A General Optimal Investment Model in the Presence of Background Risk

Alghalith, Moawia and Guo, Xu and Wong, Wing-Keung and Zhu, Lixing

Department of Economics, University of St. Andrews, Nanjing University of Aeronautics and Astronautics, China, Department of Economics, Hong Kong Baptist University, Department of Mathematics, Hong Kong Baptist University

April 2016

A General Optimal Investment Model in the Presence of Background Risk

Moawia Alghalith

Department of Economics, University of St. Andrews

Xu Guo

Nanjing University of Aeronautics and Astronautics, China

Wing-Keung Wong*

Department of Economics, Hong Kong Baptist University

Lixing Zhu

Department of Mathematics, Hong Kong Baptist University

Acknowledgement: The research is partially supported by University of St. Andrews, Nanjing University of Aeronautics and Astronautics, Hong Kong Baptist University, and Research Grants Council of Hong Kong.

April 11, 2016

Abstract: In this paper we present two dynamic models of background risk. We first present a stochastic factor model with an additive background risk. Thereafter, we present a dynamic model of simultaneous (correlated) multiplicative background risk and additive background risk. In so doing, we use a general utility function.

* *Correspondence:* Wing-Keung Wong, Department of Economics, Hong Kong Baptist University, Kowloon Tong, Hong Kong. Tel: (852)-3411-7542, Fax: (852)-3411-5580, Email: awong@hkbu.edu.hk

1 Introduction

When investors make decision on which asset or portfolio to invest, they encounter not only portfolio risk but also background risk that comes from different sources, including variations in labor income, proprietary income, investments in real estate, and unexpected expenses due to health or other issues (see, e.g., Gollier and Pratt, 1996). In this paper we follow Jiang, et al. (2010) and others to refer the assets that exposed heavily to background risk as background assets and others as financial assets or portfolio assets. It is nearly impossible for investors to reduce background risk in a short run because background assets are usually illiquid and non-tradable. In this paper we will evaluate the total risk which is consisted of the portfolio risk and the background risk while the latter could affect investors' investments in financial assets greatly.

Classical portfolio theory (Markowitz, 1952; Merton, 1969, 1971; Samuelson, 1969) do not include background risk because the market is assumed to be complete. This assumption infers that background assets can be spanned and priced by tradable financial assets, and thus, the finding of their theory depends on the assumption of market completeness. Heaton and Lucas (2000) observe that for those investors who hold a significant fraction of their wealth in stocks, proprietary business income is a large and more correlated background risk factor. Nonetheless, Campbell (2006) shows that standard portfolio theory fails to explain household investment decisions in practice. To circumvent this limitation of the classical portfolio theory, academics introduce background risk in the study of portfolio compositions. For example, Rosen and Wu (2004), Edwards (2008), and others find that there are strong cross-sectional correlations between health and both financial and non-financial assets, and that adverse health shocks discourage risky asset holdings. In addition, Cocco (2005) concludes that the investment in housing plays an important role in asset accumulation and in portfolio choice among stocks and Treasury bills. Fan and Zhao (2009) document that there are strong cross-sectional correlations between health and both financial and non-financial assets, and that adverse health shocks discourage risky asset holdings. Pelizzon and Weber (2009) conclude that the investment in housing plays an important role in asset accumulation and in portfolio choice among financial assets. Cardak and Wilkins (2009) further demonstrate that risky asset holdings are discouraged by both labour income risk and health risk. These empirical studies primarily investigate patterns of cross-sectional variations in the composition of a household's total wealth or the quantitative importance of a particular background risk in affecting portfolio choices. Li (2011) examines a static model of investment with background risk. Franke et al (2011) consider a static model of multiplicative background risk and additive background risk under CRRA and HARA preferences while Alghalith (2012) develops a model of investment with an additive background risk.

Previous literature on background risk have two major limitations: (1) they employ static analysis, and (2) they adopt restrictive assumptions at least regarding the types of preferences. To circumvent their limitations, in this paper we present two dynamic models of background risk. We first present a stochastic factor model with an additive background risk. Thereafter, we present a dynamic model of simultaneous (correlated) multiplicative background risk and additive background risk. In addition, we use a general utility function to develop the theory.

The rest of the paper is organized as follows. We will present a stochastic factor model with an additive background risk in next section and develop a model of simultaneous additive and multiplicative background risks in Section 3. The final section gives concluding remarks.

