The Welfare Economics of Tactical Voting in Democracies: A Partial Identification Equilibrium Analysis

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The Welfare Economics of Tactical Voting in Democracies: A Partial Identification Equilibrium Analysis

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Abstract

The fact that voters can manipulate election outcomes by misrepresenting their true preferences over competing political parties or candidates is commonly viewed as a major flaw of democratic voting systems. It is argued that insincere voting typically leads to suboptimal voting outcomes. However, it is also understood that insincere voting is rational behavior as it may result in the election of a candidate preferred by the voter to the candidate who would otherwise be selected. The relative magnitude of the welfare gains and losses of those who benefit from and those adversely affected by insincere voting behavior is consequently an important empirical issue. We address this question by providing exact asymptotic bounds on the welfare effects, in equilibrium, of insincere voting for an infinite class of democratic rules. We find, for instance, that preference manipulation benefits one-half to two-thirds of the population in three-candidate elections held under first-past-the-post, and one-third to one-hundred percent of the population in antiplurality elections. These bounds differ from those obtained under out-of-equilibrium manipulation. Our partial identification analysis provides a novel approach to evaluating mechanisms as a function of attitude towards risk, and it has practical implications for the choice of election rules by a mechanism designer facing a worst-case or a best-case objective. It also provides a new answer to the longstanding question of why certain rules, such as first-past-the-post, are more common in practice.

Keywords: Democracy, tactical voting, political equilibrium, social welfare, mechanism design, worst-case-scenario, best-case-scenario, partial identification.

JEL codes: D60, D72, D81, H41, P48

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1 Introduction

In tightly contested elections, voters whose true preferences are best reflected by candidates with no realistic hope of winning are commonly entreated to vote for a candidate who has a better chance of success, and whom they prefer to the candidate who would win if they voted sincerely. Some scholars have argued that such strategic or tactical voting may lead to the selection of a bad candidate, whereas others take the view that this is rational behavior and may lead to a socially preferred election outcome. The measurement of the social welfare gains or losses resulting from insincere voting is consequently an empirically relevant question and is as yet unresolved. We contribute to this debate by providing exact asymptotic bounds on the *equilibrium welfare effects* of tactical voting for an infinite class of democratic rules. Our results have practical implications for the choice of election rules by a mechanism designer facing a worst-case or a best-case objective, and offer a new rationale for why certain rules— such as first-past-the-post (or simple plurality)— are more common in practice than others.

The notion that, under most voting rules, voters can sometimes achieve a more preferred voting outcome for themselves by misrepresenting their preferences when casting their ballots is perceived as a major weakness of democratic systems. Charles L. Dodgson complained that strategic behavior "makes an election more a game of skill than a real test of wishes of electors" (Black, 1958, p. 232). The underlying concern is that insincere voting will result in welfare-inferior outcomes. Summarizing this view, Barberà (2011) writes:

> "As for the consequences of manipulation, if they occur, there may be many, but the possible loss of efficiency is particularly worrisome from the point of view of the designer. Social choice functions that would always select an efficient outcome if voters provide truthful information may end up recommending an inefficient alternative after voters distort their preferences in order to manipulate." (p. 739).

The view that preference misrepresentation results in social welfare loss is widespread, but attempts to establish this view formally have proven difficult. Individuals who engage in manipulation are acting in their self-interest, and expect to gain from doing so (e.g., Ehlers et al. (2004), Schummer (2004), Campbell and Kelly (2009), Moyouwou and Pongou (2012), Carroll (2013)). Insincere voting is therefore likely to lead to outcomes which increase the utility of certain voters, but to be harmful to others as compared to the outcome that would be selected if each individual expressed his true preferences (Gibbard (1973)). The main difficulty in studying the social welfare consequences of insincere voting therefore stems from it potentially having both positive and negative effects, rendering the alternative outcomes Pareto non-comparable.

In the absence of a particular specification of the social-welfare function, or individual utility functions, it is unrealistic to expect that much can be said about whether or not strategic manipulation of preferences under the most widely used voting procedures will definitely lead to outcomes which are welfare-inferior - or, alternatively, welfare-superior - to the outcomes that would prevail under sincere voting. We consider intensity or positional voting rules, which constitute an infinite class of democratic rules. Importantly, this class includes well-known rules such as simple plurality, antiplurality (a particular form of approval voting), and the Borda rule. Under these rules, assuming that an election involves three competing political candidates, each voter submits a ballot consisting of his ranking of the candidates, the first-ranked candidate receiving one point, the second receiving λ points ($0 \leq \lambda \leq 1$), and the third receiving 0 points; the candidate who receives the most points wins. Under simple plurality, λ is 0, which means that each voter votes only for one candidate, and the candidate who receives the most votes wins. Under antiplurality ($\lambda = 1$), each voter votes against one candidate, and the candidate with the fewest negative votes wins. Unlike these two rules, the traditional Borda rule ($\lambda = \frac{1}{2}$) requires that each voter fully rank the candidates. Obviously, our focus on these rules is justified by the fact that they are the most widely used in elections around the world. In particular, simple plurality is used in most countries to select presidents, legislators, and mayors. It follows that our findings about the welfare consequences of preference misrepresentation in elections held under these rules have testable implications.

For each of these democratic rules, we compare two different behavioral scenarios. In the first, citizens vote strategically, possibly misrepresenting their preferences in equilibrium. In the "counterfactual" scenario, they vote sincerely.[1] The outcomes of these two scenarios are then compared for each individual, enabling us to calculate the proportion of voters who benefit or lose when the insincere voting outcome emerges as the winner as opposed to the honest outcome. This way to measure the welfare effect of voting manipulation is natural under our assumption that voters have ordinal utility.[2]

The following example illustrates our purpose. Consider an election involving seven voters 1-7 and three political parties a, b and c. Voters 1 to 3 prefer a to b to c; voters 4 and 5 prefer b to c to a, and voters 6 and 7 prefer c to b to a. If all voters vote sincerely, and the electoral rule is first-past-the-post, then party a will be elected, as it will attract three votes, which is more than the two votes for each of b and c. In contrast, if voters 4 and 5 vote strategically, and cast their ballots for c rather than b, then c will prevail, which is an outcome preferred by voters 4 through 7, but which makes voters 1 through 3 worse off. It is immediately evident that the outcome under strategic voting is Pareto non-comparable to the outcome under sincere voting, as 57% of voters are better off if the strategic outcome wins compared to 43% under the sincere outcome.

This example leads to the following important questions which we answer in this paper:

1. How does the proportion of voters who benefit or lose from manipulation *in equilibrium* vary depending on voters true preferences and the voting rule? What are the limits to these effects under each rule?

2. What are the practical implications for the choice of an election rule by a social planner or a mechanism designer?

The analysis below addresses these issues. We show that the equilibrium welfare effects of prefer-

[1] These two scenarios are natural (e.g., Barberà (2011)).

[2] In general, the measurement of social welfare depends on whether or not utility is assumed to be comparable across individuals. If interpersonal utility comparison is allowed, welfare is usually measured by a weighted sum of individual utilities. The welfare effect of voting manipulation would then be obtained by subtracting aggregate utility under the strategic winner from aggregate utility under the sincere winner. Our ordinal utility assumption, which is also the assumption made in the classical papers of Gibbard (1973) and Sattherwaite (1975), clearly precludes such an approach. Vickrey (1960) and several subsequent scholars have highlighted the difficulty of using cardinal utility in the evaluation of social welfare. The ordinal approach followed by our analysis also has the advantage of imposing very little structure on preferences.

ence misrepresentation depend both on voters' true preferences and the voting rule. This naturally implies that the determination of these effects for each rule should follow a *partial identification* approach, as the effects vary across elections depending on voters' possibly unknown preferences. Indeed, we show that it is possible to obtain sharp upper and lower bounds for the proportion of voters who — in equilibrium — benefit from and, conversely, who are adversely affected by the strategic manipulation of individual preferences. For some rules, strategic misrepresentation may benefit the majority of voters, and under a number of different social welfare functions it could then be expected that the equilibrium outcome with strategic voting would be socially-preferred to the equilibrium outcome with sincere voting. From a policy perspective, and given that the choice is between electoral mechanisms that are known to be subject to strategic manipulation, there might therefore be reason to prefer using a mechanism which seems more likely to deliver *better* outcomes when citizens vote strategically, rather than one where the expectation is that strategic voting will lead to an aggregate loss in social welfare. In this sense, our analysis fits into a new research agenda on mechanism design. Whereas the traditional agenda has been concerned with the construction of mechanisms that minimize manipulation (see Serrano (2004) for an excellent literature review), the new agenda investigates mechanisms that guarantee good outcomes in equilibrium, even if these mechanisms are susceptible to manipulation (see, e.g., Bergemann and Morris (2005), Chung and Ely (2007), Barberà (2011)).

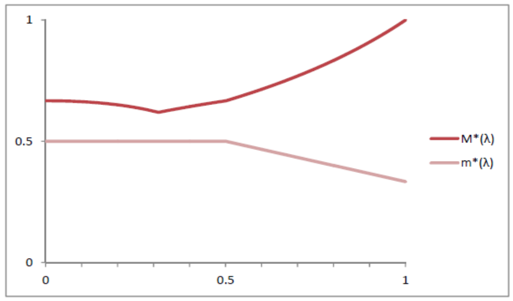
1.1 Equilibrium Welfare Gains and Losses from Manipulation

For each of the democratic rules analyzed in this study, we determine the minimum and maximum proportion of voters who gain or lose from a manipulation leading to a strong Nash equilibrium. In a static political game like the one investigated in our analysis, there are several reasons for preferring this notion of equilibrium to the notion of Nash equilibrium. From a theoretical viewpoint, the notion of strong Nash equilibrium is a useful refinement of Nash equilibrium.[3] The notion of strong Nash equilibrium (henceforth equilibrium) also more realistically describes voters' behavior in modern societies because it is based on the premise that voters have the ability to act either individually or coalitionally as is generally observed in ethnic societies or in contexts where a leader may act as a political coordinator by instructing his followers to vote in a certain manner (Posner (2004, 2005), Fish (2008), Ishiyama (2012), Diffo Lambo et al. (2015)). In such contexts, members of a group generally vote as a bloc as opposed to spreading their votes among several competing parties (Ishiyama (2012)).

For three-party elections, the minimum and maximum proportion of voters who gain or lose from manipulation in equilibrium are functions of the political rule λ and the number of voters n. For sufficiently large electorates (that is, as n goes to infinity), these proportions, denoted respectively by $m^*(\lambda)$ and $M^*(\lambda)$, are explicitly derived and presented in Figure 1.

[3] The concept of Nash equilibrium is often criticized because its predictions are sometimes unrealistic in politics. For instance, in a plurality election involving any number of voters greater than two and two competing candidates a and b, and where each voter prefers a to b, the strategy profile in which each voter casts a ballot for b instead of a is a Nash equilibrium, but it is not a strong Nash equilibrium. The more realistic voting profile in which everybody votes for a is a strong Nash equilibrium and hence a Nash equilibrium.

Figure 1: Maximum and Minimum Gains from Manipulation in Equilibrium

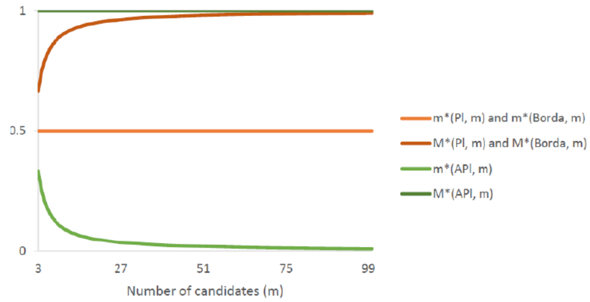


The proportion of voters who benefit from the strategic voting equilibrium outcome winning at the expense of the sincere voting outcome belongs to the interval $[\frac{1}{2}, \frac{2-2\lambda-\lambda^2}{3-3\lambda}]$ if $0 \leq \lambda \leq \lambda^* \approx 0.312$, $[\frac{1}{2}, \frac{1+2\lambda}{2+2\lambda}]$ if $\lambda^* \leq \lambda \leq \frac{1}{2}$, and $[\frac{2-\lambda}{3}, \frac{1}{2-\lambda}]$ if $\frac{1}{2} < \lambda \leq 1$. It follows that manipulation benefits one-half to two-thirds of the population under simple plurality ($\lambda = 0$) and the traditional Borda rule ($\lambda = \frac{1}{2}$), and one-third to the one-hundred percent of the population under antiplurality ($\lambda = 1$). It follows from these results that it is only under antiplurality that manipulation can lead to a Pareto improvement.

We also quantify the minimum and maximum proportions of voters who lose from manipulation in equilibrium. The share of voters who lose from manipulation ranges from $1-M^*(\lambda)$ to $1-m^*(\lambda)$, as, for each rule λ, the proportion of voters positively affected by manipulation attains the bounds $m^*(\lambda)$ and $M^*(\lambda)$. In particular, it follows that manipulation hurts one-third to one-half of the population under simple plurality and the Borda rule, and zero percent to two-thirds of the population under antiplurality.

Although elections involving at most three major political parties are highly prevalent in real-life politics, we generalize our findings to elections involving any number of competing candidates. Here, we consider simple plurality, the Borda rule, and antiplurality. For each of these rules, we are able to derive exact bounds on the equilibrium welfare effects of manipulation as a function of the number of voters and the number of candidates, where the number of candidates is at least equal to three. In particular, for each of these rules, the exact asymptotic bounds are represented in Figure 2. In this figure, $M^*(F, m)$ and $m^*(F, m)$ are respectively is the maximum and the minimum proportions of voters who benefit from manipulation in a large m-candidate election held under the rule $F \in \{Pl, APl, Borda\}$, where Pl is for plurality, $Borda$ for the traditional Borda rule, and APl is for antiplurality. For a large enough electorate, we find that, in equilibrium, manipulation benefits from half to a fraction of $1 - \frac{1}{m}$ of the population under simple plurality and Borda, and from a fraction of $\frac{1}{m}$ of the electorate to the entire population under antiplurality. It follows that, whereas the minimum welfare gain from manipulation is insensitive to the level of political competition under first-past-the-post and Borda, it goes to zero under antiplurality as political competition increases.

Figure2: Maximum and Minimum Gains from Manipulation ($m \geq 3$)



We argue that our partial identification analysis offers practical advice to a social planner faced with the task of choosing an election rule for a society when the planner is ignorant of the preferences of its members. Two main approaches are generally followed in order to compare the performance of rules or mechanisms, with each approach making a different assumption on the attitude of the social planner towards risk. The first approach is the "worst-case" approach, in which the social planner assumes that manipulation will always lead to the worst possible outcome. His goal is then to choose the rule that minimizes this negative effect. This approach has been followed in several important papers on mechanism designs (see, e.g., Hurwicz and Shapiro (1978), Segal (2003), Bergemann and Morris (2005), Chung and Ely (2007), Carroll (2013)). The second approach is the "best-case" approach, in which the social planner, considering the uncertain welfare effect of manipulation, assumes that manipulation will always lead to the best possible outcome for the society. This approach makes an assumption that is similar to the optimistic behavioral assumption proposed in Greenberg (1996). It is however not very popular in the mechanism design literature. We also consider a third approach which combines these two classical approaches under a lexicographic ordering (see below).

Under the best-case approach, the social planner will choose the rule that yields the highest possible gain from manipulation. In the class of rules we analyze in this paper, antiplurality is the only rule under which manipulation can benefit everybody (see Figure 1 for $\lambda = 1$ and Figure 2), and so is the only rule that satisfies the best-case objective in equilibrium. Under the worst-case approach, the social planner chooses the rule that minimizes the maximum loss due to manipulation. Within the class of rules we analyze, the plurality rule, the Borda rule, and all the rules λ such that $0 < \lambda < \frac{1}{2}$ meet this objective. Our third approach combines the best-case and the worst-case approaches. Under this approach, the social planner chooses among the rules that minimize the maximum loss due to manipulation the rule that leads to the highest possible gain from manipulation. In our context, only simple plurality ($\lambda = 0$) and the Borda count ($\lambda = \frac{1}{2}$) satisfy these requirements. Our analysis therefore offers a new rationale for why these two rules are far more common than others in real-life politics and organizations.[4] Interestingly, our third approach to evaluating voting rules reveals that the rule $\lambda^* \approx 0.312$ should never be used, as it is the worst-performing rule in three-party elections.

[4] As already noted, simple plurality is used in most countries to select leaders. A trivial justification for the popularity of this rule in large elections is that it minimizes voting cost, as choosing one candidate is less costly than ranking all the candidates, especially when they are many. Our analysis shows that simple plurality may also be justified on the ground that it minimizes welfare loss from manipulation.

1.2 Contributions to the Closely Related Literature

Following the canonical contributions of Vickrey (1960) and Dummett and Farquharson (1961) who argued that any reasonable voting mechanism is manipulable, and the classical works of Gibbard (1973) and Sattherwaite (1975) who formally established this conjecture for any non-dictatorial deterministic mechanism, a number of studies have drawn attention to the potential loss of efficiency that manipulation can engender (see, e.g., Mas-Colell et al. (1995), Sonnenschein (1998), Serrano (2004), Barberà (2011)). However, the longstanding question of whether or not, *in equilibrium*, voting manipulation in a democracy is likely to benefit more individuals than are harmed had not been answered. Also, the question of the exact number of individuals who gain or lose from manipulation leading to a political equilibrium had not been answered. To our knowledge, this paper is the first to address these questions. Our analysis further provides a novel approach to evaluating voting mechanisms based on the quantification of the equilibrium welfare gains and losses that manipulation can cause under each mechanism. An interesting feature of our analysis is the assumption that agents have ordinal utility. We therefore impose very little structure on utility, and so the sharp bounds on the welfare externality of manipulation that we derive for each rule do not depend on a particular functional form. We also note that, although we consider only voting mechanisms in this paper, our approach can potentially apply to a wide range of other mechanisms.

Our analysis fits into a broader research agenda on mechanism design. This literature has been traditionally concerned with the construction of mechanisms that prevent or minimize manipulation. Launching this research agenda, Vickrey (1960) wrote:

> "An analysis of the ways in which a social welfare function might be set up so as to minimize the probable influence of strategy might be interesting, but appears ... to present formidable difficulties." (p. 519).

However, as noted earlier, a recent literature has advocated for a paradigm shift, arguing that the goal of a social planner should instead be to select a mechanism that guarantees a good outcome in equilibrium, even if it is subject to manipulation (see, e.g., Bergemann and Morris (2005), Chung and Ely (2007)). Indeed, if no interesting voting mechanism is immune to manipulation — as shown by Gibbard (1973) and Satterthwaite (1975) — then a more appropriate goal should be to identify those mechanisms that yield the greatest social welfare gains or that minimize welfare losses in equilibrium; this is precisely one of the achievements of our analysis.

Our paper is also related to a recent literature on the effect of manipulation. The closest paper to ours is Moyouwou and Pongou (2012), who determine both the minimum and the maximum proportion of voters who gain or lose from manipulation by one individual under the infinite class of rules that we consider in this paper. But this paper limits its analysis to only three-candidate elections. Campbell and Kelly (2014) provide an upper bound on the number of voters who are harmed by manipulation, but unlike Moyouwou and Pongou (2012), they are not interested in the number of voters who gain from it. Caroll (2013) quantifies the susceptibility of a voting rule to manipulation, where susceptibility is measured by the maximum expected utility an individual can gain by misrepresenting his preferences.

A common assumption made in all these studies is that there is only one manipulator. In Moyouwou and Pongou (2012) and in Campbell and Kelly (2014), the manipulator believes that other voters behave sincerely. Therefore, deviation from sincere voting does not necessarily lead to an equilibrium outcome. Caroll (2013) assumes that the manipulator holds a belief over the preferences of other voters and manipulates only if the expected gain from doing so is greater than a certain fixed cost. He then derives his measure of susceptibility under the assumption that the manipulator believes that the preferences of other voters are identically and independently distributed.

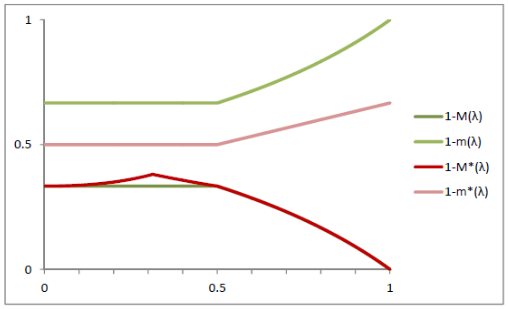
Our paper takes an entirely different approach, in which preference misrepresentation only occurs as an *equilibrium* strategy. This approach would be justified under the rational voter model (see, e.g., Myerson and Weber (1993)). Indeed, if one voter manipulates, this may induce others to manipulate as well. It follows that several voters can manipulate at the same, acting either individually or jointly in order to best-respond to each other's action. Vickrey (1960) already made the same point in the following statement:

> "In general, whenever intensity of preference is given effect in the social welfare function, whether directly as such or through considering the number of intervening ranks, it will be to the advantage of an individual or a group, whenever it can be discerned in advance which alternatives are likely to be close rivals for selection as the social choice and which alternatives are almost certain to be defeated, to exaggerate preferences among the close rivals, at the expense, if necessary, of understating the relative intensity of preferences for or against the less promising ("irrelevant") alternatives, whether this lack of promise is due to technical difficulty or impossibility or simply to lack of general appeal. Such a strategy could, of course, lead to counterstrategy, and the process of arriving at a social decision could readily turn into a "game" in the technical sense. It is thus not for nothing that we often hear references to "the game of politics"." (p. 517-518).

Consistent with the point made by Vickrey (1960), our analysis views a voting economy as a "game". It therefore differs from the aforementioned studies on the welfare effect of manipulation in that it allows for several manipulators who only misrepresent their preferences as an equilibrium strategy.

Owing to this fundamental difference in our approach, we obtain results that are very different from those in these papers. For instance, Moyouwou and Pongou (2012) find that, in large three-candidate elections, the proportion of voters who benefit (or lose) from the strategic outcome winning the election at the expense of the honest outcome ranges from $\frac{1}{3}$ to $\frac{2}{3}$ if $0 \leq \lambda \leq \frac{1}{2}$, and from $\frac{1-\lambda}{2-\lambda}$ to $\frac{1}{2-\lambda}$ if $\frac{1}{2} \leq \lambda \leq 1$. Campbell and Kelly (2014) examine a different set of rules than the one we analyze. Their set however includes simple plurality and the Borda rule, enabling a limited comparison with our results. Their findings imply that, in a three-candidate election, manipulation hurts at most two-thirds of a large electorate under simple plurality and at least the same proportion under the Borda rule. These results compare with those found by Moyouwou and Pongou (2012) for these two rules, but they clearly differ from the findings of the present paper. Our analysis implies that manipulation hurts at most one-half of a large electorate under simple plurality and the Borda rule. Importantly, this finding is also true for elections involving more than three candidates. Figure 3 provides a better comparison of our findings with those in Moyouwou and Pongou (2012) (the functions $m(\lambda)$ and $M(\lambda)$ are respectively the minimum and maximum shares of voters who benefit from non-equilibrium manipulation by only one voter). As the reader can observe, our bounds are completely different.

Figure 3: Maximum and Minimum Losses from Equilibrium and Non-Equilibrium Manipulation



Like our paper, a number of other studies have analyzed positional voting rules in elections involving three alternatives (see, e.g., Sengupta (1978), Saari (1999), Myerson (2002), Myatt (2007), Goertz and Maniquet (2011)). There is also a literature on the efficient aggregation of private information in elections (e.g., Feddersen and Pesendorfer (1997), Myerson (2000), Koriyama and Szentes (2009), Goertz and Maniquet (2011), Bhattacharya (2013)). These papers, however, are not concerned with the social welfare consequences of manipulation. Also, from a purely conceptual point of view, we differ from most of these studies by assuming ordinal preferences, which precludes interpersonal utility comparison. In this respect, we also view our partial identification approach and our use of linear programming in an ordinal framework as a contribution.

