The perils of first-order conditions of New Keynesian models

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14 April 2016

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2016/04/14

Abstract

For deriving equilibrium of sticky-price/monopolistic competition New Keynesian models, first-order conditions are often used. This paper shows that they may not be sufficient and presents a case out of a simple model where there exists no equilibrium. This is true even when Taylor rule is assumed.

1 Modified Rotemberg sticky-price model

In Kim (2016) [3], I discussed how a Rotemberg model [5] leads to accounting inconsistency problems. But there exists modification [4] of the Rotemberg model, where the firm pays the inflation costs to the household.
The household has the following optimization problem:

$$\max_{C_t, B_t, N_t} U(C_t, N_t) = \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right] \tag{1}$$

with the modified budget constraint:

$$P_t C_t + \frac{B_t}{R_t} \leq W_t N_t + D_t + B_{t-1} + P_t \frac{\chi}{2} \left(\frac{P_t}{P_{t-1}} - 1 \right)^2 C_t \tag{2}$$

where P_t refers to price level, B_t refers to a one-time bond, C_t refers to consumption, W_t refers to nominal wage, N_t is labor, D_t is profit dividends receives from a continuum of firms.
χ is the degree of price rigidity, and this form allows $C_t = Y_t$ in equilibrium, unlike the standard Rotemberg model.
Notice the additional payment made by the firm to the household. One could have assumed that this payment is given to the household optimization problem rather than being dependent on C_t, which means that the household has the control over this payment. But doing so makes less sense, since the payment made by the firm is due to some service/household products relating to inflation/deflation.

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I will only derive the optimality condition for labor/consumption, since the IS equation is not relevant for rest of this paper.

$$\frac{W_t}{P_t} = C_t^\sigma N_t^\varphi \left[1 - \frac{\chi}{2} \left(\frac{P_t}{P_{t-1}} - 1 \right)^2 \right] = u_t C_t^\sigma N_t^\varphi \tag{3}$$

One immediately sees one problem with this specification: u_t can go negative for high inflation, which means negative real wage. I will restrict to $0 < u_t \leq 1$. For simplification, I will set $P_0 = 1$.

Let there be a continuum of firms $[0, 1]$. The production function of each firm is:

$$C_t(i) = A_t N_t(i) \tag{4}$$

Aggregating over,

$$C_t = s_t A_t N_t \tag{5}$$

where $0 < s_t \leq 1$, which is inverse price dispersion factor. In this modified Rotemberg model, $s_t = 1$.

Let the classical equilibrium at time $t = 0$ be C_e, N_e. Since $W_e N_e = C_e$ in the classical equilibrium with $W_e = C_e^\sigma N_e^\varphi$,

$$C_e = W_e N_e = C_e^\sigma N_e^{\varphi+1} = \frac{C_e^{1+\sigma+\varphi}}{A_0^{1+\varphi}} \tag{6}$$

$$C_e = A_e^{\frac{1+\varphi}{\sigma+\varphi}} \tag{7}$$

Let the sticky-price monopolistic-competition equilibrium be C_0, N_0 at $t = 0$. Relating C_0 and C_e by $C_0 = kC_e$,

$$D_0 = C_0 - W_0 N_0 = kC_e - u_0 \frac{k^{1+\sigma+\varphi} C_e^{1+\sigma+\varphi}}{A_0^{1+\varphi}} \tag{8}$$

$$W_0 N_e = u_0 \frac{k^{\sigma+\varphi} C_e^{\sigma+\varphi}}{A_0^\varphi} \frac{C_e}{A_0} = u_0 \frac{k^{\sigma+\varphi} C_e^{1+\sigma+\varphi}}{A_0^{1+\varphi}} \tag{9}$$

$$W_0 N_e + D_0 = kC_e - u_0 \frac{k^{1+\sigma+\varphi} C_e^{1+\sigma+\varphi}}{A_0^{1+\varphi}} + u_0 \frac{k^{\sigma+\varphi} C_e^{1+\sigma+\varphi}}{A_0^{1+\varphi}} \tag{10}$$

The idea is that for the household problem W_0 and N_0 are given. If the household wants C_e and N_e as an outcome, would this satisfy the budget constraint? And if so, would (C_e, N_e) be preferred over (C_0, N_0)? First, it seems that the answer to the latter question be no, because the first-order optimality condition is not satisfied by (C_e, N_e). But what this paper will show is that the condition is not sufficient. After all, the FOC is only the necessary condition.