2 A stochastic factor model with an additive background risk

We adopt a three-dimensional Brownian motion $\{W_{1s}, W_{2s}, W_{3s}, \mathcal{F}_s\}_{t \leq s \leq T}$ on the probability space $(\Omega, \mathcal{F}_s, P)$, where $\{\mathcal{F}_s\}_{t \leq s \leq T}$ is the augmentation of filtration (Spitzer, 1958), W_{1s} and W_{2s} are independent, and W_{1s} and W_{3s} are correlated with ρ_{2s} be their correlation coefficient. Similar to previous stochastic factor models in the absence of background risk (see, e.g. Alghalith (2009, 2012), among others), the securities market is modeled by using a risky asset (portfolio), a risk-free bond, and an external economic factor such that, for $t \leq s \leq T$:

1. The process of the bond price is given by $dS_s^B = r(Y_s)S_s^B ds$, where $r(Y_s) \in C_b^2(R)$ is the return of bond and Y_s is the economic factor (stochastic factor).

2. The price process of a risky asset/portofolio S_s^S satisfies the stochastic differential equation (SDE)

$$dS_s^S = S_s^S[\mu(Y_s)ds + \sigma(Y_s)dW_{1s}], S_0^S = 1, \tag{2.1}$$

   where $\mu(\cdot)$ and $\sigma(\cdot)$ are the mean and volatility, respectively, for the return of the risky asset. It is assumed that the functions $\mu(Y_s)$ and $\sigma(Y_s)$ belong to $C_b^2(R)$. From the SDE in (2.1), S_s^S follows a geometric Brownian motion.

3. The dynamics of the external factor Y_s is modelled as a diffusion process by solving the following SDE

$$dY_s = g(Y_s)ds + \rho_{1s}dW_{1s} + \sqrt{1 - \rho_{1s}^2}dW_{2s},$$

   where $|\rho_{1s}| \leq 1$ and $g(\cdot)$ belongs to $C^1(R)$ with a bounded derivative.

We define $\tilde{W}_s$ such that

$$d\tilde{W}_s = \rho_{1s}dW_{1s} + \sqrt{1 - \rho_{1s}^2}dW_{2s}.$$

Since W_{1s} and W_{2s} are independent, using it$\hat{o}$'s formula, we can have

$$E(dW_{1s} \cdot d\tilde{W}_s) = \rho_{1s} ds.$$

Thus, ρ_{1s} is the correlation coefficient between the Brownian motion W_{1s} driving the asset price and the Brownian motion $\tilde{W}_s = \rho_{1s} W_{1s} + \sqrt{1-\rho_{1s}^2} W_{2s}$. Except when $\rho_{1s} = \pm 1$, the securities market is incomplete because the external factor Y_s cannot be traded.

Let π_s be the net amount of capital allocated in the risky asset or portfolio. Then, the process of investor's wealth evolves to

$$dX_s = \frac{X_s - \pi_s}{S_s^B} dS_s^B + \frac{\pi_s}{S_s^S} dS_s^S$$

with initial wealth $X_t = x > 0$. Formally, $\{\pi_s, \mathcal{F}_s\}_{t \leq s \leq T}$ is a trading portfolio process if it is progressively measurable and $E \int_t^T \pi_s^2 ds < \infty$. Their associated wealth process, denoted by X_s^π, is the solution to the integral equation

$$X_T^\pi = x + \int_t^T \{r(Y_s) X_s^\pi + (\mu(Y_s) - r(Y_s)) \pi_s\} ds + \int_t^T \pi_s \sigma(Y_s) dW_{1s}. \tag{2.2}$$

We say that a trading strategy π is admissible if $X_s^\pi \geq 0$; the set of such strategies is denoted as $A(x, y)$.