Our work also contributes to the broad literature that evaluates and compares mechanisms on the basis of their performance. In particular, evaluation based on worst-case performance has been explored in mechanism design theory (Hurwicz and Shapiro (1978), Bergemann and Morris (2005), Chung and Ely (2007)), voting (Sprumont (1995), Carroll (2013)), matching (Sönmez and Ünver (2011)), contract theory (Chassang (2013), Carroll (2015), Frankel (2015)), and pricing theory (Segal (2003)) among many other applications. The worst-case paradigm, also known as the "conservative approach", has also been applied to study rational behavior in sequential games (see, e.g., Harsanyi (1974), Greenberg (1996), Ray (2015), Xue (1998), Moyouwou et al. (2015)). Greenberg (1996) also advanced the optimistic approach, in which the decision-maker hopes for the best outcome possible when making a decision that has an uncertain consequence. We have shown that evaluating voting rules using the worst-case and best-case approaches lead to different conclusions about their performance. For instance, a social planner who chooses a rule based on the minimization of the worst-case-scenario loss should prefer the plurality rule over antiplurality, whereas a social planner who chooses a rule based on the maximization of the best-case-scenario gain should prefer antiplurality over all the other rules. These findings mean that our analysis has empirical relevance, as it concerns rules which are extremely popular in real-life political settings as well as in organizations.

The remainder of this paper is as follows. Section 2 formalizes the problem we study. Section 3 states the main results for three-candidate elections held under the infinite class of intensity or positional voting rules. Section 4 studies an extension to elections involving more than three candidates. Section 5 derives the implications of our findings for the choice of an election rule by a social planner, and Section 6 concludes. For clarity in the exposition, all the proofs are collected in Section 7.

2 The Problem: Manipulation and Social Benefit and Harm

2.1 Notation and Definitions

A society $N = \{1, ..., n\}$ of n individuals is considering choosing a ruling party or a leader from a finite set A of political parties or candidates. Following the literature, we first assume that there are only three candidates a, b and c, and analyze the infinite class of the democratic rules λ.[5] In Section 4, we consider elections involving more than three candidates. We assume n to be sufficiently large. In particular, we derive our main results under the assumption that n goes to infinity.

Any nonempty subset of the set N is called a coalition or a group, and the set of all possible coalitions is denoted by 2^N. For any finite set X, $|X|$ represents the cardinality of X.

Each individual $i \in N$ has a **preference relation** R^i over A which we assume to be linear, that is, reflexive, transitive, antisymmetric, and complete. Each individual's preference relation therefore belongs to the set $L = \{abc, acb, bac, bca, cab, cba\}$, where $R^i = abc$, for instance, means that individual i prefers a over b over c. An individual i who has preference R^i may nonetheless choose to cast a different ballot $Q^i \in L$, and in this case, we refer to Q^i as an insincere ballot or as a misrepresentation of i's true preferences.

A **preference profile** is a collection of individual preferences. We denote by L^N the set of all the preference profiles. Given a preference profile R^N and a coalition S, R^{-S} denotes the preference profile obtained from R^N by omitting the preferences of all the individuals in S. It follows that any profile R^N can be rewritten as $\left(R^S, R^{-S}\right)$. To simplify notation, we write R^{-i} for $R^{-\{i\}}$. The profile denoted $\left(Q^S, R^{-S}\right)$ is the profile obtained from R^N by substituting Q^i to R^i for all $i \in S$.

Given a linear order R and two candidates x and y, the relation denoted $R_{[y]}$ is obtained from R only by moving y to the top, and the relation denoted $R_{[yx]}$ is obtained from R only by moving y to the top and x to the second position.

A deterministic **social choice function** (SCF) is a voting rule F which maps each voting profile R^N into a single party or candidate $F\left(R^N\right)$, which is the election winner at R^N. In this paper, as already mentioned, the only SCFs in which we are interested are the voting rules λ described in the Introduction, and denoted by F_λ. These rules include the plurality rule ($\lambda = 0$), the antiplurality (or negative plurality) rule ($\lambda = 1$), and the Borda rule ($\lambda = \frac{1}{2}$); these are the most frequently used rules in actual elections. Under the plurality rule, each voter votes for only one party, which receives 1 point, each of the remaining parties receiving 0 points. Under the antiplurality rule, each voter votes against one party, which receives 0 points, each of the remaining parties receiving 1 point. The other rules, including the Borda rule, better reveal the (strategic) rankings of candidates by voters than the plurality rule and the antiplurality rule. Under any of these rules, the candidate who receives the most points wins.

Manipulability. Gibbard (1973) and Satterthwaite (1975) have shown that any non-dictatorial deterministic social choice function is **manipulable**: under complete information of voters' preferences, a voter can obtain a more preferred election outcome by not voting according to his true preferences. A SCF F is said to be **strategy-proof** if it is **non-manipulable**, that is, if no voter

[5] Three-candidate elections under the class of rules λ have been extensively studied (see, e.g., Saari (1999), Myerson (2002), Goertz and Maniquet (2011)), but the questions addressed in the available literature are not those we answer in this paper.

can profitably misrepresent his preferences. More formally, F is said to be **strategy-proof** if for all $i \in N$, $R^N \in L^N$ and $Q^i \in L$, $F\left(R^N\right) R^i F\left(Q^i, R^{-i}\right)$. A SCF F is **strongly strategy-proof** if no group of voters can profitably misrepresent their preferences: for all $S \in 2^N$, $R^N \in L^N$ and $Q^S \in L^S$, $F\left(R^N\right) R^i F\left(Q^S, R^{-S}\right)$ for some $i \in S$. Strong strategy-proofness is preferred to (individual) strategy-proofness in many real-world contexts. As noted by Barberà, Berga, and Moreno (2016), "individual strategy-proofness is a rather fragile property, unless one can also preclude manipulations of the social outcome by potential coalitions..." (p. 1073). Following this pertinent observation, the equilibrium concept we use in this paper is related to strong strategy-proofness, as this implies that the outcome can be manipulated by individual voters as well as by coordinated groups of voters.

2.2 Formalizing the Problem

In this paper, we determine the minimum and the maximum proportion of voters who benefit or lose from some voters misrepresenting their preferences. We consider only manipulations that lead to a political equilibrium.

Consider an election with a status quo. Without loss of generality, assume the status quo to be a. The election is a **one-shot game** held under a political rule F_λ biased towards the status quo.[6] Let $R^N \in L^N$ be a preference profile. Let $S \subset N$ be a group of voters and $T^N = \left(T^S, R^{-S}\right) \in L^N$ a preference profile such that $F_\lambda\left(T^S, R^{-S}\right) R^i F_\lambda\left(R^N\right)$ for all $i \in S$. We say that R^N is manipulable under the rule F_λ and that T^N is an **effective manipulation** of R^N. Denote by $P(F_\lambda)$ the set of manipulable preference profiles under F_λ and by $U(F_\lambda|R^N)$ the set of effective manipulations of R^N under F_λ.

A preference profile R^N is an **equilibrium** if for any coalition of voters $S \subset N$ and any profile $Q^S \in L^S$, we have $F_\lambda\left(R^N\right) R^i F_\lambda\left(Q^S, R^{-S}\right)$ for some individual $i \in S$. If R^N is not an equilibrium (because members of some coalition find it in their interest to deviate from their true preferences), we say that R^N is (jointly) **manipulable** or **unstable**. Denote by $SN(F_\lambda|R^N)$ the set of equilibria that are effective manipulations of R^N when R^N is the profile of true preferences. Assume that R^N is not an equilibrium and let $T^N \in SN(F_\lambda|R^N)$ be an effective manipulation of R^N that is an equilibrium. We call $F_\lambda(T^N)$ a **political equilibrium** at R^N supported by T^N. Denote by $E_\lambda(R^N, T^N)$ the set of all the voters who benefit from the effective manipulation of R^N leading to the equilibrium T^N:

$$E_\lambda(R^N, T^N) = \{i \in N : F_\lambda\left(T^N\right) R^i F_\lambda\left(R^N\right)\}.$$

Our goal is to determine the maximum and the minimum of $\frac{|E_\lambda(R^N,T^N)|}{n}$ for large electorates. For an electorate of size n, these maximum and minimum are respectively:

$$M^*(\lambda, n) = \max_{(R^N,T^N)\in P(F_\lambda)\times SN(F_\lambda|R^N)} \frac{|E_\lambda(R^N,T^N)|}{n} \qquad (P1)$$

and

[6] A *one-shot* game in our context is a game in which each voter votes only once, the ballots are counted, and a winner is proclaimed in accordance with the rule F_λ. As is usual in the literature, the status quo bias implies that the status quo is replaced if and only if it is beaten by one of the challengers under the election rule. Therefore, if there is a tie, the status quo is proclaimed the winner. This way to break the tie is the alphabetic tie-break rule in our context.

$$m^*(\lambda, n) = \min_{(R^N, T^N) \in P(F_\lambda) \times SN(F_\lambda | R^N)} \frac{|E_\lambda(R^N, T^N)|}{n} \qquad (P2)$$

$M^*(\lambda, n)$ and $m^*(\lambda, n)$ are, respectively, the maximum and the minimum fraction of the voting population which benefits from some group of voters misrepresenting their preferences under F_λ, and where this manipulation leads to an equilibrium. We would like to evaluate $M^*(\lambda, n)$ and $m^*(\lambda, n)$ as n goes to infinity. Let:

$$M^*(\lambda) = \lim_{n \longrightarrow +\infty} M^*(\lambda, n) \qquad (P3)$$

and

$$m^*(\lambda) = \lim_{n \longrightarrow +\infty} m^*(\lambda, n) \qquad (P4)$$

We solve problems (P3) and (P4) by bounding $M^*(\lambda, n)$ and $m^*(\lambda, n)$ below and above and taking the limit of these bounds when n goes to infinity (Theorems 1 and 2).We also determine the minimum and maximum proportion of voters who are adversely affected by manipulation. Given that voters have strict preferences, these are $1 - M^*(\lambda, n)$ and $1 - m^*(\lambda, n)$, respectively. This *partial identification* approach essentially acknowledges that the equilibrium welfare effects of tactical voting varies depending on voters' true preferences.

2.3 A Simple Illustrative Example

To illustrate our purpose, we would like to analyze the equilibrium welfare effects of preference misrepresentation for each of the three canonical rules in an election that involves 8 voters and 3 political parties. Preferences are as follows: $R^1 = R^2 = R^3 = bac$, $R^4 = bca$, $R^5 = R^6 = R^7 = abc$ and $R^8 = cab$.

2.3.1 Plurality Rule

Assume that the election is held under the plurality rule. If all voters cast a sincere ballot, then b will win with four votes. But if voter 8 submits an insincere ballot $Q^8 = acb$, then a and b will tie with four favorable votes each, and so the tie-breaking rule means that a will win. Observe that the profile R^N is not an equilibrium, whereas the profile (Q^8, R^{-8}) is. By misrepresenting his true preferences, voter 8 gets his second-ranked party elected, which is better for him than the honest outcome b. This preference misrepresentation benefits not only voter 8, but voters 5, 6 and 7 as well. The share of voters who benefit from the manipulation is therefore $\frac{1}{2}$, whereas the share of voters who are harmed is $\frac{1}{2}$. We will see in the next sections that manipulation, in equilibrium, always benefits at least half of the population under the plurality rule, regardless of the number of candidates.

2.3.2 Borda Rule

Assume now that the election is held under the Borda rule. If all voters are honest, then b will win with 5.5 points versus 5 points for a and 1.5 points for c. But if voters 5, 6, 7 and 8 submit an insincere ballot $Q^5 = Q^6 = Q^7 = Q^8 = acb$, then a will win with 5.5 points versus 4 points for b

and 2.5 points for c. We note that the voting profile $Q^N = (Q^S, R^{-S})$ where $S = \{5, 6, 7, 8\}$ is an equilibrium. By misrepresenting their true preferences, voters 5, 6, and 7 get their first-ranked party elected and voter 8 gets his second-ranked party elected, which is better for them than the honest outcome b. As under the plurality rule, the share of voters who benefit from manipulation is $\frac{1}{2}$, whereas the share of voters who are harmed is $\frac{1}{2}$. We will formally show that manipulation always benefits at least half of the population in equilibrium under the Borda rule.

2.3.3 Antiplurality Rule

Assume now that the election is held under the antiplurality rule. If all voters cast a sincere ballot, then a and b will tie with one negative vote each, and a will win following the alphabetical tie-breaking rule. But if voters 1 and 2 submit an insincere ballot $Q^1 = Q^2 = bca$, then b will win with only one negative vote against three negative votes for a and two negative votes for c. Note that the profile R^N is not an equilibrium, whereas the profile $Q^N = (Q^S, R^{-S})$, where $S = \{1, 2\}$, is. By misrepresenting their true preferences, voters 1 and 2 get their first-ranked party elected, and their tactical voting also benefits voters 3 and 4 who also see their first-ranked party elected, but the other voters are harmed. We see that voting manipulation benefits half of the population in equilibrium. However, we will see that the minimum fraction of the population that benefits from voting manipulation under antiplurality is less than half in general, and decreases with the number of competing candidates in large elections.

Importantly, this example also shows that voting behavior depends on the voting rule. We have also seen that the welfare effects of tactical voting varies across rules. A more formal and comprehensive analysis is provided below.

3 Welfare Gains and Losses from Manipulation in Equilibrium

In this section, our goal is to identify, for large populations, the maximum and minimum proportion of voters who, in equilibrium, experience welfare gains and losses from the strategic manipulation of the electoral outcome. We focus on the infinite class of rules λ for three-candidate elections. We first identify bounds around the minimum ($m^*(\lambda, n)$) and the maximum ($M^*(\lambda, n)$) proportion of those who experience welfare gain as a result of insincere voting for any value of λ and for sufficiently large values of n. From these results, we derive the asymptotic bounds. We present a step-by-step proof of each result in the appendix.

3.1 Equilibrium Maximum Gains

Our first result identifies an upper bound of the maximum proportion of voters who are positively affected by insincere voting when $n \geq 2$ and $0 \leq \lambda \leq \dfrac{1}{2}$. The range of voting rules covered by this result therefore includes the plurality rule and the traditional Borda rule.

Proposition 1 *Assume that* $0 \leq \lambda \leq \frac{1}{2}$ *and* $n \geq 2$. *Let* $\lambda^* = \frac{1}{2}\sqrt[3]{2\sqrt{3}+2} - \frac{1}{\sqrt[3]{2\sqrt{3}+2}} \approx 0.312$. *Then:*

1. *If* $0 \leq \lambda \leq \lambda^*$, $M^*(\lambda, n) \leq \frac{2-2\lambda-\lambda^2}{3-3\lambda}$.

2. *If* $\lambda^* \leq \lambda \leq \frac{1}{2}$, $M^*(\lambda, n) \leq \frac{1+2\lambda}{2+2\lambda}$.

The second result bounds the maximum proportion of voters who benefit from insincere voting in large elections held under the rules $0 \leq \lambda \leq \dfrac{1}{2}$.

Proposition 2 *Assume that* $0 \leq \lambda \leq \frac{1}{2}$. *Then:*

1. *If* $0 \leq \lambda \leq \lambda^*$ *and* $n \geq 21$,

$$\frac{1}{n}\left\lfloor \frac{2-2\lambda-\lambda^2}{3-3\lambda} n \right\rfloor - \frac{2}{n} < M^*(\lambda, n) \leq \frac{2-2\lambda-\lambda^2}{3-3\lambda}. \tag{1}$$

2. *If* $\lambda^* \leq \lambda \leq \frac{1}{2}$ *and* $n \geq 36$,

$$\frac{1}{n}\left\lfloor \frac{1+2\lambda}{2+2\lambda} n \right\rfloor \leq M^*(\lambda, n) \leq \frac{1+2\lambda}{2+2\lambda}. \tag{2}$$

The third result establishes bounds around the maximum welfare benefit of insincere voting for sufficiently large n when $\dfrac{1}{2} < \lambda \leq 1$.

Proposition 3 *Assume that* $n \geq 15$ *and* $\dfrac{1}{2} < \lambda \leq 1$. *Then* $1 - \dfrac{1}{n}\left\lceil \dfrac{1-\lambda}{2-\lambda} n \right\rceil \leq M^*(\lambda, n) \leq \dfrac{1}{2-\lambda}$.

From Propositions 1-3, we obtain our first main result, which identifies the maximum proportion of voters who, in equilibrium, will gain from manipulation in large elections.

Theorem 1 *For* $0 \leq \lambda \leq 1$,

$$M^*(\lambda) = \begin{cases} \dfrac{2-2\lambda-\lambda^2}{3-3\lambda} & \text{if } 0 \leq \lambda \leq \lambda^* \\[2ex] \dfrac{1+2\lambda}{2+2\lambda} & \text{if } \lambda^* \leq \lambda \leq \frac{1}{2} \\[2ex] \dfrac{1}{2-\lambda} & \text{if } \dfrac{1}{2} < \lambda \leq 1 \end{cases}$$

where $\lambda^* = \frac{1}{2}\sqrt[3]{2\sqrt{3}+2} - \frac{1}{\sqrt[3]{2\sqrt{3}+2}}$.

We note that the function $M^*(\lambda)$ is continuous. It is also differentiable, except at $\lambda = \lambda^*$ and at $\lambda = \frac{1}{2}$. While under the plurality rule ($\lambda = 0$), insincere voting benefits at most two-thirds of the electorate in equilibrium, under the antiplurality rule ($\lambda = 1$), it may lead to a Pareto improvement. In such circumstances, strategic behavior may be viewed as a virtue, rather than a vice.

3.2 Equilibrium Minimum Gains

In this section, we identify the minimum proportion of the population which experiences welfare gains from insincere voting. Strikingly, our first result shows that, in equilibrium, a majority of the population will benefit from strategic manipulation of the voting outcome for the rules $0 \leq \lambda \leq \dfrac{1}{2}$.

Proposition 4 *Assume that* $0 \leq \lambda \leq \frac{1}{2}$ *and* $n \geq 2$*. Then* $m^*(\lambda, n) \geq \frac{1}{2}$*.*

The second result establishes bounds around the minimum share of voters who benefit from manipulation for sufficiently large populations when $0 \leq \lambda \leq \frac{1}{2}$.

Proposition 5 *Assume that* $0 \leq \lambda \leq \frac{1}{2}$ *and* $n \geq 37$*. Then* $\frac{1}{2} \leq m^*(\lambda, n) \leq \frac{1}{2} + \frac{1}{n}$*.*

The next result is concerned with the rules $\dfrac{1}{2} < \lambda < 1$.

Proposition 6 *Assume that* $\frac{1}{2} < \lambda < 1$ *and that* $n > \max\left(\frac{6(\lambda+1)(2-\lambda)}{\lambda\left(4\lambda-\lambda^2-1\right)}, \frac{3\lambda(2\lambda-1)}{(1-\lambda)\left(4\lambda-\lambda^2-1\right)}\right)$*. Then* $\frac{2-\lambda}{3} \leq m^*(\lambda, n) \leq 1 - \frac{1}{n}\left\lfloor \frac{1+\lambda}{3} n \right\rfloor + \frac{1}{n}$*.*

We conduct a similar analysis for antiplurality below.

Proposition 7 *Assume that* $n \geq 6$*. Then* $\frac{1}{3} \leq m^*(1, n) \leq \frac{1}{3} + \frac{2}{n}$*.*

Having established bounds around the minimum proportion of the voting population which enjoys welfare gains from manipulation, we now derive our second main result of the paper.

Theorem 2 *For* $0 \leq \lambda \leq 1$*,*

$$m^*(\lambda) = \begin{cases} \dfrac{1}{2} \text{ if } 0 \leq \lambda \leq \dfrac{1}{2} \\ \\ \frac{2-\lambda}{3} \text{ if } \dfrac{1}{2} < \lambda \leq 1 \end{cases}$$

Theorem 2 allows us to study some interesting properties of the minimum welfare gain function $m^*(\lambda)$. In particular, we note that this function is continuous and is everywhere differentiable, except at $\lambda = \dfrac{1}{2}$. It also implies that insincere voting benefits at least one-half of the population under the plurality rule ($\lambda = 0$) and the traditional Borda rule ($\lambda = \dfrac{1}{2}$), and one-third of the population under antiplurality ($\lambda = 1$).

3.3 Exact Asymptotic Bounds on Equilibrium Gains and Losses from Manipulation

From Theorems 1 and 2, we deduce that the proportion of a large electorate that is positively affected by preference misrepresentation ranges from a minimum of $m^*(\lambda)$ to a maximum of $M^*(\lambda)$. Denote by $G(\lambda)$ this interval and let $\lambda^* = \frac{1}{2}\sqrt[3]{2\sqrt{3}+2} - \frac{1}{\sqrt[3]{2\sqrt{3}+2}} \approx 0.312$. It follows that:

$$G(\lambda) = \begin{cases} [\dfrac{1}{2}, \dfrac{2-2\lambda-\lambda^2}{3-3\lambda}] \text{ if } 0 \leq \lambda \leq \lambda^* \\ \\ [\dfrac{1}{2}, \dfrac{1+2\lambda}{2+2\lambda}] \text{ if } \lambda^* \leq \lambda \leq \frac{1}{2} \\ \\ [\frac{2-\lambda}{3}, \dfrac{1}{2-\lambda}] \text{ if } \dfrac{1}{2} < \lambda \leq 1. \end{cases}$$

Similarly, the proportion of a large population of voters negatively affected by preference misrepresentation ranges from $1 - M^*(\lambda)$ to $1 - m^*(\lambda)$. If we denote by $L(\lambda)$ this interval. It follows that:

$$L(\lambda) = \begin{cases} [\frac{1-\lambda+\lambda^2}{3-3\lambda}, \frac{1}{2}] \text{ if } 0 \leq \lambda \leq \lambda^* \\ \\ [\frac{1}{2+2\lambda}, \frac{1}{2}] \text{ if } \lambda^* \leq \lambda \leq \frac{1}{2} \\ \\ [\frac{1-\lambda}{2-\lambda}, \frac{1+\lambda}{3}] \text{ if } \frac{1}{2} < \lambda \leq 1. \end{cases}$$

It follows that, under simple plurality ($\lambda = 0$), manipulation benefits from one half to two-thirds of the population in equilibrium ($G(0) = [\frac{1}{2}, \frac{2}{3}]$) and hurts from one-third to one half of the population ($L(0) = [\frac{1}{3}, \frac{1}{2}]$). Under the traditional Borda rule ($\lambda = \frac{1}{2}$), manipulation benefits from one half to two-thirds of the population ($G(\frac{1}{2}) = [\frac{1}{2}, \frac{2}{3}]$) and hurts from one-third to one half of the population ($L(\frac{1}{2}) = [\frac{1}{3}, \frac{1}{2}]$). Under antiplurality ($\lambda = 1$), manipulation benefits from one-third to one-hundred percent of the population ($G(1) = [\frac{1}{3}, 1]$) and hurts from zero percent to two-thirds of the population ($L(1) = [0, \frac{2}{3}]$). As we will see later, these findings have an implication for the choice of an election rule by a social planner facing a worst-case or a best-case objective.

4 An Extension: Elections Involving More than Three Political Parties or Candidates

We now assume that there are $m \geq 3$ competing political parties or candidates and we analyze the equilibrium welfare effects of tactical voting for each of the three canonical positional voting rules: simple plurality, Borda count, and antiplurality. We again assume that each voter has a linear preference relation over candidates. Let R^i be the preference relation of voter i, we write $R^i = abc...$ to mean that i strictly prefers a over b, b over c, and so on. We write $a_j...a_k$ for any preference relation in which a_j is ranked first and a_k is ranked last. Given a linear order R and two candidates a and b, we notate by $R_{a,b}$ the linear order obtained from R by shifting only a and b so that: (i) a or b is top ranked in $R_{a,b}$; (ii) the relative ranking of a and b in R is preserved in $R_{a,b}$; (iii) the total number of candidates ranked between a and b is the same in R and in $R_{a,b}$; and (iv) R and $R_{a,b}$ coincide over $A \setminus \{a, b\}$. Similarly, we notate by $R_{[b]}$ the linear order obtained from R by only moving b to the top, and by $R_{[ba]}$ the linear order obtained from R by only moving b to the top and a to the second position. For example for $R = a_1a_2a_3a_4a_5$, we have $R_{a_3,a_5} = a_3a_1a_5a_2a_4$, $R_{[a_5]} = a_5a_1a_2a_3a_4$ and $R_{[a_5a_{a_3}]} = a_5a_3a_1a_2a_4$.

Hereafter the set of candidates is $A = \{a_1, a_2, ..., a_m\}$ and the lexicographic order is the linear ranking on A for which a_j is ranked j^{th} for any $j = 1, 2, ..., m$. A scoring vector is a vector $v = (\lambda_1, \lambda_2, ..., \lambda_m)$ of decreasing real numbers such that $\lambda_1 = 1$ and $\lambda_m = 0$. Under the scoring rule associated with the vector v, λ_k points are given to a candidate whenever he is ranked at the k^{th} position by a voter; the winner at a voting profile R^N is the candidate who receives the most points.

