# Iterated Kalai-Smorodinsky-Nash Compromise

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In this paper, we present a new $n$-person bargaining solution, which we call *Iterated Kalai-Smorodinsky-Nash Compromise*. We show that this solution is the unique solution satisfying a new axiom called *Kalai-Smorodinsky-Nash Decomposability*.

Keywords: Cooperative bargaining; Nash solution; Kalai-Smorodinsky solution

Subject Classification: C71; C78

## 1 Introduction

The bargaining problem in a cooperative setup was first introduced by the seminal work of Nash (1950), who also proposed a solution along with an axiomatization for it. Briefly, Nash's bargaining problem involves a fixed number of agents, a bargaining set of utility allocations that these agents bargain over, and a disagreement point that would prevail if the agents fail to reach an agreement. A solution in this setup is a rule that selects a feasible utility allocation for each bargaining problem. In particular, the solution proposed by Nash (1950) maximizes the product of the utility gains of each agent with respect to his/her disagreement utility. Nash (1950) showed that this solution is the only solution which possesses a set of axioms, involving Independence of Equivalent Utility Representations (IEUR), Independence of Irrelevant Alternatives (IIA), Symmetry (SY), and Weak Pareto Optimality (WPO). Of these axioms, IEUR claims that the bargaining outcome should be invariant with respect to all positive affine transformations of the bargaining set; IIA says that the bargaining outcome of a given bargaining problem should also be selected at any subproblem that contains it; SY requires that if the disagreements utilities of players are equal and the bargaining set is symmetric, then the bargaining outcome should also be symmetric; and WPO requires that the bargaining outcome should not be strictly inferior to another feasible allocation. While the characterization of the Nash solution was originally offered for 2-person games, it is also valid for $n$-person games under straightforward generalizations of the relevant axioms.

In the last 65 years since Nash (1950), many alternative bargaining solutions have been proposed and axiomatized. One of the most famous of these was proposed for 2-person games by Raiffa (1953) and axiomatized by Kalai and Smorodinsky (1975). Kalai-Smorodinsky solution is based on the *ideal point* associated with a given bargaining problem, i.e., the point at which each agent would receive maximal utility in the bargaining set given that all other agents obtain at least their disagreement utilities. Formally, the Kalai-Smorodinsky solution selects in each bargaining set the maximal point on the line segment joining the disagreement point to the ideal point. The three of the four axioms that characterize the Nash solution, namely IEUR, SY, and WPO, are satisfied by many well-known solutions, including the Kalai-Smorodinsky solution. Besides, Kalai-Smorodinsky solution satisfies an axiom called Individual Monotonicity (IM), as well. This axiom requires that if a bargaining set expands in such a way that the set of feasible utilities for all players other than player $i$ remains the same while the maximum utility player $i$ can achieve becomes higher at all levels of utilities enjoyed by others, then player $i$ should not be worse off after the expansion. Kalai and Smorodinsky (1975) showed for 2-person bargaining games that among all solutions that satisfy the so-called 'standard' axioms, IEUR, SY, and WPO, the Kalai-Smorodinsky solution is the unique solution that satisfies the axiom IM.[1] Later Thomson (1983) showed that in the $n$-person case the Kalai-Smorodinsky solution turns out to be the only solution that possesses Anonymity (AN), Continuity (CONT), IEUR, Monotonicity With Respect to Changes in the Number of Agents (MON), and WPO. Of these axioms, IEUR and WPO are generalizations of the former conditions used by Kalai and Smorodinsky (1975) for the 2-person case. The axiom AN is a strengthening of the axiom SY, and requires that not only the names of the agents in a given group do not matter but also that any other group of agents of the same size would reach the same bargaining outcome. CONT implores that a small change in the bargaining set causes only a small change in the bargaining outcome. Finally, MON says that if the expansion of a group of agents requires a sacrifice to support the entrants, then every incumbent must contribute.

The Nash solution satisfies AN, CONT, IEUR, and WPO; but it does not satisfy MON as directly shown by Thomson and Lensberg (1989, pp. 41-42).[2] Given the characterization results of the Nash and Kalai-Smorodinsky solutions, these two solutions are then distinguished from each other only by whether they possess IIA or IM in 2-person games and whether they possess IIA or MON

[1] Roth (1979) showed that in the characterization of the Kalai-Smorodinsky solution for 2-person games, the axiom IM can be replaced by a weaker condition, called *Restricted Monotonicity*, which says that all agents should weakly benefit from the expansion of the bargaining set if the ideal point remains unchanged.

