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Abstract
The first purpose of this paper is to study the dynamics of a general socially undesirable public evil and the possibility of cyclical Nash strategies in equilibrium. As a second result of the paper we found the analytical solutions of the hierarchical (Stackelberg) game for the public bad accumulation model. In both cases we use the differential game modeling, as the appropriate tool for the economic analysis that follows. The control setting is not the usual one, which assumes an accumulated stock of a public bad (e.g. pollutants, wastes or even tax evasion), but we claim that the disadvantage which is responsible for the unwished public evil accumulation is the use of the available inputs and equipment. Therefore, this could be a crucial assumption which possibly prevents the irreversibility of the public bad accumulation. As a continuation, we set as stock the available resources (inputs plus equipment) and the stress of the regulator is to reduce these resources. In the first case of Nash equilibrium, we find that the establishment of cyclical strategies, during the game between the agents in charge and the regulator, requires that the agents' discount rate must be greater than the government's discount rate, i.e., the agents in charge must be more impatient than the government (acting as the regulator). In the second case of the hierarchical setting, we provide the analytical expressions of the strategies as well as the steady state value of the resources' stock. We use the notion of a public bad as the opposite meaning to the public good.

Keywords: Public bad; cyclical policies; Nash equilibrium; Stackelberg equilibrium.

JEL Classifications: C61, C62, D43, H21; Q50; Q58.

1. Introduction

In this paper we deal with the dynamics of accumulation of a socially undesirable evil which harms prosperity of the economic agents of a country or a nation. In the undesirable socially evil we give the opposite meaning of the (desirable) public good, which impairs social welfare. For example the pollutants accumulation is harmful for the environmental services thought as a renewable resource, waste accumulation is bad for the public health, public debt accumulation produces disutility, therefore is detrimental to nations' households, because it reduces their consumption in order to meet their future tax burden (Greiner and Fincke, 2009)

Considering methodology, game theory may be used as an appropriate tool in order to design an efficient action against accumulation of a public "bad", because the regulator has to take into consideration the response of victims e.g. the pollutes, the taxpayers and so on. In the most cases, every socially undesirable stock is an irreversible fact, and therefore one of the planer's main concerns could be the discovery of effective ways to reduce the sources (inputs and equipment) which are responsible for the unwished stock accumulation. We use both Nash and Stackelberg differential game methods to study the intertemporal strategic relations between the economic agents in charge and the social planer.

The major problem of an evil stock accumulation requires to finding ways to effectively reduce this unwished stock, maintaining, at the same time, the standards of the economic process within a country. The environmental example says that the clean environment is clearly a public good. On the other hand, "dirty" production process generating for instance emissions and pollution from uncontrolled production, comprises a public bad. The primary question raised in pollutants accumulation is about which one of the productive factors are responsible for the accumulation of the public bad? As it is clear in the pollution paradigm, the polluters "dirty weapons" which damage the environment are all the shabby equipment which emits more than the allowed level and therefore the production process remains unabated. As it usually happens, in the equipments' market (which is based in technological progress), the more technological advanced machines are worked primarily in the developed (i.e. Western) countries and secondly as its technology (and therefore its value) depreciates the above equipment are transferred, for a reasonable reward, in developing Southern or Eastern countries. But, taking into account the maintenance cost since the above equipment is already long-standing, constitutes a source of pollution in this latter production stage. As a second source of pollution, during the production process, can be considered the raw materials which are not environmental friendly, for example the fossil fuels.

Returning to the model solution problems, one of the main concerns should be the irregularity of more than one points of equilibrium. By the large the existence of multiple points of equilibrium especially in economic models is an undesired result, because it confuses the policy makers. But in fact the multiple optimal equilibrium points separate the basins of attraction of this multiple equilibria. In technical terms, any small deviation from the threshold (which separates the basins of attraction) in one hand steers the optimal control vehicle to its unique trajectory but on the other hand "destroys" this position of indifference in which the rational decision maker he came.

Since Skiba's model (Skiba, 1978) cyclical policies became an interesting point of research in economic models. The relative literature grows rapidly and Wirl (1995), among others, tackling with the Clark's classical renewable resources' model

(Clark et al, 1979) offers the useful corollary which says "even with positive spillovers, in the long run harvesting equilibrium the cyclical strategies are possible".

As it is already made clear, the purpose of the present paper is to uncover principles underpinning efficient design of countermeasures against the sources of the undesired public harm. In particular, we model the optimal balance of competing parties, and we intent to find the implications of misspecification at the level for success or failure. An important aim of the first part of our research is the identification of mechanisms generating oscillations of both responsible (for the public evil) agents' activities and periodic countermeasures on behalf the regulator.