Denote by $S_v\left(a, R^N\right)$ the total number of points received by a candidate a at a voting profile R^N. Analyzing the welfare effects of tactical voting under any scoring rule as we have done for three-candidate elections is intractable. For this reason, we will only focus on the three canonical rules mentioned above. The scoring vector is $v^{Pl} = (1, 0, ..., 0)$ for simple plurality (Pl), $v^{APl} = (1, ..., 1, 0)$ for antiplurality (APl), and $v^B = \left(1, \frac{m-2}{m-1}, ..., \frac{1}{m-1}, 0\right)$ for the Borda rule ($Borda$). The winner of an election held under, say simple plurality, will be denoted by $Pl(R^N)$ if R^N is the voting profile.

Let R^N be a preference profile, T^N an effective manipulation of R^N under a rule F, and a and b two competing candidates. We denote by:

1. $E\left(a, R^N\right)$ the set of voters who rank candidate a first in the profile R^N;
2. $E(F, R^N, T^N)$ the set of voters who benefit from the manipulation from R^N to T^N; and
3. $E(a, b, R^N)$ the set of voters who prefer b to a under the profile R^N.

We denote respectively by $M^*(F, n, m)$ and $m^*(F, n, m)$ the maximum and the minimum fraction of voters who benefit from manipulation in an election involving n voters and m candidates and held under the rule $F \in \{Pl, Borda, APl\}$. As n tends to infinity, these bounds are denoted by $M^*(F, m)$ and $m^*(F, m)$, respectively.

For any real number x, we denote by $\lfloor x \rfloor$ the greatest integer weakly smaller than x and by $\lceil x \rceil$ the smallest integer weaker greater than x. Formally,

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1 \text{ and } \lceil x \rceil \geq x > \lceil x \rceil - 1 \tag{3}$$

4.1 Simple Plurality: Equilibrium Maximum and Minimum Gains from Manipulation

In this section, we study the equilibrium welfare effects of manipulation in plurality elections. First, we show that the winner of a plurality election under a voting profile R^N is the first-ranked candidate for at least $\left\lceil \frac{n}{m} \right\rceil$ voters.

Proposition 8 *Let $R^N \in L^N$. If $Pl\left(R^N\right) = a_j$ then $\left|E\left(a_j, R^N\right)\right| \geq \left\lceil \frac{n}{m} \right\rceil$.*

The next result states that if the number of voters who prefer the winner a_j of an election to any other candidate under a profile R^N is smaller than the number of voters who rank a_j first at a different profile T^N, and if that number is strictly smaller than the number of voters who rank a_j first at R^N, then T^N is an equilibrium given R^N.

Proposition 9 *Let T^N be an effective manipulation of R^N under simple plurality. Let $Pl(T^N) = a_j$. If $\left|E(a_j, a_l, R^N)\right| \leq \left|E(a_j, T^N)\right|$ $\forall l > j$ and $\left|E(a_j, a_l, R^N)\right| < \left|E(a_j, T^N)\right|$ $\forall l < j$, then T^N is an equilibrium given R^N.*

The next result identifies the maximum equilibrium welfare gain from manipulation.

Proposition 10 *Assume that $n \geq 2$ and $m \geq 3$. Then $M^*\left(Pl, n, m\right) = 1 - \frac{1}{n}\left\lceil \frac{n}{m} \right\rceil$.*

We now identify the minimum equilibrium gain from manipulation, showing that manipulation benefits at least half of the electorate in a plurality election.

Proposition 11 *Let $n \geq 2$ and $m \geq 3$. Then $m^*(Pl, n, m) = \frac{1}{n}\left\lceil \frac{n}{2} \right\rceil$.*

4.2 Borda Rule: Equilibrium Maximum and Minimum Gains from Manipulation

In this section, we determine the minimum and maximum equilibrium gains and losses from manipulation under the Borda rule. We first remark that, by definition, whenever a candidate x is preferred to another candidate y by a voter, the score of x is greater than that of y by at least $\frac{1}{m-1}$ in the ranking of that voter.

Remark 1 *Let R^N be a profile, $i \in N$ a voter, and $\{x, y\} \subseteq A$ two candidates. Then $S\left(x, R^i\right) \geq S\left(y, R^i\right) + \frac{1}{m-1}$ whenever xR^iy.*

The proposition below states that the winner of a Borda election under a voting profile R^N is preferred to any of the other candidates by at least $\left\lceil \frac{n}{m} \right\rceil$ voters.

Proposition 12 *Let $R^N \in L^N$. If $Borda\left(R^N\right) = x$, then $\left|E\left(x, y, R^N\right)\right| \geq \left\lceil \frac{n}{m} \right\rceil$ for all $\{x, y\} \subseteq A$.*

The next proposition identifies the maximum equilibrium welfare gain from manipulation, which is identical to the bound found for simple plurality.

Proposition 13 *Assume that $n \geq 2$ and $m \geq 3$. Then $M^*(Borda, n, m) = 1 - \frac{1}{n}\left\lceil \frac{n}{m} \right\rceil$.*

We now derive the minimum equilibrium welfare gain from manipulation, finding, like for simple plurality, that manipulation benefits at least half of the electorate in a Borda election.

Proposition 14 *Let $n \geq 2$ and $m \geq 3$. Then $m^*(Borda, n, m) = \frac{1}{n}\left\lceil \frac{n}{2} \right\rceil$.*

4.3 Antiplurality Rule: Equilibrium Maximum and Minimum Gains from Manipulation

This section identifies the maximum and minimum equilibrium welfare gains and losses from manipulation in an antiplurality election. We find that tactical voting might benefit the entire electorate in equilibrium.

Proposition 15 *Assume that $n \geq 2$ and $m \geq 3$. Then $M^*(APl, n, m) = 1$.*

The minimum equilibrium gain from tactical voting in an antiplurality election depends on the size of the electorate relative to the number of competing candidates or political parties. If the number of voters is strictly smaller than the number of candidates minus 1 (that is, $n + 1 < m$), then manipulation will benefit at least a fraction of $\dfrac{1}{n}$ of the population. Otherwise, manipulation will benefit a greater fraction of the population.

Proposition 16 *Assume that $n \geq 2$ and $m \geq 3$. Then* $m^*(APl, n, m) = \begin{cases} \dfrac{1}{n} & \text{if } n + 1 < m \\ \dfrac{1}{n}\left\lfloor \frac{n+1}{m} \right\rfloor & \text{otherwise.} \end{cases}$

4.4 Exact Asymptotic Bounds on Equilibrium Gains and Losses from Manipulation

From Propositions 10, 11, 13, 14, 15 and 16, we derive exact asymptotic bounds on the welfare gains and losses from tactical voting under each of the three canonical rules we have analyzed. The gains from manipulation ranges from a minimum of $m^*(F, m)$ to a maximum of $M^*(F, m)$ for an election involving m candidates held under a rule $F \in \{Pl, Borda, APl\}$. We have the following theorem.

Theorem 3 *The following assertions are true:*
1) $m^*(Pl, m) = \frac{1}{2}$ *and* $M^*(Pl, m) = 1 - \dfrac{1}{m}$.
2) $m^*(Borda, m) = \frac{1}{2}$ *and* $M^*(Borda, m) = 1 - \frac{1}{m}$.
3) $m^*(APl, m) = \frac{1}{m}$ *and* $M^*(APl, m) = 1$.

The proof simply follows by taking the limit of $m^*(F, n, m)$ and $M^*(F, n, m)$ as n tends to infinity for each rule $F \in \{Pl, Borda, APl\}$. Again, an interesting and distinctive property of antiplurality is that strategic manipulation of the voting outcome may lead to a Pareto improvement. This finding obtains for the plurality rule and the Borda rule only if the number of candidates tends to infinity.

Similarly, the proportion of a large population of voters negatively affected by manipulation ranges from $1 - M^*(F, m)$ to $1 - m^*(F, m)$. If we denote by $L(F, m)$ this interval. It follows that:

$$L(F, m) = \begin{cases} [\frac{1}{m}, \frac{1}{2}] \text{ if } F = Pl \\ \\ [\frac{1}{m}, \frac{1}{2}] \text{ if } F = Borda \\ \\ [0, 1 - \frac{1}{m}] \text{ if } F = APl. \end{cases}$$

5 A Social Planner's Problem: Choosing A Voting Mechanism

In this section, we derive the implications of our findings for the evaluation and comparison of the democratic rules we have analyzed, and address the problem of a social planner charged with the task of choosing an election rule. This analysis advances a recent literature which argues that, given the susceptibility of the most interesting mechanisms to manipulation, a social planner charged with the task of choosing a mechanism should select the one that ensures a good outcome in equilibrium. There exist two broad approaches to this problem. The first is the pessimistic or worst-case approach, in which a social planner believes that manipulation will lead to the worst loss in welfare and chooses the rule that minimizes this effect. The second approach is the optimistic or the best-case scenario approach, in which the social planner believes that manipulation will lead to the maximum gain in welfare and chooses the mechanism that maximizes this positive effect. We will consider a third approach which combines both approaches under a lexicographic ordering.

Within our framework, a social planner concerned about worst-case scenarios solves the following problem:

$$\min_{\lambda \in [0,1]} m^*(\lambda) \qquad (P5)$$

And a social planner facing a best-case objective solves the following problem:

$$\max_{\lambda \in [0,1]} M^*(\lambda) \qquad (P6)$$

From Theorems 1 and 2, it is immediately evident that the solution to problem $(P5)$ is any rule $\lambda \in [0, \frac{1}{2}]$ and the solution to problem $(P6)$ is $\lambda = 1$. Given the popularity of the worst-case approach and the multiplicity of solutions to the worst-case problem $(P5)$, one could refine it. For instance, the social planner could choose among the solutions to problem $(P5)$ the one that maximizes the gain occurring under the best-case scenario. In the class of rules we are analyzing, there are two solutions, namely simple plurality ($\lambda = 0$) and the Borda count ($\lambda = \frac{1}{2}$).

Simple plurality, the Borda rule and antiplurality therefore emerge from our analysis as having distinct properties. In general, a risk-averse social planner should choose simple plurality or Borda as the election rules, whereas a risk-lover or optimistic social planner should choose antiplurality. This suggests a new rationale for why these rules are the most commonly observed in practice, especially simple plurality.

6 Conclusion

This paper contributes to a longstanding debate concerning the social welfare consequences of tactical voting on the outcome of elections in democratic societies. Importantly, our analysis of this question provides a new approach to evaluating and comparing voting mechanisms based on a partial identification approach to the quantification of the equilibrium welfare gains and losses that manipulation can cause under each mechanism. It solves the problem of a social planner charged with the task of choosing an election rule. We have seen that the solution to this problem depends on the attitude of the social planner towards risk.

Although it is well understood that commonly-used voting rules are susceptible to manipulation by strategic voters, the extent to which such behavior is likely to harm or benefit other voters has received scant attention in the literature. Our analysis considers a very large class of rules used to select leaders and policies in democratic countries, and determines the minimum and maximum proportion of the voting population which, in equilibrium, experiences welfare gains and losses as a result of insincere voting . An interesting feature of our work is that the results are derived within the ordinal framework, which puts very little structure on preferences and does not make the assumption of interpersonal utility comparison. We are also the first to study this question under the assumption that deviation from sincere voting leads to an equilibrium. Not surprisingly, the results obtained under this assumption differ markedly from those obtained under the assumption of out-of-equilibrium manipulation (see Figure 3).

By considering a continuum of rules, our analysis enables the comparison of different rules by either the minimization of the worst-case-scenario loss or the maximization of the best-case-scenario gain. For instance, a social planner who chooses a rule based on the minimization of the worst-case scenario loss should prefer the plurality rule over antiplurality, whereas a social planner who chooses a rule based on the maximization of the best-case-scenario gain should prefer antiplurality over any other rule. In fact, we have seen that, in three-candidate elections, manipulation benefits from half to two-thirds of the papulation under simple plurality, whereas it benefits one-third to one-hundred percent of the population under antiplurality. Our conclusions do not change qualitatively when we consider elections that involve more than three candidates. These results also demonstrate the empirical relevance of our analysis, given that the rules considered in this paper are extremely popular in democratic countries and in organizations. They also suggest a new rationale for why certain rules like simple plurality are and should be more common than others.

7 Proofs

A step-by-step proof is provided for each result. A few preliminary results, of interest in themselves, are needed to prove certain propositions. In all the proofs, the six possible linear preferences on the set of alternatives $\{a,b,c\}$ are labelled as follows: $R_1 = abc$, $R_2 = acb$, $R_3 = bac$, $R_4 = bca$, $R_5 = cab$ and $R_6 = cba$. Each profile R^N is associated with its anonymous version $x = (x_1, x_2, x_3, x_4, x_5, x_6)$, where x_j is the total number of voters whose preferences are represented by R_j, $j = 1, 2, ..., 6$. Given $0 \leq \lambda \leq 1$, we denote by $S\left(u, \lambda, R^N\right)$ the score of an alternative u at R^N.

7.1 Proof of Proposition 1

In order to prove Proposition 1, we need to state preliminary results which provide necessary conditions for manipulation to occur. These results are also necessary to prove Proposition 4. The first result is stated below.

Lemma 1 *Let R^N be a preference profile whose anonymous version is $x = (x_1, x_2, x_3, x_4, x_5, x_6)$. Suppose that a is the winner at R^N. Assume that a manipulation can occur at R^N in favor of a political equilibrium b. Then there exists two non negative integers y_3 and y_6 such that x, y_3 and y_6 satisfy: (1) the constraints $(C_1 : C_6)$ (that is, the constraints (C_1) to (C_6) below); (2) at least one constraint from $(C_7 : C_9)$; and (3) at least one constraint from $(C_{10} : C_{12})$ where:*

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1 \qquad & (E_0) \\
(\lambda - 1)x_1 - x_2 + (1 - \lambda)x_3 + x_4 - \lambda x_5 + \lambda x_6 \leq 0 \qquad & (C_1) \\
-x_1 + (\lambda - 1)x_2 - \lambda x_3 + \lambda x_4 + (1 - \lambda)x_5 + x_6 \leq 0 \qquad & (C_2) \\
(1 - \lambda)(x_1 - x_3 - y_6) + x_2 - x_4 + \lambda(x_5 - x_6 - y_3) \leq 0 \qquad & (C_3) \\
\lambda(x_2 + y_3 - x_1) - x_3 + (\lambda - 1)(x_4 - x_6 + 2y_6) + x_5 \leq 0 \qquad & (C_4) \\
y_3 - x_3 \leq 0 \qquad & (C_5) \\
y_6 - x_6 \leq 0 \qquad & (C_6) \\
x_1 - (\lambda - 1)(x_2 + x_5 + y_6) + \lambda(x_3 - x_4 - 2y_3) - x_6 \leq 0 \qquad & (C_7) \\
x_1 + x_2 - (1 - \lambda)x_3 - x_4 + x_5 - \lambda x_6 - \lambda y_3 - (1 - \lambda)y_6 \leq 0 \qquad & (C_8) \\
(2 - \lambda)(x_1 + x_2 + x_5) + (2\lambda - 1)x_3 - (1 + \lambda)(x_4 + x_6) - 3\lambda y_3 \leq 0 \qquad & (C_9) \\
-x_1 + (1 - \lambda)(x_2 + x_5) - \lambda(x_3 - x_4 - 2y_3) + x_6 \leq 0 \qquad & (C_{10}) \\
-\lambda(x_1 - y_3) + x_2 - x_3 - (1 - \lambda)x_4 + x_5 + x_6 \leq 0 \qquad & (C_{11}) \\
-(1 + \lambda)(x_1 + x_3) + (2 - \lambda)(x_2 + x_5 + x_6) + (2\lambda - 1)x_4 + 3\lambda y_3 \leq 0 \qquad & (C_{12}).
\end{aligned}
$$

Proof. Consider λ such that $0 \leq \lambda \leq 1$ and a profile R^N at which a is elected and a manipulation may occur in favor of a political equilibrium b. In this proof, we denote by $\left(C'_j\right)$ the inequality obtained from (C_j) by replacing " $\leq$ " with " $\geq$ ". We shall proceed in four steps.

Step 1. Let $x = (x_1, x_2, x_3, x_4, x_5, x_6)$ be the anonymous version of R^N. Then x satisfies (E_0). Since a is elected at x, then $S\left(a, \lambda, R^N\right) - S\left(b, \lambda, R^N\right) \geq 0$ and $S\left(a, \lambda, R^N\right) - S\left(c, \lambda, R^N\right) \geq 0$. These yield (C_1) and (C_2). By hypothesis, S_λ is manipulable from R^N to an equilibrium T^N by some voters, inducing the election of b. Let y_3 be the total number of bac voters who strategically report bca and let y_6 be the total number of cba voters who now report bca. Then the new profile

T^N is described by $y = (x_1, x_2, x_3 - y_3, x_4 + y_3 + y_6, x_5, x_6 - y_6)$. Note that by definition of y_3 and y_6, we have $y_3 \leq x_3$ and $y_6 \leq x_6$. Thus (C_5) and (C_6) hold. Since b is the winner at T^N, y is such that $S\left(b, \lambda, T^N\right) - S\left(a, \lambda, T^N\right) \geq 0$ and $S\left(b, \lambda, T^N\right) - S\left(c, \lambda, T^N\right) \geq 0$. Expliciting these two constraints gives (C_3) and (C_4).

Since T^N is an equilibrium, there is no profitable deviation from T^N. We now prove that x, y_3 and y_6 should satisfy at least one constraint from $(C_7 : C_9)$ and at least one constraint from $(C_{10} : C_{12})$. We proceed as follows:

Step 2. We assume that the assumption that T^N is an equilibrium is disrupted by the existence of a profitable deviation from T^N in favor of a. This is possible only if a can be elected when all *cab* voters now report *acb* while an appropriate number z_1 of *abc* voters report *acb*. The anonymous version of the new profile, say Q^N, is $z = (x_1 - z_1, x_2 + z_1 + x_5, x_3 - y_3, x_4 + y_3 + y_6, 0, x_6 - y_6)$ with $0 \leq z_1 \leq x_1$. Since a should be elected at Q^N, z_1 is such that $S\left(a, \lambda, Q^N\right) - S\left(b, \lambda, Q^N\right) \geq 0$ and $S\left(a, \lambda, Q^N\right) - S\left(c, \lambda, Q^N\right) \geq 0$. For $\lambda > 0$, rewriting these two conditions, we obtain

$$z_1 \geq \max(0, \frac{(\lambda-1)x_1 - x_2 + (1-\lambda)x_3 + x_4 - x_5 + \lambda x_6 + \lambda y_3 + (1-\lambda)y_6}{\lambda}) = z_1^-$$

and

$$z_1 \leq \min(x_1, \frac{x_1 + (1-\lambda)x_2 + \lambda x_3 - \lambda x_4 + (1-\lambda)x_5 - x_6 - 2\lambda y_3 + (1-\lambda)y_6}{\lambda}) = z_1^+.$$

Clearly, z_1 exists only if $z_1^- \leq z_1^+$. Equivalently each term from z_1^- should be less than or equal to each term from z_1^+. Thus comparing each term from z_1^- to each term from z_1^+ yields $(C_7' - C_9')$. Now for $\lambda = 0$, (C_9') is implied by (C_7') and (C_8'); and $S\left(a, \lambda, Q^N\right) - S\left(b, \lambda, Q^N\right) \geq 0$ and $S\left(a, \lambda, Q^N\right) - S\left(c, \lambda, Q^N\right) \geq 0$ immediately yield (C_7') and (C_8'). In both cases, $(C_7' - C_9')$ hold when Q^N exists. Since there should not exist a profitable deviation from T^N, at least one inequality from $(C_7' - C_9')$ does not hold. This prove that at least one inequality from $(C_7 - C_9)$ holds.

Step 3. Similarly we assume that the assumption that T^N is an equilibrium is disrupted by the existence of a profitable deviation from T^N in favor of c. To favor the election of c instead of b, all *acb* voters should report *cab*, an appropriate number z_6 of *cba* voters should now submit *cab* while the rest of *cba* voters truthfully report *cba*. The anonymous version of the new profile, say H^N, is $z = (x_1, 0, x_3 - y_3, x_4 + y_3, x_5 + x_2 + z_6, x_6 - z_6)$ with $0 \leq z_6 \leq x_6$. Since c should be elected at H^N, z_6 satisfies $S\left(c, \lambda, H^N\right) - S\left(a, \lambda, H^N\right) \geq 0$ and $S\left(c, \lambda, H^N\right) - S\left(b, \lambda, H^N\right) \geq 0$. For $\lambda > 0$, these amount to:

$$z_6 \geq \max(0, \frac{\lambda x_1 - x_2 + x_3 + (1-\lambda)x_4 - x_5 - (1-\lambda)x_6 - \lambda y_3}{\lambda}) = z_6^-$$

and

$$z_6 \leq \min(x_6, \frac{-x_1 + (1-\lambda)x_2 - \lambda x_3 + \lambda x_4 + (1-\lambda)x_5 + x_6 + 2\lambda y_3}{\lambda}) = z_6^+.$$

Clearly, z_6 exists if and only if $z_6^- \leq z_6^+$. As above, comparing each term from z_6^- to each term from z_6^+ yields $(C_{10}' - C_{12}')$. Now for $\lambda = 0$, (C_{12}') is implied by (C_{10}') and (C_{11}'); and $S\left(c, \lambda, H^N\right) - S\left(a, \lambda, H^N\right) \geq 0$ and $S\left(c, \lambda, H^N\right) - S\left(b, \lambda, H^N\right) \geq 0$ immediately yield (C_{10}') and (C_{11}'). In both cases, $(C'_{10} - C'_{12})$ hold when H^N exists. Since there should not exist a profitable deviation from T^N, at least one inequality from $(C'_{10} - C'_{12})$ does not hold. This proves that at least one inequality from $(C_{10} - C_{12})$ holds.

Step 4. We conclude from the previous steps that x, y_3 and y_6 simultaneously satisfy: (1) $(C_1 : C_6)$; (2) at least one inequality from $(C_7 : C_9)$; (3) and at least one inequality from $(C_{10} : C_{12})$.

■

From Lemma 1, we obtain nine subdomains $D_{i,j}$ with $7 \leq i \leq 9$ and $10 \leq j \leq 12$ where $D_{i,j}$ is defined by:

$$D_{i,j} = \{x : \text{ there exist two integers } y_3 \geq 0 \text{ and } y_6 \geq 0 \text{ such that } (E_0)\,,\ (C_1 : C_6)\,,\ (C_i)\,, \text{ and } (C_j)\}\,. \tag{4}$$

For each such subdomain, we provide an upper bound of the maximum and the minimum value of $x_3 + x_4 + x_6$. This yields Lemmas 2, 3 and 4 below.

Lemma 2 *Assume that* $0 \leq \lambda \leq \frac{1}{2}$. *For each of the three subdomains* $D_{7,j}$ *for* $j = 10, 11, 12$, *the maximum value* $M^*(D_{7,j})$ *and the minimum value* $m^*(D_{7,j})$ *of* $x_3 + x_4 + x_6$ *are such that :*

	$D_{7,10}$	$D_{7,11}$	$D_{7,12}$
$M^*(D_{7,j}) \leq$	$\frac{2-2\lambda-\lambda^2}{3-3\lambda}$	$\frac{2\lambda(3-\lambda)}{1+7\lambda-3\lambda^2}, \lambda_1 \leq \lambda \leq \frac{1}{2}$	$\begin{cases} \frac{2-2\lambda-\lambda^2}{3-3\lambda}, 0 \leq \lambda \leq \lambda_2 \\ \frac{6-6\lambda+\lambda^2}{3(3-2\lambda)}, \lambda_2 < \lambda \leq \frac{1}{2} \end{cases}$
$m^*(D_{7,j}) \geq$	$\frac{2-\lambda}{3}$	$\frac{\lambda^3+\lambda^2-6\lambda+2}{3-8\lambda-\lambda^2+\lambda^3}, \lambda_1 \leq \lambda \leq \frac{1}{2}$ *with* $\lambda_1 = \frac{3-\sqrt{5}}{2}$	$\frac{2-\lambda}{3}$ *with* $\lambda_2 = \frac{3-\sqrt{3}}{3}$

Proof. In the present proof, we rewrite each inequality (C_j), $j = 1, 2, ..., 12$, as an equality (E_j) by introducing a positive slack variable e_j.[7] For example, (C_1) becomes $(E_1) : (\lambda - 1)x_1 - x_2 + (1 - \lambda)x_3 + x_4 - \lambda x_5 + \lambda x_6 + e_1 = 0$.