$$kC_e - u_0 \frac{k^{1+\sigma+\varphi} C_e^{1+\sigma+\varphi}}{A_0^{1+\varphi}} + u_0 \frac{k^{\sigma+\varphi} C_e^{1+\sigma+\varphi}}{A_0^{1+\varphi}} \geq C_e \tag{11}$$

$$C_e^{1+\sigma+\varphi} u_0 \left[k^{\sigma+\varphi} - k^{1+\sigma+\varphi} \right] \geq (1-k) A_0^{1+\varphi} C_e \tag{12}$$

$$u_0 k^{\sigma+\varphi} [1 - k] \geq 1 - k \tag{13}$$

For $k \geq 1$,

$$u_0 k^{\sigma+\varphi} \leq 1 \tag{14}$$

It is easy to find u_0 that satisfies Equation 14 given k. For a welfare evaluation example, suppose $A_0 = 1$. Then, $C_e = 1$. Let $k = 2$, which means $C_0 = 2$, with $N_0 = 2$. Let $\sigma = 0.5, \varphi = 0.5$.

Utility provided by (C_0, N_0) is $w(C_0, N_0) = w(2, 2) = 2^0.5/0.5 - 2^1.5/1.5 \approx 0.943$. $w(C_e, N_e) = 1/0.5 - 1/1.5 \approx 1.333$. Thus, $w(C_e, N_e) > w(C_0, N_0)$.

With $u = 1/2$, Equation 14 is satisfied and $0 < u \leq 1$ is retained, with inflation of 100% when $\chi = 1$.

What this welfare calibration shows is that this Rotemberg model inevitably leads to disequilibrium for some sets of inflation and output that were previously thought as possible equilibria. The household, given W_t, P_t, D_t, B_{-1}, will prefer (C_e, N_e) over (C_0, N_0) and thus will not choose (C_0, N_0). This demonstrates the fact that FOC is only a necessary one, but does not guarantee that the correct solution would be achieved.

An additional interpretation of Equation 14 would be that for each $k > 1$, u_0 must be greater than a certain point in order to be considered as a possible equilibrium - meaning that inflation rate cannot go below or above a certain level. For $k < 1$, equation 13 is not satisfied automatically.

1.1 Output asymmetry and monetary policy

The above discussion shows that whenever the output is higher than the classical level, disequilibrium is the only possible result for non-zero inflation. Even if one adjusts u_t to rule out (C_e, N_e), there always exists a nearby result (C, N) that is better relative to the original (C_t, N_t). This continues until $u_t = 1$, which rules out inflation. Getting the cue from the New Keynesian Phillips Curve (NKPC), which can also be derived from the first-order approximation solution of the Rotemberg model around zero inflation steady state:

$$\pi_t \equiv \frac{P_t}{P_{t-1}} - 1 = \beta E_t [\pi_{t+1}] + \kappa \tilde{y}_t \tag{15}$$

where $c_t = y_t = \log(Y_t)$ and $\tilde{y}_t = \log(Y_t/Y_{n,t})$ (output gap) with $Y_{n,t}$ the natural level obtained without sticky price but with monopolistic competition. $\kappa > 0$ is assumed. Thus, down to some positive output gap (because $Y_{n,0} < Y_e$), π_t is constrained to zero.

$$\beta E_t [\pi_{t+1}] = -\kappa \tilde{y}_t \tag{16}$$

This suggests that when $\tilde{y}_t$ is above the classical level, one must expect deflation at the next period.

To the first order approximation around zero inflation steady state with the value of χ small enough, one can also recover the standard IS equation:

$$E_t [c_{t+1}] = c_t + \frac{1}{\sigma} (i_t - E_t [\pi_{t+1}] - \rho) \tag{17}$$

where $\rho = -\log\beta$, $c_t = \log C_t$ with i_t being nominal interest rate. Reformulating as the output gap version,

$$E_t[c_{\tilde{t+1}}] = \tilde{c}_t + \frac{1}{\sigma}(i_t - E_t[\pi_{t+1}] - r_t^n) \tag{18}$$

where r_t^n is the natural rate of interest.

Thus, if y_t is greater than the classical level y_e, $i_t \leq r_t^n - (y_t - y_e)$ must be satisfied to retain equilibrium, assuming technology remains constant - otherwise, inflation expectation would be inconsistent. Thus, any boom would instantaneously be curbed - implying output asymmetry. If central bank has control on i_t and sets it wrong, then disequilibrium is inevitable.

2 Calvo model

The Calvo sticky-price model [1] presented here has the same household optimization problem as before except that the budget constraint is now

$$P_t C_t + \frac{B_t}{R_t} \leq W_t N_t + D_t + B_{t-1} \tag{19}$$

Thus real wage first-order condition is given by

$$\frac{W_t}{P_t} = C_t^\sigma N_t^\varphi \tag{20}$$

Again assume $P_0 = 1$.