The discussion of a threshold occurrence does not only limited in the well known (S, s) policies in inventory management, but there are however, other nonlinearities implying oscillatory behavior. We intend to study this issue by using the methodology of stable limit cycles. An intuitive explanation of cyclical policies could be the following. Since the continuous orbit of a dynamical system is bounded in a specific region, then the possible equilibrium has to be a point or a cycle. Obviously the cycle (therefore a cyclical strategy) is a richer equilibrium concept than the point equilibrium. Specifically for any cyclical policy, e.g. a taxation policy, with its trajectory bounded into a planar and since it is happens continuously then for its orbit is unavoidable to follow its previous steps.

Moreover, in higher than the two dimensional systems, sufficient conditions for the existence of limit cycles of nonlinear dynamical systems are provident. Arithmetically the sufficient conditions requires that a pair of purely imaginary eigenvalues exists, for a particular value of the bifurcation parameter, and the real part of this pair of eigenvalues changes smoothly its sign as the parameter is altered from below its actual value to above.

The stability of limit cycles is of great importance for the long run behavior of a dynamical system. Economic mechanisms that may be a source of limit cycles, as mentioned by Dockner and Feichtinger (1995) are: (i) complementarity over time, (ii) dominated cross effects with respect to capital stocks, and (iii) positive growth of equilibrium.

The contribution of the paper, in the public economics field, is that it considers the bad stock accumulation control problem not in its irreversible aspect i.e., as a stock of an accumulated public evil, but the paper focuses at the stock of resources which is responsible for the public bad accumulation and therefore immediately, the former stock, may damage the welfare of a nation. Consequently, by this perspective, one can prevent the accumulated undesirable stock by weakening the responsible agents' resources.

The problem is modeled first as a Nash differential game, for which we explore at equilibrium the possibility of limit cycles and second as a Stackelberg differential game for which we calculate the equilibrium strategies. The public economics control game takes place between the government, acting as the regulator, and the agents in charge for which the resources used in economic activities are responsible for public evil. Such stock accumulation and regulation control models can be found, among others, in Forster (1980) concerning optimal energy use model; in Xepapadeas (1992) regarding environmental policy design and non-point source pollution and so on.

The remainder of the paper is organized as follows. Section 2 comments on cyclical policies in the undesirable stock control, while Section 3 introduces the Nash differential game and gives a necessary condition for cyclical strategies. Section 4 investigates the Stickler differential game between the regulator and the agents in charge and calculates the equilibrium strategies and the players' value functions. The last section concludes the paper.

2. Cyclical Policies in stock accumulation and Impatience

An intuitive explanation of cyclical policies in a stock control setting, between the agents in charge and the government (government is the regulator), may be the following. We assume that the resources of the above agents are the only responsible in a public bad accumulation situation. Taking the example of pollutants accumulation, resources may be the inputs and the antiquated equipment used in the production process. As shabby equipment it is considered the already used production equipment, which change owner to the Southern or Eastern developing countries at a low acquisition cost. Similarly, all the extracted depletable resources which are used as inputs in the production are the resources of pollution. The power of such a "dirty" production process is based on the accumulation of the responsible resources.

The agents in charge for the public evil accumulation derive utility from the higher intensity of the responsible mechanisms, such as emissions or tax evasion, while the other side (e.g. the government or any group of agents that fight against accumulation) derive utility from the measures taken against mechanisms (e.g. abatement or counter evasion). Let us start with rather few responsible mechanisms and a rather low and decreasing accumulated stock of the agents' resources. A farsighted regulator, which only gains benefits from the resources reduction, will curb its measures against since further reducing the stock would only be possible at high costs. As a consequence the resource stock of the responsible agents starts to grow again. Now the agents in charge has to react by increasing the intensity of their "dirty" mechanisms but only moderately such the government measures would be still not very efficient, leading mainly to higher costs but moderate benefits for the government.

Moreover this would save the responsible agents' resources and the dynamical system would approach a stable steady state. If the agents in charge have a high discount rate, that is a realistic assumption, they behave myopically reacting strongly, i.e. they intensify their "dirty" mechanisms (e.g. emissions or tax evasion). This provokes measures on the regulator's side which in turn lead to an increasing reduction and decreasing agents' resources. To avoid their resources' exhaustion the agents in charge have to reduce the intensity of their mechanisms, so the cycle is closed.

3. The Nash Differential Game

Let us denote by $x(t)$ the instantaneous resources available to the representative agent in charge at time t. Without any regulator's counter action taken and also without any other actions on the agents in charge side, the stock of resources grows according to the function $g(x)$, which is considered as growth function, obviously dependent on the available resources, satisfying the conditions $g(0)=0$, $g(x)>0$ for all $x\in(0,K)$, $g(x)<0$ for all $x\in(K,\infty)$, $g''(x)\leq 0$.

