# On the Optimal Lifetime of Real Assets

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**On the Optimal Lifetime of Real Assets**

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**Abstract**

We show that the “abandonment” model emphasized by researchers in capital budgeting and the “steady state” replacement model emphasized by economic theorists constitute sub-cases of a more general class of transitory replacement models in which the horizon of reinvestments is determined endogenously along with the other decision variables. Moreover, comparisons between our model and that of steady state replacement revealed that there are considerable differences. In particular, we found that: i) the two models lead to different estimates concerning the profit horizon, the duration of replacements, the timing of abandonment or scrapping, and the impact of productive capacity and market structure on service lives, as these are determined by various parameters, ii) even though the steady state replacement policy may result in higher total profit, it does so at great expense in flexibility for the planner, because the replacements are built into the model from the beginning, and iii) the transitory replacement policy seems more realistic in that the replacements are undertaken only if forced on the planner by decreasing profits.

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Keywords: optimal service life, replacement, abandonment, scrapping, horizon of reinvestments

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## 1. Introduction

The economic life of assets is a key variable in many fields of decision sciences. In capital budgeting, for example, if the economic life of assets under consideration is unknown, their relevant cash flows and scrap values cannot be ascertained. So computing the standard criteria of net present value and internal rate of return becomes untenable and this in turn inhibits the accurate planning of investments. Similarly, in accounting and finance, if the economic life of assets is unknown, their depreciation cannot be accounted for with any precision and all approximations of, say, the user cost of capital are rendered uncertain. Lastly, in economics, if the useful life of assets is unknown, as Haavelmo (1962,) has established, we cannot derive consistent aggregates of the stock of capital of a firm, a sector or an economy. No wonder therefore that the issue of establishing the optimal life of assets occupies a central place in economic theory and policy.

Preinreich (1940) was the first to show how the optimal life of assets can be determined. More specifically, according to his theorem, in order for the economic life of a single machine to be optimal, it should be computed jointly with the economic life of each machine in the chain of future replacements extending as far into the future as the owner's profit horizon. But he formulated it on the assumption that the horizon of the reinvestment process is decided by the owner of the machine on the basis of his perception on how long the investment opportunity might remain profitable. Thus, depending on the specification of the owner's profit horizon, there emerged a continuum of models for the determination of the optimal lifetime of assets. In particular, by limiting the owner's profit horizon to a single investment cycle, researchers in the field of capital budgeting obtained the so-called "abandonment" class of models and used it to derive sharp rules regarding optimal asset life. Initially Robichek and Van Horne (1967) suggested that an asset should be abandoned in any period in which the present value of future cash flows does not exceed its abandonment value. Then, drawing on the possibility that the function of cash flows may not have a single peak, Dyl and Long (1969) argued that abandonment should not occur at the earliest possible date that the above abandonment condition is satisfied, but rather at the date that yields the highest net present value over all future abandonment opportunities. Lastly, Howe and McCabe (1983): i) highlighted the patterns of cash flows and scrap values under which the "abandonment" model leads to a unique global optimum of the abandonment time, ii) characterized the complete range of models that can be obtained by varying the owner's profit horizon, and iii) clarified the circumstances which should guide in practice the choice between "abandonment" and "replacement" models.

Research economists, on the other hand, continued to work in the tradition of

Terborgh (1949) and Smith (1962) by assuming invariably that the owner's profit horizon is infinite. This in turn led them to concentrate exclusively on a single class of "replacement" models, all of which presume that reinvestments take place at equal time intervals. Just to indicate how pervasive this conceptualization has been, it suffices to mention that it has been adopted in all significant contributions in this area from Brems (1968) to Nickel (1975), Rust (1987) and Van Hilten (1991), and to Mauer and Ott (1995), more recently. The question then arises as to the importance of the implications that may result if, instead of treating the owner's profit horizon as given while solving for the optimal lifetime of assets, we make it an endogenous variable which is computed along with all other variables in the optimization process. Our objective in this paper is to investigate the implications of the proposed generalization and characterize their significance for applications.