[2] We should also note that in the n-person case, the Nash solution also satisfies, unlike the Kalai-Smorodinsky solution, the strong Pareto optimality (PO) which requests that the bargaining outcome should not be weakly inferior to another feasible allocation. Roth (1979) was the first to note that the axiom PO is not satisfied by the Kalai-Smorodinsky solution unless the number of agents is two. In fact, in $n$-person games the Kalai-Smorodinsky solution could even select the disagreement point if utilities were not assumed to be freely disposable.

in $n$-person games. On the other hand, as recently shown by Rachmilevitch (2014) for 2-person games, the common attribute of the Nash and Kalai-Smorodinsky solutions can be characterized by a single axiom, called Kalai-Smorodinsky-Nash Robustness (KSNR), which requires that each agent receives at least the minimum of the utility payoffs he or she would have received under the Kalai-Smorodinsky and Nash solutions. Rachmilevitch (2014) shows for 2-person games that among all solutions that satisfy KSNR, the Kalai-Smorodinsky solution is the only one that satisfies IM whereas the Nash solution is the only one that satisfies IIA.[3]

The results of Rachmilevitch (2014) imply that if the agents faced with a bargaining game should make -before the bargaining set is known to themselves- a compromise between the Kalai-Smorodinsky and Nash solutions, they may focus on solutions that satisfy KSNR. In this paper, we propose that a meaningful compromise on this set of solutions can be obtained by iteratively applying the idea of KSNR infinitely many times, yielding a unique solution which we call the *Iterated Kalai-Smorodinsky-Nash Compromise* (IKSNC). More formally, in any $n$-person game that involves a bargaining set $S_0$ and a disagreement point $d_0$, the (referential) compromise point $m(S_0, d_0)$ denotes an allocation in $S_0$ where each player receives the minimum of the utility payoffs he or she would have received under the Kalai-Smorodinsky and Nash solutions. Let $U(S_0, d_0)$ be the set of allocations in $S$ such that they are at least as good as $m(S_0, d_0)$. Clearly, any solution that satisfies KSNR must select for each problem $(S_0, d_0)$ an outcome in $U(S_0, d_0)$. Now, one can set normalize $U(S_0, d_0)$ with respect to the point $m(S_0, d_0)$ to obtain the subproblem $(S_1, d_1)$ of $(S_0, d_0)$ where $d_1 = 0$. Repeating this procedure, one can construct, for each step $i$, the problem $(S_i, d_i)$ and the compromise point $m(S_i, d_i)$. Then, calculating $\lim_{i\to\infty} m(S_i, d_i)$ yields the IKSNC solution.

The solution we propose satisfies a new axiom, which we call, *Kalai-Smorodinsky-Nash Decom-*

[3] The results of Rachmilevitch (2014) are independent from the previous characterization results since KSNR neither implies nor is implied by the set of standard axioms, involving IEUR, SY, and WPO.

*posability* (KSND).[4] This axiom requires that for any bargaining problem $(S, d)$, the solution can be calculated in two steps, by first calculating the compromise point $m(S, d)$ (as defined by Rachmilevitch, 2014) in $S$ and then taking it to be the starting point for the distribution of the utilities in $S$.

The paper is organized as follows: In Section 2 we introduce the basic structures and in Section 3 we present our characterization result. Finally, Section 4 contains some concluding remarks.

## 2 Basic Structures

We consider an $n$-person bargaining problem (simply a problem) which consists of a pair $(S, d)$ where $S$ is a non-empty subset of $\mathbb{R}^n_+$, representing von Neumann-Morgenstern utilities attainable through the cooperative actions of $n$ agents and $d$ is a point in $S$, which is called the disagreement point and denotes where the bargaining is settled if the agents fail to agree on an outcome in $S$. The society of $n$ agents facing the described problem is denoted by $N = \{1, 2, ..., n\}$. We consider the domain $\Sigma^n$ of problems where

(a) $S$ is compact and convex.