In the Calvo model, production function for an individual firm remains the same, but now price dispersion affects the final output. Production function will be changed to:

$$Y_t = s_t A_t N_t^{1-\alpha} \tag{21}$$

For the Calvo model, $0 < s_t \leq 1$, and no one value can be pre-determined for s_t without specified monetary policy.

Decreasing returns to scale is required for what follows - in constant returns to scale, the method presented below would fail, as will be seen.

I will assume $P_e = 1$.

$$C_e = W_e N_e = \frac{C_e^{\sigma + \frac{1+\varphi}{1-\alpha}}}{A_0^{\frac{1+\varphi}{1-\alpha}}} \tag{22}$$

$$C_e = A_0^{\frac{1+\varphi}{\sigma - \alpha\sigma + \varphi + \alpha}} \tag{23}$$

$$N_e = A_0^{\frac{1+\varphi - \sigma + \alpha\sigma - \varphi - \alpha}{(\sigma - \alpha\sigma + \varphi + \alpha)(1-\alpha)}} \tag{24}$$

$$D_0 = A_0^{\frac{1+\varphi}{\sigma - \alpha\sigma + \varphi + \alpha}} \left[k - \frac{k^{\frac{\sigma - \alpha\sigma + 1 + \varphi}{1-\alpha}}}{s_0^{\frac{1+\varphi}{1-\alpha}}} \right] \tag{25}$$

$$W_0 N_e = A_0^{\frac{1+\varphi}{\sigma - \alpha\sigma + \varphi + \alpha}} \frac{k^{\frac{\sigma - \alpha\sigma + \varphi}{1-\alpha}}}{s_0^{\frac{\varphi}{1-\alpha}}} \tag{26}$$

$$W_0 N_e + D_0 = A_0^{\frac{1+\varphi}{\sigma - \alpha\sigma + \varphi + \alpha}} \left[\frac{k^{\frac{\sigma - \alpha\sigma + \varphi}{1-\alpha}}}{s_0^{\frac{\varphi}{1-\alpha}}} + k - \frac{k^{\frac{\sigma - \alpha\sigma + 1 + \varphi}{1-\alpha}}}{s_0^{\frac{1+\varphi}{1-\alpha}}} \right] \tag{27}$$

It is wished that $C_e \leq W_0 N_e + D_0$, thus:

$$1 \leq \left[\frac{k^{\frac{\sigma - \alpha\sigma + \varphi}{1-\alpha}}}{s_0^{\frac{\varphi}{1-\alpha}}} + k - \frac{k^{\frac{\sigma - \alpha\sigma + 1 + \varphi}{1-\alpha}}}{s_0^{\frac{1+\varphi}{1-\alpha}}} \right] \tag{28}$$

For calibration, suppose that s_0 is very close to 1 that $s_0 = 1$ can be assumed. Then,

$$1 \leq \left[k^{\frac{\sigma - \alpha\sigma + \varphi}{1-\alpha}} + k - k^{\frac{\sigma - \alpha\sigma + 1 + \varphi}{1-\alpha}} \right] \tag{29}$$

Re-arranging,

$$1 - k \leq k^{\frac{\sigma - \alpha\sigma + \varphi}{1-\alpha}} \left[1 - k^{\frac{1}{1-\alpha}} \right] \tag{30}$$

Let $k < 1$.

Again, I will assume that $0 < \alpha < 1$ is very close to 1, but not 1, and let $0 < \varphi$ be close to 0, but not 0. Similarly let $0 < \sigma$ be close to 0 but not 0. Then the RHS approaches 1 as these parameters become close to the assigned limits, with φ coming close to 0 faster than $1 - \alpha$.

This demonstrates that there exists some $(\alpha, \sigma, \varphi, k)$ that satisfies Equation 30, but it can easily be shown that in case of constant returns to scale ($\alpha = 0$), Equation 30 does not hold.

This existence proof demonstrates that some (C_0, N_0) that is supposed to be some equilibrium of the Calvo model via first-order conditions, is not actually an equilibrium, since the household definitely prefers (C_e, N_e) over (C_0, N_0).

3   Conclusion

This paper suggests that first-order conditions derived from optimization problems are not sufficient to find a sticky-price monopolistic competition (Rotemberg, Calvo) model equilibrium, unless one moves away from standard interpretations. The original Rotemberg model suffers from the accounting problem exposed in [3], and thus this paper uses the variant for demonstration. Interpretation of New Keynesian Phillips Curve may have to change - inflation is locked to zero for a range of outputs.

For the Calvo model, while not discussed in this paper, sub-utility function in terms of CES aggregator may be understood as a final good producer production function. Usually, it is assumed that by competitive assumptions zero profit results in for the final good producer. But this certainly does not have to be the case (zero profit is assumed, not derived). When the condition is relaxed, a different understanding can be reached.

References

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