The paper is organized as follows. In Section 2 we set up the model and analyze the properties of the replacement policies that emanate from its solution. The structure of the model allows for a number of reinvestment cycles, which terminate with abandonment or terminal scrapping of the asset under consideration. Thus, with all other variables and parameters given, the owner of the asset is presumed to determine the profit horizon and the dates of reinvestments, if any, so as to maximize the net present value of overall profits. In Section 3 we compare the policy from our model and that from the steady state replacement model by means of a numerical example. Then, in Section 4, we draw the implications of the analysis for the two types of replacement policies, and, finally, in Section 5 we conclude with a synopsis of the main results and a suggestion for further research.

## 2. The model

At the end of the service life of equipment, there are always two options, to replace it and continue doing so up to some profit horizon or to abandon or scrap it and terminate operations. To examine them we formulate the service life problem for a multiple series of operating periods and we compare the following two alternative approaches:

1. *Transitory replacements with terminal scraping,* where the equipment is replaced an optimally determined number of times, ending with scrapping.[1]
2. *Steady state replacements,* where the equipment is replaced at equal time intervals, indefinitely.

To keep the analysis as simple as possible we adopt the following two simplifying assumptions: 1) Time invariance, in the sense that *only relative time matters* and in particular all operating periods are similar and hence we can examine each one starting at zero time, and 2)

*Impatience* on the part of the owner in the sense that he does not accept even a temporary drop in total profits, terminating operations when this happens. Among other reasons this could be attributed to the uncertainty caused by the possibility of an obsolescence effect eliminating all operating revenue and scrap value thereafter.

We use the term equipment in a general sense, by not adopting any particular type of asset. Thus, considering first a single operating period of duration $T$, we assume that the maximum total profit in present values, is given by some *profit function*:

$$A(T) = Q(T) + e^{-\sigma T} S(T) - P, \text{ for } 0 \le T \le \infty, \tag{1}$$

where $Q(T)$ is the net operating revenue, $S(T)$ is the scrap value of the equipment, $P$ is the price of new equipment, and $\sigma$ is the discount rate. In general, as we demonstrate in Bitros and Flytzanis (2007), $A(T)$ is the result of some optimization procedure that may include uncertainties, in which case it refers to expected values. Also, $S(T)$ may be zero if we have abandonment, negative if we have disposal costs, and in special cases it may even be higher than $P$ if we have upgrading. The term:

$$A(0) = S(0) - P \le 0, \tag{2}$$

denotes the cost of new as opposed to unused equipment. It will be called *transactions cost.* For simplicity we shall ignore for the moment transactions cost by setting: $A(0) = 0$. Then we will include transactions cost as a correction term, assuming it is small compared to the other quantities involved. Given the above, we consider also the *final profit rate* in current values given by:

$$\alpha(T) = A'(T)e^{\sigma T} = Q'(T)e^{\sigma T} - \sigma S'(T) \tag{3}$$

We will assume that $\alpha(T)$ is continuous.

### 2.1 Transitory replacements

Applying the procedure of dynamic programming we reorder the operating periods, starting with the last period leading to scrapping and going backwards. As the scrapping period differs from all other replacement periods we will refer to it as $0-$period. Thus, $1-$period is the last replacement period, etc. In view of the impatience assumption, we will say that the equipment is *profitable*[2] if it starts with strictly positive profit rate:

$$A'(0) = \alpha(0) > 0.$$

In accordance with the impatience assumption the scrapping period duration $T_0$ is given by the first maximum of $A(T)$, and it is determined by the first time the function $\alpha(T)$ crosses the zero value or else it is infinite. We will set:

$$\Pi_0 = A(T_0) > 0 \text{, where } 0 < T_0 \le \infty . \tag{4}$$

Assuming profitability, we consider the last replacement period before the scrapping period, and the corresponding $1-$replacement total profit function:[3]