Starting up the mechanisms, which are responsible for the public bad accumulation, is costly for the agents in charge, e.g. compliance costs and damages in the available equipment, also reducing their capital available to the production process. This clearly affects negatively the agents' resources. The reduction of the growth of the resource stock, however, does not only depend on the intensity of mechanisms $\nu(t)$, but is also influenced by the measures $u(t)$ against, undertaken by the government. We set as instrument variables for both sides the intensity of the responsible mechanisms $\nu(t)$ and the regulator's actions $u(t)$ undertaken, which are assumed non-negatives $\nu(t) \geq 0, \quad u(t) \geq 0$.

Analogously to the models of optimal harvesting natural resources one can thought as "harvesting" the resources of the agents in charge and this reducing function is denoted by $\phi(u,\nu)$. Combining the growth $g(x)$ with the reducing function $\phi(u,\nu)$ the state dynamics can be written as

$$\dot{x} = g(x) - \phi(u,\nu), \qquad x(0) = x_0 > 0 \tag{1}$$

Along a trajectory the non negativity constraint is imposed, that is

$$x(t) \geq 0 \quad \forall t \geq 0 \tag{2}$$

With the assumption of the compliance costs and the damages incurred in equipment due to the intensive usage, a higher intensity of the responsible mechanisms and also the government measures leads to stronger reduction of the agents in charge resources and therefore we assume the partial derivatives of the function that reduces the resources $\phi(u,\nu)$ to be positive, i.e. $\phi_u > 0, \ \phi_\nu > 0$. Moreover the law of diminishing returns is applied only for the government actions undertaken, that is $\phi_{uu} < 0$ and for simplicity we assume $\phi_{\nu\nu} = 0$.

The utility functions the two players need to maximize defined as follows:

Player 1, the government (or the regulator), derive instantaneous utility, on one hand from the responsible agents mechanisms' reductions $\phi(u,\nu)$, and on the other hand from their measures effort $u(t)$ which gives rise to increasing and convex costs $a(u)$. Additionally a high stock of resources, on behalf the agents which are answerable for accumulation cause disutility, which is described by the increasing function $\delta(x)$.

Summing up, the present value of player's 1 utility is described by the following functional

$$J_1 = \int_0^\infty e^{-\rho_1 t} \left[\phi(u,\nu) - \delta(x) - a(u)\right] dt \qquad (3)$$

Player 2, the economic agents in charge, enjoy utility $\upsilon(x)$ from the available resources $x(t)$, and utility, as well, from the intensity ν of the responsible mechanisms or methods used, which is described by the function $\beta(\nu)$. For the utilities $\upsilon(x)$ and $\beta(\nu)$ we assume that are monotonically increasing functions with decreasing marginal returns, therefore for the first derivatives we have $\upsilon'(x) > 0,\ \beta'(\nu) > 0$ and for the second $\upsilon''(x) < 0,\ \beta''(\nu) < 0$. So, player's 2 utility function is defined, in its additively separable form, as:

$$J_2 = \int_0^\infty e^{-\rho_2 t} \left[\upsilon(x) + \beta(\nu)\right] dt \qquad (4)$$

3.1. Equilibrium analysis

We begin analysis with the concept of open loop Nash equilibrium, which is based on the fact that every player's strategy is the best reply to the opponent's exogenously given strategy. Obviously, equilibrium holds if both strategies are simultaneously best replies.

The current value Hamiltonians for both players, are defined as follows

$$H_1 = \phi(u,\nu) - \delta(x) - a(u) + \lambda\left(g(x) - \phi(u,\nu)\right)$$

$$H_2 = \upsilon(x) + \beta(\nu) + \mu\left(g(x) - \phi(u,\nu)\right)$$

The first order conditions, for the maximization problem, are the following system of differential equations for both players:

First, the maximized Hamiltonians are

$$\frac{\partial H_1}{\partial u}=(1-\lambda)\phi_u(u,\nu)-a'(u)=0 \tag{5}$$

$$\frac{\partial H_2}{\partial \nu}=\beta'(\nu)-\mu\phi_\nu(u,\nu)=0 \tag{6}$$

and second the costate variables are defined by the equations

$$\dot{\lambda}=\rho_1\lambda-\frac{\partial H_1}{\partial x}=\lambda\left[\rho_1-g'(x)\right]+\delta'(x) \tag{7}$$

$$\dot{\mu}=\rho_2\mu-\frac{\partial H_2}{\partial x}=\mu\left[\rho_2-g'(x)\right]-\upsilon'(x) \tag{8}$$