(b) For all $x, y \in \mathbb{R}^n_+$, i.e., if $x \in S$ and $x \geq y \geq d$, then $y \in S$ ($d$-comprehensiveness or the possibility of free disposal of utility).[5]

Given a problem $(S, d)$, $IR(S, d)$ denotes the individually rational allocations; i.e., $IR(S, d) = \{x \in S \,|\, x \geq d\}$. For any bargaining set $S \subset \mathbb{R}^n_+$ and any $y \in \mathbb{R}^n_+$, let $S - y = \{x \in \mathbb{R}^n \,|\, \exists\, z \in S$ such that $x = z - y\}$. We define the weak Pareto frontier of $S$ as $WP(S) = \{x \in S \,|\, y >$

[4]The notion of the decomposability of a solution was previously used in several characterization results by Kalai (1977), Salonen (1988), Rachmilevitch (2012), Saglam (2014), and Trockel (2014, 2015), among others.

[5]Given two vectors $x$ and $y$ in $\mathbb{R}^n_+$, $x > y$ means $x_i > y_i$ for all $i \in N$ and $x \geq y$ means $x_i \geq y_i$ for all $i \in N$.

$x$ implies $y \notin S\}$ and the strong Pareto frontier of $S$ as $P(S) = \{x \in S \,|\, y \geq x$ and $y \neq x$ implies $y \notin S\}$. Also, let $a_i(S,d)$ denote the maximal utility agent $i$ can expect in the problem $S$, given that agent $j \neq i$ obtains at least $d_j$ units of utility; i.e., $a_i(S,d) = max\{x_i : x \geq d\}$. We call $a(S,d) = (a_1(S,d), \ldots, a_n(S,d))$ the ideal point of the problem $(S,d)$.

A solution $F$ is a mapping from $\Sigma^n$ to $\mathbb{R}^n_+$ such that for each $(S,d) \in \Sigma^n$, $F(S,d) \in S$. The Nash (1950) solution maps each problem $(S,d) \in \Sigma^n$ to the point $N(S,d) = arg\max_{x \in S} \prod_{i=1}^n (x_i - d_i)$, at which the product of players' payoff gains from agreement is maximized. On the other hand, the Kalai-Smorodinsky solution (Raiffa, 1953; Kalai-Smorodinsky, 1975) selects in each bargaining set the maximal point on the line segment joining the disagreement point to the ideal point. Formally, it maps each problem $(S,d) \in \Sigma^n$ to the point $KS(S,d) \in WP(S)$ such that $[KS_i(S,d) - d_i]/[KS_j(S,d) - d_j] = [a_i(S,d) - d_i]/[a_j(S,d) - d_j]$ for all $i,j \in N$, implying that each player's payoff gain from agreement has the same proportion to his or her ideal payoff gain from agreement.

To reconcile between the Nash and Kalai-Smorodinsky solutions, we will use the compromise point in Rachmilevitch (2014). Formally, given any problem $(S,d)$, the (referential) compromise point $m(S,d) \in S$ is defined such that $m_i(S,d) = \min\{KS_i(S,d), N_i(S,d)\}$ for all $i \in N$. Now consider the sequence of points $(r^t(S,d))_{t=0}^{\infty}$ where $r^0(S,d) = d$, and $r^t(S,d) = m(S, r^{t-1}(S,d))$ for each integer $t \geq 1$. Clearly, $\lim_{t\to\infty} r^t(S,d) \in S$. The solution that maps each problem $(S,d) \in \Sigma^n$ to the allocation $\lim_{t\to\infty} r^t(S,d)$ will be called the **Iterated Kalai-Smorodinsky-Nash Compromise (IKSNC)**. This new solution will be shown to be the unique solution satisfying the following axiom.

**Kalai-Smorodinsky-Nash Decomposability (KSND)**. For all $(S,d) \in \Sigma^n$, $F(S,d) = m(S,d) + F(IR(S - m(S,d), 0), 0)$.

According to the above axiom, the solution $F(S,d)$ can be calculated in two steps, by first calculating the reference point $m(S,d)$ in $S$ and then taking it to be the starting point for the distribution of the utilities in $S$.