Subdomain $D_{7,10}$: By solving $(E_0 : E_6)$, (E_7) and (E_{10}) with respect to e_5, e_6, e_3, e_4, x_1, x_2, x_3, x_6 and y_3, we obtain

$$x_3 + x_4 + x_6 = \frac{2 - 2\lambda - \lambda^2}{3 - 3\lambda} - \lambda x_4 + (\lambda - 1)\, x_5 + \frac{\lambda^2 y_6}{2\lambda - 2} + \frac{(\lambda + 1)\, e_1}{3\lambda - 3} + \frac{(2\lambda - 1)\, e_2}{3 - 3\lambda} - \frac{\lambda^2\, (e_7 + e_{10})}{2\,(1 - \lambda)^2}$$

Since each variable has a negative coefficient, we conclude that $M^*(D_{7,10}) \leq \frac{2-2\lambda-\lambda^2}{3-3\lambda}$.

Similarly, by solving $(E_0 : E_6)$, (E_7) and (E_{10}) with respect to e_1, e_6, e_2, e_3, e_{10}, x_1, x_2, x_3 and x_4, we obtain

$$x_3 + x_4 + x_6 = \frac{2 - \lambda}{3} + \frac{1}{3}e_4 + \frac{2}{3}e_7 + \frac{1 - \lambda}{3}x_5 + \lambda e_5$$

[7] The same set of notations will also be used to prove subsequent propositions.

Since each variable has a positive coefficient, we conclude that $m^*(D_{7,10}) \geq \frac{2-\lambda}{3}$.

Subdomain $D_{7,11}$: It appears that $D_{7,11}$ is not feasible when $0 \leq \lambda < \frac{3-\sqrt{5}}{2}$. To see this, first assume that $0 \leq \lambda \leq \lambda_0$ where $\lambda_0 \approx 0.36445$ is the unique solution to $3 - 8\lambda - \lambda^2 + \lambda^3 = 0$ that belongs to $[0,1]$. By combining (C_1), (C_5), (C_7) and (C_{11}) as

$$(C_1) + \frac{\lambda\left(3+\lambda-4\lambda^2\right)}{3-2\lambda+\lambda^2}(C_5) + \frac{3-2\lambda-2\lambda^2}{3-2\lambda+\lambda^2}(C_7) + \frac{3-5\lambda}{3-2\lambda+\lambda^2}(C_{11})$$

and since each coefficient of this combination is positive, it appears that:

$$\frac{\lambda^3}{\lambda^2-2\lambda+3} + \lambda x_4 + \frac{\left(3-8\lambda-\lambda^2+\lambda^3\right)x_2 + \left(6-13\lambda+2\lambda^2\right)x_5 + (1-\lambda)\left(3-2\lambda-2\lambda^2\right)y_6}{3-2\lambda+\lambda^2} \leq 0.$$

This is a contradiction since the left hand side of this inequality is positive for $0 \leq \lambda \leq \lambda_0$.

First assume that $\lambda_0 < \lambda < \lambda_1 = \frac{3-\sqrt{5}}{2}$. By combining (E_1), (E_5), (E_6), (E_7) and (E_{11}) as

$$(C_1) + \frac{\lambda\left(4-5\lambda+\lambda^2\right)}{1+3\lambda-\lambda^2}(C_5) + \frac{8\lambda+\lambda^2-\lambda^3-3}{1+3\lambda-\lambda^2}(C_6) + \frac{2-\lambda}{1+3\lambda-\lambda^2}(C_7) + \frac{\lambda(3-\lambda)}{1+3\lambda-\lambda^2}(C_{11})$$

and since each coefficient of this combination is positive, it follows that

$$\lambda x_4 + (1-\lambda)x_5 + \frac{2\lambda^2-\lambda^3+5\lambda-1}{3\lambda-\lambda^2+1}y_6 + \frac{\lambda^2-3\lambda+1}{3\lambda-\lambda^2+1} \leq 0.$$

This is also a contradiction since the left hand side of this inequality is positive for $\lambda_0 < \lambda < \lambda_1$.

Now assume that $\lambda_1 \leq \lambda \leq \frac{1}{2}$. By solving $(E_0 : E_6)$, (E_7) and (E_{11}) with respect to e_5, e_6, e_2, e_3, e_4, x_2, x_3, x_6 and y_3, we obtain

$$\begin{aligned} x_3 + x_4 + x_6 \quad = \quad & \frac{2\lambda(3-\lambda)}{1+7\lambda-3\lambda^2} + \frac{\left(2-6\lambda-\lambda^2\right)x_1 + \lambda(\lambda-3)x_4 + (1-\lambda)(\lambda-3)x_5}{1+7\lambda-3\lambda^2} \\ & + \frac{(1-2\lambda)(\lambda-1)y_6 + (\lambda-3)e_1 + (2\lambda-1)e_7 + (4\lambda-2)e_{11}}{1+7\lambda-3\lambda^2} \end{aligned}$$

Since each variable has a negative coefficient, we conclude that $M^*(D_{7,11}) \leq \frac{2\lambda(3-\lambda)}{1+7\lambda-3\lambda^2}$.

Similarly, by solving $(E_0 : E_6)$, (E_7) and (E_{11}) with respect to e_6, e_2, e_3, e_4, x_1, x_2, x_3, x_6 and y_3, we obtain:

$$\begin{aligned} x_3 + x_4 + x_6 \quad = \quad & \frac{\lambda^3+\lambda^2-6\lambda+2}{3-8\lambda-\lambda^2+\lambda^3} + \frac{\left(\lambda-2\lambda^2\right)x_4 + \left(2\lambda^2-3\lambda+1\right)x_5 + \left(\lambda^2+3\lambda-1\right)e_7}{\lambda^2+8\lambda-3-\lambda^3} \\ & + \frac{(1-2\lambda)e_1 + \left(\lambda^3+6\lambda^2-2\lambda\right)e_5 + \left(4\lambda-1-2\lambda^2-\lambda^3\right)y_6 + \lambda^2 e_{11}}{\lambda^2+8\lambda-3-\lambda^3}. \end{aligned}$$

Since each variable has a non-negative coefficient, we conclude that $m^*(D_{7,11}) \geq \frac{\lambda^3+\lambda^2-6\lambda+2}{3-8\lambda-\lambda^2+\lambda^3}$.

Subdomain $D_{7,12}$: By solving $(E_0 : E_6)$, (E_7) and (E_{12}) with respect to e_1, e_{12}, e_6, e_2, e_3, x_1, x_2, x_3 and x_4, we obtain:

$$x_3 + x_4 + x_6 = \frac{2-\lambda}{3} + \frac{1-\lambda}{3}x_5 + \frac{1}{3}e_4 + \lambda e_5 + \frac{2}{3}e_7$$

Each variable has a non-negative coeffocient. Thus $m^*(D_{7,12}) \geq \frac{2-\lambda}{3}$.

About $M^*(D_{7,12})$, first assume that $0 \leq \lambda \leq \lambda_2 = \frac{3-\sqrt{3}}{3}$. Then by solving $(E_0 : E_6)$, (E_7) and (E_{12}) with respect to e_5, e_6, e_2, e_3, e_4, x_1, x_3, x_6 and y_3, we obtain:

$$\begin{aligned} x_3 + x_4 + x_6 &= \frac{2-2\lambda-\lambda^2}{3-3\lambda} + \frac{\left(2-6\lambda+3\lambda^2\right)x_2 + (2\lambda-1)(2\lambda-3)x_5 + (1-2\lambda)e_7}{\lambda-1} - \lambda x_4 \\ &\quad + (2\lambda-1)y_6 - e_1 - \frac{2}{3}\frac{(2\lambda-1)e_{12}}{\lambda-1}. \end{aligned}$$

Each variable has a non-positive coefficient. Thus $M^*(D_{7,12}) \leq \frac{2-2\lambda-\lambda^2}{3-3\lambda}$ for $0 \leq \lambda \leq \lambda_2$.

Now assume that $\lambda_2 < \lambda \leq \frac{1}{2}$. Similarly, by solving $(E_0 : E_6)$, (E_7) and (E_{12}) with respect to e_6, e_2, e_3, e_4, x_1, x_2, x_3, x_6 and y_3, we obtain:

$$\begin{aligned} x_3 + x_4 + x_6 &= \frac{6-6\lambda+\lambda^2}{3(3-2\lambda)} - \frac{\lambda(\lambda-1)^2 x_4 + (1-\lambda)^3 x_5 + (1-\lambda)\left(3\lambda-\lambda^2-1\right)y_6}{(2\lambda-1)(2\lambda-3)} \\ &\quad - \frac{(\lambda-1)^2 e_1 + \left(3\lambda-\lambda^2-1\right)e_7 + \lambda\left(6\lambda-3\lambda^2-2\right)e_5}{(2\lambda-1)(2\lambda-3)} - \frac{1}{3}\frac{\lambda^2 e_{12}}{(2\lambda-1)(2\lambda-3)}. \end{aligned}$$

Each variable has a non-positive coefficient. Thus $M^*(D_{7,12}) \leq \frac{6-6\lambda+\lambda^2}{3(3-2\lambda)}$ for $\lambda_2 < \lambda \leq \frac{1}{2}$. ■

The next lemma concerns the subdomains $D_{8,j}$, $j = 10, 11, 12$.

Lemma 3 *Assume that $0 \leq \lambda \leq \frac{1}{2}$. For each of the three subdomains $D_{8,j}$, $j = 10, 11, 12$, the maximum value $M^*(D_{i,j})$ and the minimum value $m^*(D_{i,j})$ of $x_3 + x_4 + x_6$ are such that:*

	$D_{8,10}$	$D_{8,11}$	$D_{8,12}$
$M^*(D_{8,j}) \leq$	$\begin{cases} \frac{2-2\lambda-\lambda^2}{3-3\lambda}, 0 \leq \lambda \leq \lambda_3 \\ \frac{2-4\lambda-\lambda^2+\lambda^3}{4-10\lambda+3\lambda^2}, \lambda_3 < \lambda \leq \lambda_1 \\ \frac{2\lambda(2-\lambda)}{3\lambda-\lambda^2+1}, \lambda_1 < \lambda \leq \frac{1}{2} \end{cases}$ *with* $\lambda_3 = 3-\sqrt{7}, \lambda_1 = \frac{3-\sqrt{5}}{2}$	$\begin{cases} \frac{\lambda+2\lambda^2-2}{(2\lambda+3)(\lambda-1)}, 0 \leq \lambda \leq \lambda_4 \\ \frac{2\lambda+1}{2\lambda+2}, \lambda_4 < \lambda \leq \frac{1}{2} \end{cases}$ $\lambda_4 = \frac{\sqrt{17}-3}{4}$	$\begin{cases} \frac{2-2\lambda-\lambda^2}{3-3\lambda}, 0 \leq \lambda \leq \frac{1}{3} \\ \frac{1+5\lambda-2\lambda^2}{3+3\lambda}, \frac{1}{3} < \lambda \leq \frac{1}{2} \end{cases}$
$m^*(D_{8,j}) \geq$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Proof. Subdomain $D_{8,10}$: By solving $(E_0 : E_6)$, (E_8) and (E_{10}) with respect to e_1, e_2, e_3, e_4, e_{10}, x_1, x_3, x_4 and x_6, we obtain:

$$x_3 + x_4 + x_6 = \frac{1}{2} + \frac{1}{2}e_8 + \frac{1}{2}\lambda e_5 + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)e_6.$$

Therefore $m^*(D_{8,10}) \geq \frac{1}{2}$. To deal with $M^*(D_{8,10})$, we distinguish three cases.

First assume that $0 \leq \lambda \leq \lambda_3 = 3 - \sqrt{7}$. By solving $(E_0 : E_6)$, (E_8) and (E_{10}) with respect to e_5, e_6, e_2, e_3, e_4, x_1,x_3, x_6 and y_6, we obtain:

$$\begin{aligned} x_3 + x_4 + x_6 &= \frac{\lambda^2 + 2\lambda - 2}{3\lambda - 3} + \frac{\left(\lambda^2 - 6\lambda + 2\right) x_2}{3\lambda - 3} - \lambda x_4 - \frac{2\lambda - 1}{\lambda - 1} x_5 \\ &\quad - \frac{2}{3} \frac{\lambda (2\lambda - 1)}{\lambda - 1} y_3 + \frac{\lambda + 1}{3\lambda - 3} e_1 - \frac{1}{3} \frac{2\lambda - 1}{\lambda - 1} e_{10}. \end{aligned}$$

Thus $M^* (D_{8,10}) \leq \frac{2-2\lambda-\lambda^2}{3-3\lambda}$ for $0 \leq \lambda \leq \lambda_3$.

Now assume that $\lambda_3 < \lambda \leq \lambda_1 = \frac{3-\sqrt{5}}{2}$. By solving $(E_0 : E_6)$, (E_8) and (E_{10}) with respect to $x_6, x_3, e_5, e_2, e_3, e_4, x_1, x_2$ and y_6, we obtain:

$$\begin{aligned} x_3 + x_4 + x_6 &= \frac{2 - 4\lambda - \lambda^2 + \lambda^3}{4 - 10\lambda + 3\lambda^2} - \frac{2\lambda (\lambda - 1)^2 x_4}{-10\lambda + 3\lambda^2 + 4} - \frac{\lambda (\lambda - 1)(\lambda - 2) x_5}{-10\lambda + 3\lambda^2 + 4} - \frac{\lambda \left(2 - 6\lambda + 3\lambda^2\right) y_3}{-10\lambda + 3\lambda^2 + 4} \\ &\quad - \frac{\lambda (2 - \lambda) e_1}{-10\lambda + 3\lambda^2 + 4} + \frac{\left(\lambda^2 - 6\lambda + 2\right) e_8}{3\lambda^2 - 10\lambda + 4} - \frac{(\lambda - 1)\left(2 - 6\lambda + \lambda^2\right) e_6}{-10\lambda + 3\lambda^2 + 4} - \frac{\lambda^2 e_{10}}{3\lambda^2 - 10\lambda + 4}. \end{aligned}$$

Thus $M^* (D_{8,10}) \leq \frac{2-4\lambda-\lambda^2+\lambda^3}{4-10\lambda+3\lambda^2}$ for $\lambda_3 < \lambda \leq \lambda_1$.

Finally, assume that $\lambda_1 < \lambda \leq \frac{1}{2}$. By solving $(E_0 : E_6)$, (E_8) and (E_{10}) with respect to e_5, e_6, e_2, e_3, e_4, e_8, x_2, x_3 and x_6, we obtain:

$$\begin{aligned} x_3 + x_4 + x_6 &= \frac{2\lambda (2 - \lambda)}{3\lambda - \lambda^2 + 1} + \frac{\lambda^2 - 6\lambda + 2}{3\lambda - \lambda^2 + 1} x_1 - \frac{3\lambda - 3\lambda^2}{3\lambda - \lambda^2 + 1} x_4 + \frac{\lambda^2 - 1}{3\lambda - \lambda^2 + 1} x_5 \\ &\quad - \frac{2\lambda - 4\lambda^2}{3\lambda - \lambda^2 + 1} y_3 - \frac{\lambda + 1}{3\lambda - \lambda^2 + 1} e_1 + \frac{2\lambda - 1}{3\lambda - \lambda^2 + 1} e_{10}. \end{aligned}$$

Subdomain $D_{8,11}$: By solving $(E_0 : E_6)$, (E_8) and (E_{11}) with respect to e_1, e_2, e_3, e_4, e_{11}, x_1, x_4, x_3 and x_6, we obtain:

$$x_3 + x_4 + x_6 = \frac{1}{2} + \frac{1}{2} e_8 + \frac{\lambda}{2} e_5 + \left(\frac{1}{2} - \frac{\lambda}{2}\right) e_6.$$

Therefore $m^* (D_{8,11}) \geq \frac{1}{2}$.

To deal with $M^* (D_{8,11})$, first assume that $0 \leq \lambda \leq \lambda_4 = \frac{\sqrt{17}-3}{4}$. By solving $(E_0 : E_6)$, (E_8) and (E_{11}) with respect to e_5, e_6, e_2, e_3, e_4, e_8, x_1, x_3 and x_6, we obtain:

$$\begin{aligned} x_3 + x_4 + x_6 &= \frac{\lambda + 2\lambda^2 - 2}{(2\lambda + 3)(\lambda - 1)} - \frac{3\lambda + 2\lambda^2 - 1}{(2\lambda + 3)(\lambda - 1)} x_2 - \frac{\lambda (2\lambda - 3)}{(2\lambda + 3)(\lambda - 1)} x_4 \\ &\quad - \frac{(\lambda + 3)(2\lambda - 1)}{(2\lambda + 3)(\lambda - 1)} x_5 - \frac{\lambda (2\lambda - 1)}{(2\lambda + 3)(\lambda - 1)} y_3 + \frac{2}{(2\lambda + 3)(\lambda - 1)} e_1 - \frac{2\lambda - 1}{2\lambda^2 + \lambda - 3} e_{11}. \end{aligned}$$

Since each variable has a non-negative coefficient, we conclude that $M^* (D_{8,11}) \leq \frac{\lambda+2\lambda^2-2}{(2\lambda+3)(\lambda-1)}$ for $0 \leq \lambda \leq \lambda_4 = \frac{\sqrt{17}-3}{4}$.

Now assume that $\lambda_4 < \lambda \leq \frac{1}{2}$. By solving $(E_0 : E_6)$, (E_8) and (E_{11}) with respect to e_5, e_6, e_2, e_3, e_4, e_8, x_2, x_3 and x_6, we obtain:

$$\begin{aligned} x_3 + x_4 + x_6 &= \frac{2\lambda+1}{2\lambda+2} - \frac{1}{2}\frac{3\lambda+2\lambda^2-1}{\lambda+1}x_1 + \frac{1}{2}\frac{\lambda(2\lambda-3)}{\lambda+1}x_4 \\ &\quad + \frac{\lambda-1}{\lambda+1}x_5 + \frac{1}{2}\frac{\lambda(2\lambda-1)}{\lambda+1}y_3 - \frac{e_1}{\lambda+1} + \frac{2\lambda-1}{2\lambda+2}e_{11}. \end{aligned}$$

Therefore $M^*(D_{8,11}) \leq \frac{2\lambda+1}{2\lambda+2}$ for $\lambda_4 < \lambda \leq \frac{1}{2}$.

Subdomain $D_{8,12}$: By solving $(E_0 : E_6)$, (E_8) and (E_{12}) with respect to e_1, e_{12}, e_2, e_3, e_4, x_1, x_3, x_4 and x_6, we obtain:

$$x_3 + x_4 + x_6 = \frac{1}{2} + \frac{1}{2}e_8 + \frac{\lambda}{2}e_5 + \left(\frac{1}{2} - \frac{\lambda}{2}\right)e_6.$$

Thus $m * (D_{8,12}) \geq \frac{1}{2}$.

About $M^*(D_{8,12})$, first assume that $0 \leq \lambda \leq \frac{1}{3}$. Then by solving $(E_0 : E_6)$, (E_8) and (E_{12}) with respect to e_5, e_6, e_2, e_3, e_4, e_8, x_1, x_3 and x_6, we obtain:

$$\begin{aligned} x_3 + x_4 + x_6 &= \frac{2-2\lambda-\lambda^2}{3-3\lambda} + \frac{1}{2}\frac{1-3\lambda}{\lambda-1}x_2 - \lambda x_4 - \frac{2\lambda-1}{\lambda-1}x_5 \\ &\quad - \frac{\lambda}{2}\frac{(2\lambda-1)}{\lambda-1}y_3 + \frac{1}{2}\frac{1}{\lambda-1}e_1 - \frac{1}{6}\frac{2\lambda-1}{\lambda-1}e_{12}. \end{aligned}$$

Thus $M^*(D_{8,12}) \leq \frac{2-2\lambda-\lambda^2}{3-3\lambda}$ for $0 \leq \lambda \leq \frac{1}{3}$.

Now assume that $\frac{1}{3} < \lambda \leq \frac{1}{2}$. By solving $(E_0 : E_6)$, (E_8) and (E_{12}) with respect to e_5, e_6, e_2, e_3, e_4, e_8, x_2, x_3 and x_6, we obtain:

$$\begin{aligned} x_3 + x_4 + x_6 &= \frac{1+5\lambda-2\lambda^2}{3+3\lambda} + \frac{1-3\lambda}{\lambda+1}x_1 + \frac{2\lambda(\lambda-1)}{\lambda+1}x_4 + \frac{\lambda-1}{\lambda+1}x_5 \\ &\quad + \frac{\lambda(2\lambda-1)}{\lambda+1}y_3 - \frac{1}{\lambda+1}e_1 + \frac{1}{3}\frac{2\lambda-1}{\lambda+1}e_{12}. \end{aligned}$$

Therefore $M^*(D_{8,12}) \leq \frac{1+5\lambda-2\lambda^2}{3+3\lambda}$ for $\frac{1}{3} < \lambda \leq \frac{1}{2}$. ■

The next lemma concerns the subdomains $D_{9,j}$, $j = 10, 11, 12$.

Lemma 4 *Assume that $0 \leq \lambda \leq \frac{1}{2}$. For each of the three subdomains $D_{9,j}$, $j = 10, 11, 12$, the maximum value $M^*(D_{i,j})$ and the minimum value $m^*(D_{i,j})$ of $x_3 + x_4 + x_6$ are such that :*

	$D_{9,10}$	$D_{9,11}$	$D_{9,12}$
$M^*(D_{9,j}) \leq$	$\begin{cases} \frac{2-2\lambda-\lambda^2}{3-3\lambda}, 0 \leq \lambda \leq \lambda_2 \\ \frac{1}{3}\frac{-10\lambda+5\lambda^2+6}{(\lambda-1)(3\lambda-4)}, \lambda_2 < \lambda \leq \frac{1}{2} \end{cases}$ *with* $\lambda_2 = \frac{3-\sqrt{3}}{3}$	$\begin{cases} \frac{1}{6}\frac{5\lambda-4}{\lambda-1}, 0 \leq \lambda \leq \lambda_5 \\ \frac{1}{3}\frac{1+8\lambda-2\lambda^2}{1+3\lambda-\lambda^2}, \lambda_5 < \lambda \leq \frac{1}{2} \end{cases}$ *with* $\lambda_5 = \frac{\sqrt{13}-3}{2}$	$\begin{cases} \frac{1}{3}\frac{2\lambda+\lambda^2-2}{\lambda-1}, 0 \leq \lambda \leq \lambda_1 \\ \frac{1}{3}\frac{3-4\lambda+2\lambda^2}{(\lambda-1)(\lambda-2)}, \lambda_1 < \lambda \leq \frac{1}{2} \end{cases}$ *with* $\lambda_1 = \frac{3-\sqrt{5}}{2}$
$m^*(D_{9,j}) \geq$	$\frac{2-\lambda}{3}$	$\frac{2-\lambda}{3}$	$\frac{2-\lambda}{3}$

Proof. Subdomain $D_{9,10}$: By solving $(E_0 : E_6)$, (E_9) and (E_{10}) with respect to e_1, e_6, e_2, e_3, e_4, e_{10}, x_1, x_3 and x_4, we obtain:

$$x_3 + x_4 + x_6 = \frac{2-\lambda}{3} + \lambda e_5 + \frac{1}{3} e_9.$$

Therefore $m^*(D_{9,10}) \geq \frac{2-\lambda}{3}$.

To deal with $M^*(D_{9,10})$, we distinguish two cases.

First assume that $0 \leq \lambda \leq \lambda_2 = \frac{3-\sqrt{3}}{3}$. By solving $(E_0 : E_6)$, (E_9) and (E_{10}) with respect to e_5, e_6, e_2, e_3, e_4, x_1, x_3, x_6 and y_3, we obtain:

$$\begin{aligned} x_3 + x_4 + x_6 &= \frac{1}{3}\frac{2\lambda+\lambda^2-2}{\lambda-1} + \frac{-6\lambda+3\lambda^2+2}{\lambda-1}x_2 - \lambda x_4 \\ &\quad + \frac{(2\lambda-1)(2\lambda-3)}{\lambda-1}x_5 - e_1 - \frac{2}{3}\frac{2\lambda-1}{\lambda-1}e_9 - \frac{2\lambda-1}{\lambda-1}e_{10}. \end{aligned}$$

Thus $M^*(D_{9,10}) \leq \frac{2-2\lambda-\lambda^2}{3-3\lambda}$ for $0 \leq \lambda \leq \lambda_2$.