$$\Pi_1(T) = A(T) + e^{-\sigma T}\Pi_0 \tag{5}$$

We will say that the equipment is $1-$*replaceable*, if this profit function starts with a strictly positive profit rate:

$$\Pi_1'(0) = \alpha(0) - \sigma\Pi_0 > 0 \Rightarrow \alpha(0) > \sigma\Pi_0$$

Then the first maximum will be strictly larger than $\Pi_0 = \Pi_1(0)$, and the corresponding replacement time $T_1$ will be determined by the first time $\Pi_1'(T)$ crosses the zero level, i.e. by the first time $\alpha(T)$ crosses the level $\sigma\Pi_0$ from above. We will set:

$$\Pi_1 = A(T_1) + e^{-\sigma T_1}\Pi_0 > \Pi_0 \tag{6}$$

In the same way we define inductively the notion of $n-$*replaceable* equipment with replacement time $T_n$, $n-$period profit $A(T_n)$ and total profit $\Pi_n$.

Classifying profitable equipment with respect to replacement properties, we will say that it is:

1. *Scrapping finitely* $N-$*replaceable,* if it has only $N$ profitable replacements, for some $N \ge 0$. In particular we will say that it is *scrapping replaceable* if $N > 0$, *scrapping non-replaceable* if $N = 0$.
2. *Scrapping infinitely replaceable* if all replacements are profitable.

Considering also the scrapping period, we will say that the equipment is:

3. *Non-scrappable* if $T_0 = \infty$, *scrapping durable* if it is both non-scrappable and non-replaceable: $T_0 = \infty$ & $N = 0$.

Moreover, by implication of the above, it should be noted that a finitely $N-$replaceable equipment may be non-replaceable: $N = 0$, or non-scrappable: $T_0 = \infty$, or even scrapping durable:

$T_0 = \infty, N = 0$.

We collect the main properties in:

***Lemma 1***

1. *Profitable replacements have successively strictly decreasing durations, strictly decreasing period profits and strictly increasing total profits:*
$$T_n < T_{n-1},\ A(T_n) < A(T_{n-1}),\ \Pi_n > \Pi_{n-1}, \text{ with } \Pi_n = A(T_n) + e^{-\sigma T_n}\Pi_{n-1}.$$
2. *If the equipment is finitely $N-$replaceable, then it satisfies:*
$$\Pi_{n-1} = \frac{\alpha(T_n)}{\sigma} < \frac{\alpha(0)}{\sigma},\ \text{for } 1 \le n \le N \ \ \&\ \Pi_N \ge \frac{\alpha(0)}{\sigma}.$$
3. *If the equipment is infinitely replaceable, then because of the monotonicities in 1, we obtain limits:*
$$T_n \downarrow T_\infty,\ A(T_n) \downarrow A(T_\infty), \Pi_n \uparrow \Pi_\infty = \frac{A(T_\infty)}{1-e^{-\sigma T_\infty}} = \frac{\alpha(T_\infty)}{\sigma}.$$
*In this case $\alpha(T)$ is constant for $0 \le T \le T_\infty$, strictly decreasing immediately afterwards, and we have:*
$$\Pi_\infty = \frac{\alpha(0)}{\sigma}.$$
*In particular we may have: $T_\infty = 0$.*

