)'\,\Omega_2^{-1}\,(Z-\mu)}{2}\right\}}{2\pi\,|\Omega_2|^{\frac{1}{2}}} d\tau dx \end{array} \right. . \tag{61}$$

and $\theta = (\theta_\tau, \theta_x')'$.

Now let

$$\Upsilon = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{62}$$

$$Z = \Upsilon Y. \tag{63}$$

then,

$$Y = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} Z, \tag{64}$$

$$\begin{aligned} \mu^* &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} (\mu + \Omega_1 \theta) \\ &= \begin{pmatrix} \mu_x + \theta_x \sigma_{x1}^2 + \theta_\tau \sigma_{\tau x1} \\ \mu_\tau - \mu_x + \theta_x \left(\sigma_{\tau x1} - \sigma_{x1}^2\right) + \theta_\tau \left(\sigma_{\tau 1}^2 - \sigma_{\tau x1}\right) \end{pmatrix}, \end{aligned} \tag{65}$$

$$\begin{aligned} \Omega_1^* &= \left(\Upsilon' \Omega_1^{-1} \Upsilon\right)^{-1} \\ &= \Upsilon^{-1} \Omega_1 \Upsilon'^{-1} \\ &= \begin{pmatrix} \sigma_{x1}^2 & \sigma_{\tau x1} - \sigma_{x1}^2 \\ \sigma_{\tau x1} - \sigma_{x1}^2 & \sigma_{\tau 1}^2 + \sigma_{x1}^2 - 2\sigma_{\tau x1} \end{pmatrix}, \end{aligned} \tag{66}$$

$$\begin{aligned} (\Upsilon Y - \mu)' \Omega_1^{-1} (\Upsilon Y - \mu) &= (\Upsilon Y - \mu - \Omega_1 \theta)' \Omega_1^{-1} (\Upsilon Y - \mu - \Omega_1 \theta) \\ &- 2\theta' \mu - \theta' \Omega_1 \theta + 2\theta' \Upsilon Y \\ &= (Y - \mu^*)' \Omega_1^{*-1} (Y - \mu^*) - 2\theta' \mu \\ &- \theta' \Omega_1 \theta + 2\theta' \Upsilon Y \\ &= \frac{(y_{\tau+1} - \mu_\tau^*)^2}{\sigma_{\tau 1}^{*2}} + \frac{(y_{x+1} - \mu_x^* + \overline{\omega})^2}{\sigma_{x1}^{*2} - \sigma_{\tau 1}^{*-2} \sigma_{\tau x1}^{*2}} - 2\theta' \mu \\ &- \theta' \Omega_1 \theta + 2\theta' \Upsilon Y, \end{aligned} \tag{67}$$

and

$$
\begin{aligned}
I_1(\theta) &= \int_{-\infty}^{+\infty}\int_{x}^{+\infty} \exp(\theta' Z) \frac{\exp\left\{-\dfrac{(Z-\mu)'\,\Omega_1^{-1}\,(Z-\mu)}{2}\right\}}{2\pi\left|\Omega_1\right|^{\frac{1}{2}}} d\tau dx \\
&= \int_{-\infty}^{+\infty}\int_{0}^{+\infty} \frac{\exp\left\{\theta'\Upsilon Y - \dfrac{(\Upsilon Y-\mu)'\,\Omega_1^{-1}\,(\Upsilon Y-\mu)}{2}\right\}}{2\pi\left|\Omega_1\right|^{\frac{1}{2}}} dy_{\tau+1}dy_{x+1} \\
&= \exp\left(\theta'\mu + \frac{\theta'\Omega_1\theta}{2}\right)\int_{-\infty}^{+\infty}\int_{0}^{+\infty} \frac{\exp\left\{-\dfrac{\left(y_{\tau+1}-\mu_\tau^*\right)^2}{2\sigma_{\tau 1}^{*2}} - \dfrac{\left(y_{x+1}-\mu_x^*+\varpi\right)^2}{2\left(\sigma_{x1}^{*2}-\sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*2}\right)}\right\}}{2\pi\left|\Omega_1\right|^{\frac{1}{2}}} dy_{\tau+1}dy_{x+1} \\
&= \exp\left(\theta'\mu + \frac{\theta'\Omega_1\theta}{2}\right)\Phi\left(\frac{\mu_\tau^*}{\sigma_{\tau 1}^*}\right). \qquad (68)
\end{aligned}
$$

where $\varpi = -\sigma_{\tau 1}^{*-2}\sigma_{\tau x1}^{*}y_{\tau+1}$, $\mu_\tau^* = \mu_\tau - \mu_x + \theta_x\left(\sigma_{\tau x1} - \sigma_{x1}^2\right) + \theta_\tau\left(\sigma_{\tau 1}^2 - \sigma_{\tau x1}\right)$ and $\sigma_{\tau 1}^* = \sqrt{\sigma_{\tau 1}^2 + \sigma_{x1}^2 - 2\sigma_{\tau x1}}$. The last equality follows from the fact that $\left|\Omega_1\right| = \left|\Omega_1^*\right|$ and from the properties of the normal distribution and where $\Phi\left(.\right)$ is the cumulative of the standard normal distribution.

Similarly, it can be shown that

$$
I_2(\theta) = \exp\left(\theta'\mu + \frac{\theta'\Omega_2\theta}{2}\right)\Phi\left(-\frac{\mu_\tau^*}{\sigma_{\tau 2}^*}\right), \qquad (69)
$$

where $\sigma_{\tau 2}^* = \sqrt{\sigma_{\tau 2}^2 + \sigma_{x2}^2 - 2\sigma_{\tau x2}}$.

Therefore, the moment generating function is given by

$$
M(\theta) = c\left\{\exp\left(\theta'\mu + \frac{\theta'\Omega_1\theta}{2}\right)\Phi\left(\frac{\mu_\tau^*}{\sigma_{\tau 1}^*}\right) + \exp\left(\theta'\mu + \frac{\theta'\Omega_2\theta}{2}\right)\Phi\left(-\frac{\mu_\tau^*}{\sigma_{\tau 2}^*}\right)\right\}, \qquad (70)
$$

By the properties of the moment generating function we have

$$
M(\theta = 0) = c(I_1(\theta = 0) + I_2(\theta = 0)) = 1, \qquad (71)
$$

where

$$
I_1(\theta = 0) = \Phi\left(\frac{\mu_\tau - \mu_x}{\sqrt{\sigma_{\tau 1}^2 + \sigma_{x1}^2 - 2\sigma_{\tau x1}}}\right), \qquad (72)
$$

and

$$
I_2(\theta = 0) = \Phi\left(-\frac{\mu_\tau - \mu_x}{\sqrt{\sigma_{\tau 2}^2 + \sigma_{x2}^2 - 2\sigma_{\tau x2}}}\right). \qquad (73)
$$

Thus

$$c = \frac{1}{\Phi\left(\frac{\mu_\tau - \mu_x}{\sqrt{\sigma_{\tau 1}^2 + \sigma_{x1}^2 - 2\sigma_{\tau x1}}}\right) + \Phi\left(-\frac{\mu_\tau - \mu_x}{\sqrt{\sigma_{\tau 2}^2 + \sigma_{x2}^2 - 2\sigma_{\tau x2}}}\right)}. \tag{74}$$

End of the proof■