3.2. Stability of equilibrium

An interior steady state $\left(x^*,\lambda^*,\mu^*\right)$ with the optimal controls $\left(u^*,\nu^*\right)$ is a solution of the following system (taking steady states):

$$\begin{aligned}
&g(x)=\phi(u,\nu)\\
&\lambda\left(\rho_1-g'(x)\right)=-\delta'(x)\\
&\mu\left(\rho_2-g'(x)\right)=\upsilon'(x)\\
&(1-\lambda)\phi_u(u,\nu)=a'(u)\\
&\mu\phi_\nu(u,\nu)=\beta'(\nu)
\end{aligned}$$

and the Jacobian matrix, evaluated at the steady state, is

$$J=\begin{pmatrix}
\frac{\partial}{\partial x}\left[g(x)-\phi(u,\nu)\right] & \frac{\partial}{\partial \lambda}\left[g(x)-\phi(u,\nu)\right] & \frac{\partial}{\partial \mu}\left[g(x)-\phi(u,\nu)\right]\\
\frac{\partial}{\partial x}\left[\lambda\left(\rho_1-g'(x)\right)+\delta'(x)\right] & \frac{\partial}{\partial \lambda}\left[\lambda\left(\rho_1-g'(x)\right)+\delta'(x)\right] & \frac{\partial}{\partial \mu}\left[\lambda\left(\rho_1-g'(x)\right)+\delta'(x)\right]\\
\frac{\partial}{\partial x}\left[\mu\left(\rho_2-g'(x)\right)-\upsilon'(x)\right] & \frac{\partial}{\partial \lambda}\left[\mu\left(\rho_2-g'(x)\right)-\upsilon'(x)\right] & \frac{\partial}{\partial \mu}\left[\mu\left(\rho_2-g'(x)\right)-\upsilon'(x)\right]
\end{pmatrix}$$

which after the simple calculations, takes the following final form:

$$J=\begin{pmatrix}
g'(x) & -\partial\phi(u,\nu)/\partial\lambda & -\partial\phi(u,\nu)/\partial\mu\\
-\lambda g''(x)+\delta''(x) & \rho_1-g'(x) & 0\\
-\mu g''(x)-\upsilon''(x) & 0 & \rho_2-g'(x)
\end{pmatrix} \tag{9}$$

The main stability analysis is focused in periodic solutions, and therefore we make use of the Hopf bifurcations. Thus, computing determinants and trace of the Jacobian matrix (9) we have

$$\mathrm{tr}J = \rho_1 + \rho_2 - g'(x)$$

$$\det J = g'\left[\rho_1 - g'(x)\right]\left[\rho_2 - g'(x)\right] - \left[\lambda g''(x) - \delta''(x)\right]\left[\rho_2 - g'(x)\right]\frac{\partial\phi(u,\nu)}{\partial\lambda} - \\ -\left[\mu g''(x) + \upsilon''(x)\right]\left[\rho_1 - g'(x)\right]\frac{\partial\phi(u,\nu)}{\partial\mu}$$

The Jacobian (9) possesses two purely imaginary eigenvalues $\pm i\sqrt{\omega}$ if the condition $\frac{\det J}{\mathrm{tr}J} = \omega > 0$ holds.

In the following we compute the value of ω as:

$$\omega = \rho_1\rho_2 - \left[g'(x)\right]^2 - \left[\lambda g''(x) - \delta''(x)\right]\frac{\partial\phi(u,\nu)}{\partial\lambda} - \left[\mu g''(x) + \upsilon''(x)\right]\frac{\partial\phi(u,\nu)}{\partial\mu}$$

A Hopf bifurcation can thus only occur if the conditions $\omega > 0$ and the following

$$\rho_1\left[\lambda g''(x) - \delta''(x)\right]\frac{\partial\phi(u,\nu)}{\partial\lambda} + \rho_2\left[\mu g''(x) + \upsilon''(x)\right]\frac{\partial\phi(u,\nu)}{\partial\mu} = \\ = \rho_1\rho_2\left[\rho_1 + \rho_2 - 2g'(x)\right] \qquad (10)$$

has to hold.

In what follows we give specific forms in the functions of the model in order to extract some useful conclusions for periodic solutions.

3.3. Specifications of the model

We specify the functions involved as

Growth function: $$g(x) = Rx(1-x) \qquad (11.1)$$

The function that reduces the responsible agents' resources as a Cobb – Douglas type

$$\phi(u,\nu)=u^{\pi}\nu \tag{11.2}$$

The government's cost function as a linear function

$$a(u)=au \tag{11.3}$$

The government's damage function $\delta(x)$and the agents' utility derived from the resource stock $\sigma(x)$ in linear forms, respectively

$$\delta(x)=\delta x \tag{11.4}$$

$$\upsilon(x)=\sigma x \tag{11.5}$$

Finally, the agents' utility function, derived form the resources' intensive usage, we assume to be in the form

$$\beta(\nu)=\gamma-\frac{\nu^{\eta-1}}{1-\eta} \tag{11.6}$$

with $r,\ a,\ \delta,\ \sigma,\ \gamma,\ b>0,\ \ \pi,\ \eta\in(0,1)$

For the above specifications, the necessary condition for cyclical strategies is given from the next proposition.