***Proof***

*Part 1 follows from the property that $T_n$ is the first time $\alpha(T)$ crosses the level $\sigma\Pi_{n-1}$ from above, where $\sigma\Pi_{n-1}$ is a strictly increasing positive sequence. In particular, concerning the last replacement before scrapping, if we have $T_0 = \infty$, then we should have $T_1 < \infty$. Indeed in this case we have $\Pi_1(0) = \Pi_1(\infty) = \Pi_0$ with $\Pi_1'(0) > 0$, and the first maximum of $\Pi_1(T)$ will be at some $T_1 < \infty$. Part 2 is in the definitions. Finally, concerning part 3 we note first that we have $\alpha(T) \ge \alpha(T_\infty)$, because by definition in every interval $0 \le T \le T_n$ we have $\alpha(T) \ge \alpha(T_n)$ and $T_n \downarrow T_\infty$. If $\alpha(T)$ is not constant equal to $\alpha(T_\infty)$, then we will have:*
$$A(T_\infty) = \int_0^{T_\infty} \alpha(T) e^{-\sigma T} dT > \frac{\alpha(T_\infty)}{\sigma}(1-e^{-\sigma T_\infty}) \Rightarrow \Pi_\infty = \frac{A(T_\infty)}{1-e^{-\sigma T_\infty}} > \frac{\alpha(T_\infty)}{\sigma},$$
*contradicting the limiting condition $\Pi_\infty = \alpha(T_\infty)/\sigma$.*

We can summarize the above as follows:

*We keep replacing until the total profit $\Pi_n$ crosses the critical level $\alpha(0)/\sigma$, or else we replace indefinitely at equal time intervals $T_\infty$.*

## 2.2 Steady state replacements

Independently of the above limiting procedure, infinite replacements at equal time intervals can also be examined directly.[4] In this case, assuming always no transactions cost, if $T$ is the uniform duration, the *steady state profit* will be:

$$\Pi(T)=\sum_{v=0}^{\infty}e^{-v\sigma T}A(T)=\frac{A(T)}{1-e^{-\sigma T}}\ , \tag{7}$$

Where $A(T)$ is the same as above since it involves maximizing one period profits for fixed $T$. Taking the derivative, we find:

$$\Pi'(T)=\frac{\sigma e^{-\sigma T}}{1-e^{-\sigma T}}[\frac{\alpha(T)}{\sigma}-\Pi(T)]. \tag{8}$$

Clearly the steady state policy is easier to study and classify because it is determined by a single function related directly to the profit rate function $\alpha(T)$. We collect the main properties:

***Lemma 2***

*1. The steady state profit function increases, is constant, or decreases, according as its value $\Pi(T)$ is smaller, equal to, or larger respectively, than the value $\alpha(T)/\sigma$.*

*2. Initially the function $\Pi(T)$ lies between the function $\alpha(T)/\sigma$ and the constant $\alpha(0)/\sigma$. In particular, we have:*

$$\Pi(0)=\frac{\alpha(0)}{\sigma}\ \ \&\ \ \Pi'(0)=\frac{\alpha'(0)}{2\sigma} \tag{9}$$

***Proof***

*Part 1 is a consequence of the derivative formula. The relations in part 2 are obtained by applying l' Hopital rule at $T=0$, assuming that $\alpha(T)$ is continuously differentiable there.*

We will say that the equipment is *steady state profitable* if it satisfies $\Pi(0)>0$, and then the *steady state duration* $T^*$ is defined as the first time $\Pi(T)$ starts dropping, i.e. replacement is postponed as far as possible. We will set: $\Pi^*=\Pi(T^*)$ and call the equipment: *steady state, durable* if $T^*=\infty$, *replaceable* if $0<T^*<\infty$, *disposable if* $T^*=0$.

**Remark.** *The case where $\Pi(T)$ has an initial segment of constancy decreasing immediately afterwards requires special treatment. According to the definition the steady state duration $T^*$ is the time of first drop, i.e. at the right end of the interval. In fact this is also the correct assignment if we obtain it as the limit when the transactions cost goes to zero. Hence the equipment is steady state replaceable but with $\Pi^*=\Pi(0)$. Actually the steady state duration in this case would be indifferent to any value in the interval of constancy including the value $T=0$. We will differentiate this case by saying that the equipment is steady state weakly replaceable if $\Pi^*=\Pi(0)$, strongly if $\Pi^*>\Pi(0)$. Similarly, if $\Pi(T)$ is constant throughout then we could say* *that the equipment is steady state weakly durable.*

Comparing now the two policies, i.e. the transitory policy with terminal scrapping and the steady state policy, we find the following:

***Proposition***

*Ignoring transactions costs for new equipment, the profitability condition for steady state and transitory replacement policies is the same:*

$$\alpha(0) > 0 .$$

*Assuming profitability, we distinguish the following cases:*

*A. If the equipment is steady state durable, then it is also scrapping durable with:*

$$T^* = T_0 = \infty,\ N = 0,\ \&\ \Pi^* = \Pi_0 = A(\infty) .$$

*Also in this case it satisfies* $\alpha(T) \geq \alpha(0)$ *for all* $T$ *.*

*B. If the equipment is steady state strongly replaceable, then it is also scrapping finite* $N-$*replaceable for some* $N \geq 0$*, with:*

$$0 < \tau^* \leq T^* < T_N\ ,\ \frac{\alpha(0)}{\sigma} < \Pi^* \leq \frac{\bar{\alpha}}{\sigma}\ \&\ \frac{\alpha(0)}{\sigma} \leq \Pi_N < \frac{\bar{\alpha}}{\sigma},$$

*where* $\alpha^* = \alpha(\tau^*)$ *is the first maximum of* $\alpha(T)$ *and* $\bar{\alpha}$ *is the global maximum of* $\alpha(T)$*. Also in this case* $\alpha(T)$ *initially rises strictly above* $\alpha(0)$*, after a possible segment of constancy.*

*C. If the equipment is scrapping infinitely replaceable then it is also steady state disposable or steady state weakly replaceable, with:*

$$T^* = T_\infty\ \ \&\ \Pi^* = \Pi_\infty = \alpha(0)/\sigma\ .$$

*Also in this case* $\alpha(T)$ *initially falls strictly below* $\alpha(0)$*, after a possible segment of constancy.*

***Proof.***

*A. By the steady state durability assumption* $\Pi(T)$ *is increasing and then* $\alpha(T)$ *has no zeroes because:* $\alpha(T) \geq \sigma\Pi(T) \geq \sigma\Pi(0) = \alpha(0) > 0$*, by Lemma 2.1 and profitability. Here belongs also the case where* $\Pi(T)$ *and hence also* $\alpha(T)$ *are constant throughout.*

*B. Starting from the level* $\alpha(0)/\sigma$*, by the strong replaceability assumption the function* $\Pi(T)$ *increases in the interval* $0 \leq T \leq T^*$ *to a level strictly above* $\alpha(0)/\sigma$ *and starts strictly decreasing immediately after. By Lemma 2* $\alpha(T)/\sigma$ *will lie above it while* $\Pi(T)$ *increases and drop back crossing the level* $\Pi^*$ *at* $T^*$*. We conclude that it must have a maximum at some* $\tau^* \leq T^*$ *and that* $\Pi^*$ *is no greater than this maximum. Concerning transitory replacement policy we note that if it is finitely* $N-$*replaceable then* $\sigma\Pi_{N-1} < \alpha(0) < \sigma\Pi^*$*, and hence* $\alpha(T)$ *will cross the level* $\sigma\Pi_{N-1}$ *strictly after* $T^*$*, giving* $T_N > T^*$*. Finally for the bounds on the total profits for transitory replacements we have:*

$$\Pi_N = A(T_N) + e^{-\sigma T_N}\Pi_{N-1} < \frac{\bar{\alpha}}{\sigma}(1 - e^{-\sigma T_N}) + e^{-\sigma T_N}\frac{\alpha(0)}{\sigma} < \frac{\bar{\alpha}}{\sigma}$$

*C. It follows from parts A & B by exclusion. Or directly, by the assumption and the last part of Lemma 1 we have that* $\alpha(T)$ *is constant in the interval* $0 \leq T \leq T_\infty$

*strictly decreasing immediately afterwards. It follows from Lemma 2 that* $\Pi^*(T)$ *will do the same so the equipment will be steady state disposable or weakly replaceable.*