Now assume that $\lambda_2 < \lambda \leq \frac{1}{2}$. By solving $(E_0 : E_6)$, (E_9) and (E_{10}) with respect to e_5, e_6, e_2, e_3, e_4, x_1, x_2, x_3 and y_3, we obtain:

$$\begin{aligned} x_3 + x_4 + x_6 &= \frac{1}{3}\frac{-10\lambda+5\lambda^2+6}{(\lambda-1)(3\lambda-4)} - \frac{\lambda e_{10}}{(\lambda-1)(3\lambda-4)} + \frac{-6\lambda+3\lambda^2+2}{(\lambda-1)(3\lambda-4)}x_6 \\ &\quad + \frac{\lambda(\lambda-2)x_4}{(\lambda-1)(3\lambda-4)} - \frac{\lambda-2}{3\lambda-4}x_5 + \frac{\lambda-2}{(\lambda-1)(3\lambda-4)}e_1 - \frac{2}{3}\frac{\lambda e_9}{(\lambda-1)(3\lambda-4)}. \end{aligned}$$

Thus $M^*(D_{9,10}) \leq \frac{1}{3}\frac{-10\lambda+5\lambda^2+6}{(\lambda-1)(3\lambda-4)}$ for $\lambda_2 < \lambda \leq \frac{1}{2}$.

Subdomain $D_{9,11}$: By solving $(E_0 : E_6)$, (E_9) and (E_{11}) with respect to e_1, e_6, e_2, e_3, e_4, e_{11}, x_1, x_3 and x_4, we obtain:

$$x_3 + x_4 + x_6 = \frac{2-\lambda}{3} + \frac{1}{3}e_9 + \lambda e_5.$$

Therefore $m^*(D_{9,11}) \geq \frac{2-\lambda}{3}$.

To deal with $M^*(D_{9,11})$, first assume that $0 \leq \lambda \leq \lambda_5 = \frac{\sqrt{13}-3}{2}$. By solving $(E_0 : E_6)$, (E_9) and (E_{11}) with respect to e_5, e_6, e_2, e_3, e_4, x_1, x_3, x_6 and y_3, we obtain:

$$\begin{aligned} x_3 + x_4 + x_6 &= \frac{1}{6}\frac{5\lambda-4}{\lambda-1} - \frac{1}{2}\frac{\left(3\lambda+\lambda^2-1\right)x_2}{(\lambda-1)(\lambda+1)} - \frac{1}{2}\frac{\lambda(\lambda-2)x_4}{(\lambda-1)(\lambda+1)} - \frac{1}{6}\frac{(2\lambda-1)e_9}{(\lambda-1)(\lambda+1)} \\ &\quad - \frac{3}{2}\frac{2\lambda-1}{(\lambda-1)(\lambda+1)}x_5 - \frac{1}{2}\frac{\lambda-2}{(\lambda-1)(\lambda+1)}e_1 - \frac{1}{2}\frac{2\lambda-1}{(\lambda-1)(\lambda+1)}e_{11}. \end{aligned}$$

Since each variable has a non positive coefficient, we conclude that $M^*(D_{9,11}) \leq \frac{1}{6}\frac{5\lambda-4}{\lambda-1}$ for $0 \leq \lambda \leq \lambda_5 = \frac{\sqrt{13}-3}{2}$.

Now assume that $\lambda_5 < \lambda \leq \frac{1}{2}$. By solving $(E_0 : E_6)$, (E_9) and (E_{11}) with respect to e_5, e_6, e_2, e_3, e_4, x_2, x_3, x_6 and y_3, we obtain:

$$\begin{aligned} x_3 + x_4 + x_6 &= \frac{1}{3}\frac{1+8\lambda-2\lambda^2}{1+3\lambda-\lambda^2} + \frac{\left(3\lambda+\lambda^2-1\right)x_1}{-3\lambda+\lambda^2-1} + \frac{\lambda\left(\lambda-2\right)x_4}{3\lambda-\lambda^2+1} + \frac{\left(\lambda-1\right)\left(\lambda-2\right)x_5}{-3\lambda+\lambda^2-1} \\ &\quad + \frac{\lambda-2}{3\lambda-\lambda^2+1}e_1 - \frac{1}{3}\frac{2\lambda-1}{-3\lambda+\lambda^2-1}e_9 + \frac{2\lambda-1}{3\lambda-\lambda^2+1}e_{11}. \end{aligned}$$

Therefore $M^*\left(D_{9,11}\right) \leq \frac{1}{3}\frac{1+8\lambda-2\lambda^2}{1+3\lambda-\lambda^2}$ for $\lambda_5 < \lambda \leq \frac{1}{2}$.

Subdomain $D_{9,12}$: By solving $(E_0 : E_6)$, (E_9) and (E_{12}) with respect to e_1, e_{12}, e_6, e_2, e_3, e_4, x_1, x_3 and x_4, we obtain:

$$x_3 + x_4 + x_6 = \frac{2-\lambda}{3} + \frac{1}{3}e_9 + \lambda e_5.$$

Therefore $m^*\left(D_{9,12}\right) \geq \frac{2-\lambda}{3}$.

About $M^*\left(D_{9,12}\right)$, first assume that $0 \leq \lambda \leq \lambda_1 = \frac{3-\sqrt{5}}{2}$. Then by solving $(E_0 : E_6)$, (E_9) and (E_{12}) with respect to e_5, e_6, e_2, e_3, e_4, e_8, x_1, x_3, x_6 and y_3, we obtain:

$$\begin{aligned} x_3 + x_4 + x_6 &= \frac{1}{3}\frac{2\lambda+\lambda^2-2}{\lambda-1} + \frac{-3\lambda+\lambda^2+1}{\lambda-1}x_2 - \lambda x_4 - e_1 \\ &\quad + \frac{\left(2\lambda-1\right)\left(\lambda-2\right)}{\lambda-1}x_5 - \frac{1}{3}\frac{2\lambda-1}{\lambda-1}e_9 - \frac{1}{3}\frac{2\lambda-1}{\lambda-1}e_{12}. \end{aligned}$$

Thus $M^*\left(D_{9,12}\right) \leq \frac{2-2\lambda-\lambda^2}{3-3\lambda}$ for $0 \leq \lambda \leq \lambda_1$.

Now assume that $\lambda_1 < \lambda \leq \frac{1}{2}$. By solving $(E_0 : E_6)$, (E_9) and (E_{12}) with respect to e_5, e_6, e_2, e_3, e_4, x_1, x_2, x_3 and y_3, we obtain:

$$\begin{aligned} x_3 + x_4 + x_6 &= \frac{1}{3}\frac{3-4\lambda+2\lambda^2}{\left(\lambda-1\right)\left(\lambda-2\right)} - \frac{\lambda x_4}{\left(\lambda-1\right)\left(\lambda-2\right)} - \frac{1}{3}\frac{\lambda e_{12}}{\left(\lambda-1\right)\left(\lambda-2\right)} \\ &\quad + \frac{1-3\lambda+\lambda^2}{\left(\lambda-1\right)\left(\lambda-2\right)}x_6 - \frac{1}{3}\frac{\lambda}{\left(\lambda-1\right)\left(\lambda-2\right)}e_9 + \frac{x_5}{\lambda-2} - \frac{e_1}{\left(\lambda-1\right)\left(\lambda-2\right)}. \end{aligned}$$

Therefore $M^*\left(D_{9,12}\right) \leq \frac{1}{3}\frac{3-4\lambda+2\lambda^2}{(\lambda-1)(\lambda-2)}$ for $\lambda_1 < \lambda \leq \frac{1}{2}$. ■

We are now ready to prove Proposition 1.

Proof. (**Proposition 1**) Suppose that S_λ is manipulable at a given profile R^N. Without loss of generality, assume that a is elected at R^N and that a manipulation can occur in favor of a political equilibrium b. Then the anonymous version $x = (x_1, x_2, x_3, x_4, x_5, x_6)$ belongs to at least one of the nine subdomains $D_{i,j}$ described at (4). Thus:

$$M^*\left(\lambda, n\right) \leq \max_{7 \leq i \leq 9, 10 \leq j \leq 12} M^*\left(D_{i,j}\right).$$

Using bounds provided by Lemmas 2, 3 and 4, we deduce[8] that $M^*\left(\lambda, n\right) \leq \frac{2-2\lambda-\lambda^2}{3-3\lambda}$ if $\frac{2-2\lambda-\lambda^2}{3-3\lambda} \geq \frac{1+2\lambda}{2+2\lambda}$ and $M^*\left(\lambda, n\right) \leq \frac{1+2\lambda}{2+2\lambda}$ if $\frac{2-2\lambda-\lambda^2}{3-3\lambda} \leq \frac{1+2\lambda}{2+2\lambda}$. Note that for $0 \leq \lambda \leq \frac{1}{2}$, $\frac{2-2\lambda-\lambda^2}{3-3\lambda} \geq \frac{1+2\lambda}{2+2\lambda}$ is equivalent

[8] A simple way to see this is to sketch the curve of $M^*\left(D_{i,j}\right)$ for each subdomain in order to deduce the maximum bound.

to $0 \leq \lambda \leq \lambda^*$. ■

7.2 Proof of Proposition 2

Proof. The proof is done in three steps.

Step 1. We show that $M^*(\lambda, n) > \frac{1}{n}\left\lfloor \frac{2-2\lambda-\lambda^2}{3-3\lambda} n \right\rfloor - \frac{2}{n}$ for $0 \leq \lambda \leq \lambda^*$ and $n \geq 21$. Assume that $0 \leq \lambda \leq \lambda^*$ and $n \geq 21$. Let $q = \left\lfloor \frac{1+\lambda}{3} n \right\rfloor$ and $p = \left\lfloor \frac{1-2\lambda}{3-3\lambda} n + \frac{1+\lambda}{3} n \right\rfloor - q$. Note that $\frac{1-2\lambda}{3-3\lambda}$ is decreasing as λ increases from 0 to λ^*. Thus $\frac{1-2\lambda}{3-3\lambda} n \geq \frac{1-2\lambda^*}{3-3\lambda^*} n > 3$. Thus $p \geq \left\lfloor 3 + \frac{1+\lambda}{3} n \right\rfloor - q = 3$ and p is a positive integer. Consider the following profiles:

Type	Initial profile R^N			Strategic profiles T^N		
Preferences	a	b	c	a	b	b
	b	a	b	b	a	c
	c	c	a	c	c	a
Number of voters	$n+1-p-q$	$p-1$	q	$n+1-p-q$	$p-1$	q

Assume that the sincere profile R^N is described by the initial profile above. Then $S(a, \lambda, R^N) - S(b, \lambda, R^N) = n + 1 - p - q + \lambda(p-1) - p + 1 - \lambda(n+1-p) = (1-\lambda)n - (2-2\lambda)p - q + 2 - 2\lambda$.

Since $1 - \lambda > 0$ and $p \leq \frac{1-2\lambda}{3-3\lambda} n + \frac{1+\lambda}{3} n - q$, we have $S(a, \lambda, R^N) - S(b, \lambda, R^N) \geq (1-\lambda)n - (2-2\lambda)\left(\frac{1-2\lambda}{3-3\lambda} n + \frac{1+\lambda}{3} n - q\right) - q + 2 - 2\lambda = \left(\frac{2}{3}\lambda^2 + \frac{1}{3}\lambda - \frac{1}{3}\right)n + (1-2\lambda)q + (2-2\lambda)$. Moreover $q > \frac{1+\lambda}{3} n - 1$ and $1 - 2\lambda > 0$. Thus $S(a, \lambda, R^N) - S(b, \lambda, R^N) > \left(\frac{2}{3}\lambda^2 + \frac{1}{3}\lambda - \frac{1}{3}\right)n + (1-2\lambda)\left(\frac{1+\lambda}{3} n - 1\right) + (2-2\lambda) = 1$. Similarly $S(a, \lambda, R^N) - S(c, \lambda, R^N) = n + 1 - p - q + \lambda(p-1) - q = n - (1-\lambda)p - 2q - \lambda + 1 \geq n - (1-\lambda)\left(\frac{1-2\lambda}{3-3\lambda} n + \frac{1+\lambda}{3} n - q\right) - 2q + 1 - \lambda = \left(\frac{1}{3}\lambda^2 + \frac{2}{3}\lambda + \frac{1}{3}\right)n - (1+\lambda)q + (1-\lambda)$. Hence $S(a, \lambda, R^N) - S(c, \lambda, R^N) \geq \left(\frac{1}{3}\lambda^2 + \frac{2}{3}\lambda + \frac{1}{3}\right)n - (1+\lambda)\left(\frac{1+\lambda}{3} n\right) + (1-\lambda) = 1 - \lambda > 0$. Therefore a wins at R^N.

Now suppose that all cba voters strategically submit bca. The new profile is described by the strategic profile T^N above. At T^N, c is Pareto dominated by b. Moreover, $S(b, \lambda, T^N) - S(a, \lambda, T^N) = p + q - 1 + \lambda(n+1-p-q) - (n+1-p-q) - \lambda(p-1) = (\lambda-1)n + (2-2\lambda)p + (2-\lambda)q + 2\lambda - 2$. By definition of p and q, $p = \left\lfloor \frac{1-2\lambda}{3-3\lambda} n + \frac{1+\lambda}{3} n \right\rfloor - q > \frac{1-2\lambda}{3-3\lambda} n + \frac{1+\lambda}{3} n - 1 - q$. Thus $S(b, \lambda, T^N) - S(a, \lambda, T^N) > (\lambda-1)n + (2-2\lambda)\left(\frac{1-2\lambda}{3-3\lambda} n + \frac{1+\lambda}{3} n - 1 - q\right) + (2-\lambda)q + 2\lambda - 2 = \left(\frac{1}{3} - \frac{1}{3}\lambda - \frac{2}{3}\lambda^2\right)n + \lambda q + (4\lambda - 4) \geq \left(\frac{1}{3} - \frac{1}{3}\lambda - \frac{2}{3}\lambda^2\right)n + \lambda\left(\frac{1+\lambda}{3} n - 1\right) + 4\lambda - 4 = \left(\frac{1}{3} - \frac{1}{3}\lambda^2\right)n + 3\lambda - 4 \geq 20\left(\frac{1}{3} - \frac{1}{3}\lambda^2\right) + 3\lambda - 4$
$= \frac{8+9\lambda-20\lambda^2}{3} \geq \frac{8}{3}$ for $0 \leq \lambda \leq \lambda^*$. It follows that $S(b, \lambda, T^N) - S(a, \lambda, T^N) > 0$. Therefore b wins at the new profile T^N.

Finally let us prove that T^N is an equilibrium. Note that strategic voting from abc voters may occur only in favor of a while all cba voters would like to favor the election of c. We first assume that from T^N, all abc voters deviate by submitting acb. At the new profile Q^N, $S(b, \lambda, Q^N) - S(a, \lambda, Q^N) = p + q - 1 - (n+1-p-q) - \lambda(p-1) = -n + (2-\lambda)p + 2q + \lambda - 2 > -n + (2-\lambda)\left(\frac{1-2\lambda}{3-3\lambda} n + \frac{1+\lambda}{3} n - 1 - q\right) + 2q + \lambda - 2 = \frac{1-3\lambda+\lambda^3}{3-3\lambda} n + \lambda q + 2(\lambda-2) \geq \frac{1-3\lambda+\lambda^3}{3-3\lambda} n + \lambda\left(\frac{1+\lambda}{3} n - 1\right) + 2(\lambda-2) = \frac{1-2\lambda}{3-3\lambda} n + \lambda - 4 \geq 21\left(\frac{1-2\lambda}{3-3\lambda}\right) + \lambda - 4 = \frac{3-9\lambda-\lambda^2}{1-\lambda} > 0$ for $0 \leq \lambda \leq \lambda^*$. We deduce that $S(b, \lambda, Q^N) - S(a, \lambda, Q^N) > 0$. Thus b still collects more points than a at Q^N. Now we assume that some *cba* voters act strategically. Note that at the new profile H^N, preferences of *abc* voters and *bac* voters are as in the initial profile R^N. Since at R^N, c initially collects less points than a while *cba* voters were contributing the maximum for c, then c can not be elected by unilateral actions from *cba* voters. Therefore, there is no opportunity for a profitable deviation at T^N by any coalition. In other words, T^N is an equilibrium.

Since from R^N to Q^N, both *bac* voters and *cba* voters benefit from the manipulation, we conclude that $M^*(n,\lambda) \geq \frac{p+q-1}{n} > \frac{1}{n}\left\lfloor \frac{1-2\lambda}{3-3\lambda}n + \frac{1+\lambda}{3}n \right\rfloor - \frac{2}{n} = \frac{1}{n}\left\lfloor \frac{2-2\lambda-\lambda^2}{3-3\lambda}n \right\rfloor - \frac{2}{n}$.

Step 2. We show that $M^*(n,\lambda) > \frac{1}{n}\left\lfloor \frac{1}{2}\frac{2\lambda+1}{\lambda+1}n \right\rfloor$ for $\lambda^* \leq \lambda \leq \frac{1}{2}$ and $n \geq 36$. Assume that $\lambda^* \leq \lambda \leq \frac{1}{2}$ and $n \geq 36$. Let $n = 2k + r$ with $r \in \{0,1\}$, $p = \frac{n-r}{2} + 1$ and $q = \left\lfloor \frac{2\lambda+1}{2+2\lambda}n \right\rfloor - p$. Note that $p+q = \left\lfloor \frac{2\lambda+1}{2+2\lambda}n \right\rfloor \leq \frac{1}{2}\frac{2\lambda+1}{\lambda+1}n < n$ and $q \geq \frac{1}{2}\frac{2\lambda+1}{\lambda+1}n - 1 - \frac{n-r}{2} - 1 = \frac{\lambda}{2\lambda+2}n - 2 + \frac{\lambda+1}{2\lambda+2}r \geq \frac{\lambda^*}{2+2\lambda^*}n - 2 > 0$. Therefore $n+1-p-q$ and $q-1$ are non negative integers. Consider the following profiles:

Type	Initial profile R^N			Strategic profiles T^N		
Preferences	a	b	c	a	b	b
	c	a	b	c	a	c
	b	c	a	b	c	a
Number of voters	$n+1-p-q$	p	$q-1$	$n+1-p-q$	$p-2$	$q+1$

Assume that the sincere profile R^N is described by the initial profile above. Then $S(a,\lambda,R^N) - S(b,\lambda,R^N) = n-(p+q)-(1-\lambda)p-\lambda q+\lambda+1$. Since $q = \left\lfloor \frac{2\lambda+1}{2+2\lambda}n \right\rfloor - p$, it follows that $S(a,\lambda,R^N) - S(b,\lambda,R^N) = n-(1+\lambda)\left\lfloor \frac{2\lambda+1}{2+2\lambda}n \right\rfloor - (1-2\lambda)p + \lambda + 1$. Taking into account that $\left\lfloor \frac{2\lambda+1}{2+2\lambda}n \right\rfloor \leq \frac{2\lambda+1}{2+2\lambda}n$ and $p = \frac{n-r}{2} + 1$, we obtain $S(a,\lambda,R^N) - S(b,\lambda,R^N) \geq \frac{1}{2}r + (3-r)\lambda > 0$. In the same way, $S(a,\lambda,R^N) - S(c,\lambda,R^N) = n-(1-\lambda)(p+q)-q-\lambda-n\lambda+p\lambda+2$. Thus $S(a,\lambda,R^N) - S(c,\lambda,R^N) = n-(2-\lambda)\left\lfloor \frac{2\lambda+1}{2+2\lambda}n \right\rfloor - n\lambda + (1+\lambda)p + 2 - \lambda$. We deduce that $S(a,\lambda,R^N) - S(c,\lambda,R^N) \geq n - (2-\lambda)\frac{2\lambda+1}{2+2\lambda}n - n\lambda + (1+\lambda)\left(\frac{n-r}{2}+1\right) + 2 - \lambda = \frac{\lambda^2-\lambda+1}{2\lambda+2}n + 3 - \frac{r(\lambda+1)}{2} > 0$. Therefore a wins at R^N.

Now suppose that all *cba* voters and exactly two *bac* voters strategically submit *bca*. The new profile is described by the strategic profile T^N above. At T^N, $S(b,\lambda,T^N) - S(a,\lambda,T^N) = 2(p+q) - n + 2\lambda - p\lambda - 2$. Since $\left\lfloor \frac{2\lambda+1}{2+2\lambda}n \right\rfloor > \frac{2\lambda+1}{2+2\lambda}n - 1$ and $p = \frac{n-r}{2}+1$, $S(b,\lambda,T^N) - S(a,\lambda,T^N) > -\frac{1}{2}\lambda\frac{\lambda-1}{\lambda+1}n + \lambda + \frac{1}{2}r\lambda - 4$. Then for $\lambda^* \leq \lambda \leq \frac{1}{2}$, $S(b,\lambda,T^N) - S(a,\lambda,T^N) > -\frac{1}{2}\frac{\lambda^*(\lambda^*-1)}{\lambda^*+1}n + \left(1+\frac{1}{2}r\right)\lambda^* - 4 = F(n)$. Noting that $-\frac{1}{2}\frac{\lambda^*(\lambda^*-1)}{\lambda^*+1} > 0$, we deduce that $F(n)$ is increasing as n increases for both even values or odd values of n. Moreover, $F(36) > 0$ and $F(37) > 0$. Therefore, $S(b,\lambda,T^N) - S(a,\lambda,T^N) > 0$ for $n \geq 36$. Similarly, $S(b,\lambda,T^N) - S(c,\lambda,T^N) = p + q - 2\lambda - n\lambda + p\lambda - 1$. Thus $S(b,\lambda,T^N) - S(c,\lambda,T^N) > \frac{2\lambda+1}{2+2\lambda}n - 1 - 2\lambda - n\lambda + \left(\frac{n-r}{2}+1\right)\lambda - 1 = \frac{1}{2}\frac{\lambda-\lambda^2+1}{\lambda+1}n - \frac{1}{2}\lambda r - (\lambda+2)$. For $\lambda^* \leq \lambda \leq \frac{1}{2}$, $\frac{1}{2}\frac{\lambda-\lambda^2+1}{\lambda+1}$ decreases as λ increases. Thus $S(b,\lambda,T^N) - S(c,\lambda,T^N) \geq \frac{5}{12}n - \frac{1}{2}\lambda r - (\lambda+2) > 0$ for $n \geq 36$. Therefore b wins at the new profile T^N.

Finally let us prove that T^N is an equilibrium. Strategic voting in favor of a by only *acb* voters is not achievable since those voters are already contributing the maximum for a and nothing for b. In order to advantage c against b, assume that *acb* voters and *cba* voters now submit *cab*. At the new profile, say Q^N obtained from T^N, $p-2$ voters report *bac*, 2 voters report *bca* and $n-p$ voters report *cab*. It follows that $S(b,\lambda,Q^N) - S(c,\lambda,Q^N) = p - (n-p+2\lambda)$. Since $p = \frac{n-r}{2}+1$,

$S\left(b,\lambda,Q^N\right) - S\left(c,\lambda,Q^N\right) = 2 - 2\lambda - r \geq 0$. By assumption, $r \in \{0,1\}$ and $\lambda \leq \frac{1}{2}$. Thus $S\left(b,\lambda,Q^N\right) - S\left(c,\lambda,Q^N\right) \geq 0$. Therefore c does not win at Q^N. We conclude that T^N is an equilibrium.

Since from R^N to T^N, both *bac* voters and *cba* voters benefit from the manipulation, we conclude that $M^*(n,\lambda) \geq \frac{p+q}{n} = \frac{1}{n}\left\lfloor \frac{1}{2}\frac{2\lambda+1}{\lambda+1}n \right\rfloor$.

Step 3. The proof is completed by taking into consideration Proposition 1. ■

7.3 Proof of Proposition 3

Proof. Assume that $n \geq 15$ and $\dfrac{1}{2} < \lambda \leq 1$. For $\lambda = 1$, suppose that individuals unanimously rank b first, a second, and c third. Then under F_1, a wins but b wins if an individual strategically submits *bca* instead of *bac*. Since b is unanimously preferred to a, no profitable deviation is possible. The new profile is an equilibrium and $M^*(1,n) \geq 1$. Hence $M^*(1,n) = 1$.