Proposition 1

Given the specifications (11.1) - (11.6) *for the functions of the model, a necessary condition for cyclical strategies is that the regulator is less farsighted than the agents who are responsible for the unwished public evil; therefore the condition* $\rho_2>\rho_1$ *has to hold.*

Proof

In the appendix A

4. The Stackelberg Setting

In this section we analyze the case in which the two players of the game move hierarchically and the rate of measures taken against accumulation is chosen by the government before the responsible agents decides on the rate of their methods, thus the government is the leader.

4.1. Responsible agents as follower

We first consider the optimization problem for the follower, i.e. the responsible agents, which take the action of the leader as given. The responsible agents face the following objective which is maximized, that is

$$\max_{\nu} \int_0^{\infty} e^{-\rho_2 t} \left(\upsilon(x) + \beta(\nu) \right) dt$$

Note that the agents' maximization takes place with respect to the intensity of the resources utilization, which means higher intensive use. The state variable evolves according to (1), for which the growth function is simplified in linear form, i.e. $g(x) = rx$. Thus, the resources equation of motion becomes $\dot{x} = rx - \phi(u,\nu)$. Moreover we assume separability of the model through the agents' in charge separable utility function. Therefore we assume that the utility enjoyed by the resources in the form, $\upsilon(x) = \sigma x$ and the utility derived from the intensive use of the responsible resources, in the linear form $\beta(\nu) = \beta\nu$, as well.

The responsible agents' Hamiltonian current value, after the above simplifications is

$$H_2 = \sigma x + \beta\nu + \mu\left[rx - \phi(u,\nu) \right]$$

The first order conditions for an interior solution w.r.t. the control ν is therefore,

$$\frac{\partial H_2}{\partial \nu} = \beta - \mu\phi_{\nu}(u,\nu) = 0$$

and the specification for the function that reduces the responsible resources: $\phi(u,\nu)=u^{\pi}\nu^{\varepsilon}$, $\varepsilon>1$, we get the optimal control function for the responsible agents, as follows

$$\phi_{\nu}=\varepsilon u^{\pi}\nu^{\varepsilon-1}=\frac{\beta}{\mu} \quad \Rightarrow \quad \nu^{*}(u)=\left[\frac{\beta}{\varepsilon\mu}\right]^{\frac{1}{\varepsilon-1}} u^{\frac{\pi}{1-\varepsilon}} \tag{12}$$

Now, the adjoint variable μ has to follow the differential equation

$$\dot{\mu}=\rho_2\mu-\frac{\partial H_2}{\partial x}=\mu[\rho_2-r]-\sigma \tag{13}$$

Substituting the follower's optimal control function (12) into the function that reduces the resources, we take the analytical form of that function, as:

$$\phi(u,\nu^{*}(u))=u^{\pi}\left(\nu^{*}(u)\right)^{\varepsilon}=u^{\frac{\pi}{1-\varepsilon}}\left[\frac{\varepsilon\mu}{\beta}\right]^{\frac{\varepsilon}{1-\varepsilon}} \tag{14}$$

The readable expressions of (12) and (14) leads to the conclusion which says that, since $\varepsilon>1$ and therefore $\frac{\pi}{1-\varepsilon}<0$, an increase to the counter accumulation measures on behalf the government, u, results in a more cautious control on behalf the followers', $\nu^{*}(u)$. Similarly, (14) leads to a lower reduction of the responsible agents' resources, $\phi(u,\nu^{*}(u))$.

4.2. The Government as regulator

Following Dockner *et al.* (2000) (especially Chapter 5) we formulate the government's problem, for which the leader has to take into account the dynamics of the optimal decisions of the follower, expressed by the adjoint equation (13). Equation (13) now becomes the second state's evolution, so the leader's problem now is treated as an optimal control problem with two state variables. Moreover, combining with the early calculated analytical form (14), the leader's objective functional becomes (we assume the damage function $\psi(\nu)$, due to intensive use of resources, in the form $\psi(\nu)=\psi\nu$)

$$\max_{u}\int_{0}^{\infty} e^{-\rho_1 t}\left(u^{\frac{\pi}{1-\varepsilon}}\left[\frac{\varepsilon\mu}{\beta}\right]^{\frac{\varepsilon}{1-\varepsilon}}-au-\delta x-\psi\left[\frac{\varepsilon\mu}{\beta}\right]^{\frac{1}{1-\varepsilon}}u^{\frac{\pi}{1-\varepsilon}}\right)dt \qquad (15)$$

which is subject to both state dynamics, the original resource's dynamics plus the intensity's shadow price dynamics which stems from the follower's maximization problem, i.e. the following dynamics

$$\dot{x}=rx-u^{\frac{\pi}{1-\varepsilon}}\left[\frac{\varepsilon\mu}{\beta}\right]^{\frac{\varepsilon}{1-\varepsilon}} \qquad (16)$$