The background risk dynamics are given by

$$d\xi_s = \delta(Y_s) dW_{3s} \quad \text{and} \quad \xi_t = \epsilon,$$

where $\delta(Y_s)$ is the volatility of the process ξ_s. Let $\Phi_s = X_s^\pi + \xi_s$ be the final wealth. The objective of an risk-averse investor is to maximize the following expected utility of the terminal wealth

$$V(t, x, y, \epsilon) = \underset{\pi_t}{Sup} E[U(\Phi_T) \mid \mathcal{F}_t], \tag{2.3}$$

where $V(.)$ is the value function, $U(.)$ is a continuous, bounded and strictly concave utility function and π_t^* is the optimal π_t by solving (2.3). We first establish the follow theorem:

Theorem 2.1 *For the wealth process X_s^π defined in (2.2), the optimal solution that maximizes the expected utility $V(t, x, y, \epsilon)$ of the terminal wealth in (2.3) is*

$$\pi_t^* = -\frac{\sigma(y)\rho_{1t}V_{xy} + \sigma(y)\delta_t\rho_{2t}V_{x\epsilon} + (\mu(y) - r(y))V_x}{\sigma^2(y)V_{xx}} \ . \tag{2.4}$$

In addition, we have

$$sign\left(\frac{\partial \pi_t^*}{\partial \delta_t}\right) = sign\left(-\rho_{2t}\right) \ . \tag{2.5}$$

From Theorem 2.1, we find that the optimal investment π_t^* depends on the return of bond $r(y)$, the mean $\mu(y)$ and volatility $\sigma(y)$ of the returns of the risky asset, the correlation coefficient ρ_{1s} between W_{1s} and $\tilde{W}_s = \rho_{1s}W_{1s} + \sqrt{1-\rho_{1s}^2}W_{2s}$, and the correlation coefficient ρ_{2s} between the W_{1s} and W_{3s}. However, the correlation coefficient between the W_{2s} and W_{3s} has no impact on the optimal investment π_t^*.

As for the impact of background risk on the optimal portfolio, we can conclude that only the volatility of the returns of the risky asset $\sigma(y)$ and the correlation coefficient between the W_{1s} and W_{3s} (the main risk and background risk) ρ_{2s} has impact. To be precise, the larger the volatility $\sigma(y)$ is, the bigger the impact can be. Furthermore, an independent background risk has no impact on the optimal portfolio. However, an increase in background risk will increase (decrease) the optimal portfolio if it is negatively (positively) correlated with the portfolio risk. We also note that Gollier and Pratt (1996), Quiggin (2003), and others define vulnerability/aversion in the *weak* sense (they call it a *weak* inequality), and thus, our result is consistent with their findings.

3 Multiplicative background risk and additive background risk

In this section, we present a model of simultaneous additive and multiplicative background risks (without the stochastic factor). As before, the additive background risk dynamics are given by

$$d\xi_s = \delta_s dW_{s2} \quad \text{and} \quad \xi_t = \epsilon. \tag{3.1}$$

The multiplicative background risk dynamics are given by

$$d\eta_s = \theta_s dW_{s3}, \eta_t = \eta, \eta_s > 0, \tag{3.2}$$

where θ_s is the volatility. The wealth process is specified as

$$X_T = x + \int_t^T \{r_s X_s + ((\mu_s - r_s)\,\pi_s)\}\, ds + \int_t^T \pi_s \sigma_s dW_{s1}. \tag{3.3}$$

Here, $\{W_{s1}, W_{s2}, W_{s3}, \mathcal{F}_s\}_{t \le s \le T}$ is a three-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}_s, P)$. Let ρ_{12t} be the correlation factor between the main risk and the additive background risk, ρ_{13t} be the correlation factor between the main risk and the multiplicative background risk, and ρ_{23t} be the correlation factor between the two background risks. In addition, we let $\Psi_s \triangleq \eta_s X_s + \xi_s$ be the total wealth.

The objective of the investor is to maximize the following expected utility of the terminal wealth:

$$V(t, x, \epsilon, \eta) = \underset{\pi_t}{Sup} E\left[U(\Psi_T) \mid \mathcal{F}_t\right]. \tag{3.4}$$

We obtain the following theorem to maximize the expected utility $V(t, x, \epsilon, \eta)$ of the terminal wealth in (3.4):

Theorem 3.1 *For the wealth process X_T defined in (3.3), the optimal solution that maximizes the expected utility $V(t, x, \epsilon, \eta)$ of the terminal wealth in (3.4) is*

$$\pi_t^* = -\frac{(\mu_t - r_t)\, V_x + \sigma_t \theta_t \rho_{13t} V_{x\eta} + \sigma_t \delta_t \rho_{12t} V_{x\epsilon}}{\eta_t \sigma_t^2 V_{xx}}. \tag{3.5}$$

In addition, we get

1. *sign* $(\partial \pi_t^* / \partial \theta_t) = sign\,(-\rho_{13t})$, *and*
2. *sign* $(\partial \pi_t^* / \partial \delta_t) = sign\,(-\rho_{12t})$.