Now suppose that $\dfrac{1}{2} < \lambda < 1$.

1. We first prove that $M^*(\lambda,n) \geq 1 - \dfrac{1}{n}\left\lceil n\dfrac{1-\lambda}{2-\lambda}\right\rceil$. For this purpose, pose $k = \left\lceil n\dfrac{1-\lambda}{2-\lambda}\right\rceil$. Since $\dfrac{1}{2} < \lambda < 1$, it follows that $0 < \dfrac{1-\lambda}{2-\lambda} < \dfrac{1}{3}$ and that $1 \leq k \leq \left\lceil \dfrac{n}{3}\right\rceil$. Moreover $0 \leq k-1 < n\dfrac{1-\lambda}{2-\lambda}$. Consida profile R^N and T^N described as follows:

	Initial profile R^N		strategic profile T^N		
Preferences	a	b	a	b	**b**
	c	a	c	a	**c**
	b	c	b	c	**a**
Number of voters	k	$n-k$	k	$n-k-3$	**3**

With sincere votes, $S\left(a,\lambda,R^N\right) - S\left(b,\lambda,R^N\right) = (2-\lambda)\left(k - n\dfrac{1-\lambda}{2-\lambda}\right) \geq 0$ and $S\left(a,\lambda,R^N\right) - S\left(c,\lambda,R^N\right) = n\lambda - (2\lambda-1)k \geq n\lambda - (2\lambda-1)\left(n\dfrac{1-\lambda}{2-\lambda}+1\right) = n\frac{\lambda-\lambda^2-1}{\lambda-2} + (1-2\lambda) \geq \frac{n}{2} - 1 > 0$ for $\dfrac{1}{2} < \lambda < 1$ and $n \geq 15$. Then a is elected at R^N.

Note that $n - k \geq n - \left\lceil\dfrac{n}{3}\right\rceil > 3$. Suppose that three *bac* voters now strategically submit *bca* (instead of *bac*). We obtain a new profile T^N at which $S\left(b,\lambda,T^N\right) - S\left(a,\lambda,T^N\right) = 3\lambda - (2-\lambda)\left(k - n\dfrac{1-\lambda}{2-\lambda}\right)$ and $S\left(b,\lambda,T^N\right) - S\left(c,\lambda,T^N\right) = n - (1+\lambda)k - 3\lambda$. Since $k - 1 < n\dfrac{1-\lambda}{2-\lambda}$, we deduce that $S\left(b,\lambda,T^N\right) - S\left(a,\lambda,T^N\right) > 4\lambda - 2 > 0$ for $\dfrac{1}{2} < \lambda < 1$. Moreover, $k - 1 < n\dfrac{1-\lambda}{2-\lambda}$ implies that $n - (1+\lambda)k - 3\lambda > n - 2\left(\dfrac{n}{3}+1\right) - 3 = \dfrac{n-15}{3} \geq 0$. That is $S\left(b,\lambda,T^N\right) - S\left(c,\lambda,T^N\right)$. Therefore b is elected at T^N in favor of $n-k$ voters. We claim that T^N is an equilibrium.

In fact, b is elected at T^N. Thus any deviation should be in favor of a or c by only *acb* voters. But those voters are already contributing the maximum for a and nothing for b. Thus any strategic voting occurs in favor of c by submitting cab instead of acb. Even when we assume that all acb voters carry such a strategic action, we obtain a new profile Q^N at which $S\left(b,\lambda,T^N\right)-S\left(c,\lambda,T^N\right)=n-2k-3\lambda\geq\dfrac{n-15}{3}\geq 0$. Thus b is still elected at Q^N. This proves that T^N is an equilibrium. Therefore $M^*(1,n)\geq\frac{x_3+x_4+x_6}{n}=1-\dfrac{1}{n}\left\lceil n\dfrac{1-\lambda}{2-\lambda}\right\rceil$.

2. Now we prove that $M^*(1,n)\leq\dfrac{1}{2-\lambda}$. To see this, we solve $(E_0:E_4)$ with respect to e_2,e_3,e_4,x_2 and x_3 to obtain

$$x_3+x_4+x_6=\frac{1}{2-\lambda}-\frac{\lambda}{2-\lambda}x_1-\frac{\lambda}{2-\lambda}x_4-\frac{1-\lambda}{2-\lambda}x_5-\frac{2\lambda-1}{2-\lambda}x_6-\frac{1}{2-\lambda}e_1$$

Since for $\frac{1}{2}<\lambda<1$, each coefficient that appears is non negative, we deduce that $M^*(1,n)\leq\dfrac{1}{2-\lambda}$.

■

7.4 Proof of Theorem 1

Proof. Let $0\leq\lambda\leq\lambda^*$. From Proposition 2, we know that $\frac{1}{n}\left\lfloor\frac{2-2\lambda-\lambda^2}{3-3\lambda}n\right\rfloor-\frac{2}{n}<M^*(\lambda,n)\leq\frac{2-2\lambda-\lambda^2}{3-3\lambda}$ for $n\geq 21$. When n tends to infinity, the lower bound of $M^*(\lambda,n)$, $\frac{1}{n}\left\lfloor\frac{2-2\lambda-\lambda^2}{3-3\lambda}n\right\rfloor-\frac{2}{n}$, tends to $\frac{2-2\lambda-\lambda^2}{3-3\lambda}$, which is an upper bound of $M^*(\lambda,n)$. It follows that, when n tends to infinity, $M^*(\lambda,n)$ tends to $M^*(\lambda)=\frac{2-2\lambda-\lambda^2}{3-3\lambda}$.

Let $\lambda^*\leq\lambda\leq\frac{1}{2}$. From Proposition 2, we know that $\frac{1}{n}\left\lfloor\frac{1+2\lambda}{2+2\lambda}n\right\rfloor\leq M^*(\lambda,n)\leq\frac{1+2\lambda}{2+2\lambda}$ for $n\geq 25$. When n tends to infinity, $\frac{1}{n}\left\lfloor\frac{1+2\lambda}{2+2\lambda}n\right\rfloor$ tends to $\frac{1+2\lambda}{2+2\lambda}$, and it follows that $M^*(\lambda,n)$ tends to $M^*(\lambda)=\frac{1+2\lambda}{2+2\lambda}$.

Let $\dfrac{1}{2}<\lambda\leq 1$. We know from Proposition 3 that $1-\dfrac{1}{n}\left\lceil\dfrac{1-\lambda}{2-\lambda}n\right\rceil\leq M^*(\lambda,n)\leq\dfrac{1}{2-\lambda}$ for $n\geq 15$. Then when n tends to infinity, $1-\dfrac{1}{n}\left\lceil\dfrac{1-\lambda}{2-\lambda}n\right\rceil$ tends to $\dfrac{1}{2-\lambda}$, and therefore $M^*(\lambda,n)$ tends to $M^*(\lambda)=\dfrac{1}{2-\lambda}$. ■

7.5 Proof of Proposition 4

Proof. Assume that $0\leq\lambda\leq\frac{1}{2}$. It follows from Lemmas 2, 3 and 4 that

$$m^*(\lambda,n)\geq\min_{7\leq i\leq 9,10\leq j\leq 12}m^*(D_{i,j})=\frac{1}{2}.$$

■

7.6 Proof of Proposition 5

Proof. Assume that $0\leq\lambda\leq\frac{1}{2}$. We show that $m^*(\lambda,n)\leq\frac{1}{2}+\frac{1}{n}$ for $0\leq\lambda\leq\lambda^*$ and $n\geq 37$. Assume that $0\leq\lambda\leq\lambda^*$ and $n\geq 37$. Let $p=\left\lfloor\frac{1+\lambda}{6}n\right\rfloor$ and $q=\frac{n-\varepsilon}{2}-p$ where $\left\lfloor\frac{n}{2}\right\rfloor=\frac{n-\varepsilon}{2}$. Note that $\varepsilon=0$ if n is even and $\varepsilon = 1$ if n is odd. Moreover $p \leq \left\lfloor \frac{1}{4}n \right\rfloor$, and $q = \left\lfloor \frac{n}{2} \right\rfloor - p \geq \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{1}{4}n \right\rfloor \geq \frac{n-\varepsilon}{2} - \frac{1}{4}n = \frac{n-2\varepsilon}{4} \geq 7$. Consider the following profiles

Type	Initial profile R^N					Strategic profiles T^N		
Preferences	a	a	b	b	c	a	a	b
	b	c	a	c	b	b	c	c
	c	b	c	a	a	c	b	a
Number of voters	$q+\varepsilon$	$p-1$	q	$p-3$	4	$q+\varepsilon$	$p-1$	$p+q+1$

Assume that the sincere profile R^N is described by the initial profile above. Then:

$$S\left(a, \lambda, R^N\right) - S\left(b, \lambda, R^N\right) = (1-\lambda)\,\varepsilon + (2-4\lambda) \geq 0$$

and

$$S\left(a, \lambda, R^N\right) - S\left(c, \lambda, R^N\right) = (1-2\lambda)\,p + (q-5) + \lambda\,(4+q) + \varepsilon > 0.$$

Therefore a wins at R^N.

Now suppose that all cba voters and all bac voters strategically submit bca. The new profile is described by the strategic profile T^N above. At T^N,

$$S\left(b, \lambda, T^N\right) - S\left(a, \lambda, T^N\right) = (p+q+1+(q+\varepsilon)\,\lambda) - (q+\varepsilon+p-1) = 2-(1-\lambda)\,\varepsilon + q\lambda > 0$$

and

$$S\left(b, \lambda, T^N\right) - S\left(c, \lambda, T^N\right) = (p+q+1+(q+\varepsilon)\,\lambda) - \lambda\,(p-1+p+q+1) = (1-2\lambda)\,p + q + \lambda\varepsilon + 1 > 0.$$

Thus b wins at the new profile T^N.

Let us prove that T^N is an equilibrium. In fact, given any collective deviation by abc voters together with acb voters, b scores at least $p+q+1$ points while a scores at most $p+q$ points. Thus those voters can not profitably deviate from T^N in favor of a. In the same way, when acb voters and cba voters contribute the maximum for c and nothing for b, c scores at most $p+3+\lambda\,(p+q-3)$ and b scores at least $p+q-3+\lambda\,(q+\varepsilon)$. And

$$\begin{aligned} p+q-3+\lambda\,(q+\varepsilon) - (p+3+\lambda\,(p+q-3)) &= q-(p-3-\varepsilon)\,\lambda - 6 \geq q - (p-\varepsilon-3)\,\frac{1}{2} - 6 \\ &\geq \frac{n-\varepsilon}{8} - \frac{9}{2} + \frac{\varepsilon}{2} > 0 \text{ for } n \geq 37. \end{aligned}$$

Therefore there is no possible profitable deviation from T^N in favor of c. We then conclude that T^N is an equilibrium and that $m^*(n, \lambda) \leq \frac{q+p+1}{n} = \frac{n-\varepsilon}{2n} + \frac{1}{n} \leq \frac{1}{2} + \frac{1}{n}$.

The proof is completed by considering Proposition 4. ■

7.7 Proof of Proposition 6

In order to prove Proposition 6, we need to prove three preliminary results which identify an upper bound and a lower bound of $x_3 + x_4 + x_6$ for each of the domains defined in Section 5.1. The first lemma below concerns the sudomians $D_{7,j}$ for $j = 10, 11, 12$.

Lemma 5 *Assume that $\frac{1}{2} < \lambda \leq 1$. For each of the three subdomains $D_{7,j}$ for $j = 10, 11, 12$, the minimum value $m^*(D_{7,j})$ of $x_3 + x_4 + x_6$ is such that :*

	$D_{7,10}$	$D_{7,11}$	$D_{7,12}$
$m^*(D_{7,j}) \geq$	$\begin{cases} \textit{not feasible} & \textit{for } \frac{1}{2} < \lambda < 1 \\ \frac{1}{3} & \textit{for } \lambda = 1 \end{cases}$	$\frac{\lambda^2-\lambda-1}{\lambda^2-3\lambda-1}$	$\frac{1}{3}\frac{-6\lambda+\lambda^2+6}{3-2\lambda}$

Proof. Subdomain $D_{7,10}$: We show that $D_{7,10}$ is not feasible for $\frac{1}{2} < \lambda < 1$. First assume that $\frac{1}{2} < \lambda \leq \frac{3}{5}$. By combining (C_1), (C_7), (C_{10}) and (C_5) as

$$(C_1) + \frac{2+\lambda}{4-4\lambda}(C_7) + \frac{3\lambda}{4-4\lambda}(C_{10}) + \frac{3-3\lambda}{2}(C_5)$$

and by setting $x_6 = 1 - x_1 - x_2 - x_3 - x_4 - x_5$, it appears, that

$$\frac{3(1-\lambda)}{2}x_4 + (1-\lambda)x_5 + \frac{3-5\lambda}{2}y_3 + \frac{2+\lambda}{4}y_6 + \lambda - \frac{1}{2} \leq 0.$$

This is a contradiction since each variable has a positive coefficient for $\frac{1}{2} < \lambda \leq \frac{3}{5}$ and $\lambda - \frac{1}{2} > 0$. Now assume that $\frac{3}{5} < \lambda < 1$. By combining (C_1), (C_7), (C_{10}) and (C_5) as

$$(C_1) + \frac{3\lambda^2 - 5\lambda + 4}{2 - 2\lambda^2}(C_7) + \frac{3\lambda - \lambda^2}{2 - 2\lambda^2}(C_{10}) + \frac{4\lambda - 4\lambda^2}{1+\lambda}(C_5)$$

it appears that

$$\lambda x_4 + (1-\lambda)x_5 + \frac{5\lambda - 3}{1+\lambda}x_6 + \frac{3\lambda^2 + 4 - 5\lambda}{2+2\lambda}y_6 + \frac{(\lambda-1)^2}{\lambda+1} \leq 0.$$

This is a contradiction since variables in the left hand side all have positive coefficients and $\frac{(\lambda-1)^2}{\lambda+1} > 0$ for $\frac{3}{5} < \lambda < 1$. For $\lambda = 1$, by solving $(E_0 : E_6)$, (E_7) and (E_{10}) with respect to e_1, e_6, e_2, e_3, e_{10}, x_1, x_2, x_3 and x_4, we obtain:

$$x_3 + x_4 + x_6 = \frac{1}{3}e_4 + e_5 + \frac{2}{3}e_7 + \frac{1}{3}.$$

Therefore $m^*(D_{7,10}) \geq \frac{1}{3}$.

Subdomain $D_{7,11}$: By solving $(E_0 : E_6)$, (E_7) and (E_{11}) with respect to e_1, e_6, e_2, e_3, e_4, x_1, x_2, x_3 and x_4, we obtain:

$$\begin{aligned} x_3 + x_4 + x_6 &= \frac{\lambda^2 - \lambda - 1}{\lambda^2 - 3\lambda - 1} + \frac{(2\lambda - 1)x_6 + \left(1 - \lambda^2\right)y_6 + \lambda(\lambda+2)e_5}{3\lambda + 1 - \lambda^2} \\ &\quad + \frac{(\lambda+1)e_7 + \lambda e_{11}}{3\lambda + 1 - \lambda^2}. \end{aligned}$$

All variables have non-negative coefficients for $\frac{1}{2} < \lambda \leq 1$. Thus $m^*(D_{7,11}) \geq \frac{\lambda^2-\lambda-1}{\lambda^2-3\lambda-1}$.

Subdomain $D_{7,12}$: By solving $(E_0 : E_6)$, (E_7) and (E_{12}) with respect to e_6, e_2, e_3, e_4, x_1, x_2, x_3, x_6 and y_3, we obtain:

$$\begin{aligned} x_3 + x_4 + x_6 &= \frac{1}{3}\frac{-6\lambda + \lambda^2 + 6}{3 - 2\lambda} + \frac{\lambda(\lambda-1)^2 x_4 + (1-\lambda)^3 x_5 + (1-\lambda)\left(3\lambda - \lambda^2 - 1\right) y_6}{(2\lambda - 1)(3 - 2\lambda)} \\ &\quad + \frac{(\lambda-1)^2 e_1 + \lambda\left(6\lambda - 3\lambda^2 - 2\right) e_5 + \left(3\lambda - \lambda^2 - 1\right) e_7 + \frac{1}{3}\lambda^2 e_{12}}{(2\lambda - 1)(3 - 2\lambda)}. \end{aligned}$$

All variables have non-negative coefficients for $\frac{1}{2} < \lambda \leq 1$. Thus $m^*(D_{7,11}) \geq \frac{1}{3}\frac{-6\lambda+\lambda^2+6}{3-2\lambda}$. ∎

The next lemma concerns the subdomains $D_{8,j}$ for $j = 10, 11, 12$.

Lemma 6 *Assume that* $\frac{1}{2} < \lambda \leq 1$. *For each of the three subdomains* $D_{8,j}$ *for* $j = 10, 11, 12$, $m^*(D_{8,j}) \geq \frac{1}{2}$.

Proof. Subdomain $D_{8,10}$: By solving $(E_0 : E_6)$, (E_8) and (E_{10}) with respect to e_1, e_2, e_3, e_4, e_{10}, x_1, x_3, x_4 and x_6, we obtain:

$$x_3 + x_4 + x_6 = \frac{1}{2} + \frac{1}{2}\lambda e_5 + \frac{1-\lambda}{2} e_6 + \frac{1}{2} e_8.$$

This proves that $m^*(D_{7,11}) \geq \frac{1}{2}$.

Subdomain $D_{8,11}$: By solving $(E_0 : E_6)$, (E_8) and (E_{11}) with respect to e_1, e_2, e_3, e_4, e_{11}, x_1, x_3, x_4 and x_6, we obtain:

$$x_3 + x_4 + x_6 = \frac{1}{2} + \frac{1}{2} e_8 + \frac{1-\lambda}{2} e_6 + \frac{1}{2}\lambda e_5.$$

Thus $m^*(D_{7,11}) \geq \frac{1}{2}$.

Subdomain $D_{8,12}$: By solving $(E_0 : E_6)$, (E_8) and (E_{12}) with respect to e_1, e_{12}, e_2, e_3, e_4, x_1, x_3, x_4 and x_6, we obtain:

$$x_3 + x_4 + x_6 = \frac{1}{2} + \frac{1-\lambda}{2} e_6 + \frac{1}{2} e_8 + \frac{\lambda}{2} e_5.$$

which proves that $m^*(D_{8,12}) \geq \frac{1}{2}$. ∎

Lemma 7 below concerns the subdomains $D_{9,j}$ for $j = 10, 11, 12$.

Lemma 7 *Assume that* $\frac{1}{2} < \lambda \leq 1$. *For each of the three subdomains* $D_{9,j}$ *for* $j = 10, 11, 12$, $m^*(D_{9,j}) \geq \frac{2-\lambda}{3}$.

Proof. Subdomain $D_{9,10}$: By solving $(E_0 : E_6)$, (E_9) and (E_{10}) with respect to e_1, e_6, e_2, e_3, e_4, e_{10}, x_1, x_3 and x_4, we obtain:

$$x_3 + x_4 + x_6 = \frac{2-\lambda}{3} + \frac{1}{3} e_9 + \lambda e_5.$$

This proves that $m^*(D_{7,11}) \geq \frac{2-\lambda}{3}$.

Subdomain $D_{9,11}$: By solving $(E_0 : E_6)$, (E_9) and (E_{11}) with respect to e_1, e_6, e_2, e_3, e_4, e_{11}, x_1, x_3 and x_4, we obtain:

$$x_3 + x_4 + x_6 = \frac{2-\lambda}{3} + \lambda e_5 + \frac{1}{3} e_9.$$

Thus $m^*(D_{9,11}) \geq \frac{2-\lambda}{3}$.

Subdomain $D_{9,12}$: By solving $(E_0 : E_6)$, (E_9) and (E_{12}) with respect to e_1, e_6, e_2, e_3, e_4, e_{12}, x_1, x_3 and x_4, we obtain:

$$x_3 + x_4 + x_6 = \frac{2-\lambda}{3} + \lambda e_5 + \frac{1}{3} e_9.$$

which proves that $m^*(D_{9,12}) \geq \frac{2-\lambda}{3}$. ■

We are now ready to prove Proposition 6.

Proof. (**Proposition** 6) Assume that $\frac{1}{2} < \lambda < 1$ and that $n > \max\left(\frac{6(\lambda+1)(2-\lambda)}{\lambda(4\lambda-\lambda^2-1)}, \frac{3\lambda(2\lambda-1)}{(1-\lambda)(4\lambda-\lambda^2-1)}\right) = n^*$. Let $p_1 = \left\lfloor \frac{4\lambda-\lambda^2-1}{3\lambda+3\lambda^2} n \right\rfloor$, $p_2 = \left\lfloor \frac{1+\lambda}{3} n \right\rfloor - p_1 - 1$, $p_3 = n - p_1 - p_2$. Note that $p_1 + p_2 = \left\lfloor \frac{1+\lambda}{3} n \right\rfloor - 1 < n$ for $\frac{1}{2} < \lambda < 1$. Thus p_3 is a positive integer. Since $\left\lfloor \frac{1+\lambda}{3} n \right\rfloor > \frac{1+\lambda}{3} n - 1$ and $p_1 \leq \frac{4\lambda-\lambda^2-1}{3\lambda+3\lambda^2} n$, it follows that $p_2 > \frac{1+\lambda}{3} n - 1 - \frac{4\lambda-\lambda^2-1}{3\lambda+3\lambda^2} n - 1 = \frac{1}{3} n \frac{-3\lambda+3\lambda^2+\lambda^3+1}{\lambda(\lambda+1)} - 2$. By assumption, $n > \frac{6(\lambda+1)(2-\lambda)}{\lambda(4\lambda-\lambda^2-1)}$. Thus $p_2 > \frac{1}{3} \frac{6(\lambda+1)(2-\lambda)}{\lambda(4\lambda-\lambda^2-1)} \frac{-3\lambda+3\lambda^2+\lambda^3+1}{\lambda(\lambda+1)} - 2 = 2(1-\lambda) \frac{2-5\lambda+5\lambda^2}{\lambda^2(4\lambda-\lambda^2-1)} > 0$ for $\frac{1}{2} < \lambda < 1$. Therefore, p_2 is a positive integer. Consider the following profiles

Type	Initial profile R^N				Strategic profile T^N		
Preferences	a	a	b		a	a	b
	b	c	a		b	c	c
	c	b	c		c	b	a
Number of voters	p_1	p_2	p_3		p_1	p_2	p_3

Assume that the sincere profile R^N is described by the initial profile above. Then

$$\begin{aligned}
S\left(a,\lambda,R^N\right) - S\left(b,\lambda,R^N\right) &= (p_1+p_2) - p_3 + \lambda p_3 - \lambda p_1 \\
&= (2-\lambda)\left\lfloor \frac{1+\lambda}{3} n \right\rfloor - (1-\lambda) n - \lambda \left\lfloor \frac{4\lambda-\lambda^2-1}{3\lambda+3\lambda^2} n \right\rfloor + \lambda - 2 \\
&> (2-\lambda)\left(\frac{1+\lambda}{3} n - 1\right) - (1-\lambda) n - \lambda \left(\frac{4\lambda-\lambda^2-1}{3\lambda+3\lambda^2} n\right) + \lambda - 2 \\
&= \frac{1}{3} \frac{\lambda\left(4\lambda-\lambda^2-1\right)}{\lambda+1} \left(n - \frac{6(\lambda+1)(2-\lambda)}{\lambda\left(4\lambda-\lambda^2-1\right)}\right)
\end{aligned}$$

and

$$\begin{aligned}
S\left(a,\lambda,R^N\right)-S\left(c,\lambda,R^N\right) &= (p_1+p_2)+\lambda p_3-\lambda p_2\\
&= (1-2\lambda)\left\lfloor\frac{1+\lambda}{3}n\right\rfloor+\lambda n+\lambda\left\lfloor\frac{4\lambda-\lambda^2-1}{3\lambda+3\lambda^2}n\right\rfloor+2\lambda-1\\
&> (1-2\lambda)\left(\frac{1+\lambda}{3}n\right)+\lambda n+\lambda\left(\frac{4\lambda-\lambda^2-1}{3\lambda+3\lambda^2}n-1\right)+2\lambda-1\\
&= \frac{1}{3}\frac{\lambda\left(7-\lambda-2\lambda^2\right)}{\lambda+1}\left(n-\frac{3(1-\lambda)(\lambda+1)}{\lambda\left(7-\lambda-2\lambda^2\right)}\right).
\end{aligned}$$

For $\frac{1}{2}<\lambda<1$ and $n>n^*$, $S\left(a,\lambda,R^N\right)-S\left(b,\lambda,R^N\right)>0$ and $S\left(a,\lambda,R^N\right)-S\left(c,\lambda,R^N\right)>0$. Therefore a wins at R^N.