$$\dot{\mu}=\mu[\rho_2-r]-\sigma \qquad (17)$$

The Hamiltonian current value of the above system (15) - (17) becomes

$$H_1=u^{\frac{\pi}{1-\varepsilon}}\left\{\left[\frac{\varepsilon\mu}{\beta}\right]^{\frac{\varepsilon}{1-\varepsilon}}(1-\lambda)-\psi\left[\frac{\varepsilon\mu}{\beta}\right]^{\frac{1}{1-\varepsilon}}\right\}-au-\delta x+\lambda rx+\xi\left[\mu(\rho_2-r)-\sigma\right]$$

with $\lambda,\ \xi$ to denote the adjoint variables of the states $x,\ \mu$ respectively. We note again that the responsible agents' shadow price μ now becomes the new state variable for the government's problem.

Taking first order conditions we are able to express analytically the leader's optimal control u^* as a function of the adjoints $\lambda,\ \xi$, that is,

$$\frac{\partial H_1}{\partial u}=\frac{\pi}{1-\varepsilon}u^{\frac{\pi+\varepsilon-1}{1-\varepsilon}}\left[\frac{\varepsilon\mu}{\beta}\right]^{\frac{\varepsilon}{1-\varepsilon}}\left(1-\lambda-\frac{\psi\varepsilon\mu}{\beta}\right)-a=0 \iff$$

$$\iff u^*(\lambda,\xi)=\left[\frac{a\beta(\varepsilon-1)}{\pi\left[\psi\varepsilon\mu+\beta(\lambda-1)\right]}\right]^{\frac{1-\varepsilon}{\pi+\varepsilon-1}}\left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{\varepsilon}{\pi+\varepsilon-1}} \qquad (18)$$

Moreover, the adjoints follow the differential equations

$$\dot{\lambda} = \rho_1 \lambda - \frac{\partial H_1}{\partial x} = \lambda(\rho_1 - r) + \delta \tag{19}$$

$$\dot{\xi} = \rho_1 \xi - \frac{\partial H_1}{\partial \mu} = \xi(\rho_1 - \rho_2 + r) - \frac{au}{\pi\mu} \tag{20}$$

Note that, thanks to state separability, the government's adjoint variable ξ, with respect to the responsible agents' adjoint variable μ, has no influence on the leader's optimization problem.

The findings in the Stackelberg game are summarized in the following proposition.

Proposition 2.

In the Stackelberg game with the government as leader and responsible agents as follower a feasible solution exists, iff ψ *is sufficient large, i.e. iff*

$$\psi > \frac{\beta(1-\lambda)}{\varepsilon\mu} = \frac{\beta(\rho_1 - r + \delta)(\rho_2 - r)}{\varepsilon\sigma(\rho_1 - r)} \quad \text{or} \quad \varepsilon > \frac{\beta(1-\lambda)}{\psi\mu} \tag{21}$$

The optimal strategies are then given by

$$u_S^* = \left[\frac{a\beta(\varepsilon-1)}{\pi[\psi\varepsilon\mu + \beta(\lambda-1)]}\right]^{\frac{1-\varepsilon}{\pi+\varepsilon-1}} \left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{\varepsilon}{\pi+\varepsilon-1}} \tag{22}$$

$$\nu_S^* = \left[\frac{a\beta(\varepsilon-1)}{\pi[\psi\varepsilon\mu + \beta(\lambda-1)]}\right]^{\frac{\pi}{\pi+\varepsilon-1}} \left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{1-\pi}{\pi+\varepsilon-1}} \tag{23}$$

the optimal reducing function of the responsible resources is given by

$$\phi_S^*(u,\nu) = \left[\frac{a\beta(\varepsilon-1)}{\pi[\psi\varepsilon\mu + \beta(\lambda-1)]}\right]^{\frac{\pi}{\pi+\varepsilon-1}} \left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{\varepsilon}{\pi+\varepsilon-1}} \tag{24}$$

The steady state value of the resources' stock is given by

$$x_S^{\infty}(u,v)=\frac{1}{r}\left[\frac{a\beta(\varepsilon-1)}{\pi\left[\psi\varepsilon\mu+\beta(\lambda-1)\right]}\right]^{\frac{\pi}{\pi+\varepsilon-1}}\left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{\varepsilon}{\pi+\varepsilon-1}} \tag{25}$$

Proof

The values $(22)-(25)$ follow, from further substitutions of (18), from the maximization condition $\frac{\partial H_1}{\partial \nu}=0$ and from the steady state condition $\dot{x}=0 \;\Leftrightarrow\; rx-h_S^*(u,\nu)=0$.

Proposition 3.