Theorem 3.1 tells us that the correlation factor ρ_{23t} between the two background risks has no impact on the optimal investment. As for the impact of additive background risk on the optimal portfolio, we can conclude from Theorem 3.1 that the volatility σ_t of the return of the risky asset and the correlation coefficient ρ_{12t} between the main risk and additive background risk are two important factors. Moreover, the larger the volatility σ_t, the bigger the impact. Furthermore, an independent additive background risk also has no impact on the optimal portfolio. In addition, the larger the absolute value of ρ_{12t}, the bigger the impact of the additive background risk on the optimal investment. Moveover, the sign of this impact is opposite to the sign of ρ_{12t}. In other words, the additive background risk will increase the optimal portfolio if it is negatively correlated with price risk. Similar conclusions can be drawn for the impact of multiplicative background risk on the optimal portfolio.

4 Conclusion

In this paper, using general preferences, we introduced two dynamic models of correlated background risk. The first model involves a risky asset and an additive background risk, while the second model includes multiplicative background risk and additive background risk. We find that the impact of the background risk on the optimal portfolio is determined by the sign of the correlation factor between the main risk and the background risk. Our findings also conclude that an increase in background risk will increase (decrease) the optimal portfolio if it is negatively (positively) correlated with the portfolio risk.

References

[1] M. Alghalith, A new stochastic factor model: general explicit solutions, Applied Mathematics Letters 22 (2009) 1852-1854.

[2] M. Alghalith, The impact of the background risk, Physica A, forthcoming.

[3] J.Y. Campbell, Household finance, Journal of Finance 61 (2006) 1553-1604.

[4] B.A. Cardak, R. Wilkins, The determinants of household risky asset holdings: Australian evidence on background risk and other factors, Journal of Banking and Finance 33 (2009) 850-860.

[5] J.F. Cocco, Portfolio choice in the presence of housing, Review of Financial Studies 18 (2005) 535-567.

[6] R.D. Edwards, Health risk and portfolio choice, Journal of Business and Economic Statistics 26 (2008) 472-485.

[7] E. Fan, R. Zhao, Health status and portfolio choice: causality or heterogeneity, Journal of Banking and Finance 33 (2009) 1079-1088.

[8] G. Franke, H. Schlesinger, R.C. Stapleton, Risk Taking with Additive and Multiplicative Background Risks, Journal of Economic Theory 146 (2011) 1547-1568.

[9] C. Gollier, J.W. Pratt, Risk vulnerability and the tempering effect of background risk, Econometrica 64 (1996) 1109-1123.

[10] J. Heaton, D. Lucas, Portfolio choice in the presence of background risk, Economic Journal 110 (2000) 1-26.

[11] C.H. Jiang, Y.K. Ma, Y.B. An, An analysis of portfolio selection with background risk, Journal of Banking and Finance 34 (2010) 3055-3060.

[12] J. Li, The Demand for a Risky Asset in the Presence of a Background Risk, Journal of Economic Theory 146 (2011) 372-391.

[13] H. Markowitz, Portfolio selection, Journal of Finance 7 (1952) 77-91.

[14] R.C. Merton, Lifetime portfolio selection under uncertainty: the continuous time case, Review of Economics and Statistics 51 (1969) 247-257.

[15] R.C. Merton, Optimum consumption and portfolio rules in a continuous-time model, Journal of Economic Theory 3 (1971) 373-413.

[16] L. Pelizzon, G. Weber, Efficient portfolios when housing needs change over the life cycle, Journal of Banking and Finance 33 (2009) 2110-2121.

[17] J. Quiggin, Background risk in generalized expected utility theory, Economic Theory 22 (2003) 607-611.

[18] H.S. Rosen, S. Wu, Portfolio choice and health status, Journal of Financial Economics 72 (2004) 457-484.

[19] P.A. Samuelson, Lifetime portfolio selection by dynamic stochastic programming, Review of Economics and Statistics 51 (1969) 239-246.

[20] F. Spitzer, Some theorems concerning 2-dimensional Brownian motion, Transactions of the American Mathematical Society 87 (1958) 187-197.