Now suppose that all *bac* voters strategically submit *bca*. The new profile is described by the strategic profile T^N above. At T^N,

$$\begin{aligned}
S\left(b,\lambda,T^N\right)-S\left(a,\lambda,T^N\right) &= p_3+\lambda p_1-(p_1+p_2)\\
&= n-2\left\lfloor\frac{1+\lambda}{3}n\right\rfloor+\lambda\left\lfloor\frac{4\lambda-\lambda^2-1}{3\lambda+3\lambda^2}n\right\rfloor+2\\
&> n-2\left(\frac{1+\lambda}{3}n\right)+\lambda\left(\frac{4\lambda-\lambda^2-1}{3\lambda+3\lambda^2}n-1\right)+2\\
&= \frac{\lambda-\lambda^2}{\lambda+1}n+2-\lambda
\end{aligned}$$

and

$$\begin{aligned}
S\left(b,\lambda,T^N\right)-S\left(c,\lambda,T^N\right) &= p_3+\lambda p_1-\lambda(n-p_1)\\
&= (1-\lambda)n-\left\lfloor\frac{1+\lambda}{3}n\right\rfloor+2\lambda\left\lfloor\frac{4\lambda-\lambda^2-1}{3\lambda+3\lambda^2}n\right\rfloor+1\\
&> (1-\lambda)n-\frac{1+\lambda}{3}n+2\lambda\left(\frac{4\lambda-\lambda^2-1}{3\lambda+3\lambda^2}n-1\right)+1\\
&= \frac{2\lambda(1-\lambda)}{\lambda+1}\left(n-\frac{1}{2}\frac{(\lambda+1)(2\lambda-1)}{\lambda(1-\lambda)}\right).
\end{aligned}$$

For $\frac{1}{2}<\lambda<1$ and $n>n^*$, $S\left(b,\lambda,T^N\right)-S\left(a,\lambda,T^N\right)>0$ and $S\left(b,\lambda,T^N\right)-S\left(c,\lambda,T^N\right)>0$. Thus b wins at the new profile T^N.

Let us prove that T^N is an equilibrium. In fact, suppose that among *abc* voters, s voters deviate and now submit *acb*. Suppose that at the new profile, say H^N, the score of a is greater or equal to the score of b. That is

$$S\left(a,\lambda,H^N\right)-S\left(b,\lambda,H^N\right)=p_1+p_2-p_3-\lambda(p_1-s)\geq 0$$

or equivalently,

$$\lambda s\geq p_3+\lambda p_1-\left\lfloor\frac{1+\lambda}{3}n\right\rfloor+1=n+\lambda p_1-2\left\lfloor\frac{1+\lambda}{3}n\right\rfloor+2.$$

Between a and c, we have:

$$\begin{aligned} S\left(c,\lambda,H^N\right)-S\left(a,\lambda,H^N\right) &= \lambda\left(p_2+s+p_3\right)-p_1-p_2 \\ &= \lambda n-\lambda p_1+\lambda s-\left\lfloor\frac{1+\lambda}{3}n\right\rfloor+1 \\ &\geq \left(1+\lambda\right)n-3\left\lfloor\frac{1}{3}n\left(\lambda+1\right)\right\rfloor+3 \\ &\geq 3 \end{aligned}$$

This proves that a can not win at T^N. Therefore there is no profitable deviation in favor of a from T^N. Moreover even if all acb voters decide to submit cab, we obtain a new profile, say S^N, at which:

$$\begin{aligned} S\left(b,\lambda,S^N\right)-S\left(c,\lambda,S^N\right) &= p_3+\lambda p_1-p_2-\lambda p_3 \\ &= \left(1-\lambda\right)n-\left(2-\lambda\right)\left\lfloor\frac{\lambda+1}{3}n\right\rfloor+\left(1+\lambda\right)\left\lfloor\frac{4\lambda-\lambda^2-1}{3\lambda+3\lambda^2}n\right\rfloor+2-\lambda \\ &> \left(1-\lambda\right)n-\frac{\left(2-\lambda\right)\left(\lambda+1\right)}{3}n+\left(1+\lambda\right)\left(\frac{4\lambda-\lambda^2-1}{3\lambda+3\lambda^2}n-1\right)+2-\lambda \\ &= \frac{1}{3}\frac{\left(1-\lambda\right)\left(4\lambda-\lambda^2-1\right)}{\lambda}\left(n-\frac{3\lambda\left(2\lambda-1\right)}{\left(1-\lambda\right)\left(4\lambda-\lambda^2-1\right)}\right). \end{aligned}$$

For $\frac{1}{2}<\lambda<1$ and $n>n^*$, $S\left(b,\lambda,S^N\right)-S\left(c,\lambda,S^N\right)>0$. Therefore c can not win at S^N. Thus there is no profitable deviation in favor of c from T^N. In conclusion, T^N is an equilibrium. By definition, $m^*\left(\lambda,n\right)\leq\frac{p_3}{n}=1-\frac{1}{n}\left\lfloor\frac{\lambda+1}{3}n\right\rfloor+\frac{1}{n}$.

The proof is completed by taking into consideration Lemmas 5, 6 and 7, which prove that $m^*\left(\lambda,n\right)\geq\frac{2-\lambda}{3}$ over all the subdomains $D_{i,j}$ above. ■

7.8 Proof of Proposition 7

Proof. Assume that $n\geq 6$ and let $n=3p+\varepsilon$ with $\varepsilon\in\{0,1,2\}$. Consider the following profiles

Type	Initial profile R^N			Strategic profile T^N		
Preferences	a	a	b	a	a	b
	b	c	a	b	c	c
	c	b	c	c	b	a
Number of voters	$p+\varepsilon$	$p-2$	$p+2$	$p+\varepsilon$	$p-2$	$p+2$

Assume that the sincere profile R^N is described by the initial profile above. Then

$$S\left(a,1,R^N\right)-S\left(b,1,R^N\right)=p-2 \text{ and } S\left(a,1,R^N\right)-S\left(c,1,R^N\right)=2p+2+\varepsilon.$$

Therefore a wins at R^N.

Now suppose that all bac voters strategically submit bca. The new profile is described by the strategic profile T^N above. At T^N,

$$S\left(b,1,T^N\right)-S\left(a,1,T^N\right)=4 \text{ and } S\left(b,1,T^N\right)-S\left(a,1,T^N\right)=2+\varepsilon.$$

Thus b wins at the new profile T^N.

Let us prove that T^N is an equilibrium. In fact, at T^N, $S\left(a,\lambda,T^N\right)\leq S\left(c,\lambda,T^N\right)<S\left(b,\lambda,T^N\right)$ and any deterioration of the score of b by abc voters contributes to increasing the score of c. Thus there is no profitable deviation in favor of a from T^N. Moreover acb voters have no way to favor the election of c since they are already giving c the maximum and nothing for b. Clearly, there is no profitable deviation at T^N and T^N is therefore an equilibrium. By definition $m^*\left(\lambda,n\right)\leq\frac{p+2}{3p+\varepsilon}\leq\frac{p}{3p+\varepsilon}+\frac{2}{3p+\varepsilon}\leq\frac{1}{3}+\frac{2}{n}$.

The proof is completed by taking into consideration Lemmas 5, 6 and 7, which prove that $m^*\left(1,n\right)\geq\frac{1}{3}$ over all the subdomains $D_{i,j}$ above. ■

7.9 Proof of Theorem 2

Proof. By Proposition 6, $\frac{2-\lambda}{3}\leq m\left(\lambda,n\right)\leq 1-\frac{1}{n}\left\lfloor\frac{1+\lambda}{3}n\right\rfloor+\frac{1}{n}$ for $\frac{1}{2}<\lambda<1$ and $n>\max\left(\frac{6(\lambda+1)(2-\lambda)}{\lambda\left(4\lambda-\lambda^2-1\right)},\frac{3\lambda(2\lambda-1)}{(1-\lambda)\left(4\lambda-\lambda^2-1\right)}\right)$. Since $\left\lfloor\frac{1+\lambda}{3}n\right\rfloor\leq\frac{1+\lambda}{3}n$, if follows that $\frac{2-\lambda}{3}\leq m\left(\lambda,n\right)\leq\frac{2-\lambda}{3}+\frac{1}{n}$. As n tends to infinity, we deduce that $m^*\left(\lambda\right)=\frac{2-\lambda}{3}$. For $\lambda=1$, Proposition 7 shows that $\frac{1}{3}\leq m^*\left(1,n\right)\leq\frac{1}{3}+\frac{2}{n}$. As n tends to infinity, it follows that $m^*\left(1\right)=\frac{1}{3}$. Therefore $m^*\left(\lambda\right)=\frac{2-\lambda}{3}$ for $\frac{1}{2}<\lambda\leq 1$. ■

7.10 Proof of Proposition 8

Proof. Let $R^N\in L^N$. $\left\{E\left(a,R^N\right),a\in A\right\}$ is a partition of N. Consequently, $\left|E\left(a_1,R^N\right)\right|+\ldots+\left|E\left(a_m,R^N\right)\right|=n$. $Pl\left(R^N\right)=a_j$ implies that $\left|E\left(a_j,R^N\right)\right|\geq\left|E\left(a_l,R^N\right)\right|$ for all $l=1,..,m$. Therefore $m\times\left|E\left(a_j,R^N\right)\right|\geq n$. That is $\left|E\left(a_j,R^N\right)\right|\geq\left\lceil\frac{n}{m}\right\rceil$. ■

7.11 Proof of Proposition 9

Proof. The proof is left to the reader. ■

7.12 Proof of Proposition 10

Proof. Part 1. Let us prove that $M^*\left(Pl,n,m\right)\leq 1-\frac{1}{n}\left\lceil\frac{n}{m}\right\rceil$. Take $R^N\in P(Pl)$ and $T^N\in SN(Pl\mid R^N)$. Pose $x=Pl\left(R^N\right)$ and $y=Pl\left(T^N\right)$. We have $E(Pl,R^N,T^N)=E(x,y,R^N)\subseteq N\backslash E(x,R^N)$. In addition, $\left|E\left(x,R^N\right)\right|\geq\left\lceil\frac{n}{m}\right\rceil$ by Proposition 8. This implies that $\left|E(Pl,R^N,T^N)\right|\leq n-\left\lceil\frac{n}{m}\right\rceil$. Consequently, $M^*\left(Pl,n,m\right)\leq 1-\frac{1}{n}\left\lceil\frac{n}{m}\right\rceil$.

Part 2. Now we show that $M^*\left(Pl,n,m\right)\geq 1-\frac{1}{n}\left\lceil\frac{n}{m}\right\rceil$. We construct two profiles R^N and T^N such that $R^N\in P(Pl)$, $T^N\in SN(Pl\mid R^N)$ and $\left|E(Pl,R^N,T^N)\right|=n-\left\lceil\frac{n}{m}\right\rceil$. Pose $n=qm+r$ with $r\in\{0,1,...,m-1\}$.

Case 1 : $r=0$. Consider a partition $\{N_1,N_2,...,N_m\}$ of N in m subsets such that voter 2 belongs to N_2 and for all $k\in\{1,2,...,m\}$, $|N_k|=q$. Let R^N be a profile such that

$$\forall i\in N_3,R^i=a_3...\text{ and }\forall i\in N_k,R^i=a_ka_3....\text{ for }k\neq 3$$

We have $\left|E\left(a,R^N\right)\right|=q$ for all $a\in A$. So $Pl(R^N)=a_1$. $E(a_1,a_3,R^N)=N\backslash N_1$. Now pose $Q^2=a_3a_2...$ and $T^N=(Q^2,R^{-2})$. We have T^N is an effective manipulation of R^N since $Pl(T^N)=a_3$ and

$a_3R^2a_1$. Moreover $\left|E(a_3, a_j, R^N)\right| \le q, \forall a_j \in A$ and $\left|E(a_3, T^N)\right| = q+1$. Therefore by Proposition 9, T^N is an equilibrium. Note that $E(Pl, R^N, T^N) = N \backslash N_1$. So $\left|E(Pl, R^N, T^N)\right| = n - \lceil \frac{n}{m} \rceil$.

Case 2 : $q = 0$. Then $n < m$. Let R be the lexicographic order on A and define the profile R^N as follows:

$$R^1 = R_{[a_2]} \text{ and } R^i = R_{[a_{i+1}a_1]} \text{ for all } i \neq 1.$$

We have $\left|E(a_j, R^N)\right| = 1$ for all $j = 2, ..., n+1$ and $\left|E(a_1, R^N)\right| = 0$. Furthermore $Pl(R^N) = a_2$ and $E(a_2, a_1, R^N) = N \backslash \{1\}$. Now pose $Q^2 = R_{[a_1]}$ and $T^N = (Q^2, R^{-2})$. The new profile T^N is an effective manipulation of R^N since $Pl(T^N) = a_1$ and $a_1R^2a_2$. Moreover $\left|E(a_1, a_j, R^N)\right| \le 1$ for all $a_j \neq a_1$ and $\left|E(a_1, T^N)\right| = 1$. Thus T^N is an equilibrium by Proposition 9. Since $E(Pl, R^N, T^N) = N \backslash \{1\}$, then $\left|E(Pl, R^N, T^N)\right| = n - 1 = n - \lceil \frac{n}{m} \rceil$.

Case 3 : $r \ge 1$ and $q \ge 1$. Consider a partition $\{N_1, N_2, ..., N_m\}$ of N in m subsets such that voter 3 belongs to N_3, $|N_k| = q+1$ if $k \in \{2, 3, ..., r+1\}$ and $|N_k| = q$ otherwise. Let R^N be the profile defined by

$$\forall i \in N_1, R^i = R_{[a_1]} \text{ and } \forall i \in N_k, R^i = R_{[a_k a_1]} \text{ for all } k \neq 1.$$

We have $\left|E(a_1, R^N)\right| = q$, $\left|E(a_2, R^N)\right| = q+1$ and $\left|E(a_j, R^N)\right| \le q+1$ for all $j \ge 3$. Moreover $Pl(R^N) = a_2$ and $E(a_2, a_1, R^N) = N \backslash N_2$. Now pose $Q^3 = R_{[a_1]}$ and $T^N = (Q^3, R^{-3})$. We have T^N is an effective manipulation of R^N. In fact, $Pl(T^N) = a_1$ and $a_1R^3a_2$. Since $\left|E(a_1, a_j, R^N)\right| \le q+1$ for all $a_j \neq a_1$ and $\left|E(a_1, R^N)\right| = q+1$, it follows from Proposition 9 that T^N is an equilibrium given R^N. Thus $T^N \in SN(Pl \mid R^N)$. Note that $E(Pl, R^N, T^N) = E(a_2, a_1, R^N) = N \backslash N_2$. So $\left|E(Pl, R^N, T^N)\right| = n - \lceil \frac{n}{m} \rceil$. ■

7.13 Proof of Proposition 11

Proof. Pose $n = 2k + r$ with $r \in \{0, 1\}$. Take $R^N \in P(Pl)$ and $T^N \in SN(Pl \mid R^N)$. Pose $x = Pl\left(R^N\right)$ and $y = Pl\left(T^N\right)$. We have $E(Pl, R^N, T^N) = E(x, y, R^N)$. Suppose that $\left|E(x, y, R^N)\right| < \lceil \frac{n}{2} \rceil = \frac{n+r}{2}$; that is $\left|E(x, y, R^N)\right| \le \frac{n+r}{2} - 1$. Pose $S = E(x, y, R^N)$ and $Z^i = R_{[xy]}$ for all $i \in N \backslash S$. Pose $Z^N = (Z^{-S}, T^S)$. Since $|N \backslash S| = n - |S| \ge n - \left(\frac{n+r}{2} - 1\right) = \frac{n+2-r}{2}$. Thus $|N \backslash S| > \frac{n}{2}$ and we have $Pl(Z^N) = x$ and xR^iy for all $i \in N \backslash S$. Therefore T^N is not an equilibrium given R^N. We conclude that $\left|E(x, y, R^N)\right| \ge \lceil \frac{n}{2} \rceil$. That is $m^*\left(Pl, n, m\right) \ge \frac{1}{n} \lceil \frac{n}{2} \rceil$.

To prove that $m^*\left(Pl, n, m\right) \le \frac{1}{n} \lceil \frac{n}{2} \rceil$, we construct $R^N \in P(Pl)$ and $T^N \in SN(Pl \mid R^N)$ such that $\left|E(Pl, R^N, T^N)\right| = \lceil \frac{n}{2} \rceil$. We consider 2 cases:

Case 1 : $r = 0$. That is $n = 2k$. Consider a profile R^N such that

$$R^i = a_1a_2a_3...a_m \text{ for all } i < k;\ R^k = a_3a_1a_2... \text{ and } R^i = a_2a_1a_3...a_m \text{for all } i > k.$$

We have $Pl(R^N) = a_2$ and $E(a_2, a_1, R^N) = \{1, ..., k\}$. Pose $T^k = a_1a_m...a_2$ and $T^N = (T^k, R^{-k})$. We have $Pl(T^N) = a_1$ and $a_1R^ka_2$. That is T^N is an effective manipulation of R^N. In addition, $\left|E(a_1, T^N)\right| = k \ge \left|E(a_l, a_1, R^N)\right|$ for all $l > 1$. By Proposition 9, we conclude that T^N is an equilibrium given R^N. We have $\left|E(Pl, R^N, T^N)\right| = \left|E(a_2, a_1, R^N)\right| = \lceil \frac{n}{2} \rceil$.

Case 2 : $r = 1$. That is $n = 2k+1$. Let $\{\{1\}, N_1, N_2\}$ be a partition of N such that $|N_2| = |N_2| = k$. Consider a profile R^N such that

$$R^1 = a_3a_2a_1...; \forall i \in N_1, R^i = a_2a_1a_3... \text{ and } \forall i \in N_2, R^i = a_1a_2a_3...$$

Obviously, $Pl(R^N) = a_1$ and $E(a_1, a_2, R^N) = N_1 \cup \{1\}$. Pose $T^1 = a_2a_3a_1...$ and $T^N = (T^1, R^{-1})$. Then $Pl(T^N) = a_2$ and $a_2R^1a_1$. Thus T^N is an effective manipulation of R^N. Moreover $\left|E(a_2, T^N)\right| = k+1 > \left|E(a_l, a_2, R^N)\right|$ for all $l \neq 2$. By Proposition 9, T^N is an equilibrium. We have $\left|E(Pl, R^N, T^N)\right| = \left|E(a_1, a_2, R^N)\right| = \left\lceil \frac{n}{2} \right\rceil$. ■

7.14 Proof of Proposition 12

Proof. Assume that there exist $R^N \in L^N$ and $\{x, y\} \subseteq A$ such that $Borda\left(R^N\right) = x$ and $\left|E\left(y, x, R^N\right)\right| < \left\lceil \frac{n}{m} \right\rceil$. Pose $S = E\left(y, x, R^N\right)$. Note that $|S| \geq n - \left(\left\lceil \frac{n}{m} \right\rceil - 1\right)$. We have $S(x, R^N) = S(x, R^S) + S(x, R^{N \setminus S}) \leq \left\lceil \frac{n}{m} \right\rceil - 1 + S(x, R^{E(x,y,R^N)})$. Moreover $S(y, R^N) \geq S(y, R^{N \setminus S})$ and by Remark 1, $S(y, R^N) \geq S(x, R^{N \setminus S}) + \frac{|N \setminus S|}{m-1}$. We deduce that $S(y, R^N) - S(x, R^N) \geq \frac{n - \left(\left\lceil \frac{n}{m} \right\rceil - 1\right)}{m-1} - \left(\left\lceil \frac{n}{m} \right\rceil - 1\right)$. Since $\frac{n}{m} \leq \left\lceil \frac{n}{m} \right\rceil \leq \frac{n}{m} + \frac{m-1}{m}$, we have $1 \leq n - \left(\left\lceil \frac{n}{m} \right\rceil - 1\right) - (m-1)\left(\left\lceil \frac{n}{m} \right\rceil - 1\right) \leq m$. Therefore $S(y, R^N) - S(x, R^N) > 0$. This contradicts the fact that $Borda\left(R^N\right) = x$. ■

7.15 Proof of Proposition 13

Proof. We first prove that $M^*\left(Borda, n, m\right) \leq 1 - \frac{1}{n}\left\lceil \frac{n}{m} \right\rceil$. Consider $R^N \in P(Borda)$ and $T^N \in SN(Borda \mid R^N)$. Pose $a = Borda(R^N)$ and $b = Borda(T^N)$. By Proposition 12, we have $\left|E(b, a, R^N)\right| \geq \left\lceil \frac{n}{m} \right\rceil$. That is $\left|E(Borda, R^N, T^N)\right| = \left|E(a, b, R^N)\right| \leq n - \left\lceil \frac{n}{m} \right\rceil$. It follows that $M^*\left(Borda, n, m\right) \leq 1 - \frac{1}{n}\left\lceil \frac{n}{m} \right\rceil$.

Conversely, to prove that $M^*\left(Borda, n, m\right) \geq 1 - \frac{1}{n}\left\lceil \frac{n}{m} \right\rceil$, we construct two profiles R^N and T^N such that $R^N \in P(Borda)$, $T^N \in SN(Borda \mid R^N)$ and $\left|E(Borda, R^N, T^N)\right| = n - \left\lceil \frac{n}{m} \right\rceil$. For this purpose, pose $n = km + r$ with $0 \leq r < m$. We consider 3 cases.

Case 1 : $k \geq 1$ and $r \neq 1$. Consider a partition N_j, $j = 1, 2, ..., m$ of N such that $|N_j| = k+1$ if $j \leq r$ and $|N_j| = k$ if $j > r$. Note that $|N_1| = |N_2| = k + t$ with $t = \min(r, 1)$. Let R^N be a profile such that

$$\begin{aligned}
&\forall i \in N_1,\ R^i = a_1...a_2,\\
&\forall i \in N_2,\ R^i = a_2a_1...,\\
&\forall i \in N_j,\ R^i = a_2a_1...a_{j+1}a_j \text{ if } 3 \leq j \leq m-1,\\
&\forall i \in N_m,\ R^i = a_2a_1...a_m.
\end{aligned}$$

We prove that $Borda(R^N) = a_1$. Note that $E\left(a_1, a_j, R^N\right) = N$ for each $j \geq 3$. Moreover $S\left(a_1, R^{N_1}\right) - S\left(a_2, R^{N_1}\right) = k + t$ and $S\left(a_1, R^{N_j}\right) - S\left(a_2, R^{N_j}\right) \geq -\frac{k+t}{m-1}$ if $j = 2, 3, ..., m$. Thus $S\left(a_1, R^N\right) - S\left(a_2, R^N\right) \geq (k+t) - (k+t) = 0$. Since $E\left(a_1, a_j, R^N\right) = N$ for each a_j with $j \geq 3$ and $S\left(a_1, R^N\right) \geq S\left(a_2, R^N\right)$, it follows that $Borda\left(R^N\right) = a_1$. Pose $T^i = a_2...a_3a_1$ for all $i \in N_2$ and $T^N = \left(R^{-N_2}, T^{N_2}\right)$.

To prove that T^N is an effective manipulation of R^N, we simply show that $Bord\left(T^N\right) = a_2$ since $a_2R^ia_1$ for all $i \in N_2$. We have $S\left(a_2, T^{N_1}\right) - S\left(a_1, T^{N_1}\right) = -(k+t)$, $S\left(a_2, T^{N_2}\right) - S\left(a_1, T^{N_2}\right) = k+t$ and $S\left(a_2, T^{N_j}\right) - S\left(a_1, T^{N_j}\right) \geq \frac{k}{m-1}$ if $j = 3, ..., m$. Thus $S\left(a_2, T^N\right) - S\left(a_1, T^N\right) \geq \frac{k(m-2)}{m-1} > 0$. In the same way, for each $l \in \{3,4,...,m\}$, we have $S\left(a_2,T^{N_1}\right)-S\left(a_l,T^{N_1}\right) \geq \frac{-(k+t)(m-2)}{m-1}$, $S\left(a_2,T^{N_l}\right)-S\left(a_l,T^{N_l}\right) \geq k$ and $S\left(a_2,T^{N_j}\right)-S\left(a_l,T^{N_j}\right) \geq \frac{k}{m-1}$ if $j \neq 1,l$. Thus $S\left(a_2,T^{N}\right)-S\left(a_l,T^{N}\right) \geq k-t+\frac{t}{m-1} \geq 0$. Therefore $Bord\left(T^N\right)=a_2$.