In the Stackelberg game the analytic forms of the objective functionals are given by

$$J_S^1=\frac{a(1-\pi-\varepsilon)}{\pi\rho_1}\left[\frac{a\beta(\varepsilon-1)}{\pi\left[\psi\varepsilon\mu+\beta(\lambda-1)\right]}\right]^{\frac{1-\varepsilon}{\pi+\varepsilon-1}}\left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{\varepsilon}{\pi+\varepsilon-1}}-\frac{\delta x_0}{\rho_1-r} \tag{26}$$

$$J_S^2=\frac{\beta(\varepsilon-1)}{\varepsilon\rho_2}\left[\frac{a\beta(\varepsilon-1)}{\pi\left[\psi\varepsilon\mu+\beta(\lambda-1)\right]}\right]^{\frac{\pi}{\pi+\varepsilon-1}}\left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{1-\pi}{\pi+\varepsilon-1}}-\frac{\sigma x_0}{\rho_2-r} \tag{27}$$

Proof

On request

Proposition 4.

For the values of ψ such that

$$\frac{\beta(1-\lambda)}{\varepsilon\mu} < \psi < \frac{\beta(1-\lambda)}{\mu}$$

the regulator as leader act more cautiously and the responsible agents more aggressively compared to the Nash case[1]. *This leads to a higher reducing function and a higher profit of the Stackelberg follower compared to the Nash case. For values of* ψ *larger than* $\frac{\beta(1-\lambda)}{\mu}$, *i.e.* $\left(\psi > \frac{\beta(1-\lambda)}{\mu}\right)$, *the government acts more aggressively and the responsible agents more cautiously compared to the Nash case, leading to a lower reducing function and a lower objective value for the follower compared to the Nash case.*

Proof

Upon request due to its extension.

Remark

Since ψ is the crucial variable which measures damages to the responsible resources due to the intensive use, it is obvious (from proposition 4) that for large values of ψ the regulator follows more truculent policy, but for small values of ψ the leader's policy is holding back.

5. Conclusions

The purpose of this paper was to investigate the dynamics of a public evil accumulation together with the actions undertaken for counter accumulation. For this purpose we setup a very simple accumulation model. In this model, we claim that the disadvantage in the public evil control is not the accumulated stock of the harms, an irreversible fact, but rather the use of the available "bad" inputs together with the antiquated equipment, used in the production process. We called the production inputs and the available equipment used "the agents' resources", as the causality of the public evil accumulation.

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[1] For the Nash differential game exposition, see Halkos and Papageorgiou, 2011.

We model the responsible agents' resources as an accumulated stock first in a simultaneous (Nash) game. The Nash game takes place between the government which uses as control a counter–accumulation policy and the responsible agents using the intensity of the resources' usage as their control. The economic analysis that follows in the game's solution, focused on cyclical policies, reveals the possibility of limit cycles.

As a result we found the sufficient condition for the cyclical policies existence. According to that result it suffices, assuming different discount rates, the responsible agents' discount rate to be greater than the government's discount rate. In the second setting, we extend the simultaneous move in a hierarchical (Stackelberg) differential game, for which the government undertakes the role of leader, while the answerable agents undertake the follower's role. In the above Stackelberg game, we first calculate the analytical expressions of the player's strategies and the analytical expression of the reducing resources function.

We also found the steady state of the agents' resources stock. The analytical expressions of the value functions are finally calculated. The last proposition of the paper concerns about the behavior of the reducing resources' function. To be more precise, we found the interval between one crucial parameter of the model lies. If this parameter lies between certain values the reducing function takes higher values, leading therefore to higher profits for the follower, compared with the Nash case. One the other hand, if the parameter takes a higher value than the threshold, the government acts more aggressively and the polluters more cautiously, leading to a lower reducing function and therefore to a lower objective value for the follower, in comparison to the Nash case.

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Appendix A

Proof of proposition 1.

With the specifications, given in subsection entitled "*3.3. Specifications of the model*", one can compute

$g'(x) = R(1-2x)$, $g''(x) = -2R$, $\phi_u(u,\nu) = \gamma u^{\gamma-1}$, $\phi_\nu(u,\nu) = u^\pi$, $a'(u) = a$, $\beta'(\nu) = \nu^{\eta-2}$, $\delta'(x) = \delta$, $\upsilon'(x) = \upsilon$

$$\frac{\partial H_1}{\partial u} = 0 \Leftrightarrow (1-\lambda)\phi_u(u,\nu) = a'(u) \Leftrightarrow (1-\lambda)\gamma u^{\pi-1}\nu = a \quad (A.1)$$

$$\frac{\partial H_2}{\partial \nu} = 0 \Leftrightarrow \beta'(\nu) = \mu\phi_\nu(u,\nu) \Leftrightarrow \mu u^\pi = \nu^{\eta-2} \quad (A.2)$$