# 3 Characterization Result

**Theorem 1.** *A solution satisfies KSND if and only if it is the IKSNC solution.*

**Proof.** We will first check the "if" part, i.e., the IKSNC solution satisfies KSND. For this, we will need to verify that for all $t \geq 2$ we have

$$r^t(S,d) = m(S,d) + r^{t-1}(S^1, 0), \tag{1}$$

where $S^1 = IR(S - r^1(S,d), 0)$. We will proceed with a proof by induction. Clearly,

$$r^2(S,d) = m(S,d) + r^1(S^1, 0) = m(S,d) + m(S^1, 0). \tag{2}$$

Next, assume that for the integer $t = n$ the desired equality holds, i.e.,

$$r^n(S,d) = m(S,d) + r^{n-1}(S^1, 0). \tag{3}$$

Note that the above equality can be iterated to yield

$$r^n(S,d) = m(S,d) + \sum_{j=1}^{n-1} m(S^j, 0), \tag{4}$$

where $S^j = IR(S - r^j(S,d), 0)$. By definition we have $r^{n+1}(S,d) = m(S, r^n(S,d))$, implying

$$r^{n+1}(S,d) = r^n(S,d) + m(S^n, 0). \tag{5}$$

Using (4), we can rewrite the above equation as

$$r^{n+1}(S,d) \quad = \quad m(S,d) + \sum_{j=1}^{n-1} m(S^j, 0) + m(S^n, 0),$$

$$= \quad m(S,d) + \sum_{j=1}^{n} m(S^j,0), \tag{6}$$

implying

$$r^{n+1}(S,d) = m(S,d) + r^n(S^1,0). \tag{7}$$

Thus, we have proved that (1) holds for any integer $t \geq 2$. Now writing (1) in the limit as the integer $t$ approaches $\infty$, we obtain

$$\begin{aligned} \lim_{t\to\infty} r^t(S,d) &= m(S,d) + \lim_{t\to\infty} r^{t-1}(S^1,0), \\ &= m(S,d) + \lim_{t\to\infty} r^t(S,d), \end{aligned} \tag{8}$$

implying that $\lim_{t\to\infty} r^t(S,d)$ satisfies KSND.

To check the "only if" part, let $F$ be a solution on $\Sigma^n$ satisfying KSND. Then, $F(S,d) = m(S,d) + F(IR(S - m(S,d,),0),0)$. Consider the sequence of problems $(S^t,d^t)_{t=0}^{\infty}$ where $S^0 = S$, $d^0 = d$, and $d^t = 0$, $S^t = IR(S^{t-1} - m(S^{t-1},d^{t-1}),0)$ for every integer $t \geq 1$. It is clear that $F(S^{t+1},d^{t+1}) = F(S^0,d^0) - \sum_{j=0}^{t} m(S^j,d^j)$ for every integer $t \geq 0$ by KSND. Now suppose that $F(S^0,d^0) \neq \lim_{t\to\infty} r^t(S^0,d^0)$. Then, one can easily show by the geometry of the rule $F$ that there exists $k \geq 0$ such that $F(S^{k+1},d^{k+1}) \notin S^{k+1}$, a contradiction. Therefore, $F(S^0,d^0) = \lim_{t\to\infty} r^t(S^0,d^0)$. Since $(S,d) \in \Sigma^n$ was arbitrarily picked, $F$ must coincide with the IKSNC solution. □

# 4 Conclusion

In this paper, we have proposed a new bargaining solution, called Iterated Kalai-Smorodinsky-Nash Compromise, to reconcile between the two most well-known bargaining solutions, namely the Nash and Kalai-Smorodinsky solutions. This solution can be characterized by a single axiom called Kalai-Smorodinsky-Nash Decomposability, which requires that the outcome of the solution on any bargaining problem can be obtained by first calculating the referential *compromise* point at which each player receives the minimum of the utility payoffs he or she would have received under the Kalai-Smorodinsky and Nash solutions, and then adding this point to the solution of the subproblem admitting this compromise point as both its starting and disagreement point.

Obviously, one can find infinitely many ways to make a reconciliation between the Kalai-Smorodinsky and Nash solutions, as there exists an uncountable number of solutions that satisfy Kalai-Smorodinsky Nash Robustness, introduced by Rachmilevitch (2014). However, of all these potentially compromising solutions, our solution is the only one that requires no more information than is already contained in the Nash and Kalai Smorodinsky solutions; hence we reach a reconciliation in the most natural way.

Finally, as our solution is (iteratively) constructed using the Nash and Kalai-Smorodinsky solutions only, one can easily check that it trivially satisfies the common axioms in the characterization results of Nash and Kalai-Smorodinsky solutions, i.e., IEUR, SY, and WPO for the case of $n = 2$, and AN, CONT, IEUR, and WPO for the case of $n \geq 3$.

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