We now prove that T^N is an equilibrium given R^N. Since $Bord\left(T^N\right)=a_2$ and $E\left(a_2,R^N\right)=N\backslash N_1$; we only show that $\forall S \subseteq N_1$, $\forall Q^S \in L^S$, $Borda(Q^S,T^{-S})=a_2$. Take $S \subseteq N_1, Q^S \in L^S$, and pose $Q^N=(Q^S,T^{-S})$. We have $S\left(a_2,Q^N\right)-S\left(a_1,Q^N\right) \geq S\left(a_2,T^N\right)-S\left(a_1,T^N\right) > 0$ since $S\left(a_2,T^{N_1}\right)=0$ and $E\left(a_1,T^{N_1}\right)=N_1$. Moreover $S\left(a_2,Q^{N_1}\right)-S\left(a_3,Q^{N_1}\right) \geq -(k+t)$; $S\left(a_2,Q^{N_2}\right)-S\left(a_3,Q^{N_2}\right)=\frac{(k+t)(m-2)}{m-1}$; $S\left(a_2,Q^{N_j}\right)-S\left(a_3,Q^{N_j}\right) \geq \frac{2k}{m-1}$ if $j \neq 1,2,3$ and $S\left(a_2,Q^{N_3}\right)-S\left(a_3,Q^{N_3}\right) \geq k$. Thus $S\left(a_2,Q^N\right)-S\left(a_3,Q^N\right) \geq \frac{k-t+3k(m-3)}{m-1} \geq 0$. In the same way, for each $l \in \{4,5,...,m\}$, that is only for $m \geq 4$, we have $S\left(a_2,Q^{N_1}\right)-S\left(a_l,Q^{N_1}\right) \geq -(k+t)$; $S\left(a_2,Q^{N_{l-1}}\right)-S\left(a_l,Q^{N_{l-1}}\right) \geq \frac{k(m-2)}{m-1}$; $S\left(a_2,Q^{N_l}\right)-S\left(a_l,Q^{N_l}\right) \geq k$; $S\left(a_2,Q^{N_2}\right)-S\left(a_l,Q^{N_2}\right) \geq \frac{k+t}{m-1}$ and $S\left(a_2,Q^{N_j}\right)-S\left(a_l,Q^{T_j}\right) \geq \frac{2k}{m-1}$ for all $j \notin \{1,2,l-1,l\}$. Thus $S\left(a_2,Q^N\right)-S\left(a_l,Q^N\right) \geq \frac{3k-2t+(3k-t)(m-4)}{m-1} \geq 0$. This proves that $Bord\left(Q^N\right)=a_2$. Clearly there is no profitable deviation from T^N.

Since $E(Borda,R^N,T^N)=E(a_1,a_2,R^N)=N\backslash N_1$, then $\left|E(Borda,R^N,T^N)\right|=n-k=n-\left\lceil \frac{m}{n}\right\rceil$.

Case 2 : $k \geq 1$ and $r=1$. Consider a partition N_j, $j=1,2,...,m$ of N such that $1 \in N_1$, $2 \in N_3$, $|N_1|=k+1$ and $|N_j|=k$ if $j>1$. Let R^N be a profile such that

$$\begin{aligned}
&R^1 = a_2a_1...a_3 \\
&\forall i \in N_1\backslash\{1\},\ R^i = a_2...a_1 \\
&\forall i \in N_2,\ R^i = a_1a_2... \\
&\forall i \in N_j,\ R^i = a_1a_2...a_j \text{ if } 3 \leq j \leq m
\end{aligned}$$

Note that for each a_j with $j \geq 3$, $E\left(a_j,a_2,R^N\right)=N$. As above, $S\left(a_2,R^{\{1\}}\right)-S\left(a_1,R^{\{1\}}\right)=\frac{1}{m-1}$, $S\left(a_2,R^{N_1\backslash\{1\}}\right)-S\left(a_1,R^{N_1\backslash\{1\}}\right)=k$ and $S\left(a_2,R^{N_j}\right)-S\left(a_1,R^{N_j}\right)=-\frac{k}{m-1}$ if $j=2,3,...,m$. Thus $S\left(a_2,R^N\right)-S\left(a_1,R^N\right)=\frac{1}{m-1}$. Therefore $Bord\left(R^N\right)=a_2$.

For each voter $i \in N_2\cup\{2\}$, consider T^i such that $T^i=a_1...a_2$ and let $T^N=\left(R^{-(N_2\cup\{2\})},T^{N_2\cup\{2\}}\right)$. We prove that T^N is an effective manipulation of R^N. In fact, $S\left(a_1,T^{N_1}\right)-S\left(a_2,T^{N_1}\right)=-k-\frac{1}{m-1}$, $S\left(a_1,T^{N_2}\right)-S\left(a_2,T^{N_2}\right)=k$, $S\left(a_1,T^{N_j}\right)-S\left(a_2,T^{N_j}\right) \geq \frac{k}{m-1}$ if $4 \leq j \leq m$ and by definition of T^{N_3}, $S\left(a_1,T^{N_3}\right)-S\left(a_2,T^{N_3}\right) \geq \frac{(k-1)}{m-1}+1$. Thus $S\left(a_1,T^N\right)-S\left(a_2,T^N\right) \geq \frac{(k+1)(m-3)+k}{m-1} > 0$. For each $l \in \{3,4,...,m\}$, $S\left(a_1,T^{N_1}\right)-S\left(a_l,T^{N_1}\right) \geq \frac{1-k(m-2)}{m-1}$, $S\left(a_1,T^{N_l}\right)-S\left(a_l,T^{N_l}\right) \geq k-1$ and $S\left(a_1,T^{N_j}\right)-S\left(a_l,T^{N_j}\right) \geq \frac{k}{m-1}$ if $j \notin \{1,l\}$. Thus $S\left(a_1,T^N\right)-S\left(a_l,T^N\right) \geq \frac{1}{m-1}+k-1 > 0$. Therefore $Bord\left(T^N\right)=a_1$. Since $N_2 \cup \{2\} \subseteq E(a_1,a_2,R^N)$, then T^N is an effective manipulation of R^N.

We now prove that T^N is an equilibrium given R^N. Since $E(a_1,R^N)=N\backslash N_1$; it is enough to show that $\forall S \subseteq N_1$, $\forall Q^S \in L^S$, $Borda(Q^S,T^{-S})=a_1$. Take $S \subseteq N_1, Q^S \in L^S$, and pose $Q^N=(Q^S,T^{-S})$. We have $S\left(a_1,Q^N\right)-S\left(a_2,Q^N\right) \geq S\left(a_1,T^N\right)-S\left(a_2,T^N\right)-\frac{m-2}{m-1} \geq \frac{k(m-3)+k-1}{m-1} \geq 0$ since Voter 1 can strategically decrease the score of a_1 by at most $\frac{m-2}{m-1}$ points and all voters of $N_1\backslash\{1\}$ are already contributing the maximum for a_1 and nothing for a_2. That is $Borda(Q^N) \neq a_2$. There is consequently no lost of generality to assume that $S \subseteq N_1\backslash\{1\}$. For each $l \in \{3,4,...,m\}$, we have $S\left(a_1,Q^{N_1}\right)-S\left(a_l,Q^{N_1}\right) \geq \frac{1}{m-1}-k$, $S\left(a_1,Q^{N_2}\right)-S\left(a_l,Q^{N_2}\right) \geq \frac{k}{m-1}$, $S\left(a_1,Q^{N_l}\right)-S\left(a_l,Q^{N_l}\right) \geq k-1+\frac{1}{m-1}$ and $S\left(a_1,Q^{N_j}\right)-S\left(a_l,Q^{N_j}\right) \geq \frac{2k-1}{m-1}$ if $j \neq 1,2,l$. Thus $S\left(a_1,Q^N\right)-S\left(a_l,Q^N\right) \geq \frac{(2m-5)(k-1)}{m-1} \geq 0$.

This proves that $Bord\left(Q^N\right) \neq a_l$. Clearly, T^N is an equilibrium given R^N.

From R^N to T^N, a manipulation occurs in favor of $\left|E(Borda, R^N, T^N)\right| = n-(k+1) = n - \left\lceil \frac{m}{n} \right\rceil$ voters.

Case 3 : $k = 0$. That is $2 \leq n < m$. Consider a profile R^N such that

$$R^1 = a_2a_3...a_na_{n+1}a_1a_{n+2}...a_m,\ R^2 = a_{n+1}a_2a_1... \text{ and } R^i = a_{n+1}a_2a_1...a_i \text{ if } 3 \leq i \leq n$$

Note that voter 1 ranks a_j at the $(j-1)^{th}$ position for $2 \leq j \leq n+1$, a_1 is $(n+1)^{th}$ while a_j is j^{th} for $j > n+1$. Moreover for $a_j \in A\backslash\{a_{n+1}, a_2\}$, $E\left(a_j, a_2, R^N\right) = N$. Moreover $S\left(a_2, R^N\right) - S\left(a_{n+1}, R^N\right) = \frac{n-1}{m-1} - \frac{n-1}{m-1} = 0$. It is clear that $Borda\left(R^N\right) = a_2$. Consider the strategy profile $T^{N\backslash\{1\}}$ for members of $N\backslash\{1\}$ such that $T^2 = a_{n+1}a_1...a_2$ and $T^i = a_{n+1}a_1...a_i$ for $3 \leq i \leq n$. Pose $T^N = \left(R^1, T^{N\backslash\{1\}}\right)$.

We prove that T^N is an effective manipulation of R^N. Now $S\left(a_{n+1}, T^N\right) - S\left(a_2, T^N\right) \geq -\frac{n-1}{m-1} + 1 + \frac{2(n-2)}{m-1} > 0$, $S\left(a_{n+1}, T^N\right) - S\left(a_l, T^N\right) \geq -\frac{n-2}{m-1} + 1 + \frac{2(n-2)}{m-1} > 0$ for $l = 3, ..., n$ and $E\left(a_j, a_{n+1}, T^N\right) = N$ for all $a_j \in \{a_j : j = 1 \text{ or } j > n+1\}$. This implies that $Borda(T^N) = a_{n+1}$. Finally, remark that, $E(a_2, a_{n+1}, R^N) = N\backslash\{1\}$. Therefore , T^N is an effective manipulation of R^N.

We now prove that T^N is an equilibrium given R^N. Only player 1 has incentive to deviate from T^N, and only in favor of a candidate of $\{a_2, a_3, ..., a_n\}$. To see that this is not achievable, consider a strategy Q^1 for player 1 and pose $Q^N = \left(Q^1, T^{N\backslash\{1\}}\right)$. Assume a_1 is ranked at position p by voter 1 at Q^N and that $Borda\left(Q^N\right) = a_j$ for some $a_j \in \{a_2, a_3, ..., a_n\}$. Then $S\left(a_j, Q^N\right) - S\left(a_1, Q^N\right) \leq \frac{p-1}{m-1} - \frac{m-2}{m-1} - \frac{n-2}{m-1} \leq \frac{p+1-m}{m-1}$. Since $Borda\left(Q^N\right) = a_j$ with $j > 1$, we deduce that $S\left(a_j, Q^N\right) - S\left(a_1, Q^N\right) > 0$ and that $p + 1 > m$. Therefore $p = m$. Thus a_{n+1} is ranked q^{th} by voter 1 at Q^N with $q < m$. This implies that $S\left(a_j, Q^N\right) - S\left(a_{n+1}, Q^N\right) \leq \frac{q-1}{m-1} - 1 - \frac{2(n-2)}{m-1} < 0$. A contradiction holds since $Borda\left(Q^N\right) = a_j$.

From R^N to T^N, a manipulation occurs in favor of $\left|E(Borda, R^N, T^N)\right| = n - 1 = n - \left\lceil \frac{m}{n} \right\rceil$ voters. ■

7.16 Proof of Proposition 14

Proof. Let $R^N \in P(Borda)$, $T^N \in SN(Borda \mid R^N)$ and $S = E(Borda, R^N, T^N)$. We claim that $|S| \geq \left\lceil \frac{n}{2} \right\rceil$. On the contrary assume that $|S| < \left\lceil \frac{n}{2} \right\rceil$. Pose $Borda(R^N) = a$ and $Borda(T^N) = b$. Consider $\{N_1, N_2\}$ a partition of $N\backslash S$ such that $|N_1| = |S|$ and $|N_2| = n - 2|S|$. We have $|N_2| > 0$. Let $\varphi : S \to N_1$ be a bijection. Define $Q^{N\backslash S}$ a profile of preferences of $N\backslash S$ voters as follows:

- $Q^{\varphi(i)} = Z^i$ for $i \in S$ where Z^i is defined as follows: for all $i \in S$, $a, b \in A$, aZ^ib holds if and only if bT^ia.
- $Q^i = R_{[a]}$ for all $i \in N_2$ where $R = a_1a_2...$ is the lexicographic order.

From the construction of $Q^N = \left(Q^{N\backslash S}, T^S\right)$, we have $S(a, Q^N) - S(x, Q^N) = S(a, Q^{N_2}) - S(x, Q^{N_2}) \geq \frac{1}{m-1}$ for all $x \neq a$. Consequently, $Borda(Q^{N\backslash S}, T^S) = a$. In addition, aR^ib for all $i \in N\backslash S$. That is T^N is not an equilibrium given R^N, a contradiction. We conclude that $|S| \geq \left\lceil \frac{n}{2} \right\rceil$. It follows that $m^*\left(Borda, n, m\right) \geq \frac{1}{n}\left\lceil \frac{n}{2} \right\rceil$.

To prove that $m^*\left(Borda, n, m\right) \leq \frac{1}{n}\left\lceil \frac{n}{2} \right\rceil$, we wish to construct $R^N \in P(Borda)$ and $T^N \in SN(Borda \mid R^N)$ such that $\left|E(Borda, R^N, T^N)\right| = \left\lceil \frac{n}{2} \right\rceil$. We consider two cases.

Case 1 : $n = 2k$. Take $\{N_1, N_2\}$ a partition of N such that $2 \in N_2$, $|N_1| = |N_2| = k$ and pose:

$$R^i = a_1 a_2 ... a_m \text{ for all } i \in N_1$$
$$R^i = a_2 a_1 ... a_m \text{ for all } i \in N_2 \backslash \{2\} \text{ and } R^2 = a_2 a_m a_1 ...$$
$$T^i = a_1 ... a_m a_2 \text{ for all } i \in N_1.$$

Clearly, we have $Borda(R^N) = a_2$ and $Borda(T^{N_1}, R^{N_2}) = a_1$. Since $a_1 R^i a_2$ for all $i \in N_1$, we conclude that $T^N = (T^{N_1}, R^{-N_1})$ is an effective manipulation of R^N. It is also an equilibrium given R^N. In fact, only voters of N_2 wish to replace a_1 by another candidate, candidate a_2 specially. This is not possible since $S(a_1, T^{N_1}) - S(a_2, T^{N_1}) = k \geq |N_2|$. That is $R^N \in P(Borda)$ and $T^N \in SN(Borda \mid R^N)$. In addition, $\left|E(Borda, R^N, T^N)\right| = |N_1| = \left\lceil \frac{n}{2} \right\rceil$.

Case 2 : $n = 2k + 1$. Take $\{\{1, 2, 3\}, N_1, N_2\}$ a partition of N such that $|N_1| = |N_2| = k - 1$ and pose:

$$R^i = a_1 a_2 ... a_m \text{ for all } i \in N_1,\ R^i = a_2 a_1 ... a_m \text{ for all } i \in N_2,$$
$$R^1 = a_2 a_1 a_3 ...,\ R^2 = a_2 a_1 a_3 ... \text{ and } R^3 = a_1 a_3 a_2 ...$$
$$S = N_2 \cup \{1, 2\} \text{ and } T^i = a_2 ... a_3 a_1 \text{ for all } i \in S.$$

We have $Borda(R^N) = a_1$ and $Borda(T^S, R^{-S}) = a_2$. Since $a_2 R^i a_1$ for all $i \in S$, we conclude that $T^N = (T^S, R^{-S})$ is an effective manipulation of R^N. It is also an equilibrium given R^N. In fact and as above, only voters of $N_1 \cup \{3\}$ wish to replace a_2 by candidate a_1 while a_3 is also an option for voter 3 and only him. This is not possible to realize since $S(a_2, T^S) - S(a_1, T^S) = k + 1 > |N_1 \cup \{3\}|$. Obviuosly, voter 3 could not make a_3 be elected. That is $R^N \in P(Borda)$ and $T^N \in SN(Borda \mid R^N)$. In addition, $\left|E(Borda, R^N, T^N)\right| = |N_2 \cup \{1, 2\}| = \left\lceil \frac{n}{2} \right\rceil$. ■

7.17 Proof of Proposition 15

Proof. By definition, $M^* (APl, n, m) \leq 1$. Consider a profile R^N such that $R^i = a_2 a_1 ... a_m$ for all $i \in N$. Pose $T^1 = a_2 ... a_1$ and $T^N = (T^1, R^{-1})$. Obviously, $APl\left(R^N\right) = a_1$ and $APl\left(T^N\right) = a_2$. Since $a_2 R^i a_1$ for all $i \in N$, we conclude that T^N is an effective manipulation of R^N and an equilibrium given R^N. Finally, $E(APl, R^N, T^N) = N$. So $M^* (APl, n, m) \geq 1$. ■

7.18 Proof of Proposition 16

Proof. Case 1 : Assume that $n \leq m$. By definition, $m^*(APl, n, m) \geq \dfrac{1}{n}$. To prove that $m^*(APl, n, m) \leq \dfrac{1}{n}$, we construct two profiles R^N and Q^N such that $R^N \in P(APl)$, $T^N \in SN(APl \mid R^N)$ and $\left|E(APl, R^N, T^N)\right| = 1$. Let R^N be a profile such that $R^1 = a_3 a_2 ... a_1$ and $R^i = a_2 a_3 ... a_1$ for all $i = 2, ..., n$. Pose $T^1 = a_3 ... a_1 a_2$ and $T^N = (Q^1, R^{-1})$. We have $APl(R^N) = a_2$, $E(a_2, a_3, R^N) = \{1\}$, $APl(T^N) = a_3$ and $a_3 R^1 a_1$. That is T^N is an effective manipulation of R^N.

T^N is an equilibrium given R^N. In fact, given a profile Q^N of the form (T^1, Q^{-1}), for a_2 to be elected, it is necessary for every candidate $a_j, j = 1, ..., m$ to be ranked last by at least one voter in Q^{-1} and for a_1 to be ranked last by at least 2 voters. This is not possible since $|N_1 \backslash \{1\}| \leq m - 1$.

Since $E(a_2, a_3, R^N) = \{1\}$, we have $m^*(APl, n, m) \leq \dfrac{1}{n}$. Moreover, $n \times m^*(APl, n, m) \geq 1$. We then conclude that $m^*(APl, n, m) = \dfrac{1}{n}$.

Case 2 : Assume that $n > m$.

We first prove that $m^*(APl, n, m) \geq \lfloor \frac{n+1}{m} \rfloor$. Take $R^N \in P(APl)$ and $T^N \in SN(APl \mid R^N)$. Pose $APl(R^N) = a_l$, $APl(T^N) = a_t$, $S = E(APl, R^N, T^N)$ and $k = |S|$. Assume $k < \lfloor \frac{n+1}{m} \rfloor$. Consider two integers p and r such that $n-k = (m-1)p+r$ with $r \in \{0, 1, ..., m-2\}$. We claim that $p \geq k+1$. In fact, $k < \lfloor \frac{n+1}{m} \rfloor$. Then $k \leq \lfloor \frac{n+1}{m} \rfloor - 1 \leq \frac{n+1}{m} - 1$. Since $n = k+(m-1)p+r \leq k+(m-1)p+m-2$, we deduce that $k \leq \frac{k+(m-1)p+m-1}{m} - 1$. That is $p \geq k + \frac{1}{m-1}$. Hence $p \geq k+1$. Given $l_0 \in \{1, 2, ..., m\}$ such that $l_0 \neq l$, let $\{N_j, 1 \leq j \leq m, j \neq l\}$ be a partition of $N \backslash S$ such that $|N_j| = p$ for all $j \neq l_0$ and $|N_{l_0}| = p + r$. Consider a strategy Q^S of $N \backslash S$ voters such that all N_j voters rank a_j last for $j \in \{1, 2, ..., m\} \setminus \{l\}$. Since $p \geq k+1$ and $|S| = k$, we have $APl(Q^S, T^{-S}) = a_l$. In addition, $a_l R^i a_t$ for all $i \in N \backslash S$. This is a contradiction since $S = E(APl, R^N, T^N)$ and $T^N \in SN(APl \mid R^N)$. We conclude that $k \geq \lfloor \frac{n+1}{m} \rfloor$.

We now prove that $m^*(APl, n, m) \geq \lfloor \frac{n+1}{m} \rfloor$. To do this, we construct two profiles R^N and T^N such that $R^N \in P(APl)$, $|E(APl, R^N, T^N)| = \lfloor \frac{n+1}{m} \rfloor$ and $T^N \in SN(APl \mid R^N)$. Pose $n = mq + r$ with $0 \leq r < m$. Consider a partition $\{N_1, N_2, ..., N_m\}$ of N in m subsets such that voter 1 belongs to N_m and $|N_j| = \begin{cases} q+1 \text{ if } 2 \leq j \leq r+1 \\ q \text{ otherwise} \end{cases}$ if $r < m-1$ and

if $r = n-1, |N_j| = \begin{cases} q+1 \text{ if } 1 \leq j \leq n-1 \\ q \text{ if } j = n \end{cases}$ if $r = n-1$.

Observe that $|N_1| = \lfloor \frac{n+1}{m} \rfloor$.

Let R^N be a profile such that:

$$
\begin{aligned}
R^i &= a_1 a_m ... a_2 \text{ for all } i \in N_1 \\
R^i &= a_m a_1 ... a_j \text{ for all } i \in N_j \text{ with } j \in \{2, ..., m-1\} \\
R^i &= a_m a_1 ... a_2 \text{ for all } i \in N_m \backslash \{1\} \text{ and } R^1 = a_m a_2 ... a_1.
\end{aligned}
$$

Note that a_m is never ranked last while any other alternative is ranked last by some voter. Then $APlu(R^N) = a_m$ Define $T^i = a_1 ... a_m$ for all $i \in N_1$ and pose $T^N = (T^{N_1}, R^{-N_1})$. Obviously, $APl(T^N) = a_1$ and $a_1 R^i a_m$ for all $i \in N_1$. That is, T^N is an effective manipulation of R^N.

Note that $E(Apl, R^N, T^N) = N_1$. To prove that T^N is an equilibrium given R^N, it is sufficient to prove that $APl(Q^{-N_1}, T^{N_1}) \neq a_m$ for all possible profiles Q^{-N_1} of $N \backslash N_1$ voters. Assume that $APl(Q^{-N_1}, T^{N_1}) = a_m$ for some Q^{-N_1}. Since a_m is ranked last at (Q^{-N_1}, Q^{N_1}) by $|N_1|$ voters, then by definition of APl, for all $j \in \{1, 2, ..., m-1\}$, a_j is ranked last by at least $|N_1| + 1$ voters. Therefore $|N| \geq (m-1)(|N_1| + 1) + |N_1|$. That leads to the contradiction $n + 1 \geq m \lfloor \frac{n+1}{m} \rfloor + m$. Thus $APl(Q^{-N_1}, T^{N_1}) \neq a_m$ and T^N is an equilibrium given R^N.

In addition, we have, $|E(APl, R^N, T^N)| = |N_1| = \lfloor \frac{n+1}{m} \rfloor$. ■

7.19 Proof of Theorem 3

Proof. The proof is obtained by taking the limits of the functions $m^*(F, n, m)$ and $M^*(F, n, m)$ for each $F \in \{Pl, Borda, APl\}$ obtained in Sections 4.1, 4.2, and 4.3 (see Propositions 10, 11, 13, 14, 15 and 16). ■

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