Combining $(A.1)$ and $(A.2)$ the optimal strategies take the following forms

$$u^* = \mu^{-1/[1+(1-\pi)(1-\eta)]}\left[\frac{a}{\pi(1-\lambda)}\right]^{(\eta-2)/[1+(1-\eta)(1-\pi)]} \quad (A.3),$$

$$\nu^* = \mu^{(\pi-1)/[1+(1-\pi)(1-\eta)]}\left[\frac{a}{\pi(1-\lambda)}\right]^{\pi/[1+(1-\pi)(1-\eta)]} \quad (A.4)$$

and the optimal reducing function becomes

$$\phi(u^*,\nu^*) = \mu^{-1/[1+(1-\pi)(1-\eta)]}\left[\frac{a}{\pi(1-\lambda)}\right]^{\pi(\eta-1)/[1+(1-\pi)(1-\eta)]} \quad (A.5)$$

with the following partial derivatives

$$\frac{\partial\phi}{\partial\lambda} = \frac{\mu^{-1/[1+(1-\pi)(1-\eta)]}\left[\frac{a}{\pi(1-\lambda)}\right]^{\pi(\eta-1)/[1+(1-\pi)(1-\eta)]}}{(1-\lambda)}\frac{\pi(\eta-1)}{1+(1-\eta)(1-\pi)} = \quad (A.6)$$
$$= \frac{\phi(u^*,\nu^*)}{(1-\lambda)}\frac{\pi(\mu-1)}{1+(1-\eta)(1-\pi)}$$

$$\frac{\partial\phi}{\partial\mu} = \frac{\mu^{-1/[1+(1-\pi)(1-\eta)]}\left[\frac{a}{\pi(1-\lambda)}\right]^{\pi(\eta-1)/[1+(1-\pi)(1-\eta)]}}{\lambda_2}\frac{-1}{1+(1-\eta)(1-\pi)} = \quad (A.7)$$
$$= \frac{\phi(u^*,\nu^*)}{\mu}\frac{-1}{1+(1-\eta)(1-\pi)}$$

Both derivatives $(A.6)$, $(A.7)$ are negatives due to the assumptions on the parameters $\pi,\ \eta \in (0,1)$ and on the signs of the functions derivates, that is

$\phi_u > 0,\ \phi_\nu > 0,\ \upsilon'(x) > 0,\ \delta'(x) > 0$, which ensures the positive sign of the adjoints $\lambda,\ \mu$.

Bifurcation condition $\omega = \dfrac{\det(J)}{\operatorname{tr}(J)}$ now becomes

$\rho_1\rho_2\left[\rho_1+\rho_2-2g'(x)\right]=\lambda\rho_1 g''(x)\dfrac{\partial\phi}{\partial\lambda}+\mu\rho_2 g''(x)\dfrac{\partial\phi}{\partial\mu}$, which after substituting the values from (A.6), (A.7) and making the rest of algebraic manipulations, finally yields (at the steady states)

$$\frac{\phi(u_\infty,\nu_\infty)g''(x)}{1+(1-\eta)(1-\pi)}\left[\rho_1\pi(1-\eta)\frac{\delta}{\delta+g'(x)-\rho_1}-\rho_2\right]-\rho_1\rho_2\left[\rho_1+\rho_2-2g'(x)\right]=0 \qquad (\text{A.8})$$

Where we have set $\dfrac{\lambda}{1-\lambda}=\dfrac{\delta}{\rho_1-g'(x)-\delta}$ stemming from the adjoint equation $\dot{\lambda}=\lambda\left(\rho_1-g'(x)\right)-\delta'(x)$, which at the steady states reduces into $\lambda=\delta'(x)/\left(\rho_1-g'(x)\right)$.

Condition $w>0$ after substitution the values from (A.6), (A.7) becomes

$$w=\rho_1\rho_2-\left[g'(x)\right]^2+\frac{\phi(u,\nu)g''(x)}{1+(1-\eta)(1-\pi)}\left[\pi(1-\eta)\frac{-\delta}{g'(x)+\delta-\rho_1}+1\right]>0 \qquad (\text{A.9})$$

The division of (A.8) by ρ_1 yields

$$\frac{\phi(u_\infty,\nu_\infty)g''(x)}{1+(1-\eta)(1-\pi)}\left[\pi(1-\eta)\frac{\delta}{\delta+g'(x)-\rho_1}-\frac{\rho_2}{\rho_1}\right]-\rho_2\left[\rho_1+\rho_2-2g'(x)\right]=0 \qquad (\text{A.10})$$

The sum $(\text{A.9})+(\text{A.10})$ must be positive, thus after simplifications and taking into account that (at the steady state) $\phi(u_\infty,\nu_\infty)=g(x)$, we have:

$g(x)g''(x)\dfrac{\rho_1-\rho_2}{\rho_1\left[1+(1-\eta)(1-\pi)\right]}>\left[\rho_2-g'(x)\right]^2$ and the result $\rho_2>\rho_1$ follows from the strict concavity of the logistic growth $g''<0$, since it is supposed $0<\eta<1$ and $0<\pi<1$, as well.