The above allow us to classify equipment according to the properties of $\alpha(T)$. For convenience we will consider equipment for which $\alpha(T)$ is single-peaked, i.e. it has at most two monotone sections, *not necessarily strictly monotone*. Thus we distinguish the following types:

*A:* $\alpha(T) \geq \alpha(0)$. *The equipment will be scrapping durable. Also steady state durable if it is monotone increasing, otherwise it can be steady state strongly replaceable.*

*B:* $\alpha(T)$, *at first rises strictly above* $\alpha(0)$ *but eventually drops strictly bellow. The equipment will be steady state strongly replaceable and scrapping finitely replaceable.*

*C:* $\alpha(T) \leq \alpha(0)$. *It will be steady state disposable or steady state weakly replaceable and scrapping infinitely replaceable.*

*D:* $\alpha(T)$, *after a possible segment of constancy, at first drops strictly below* $\alpha(0)$ *but eventually rises strictly above. It will be steady state disposable or steady state weakly replaceable. Also it will be scrapping infinitely replaceable unless* $A(T_0) > \alpha(0)/\sigma$ *in which case it will be scrapping non-replaceable.*

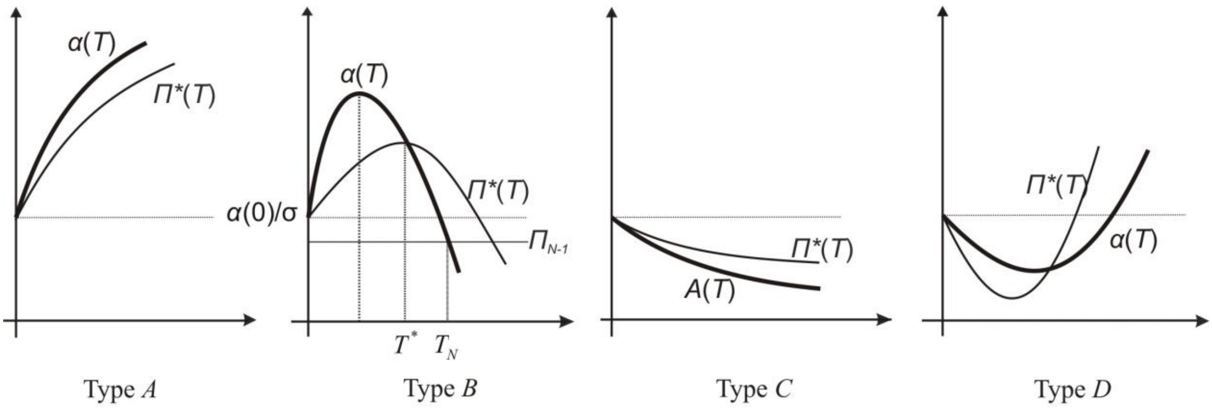


Figure 1

### 2.3 Transactions cost

Introducing now transactions cost we will examine only the case where it is small negative: $A(0) < 0 \Rightarrow S(0) < P$. In the steady state policy we have the correction to the profit given by the cost term $C(T)$:

$$\Pi(T) = \frac{A(T) - A(0)}{1 - e^{-\sigma T}} + \frac{A(0)}{1 - e^{-\sigma T}} \Rightarrow C(T) = \frac{A(0)}{1 - e^{-\sigma T}} \tag{10}$$

It is monotonically increasing in $T$, from $C(0) = -\infty$ to $C(\infty) = 0$. Thus, as $A(0)$ decreases from zero to negative values, the steady state duration increases and the steady state profit falls as indicated in Figure 2. The effect is similar for the transitory replacement policy, except that $T_0$ is not affected. Also, since $\alpha(T)$ remains the same we may have an increase in the number of replacements because $\sigma\Pi_n$ will be lower. For small transaction costs we can estimate these effects directly, as follows:

***Corollary***

*As* $C = A(0)$ *decreases from the zero to negative values, the profit and the duration are affected as follows:*

*1. For steady state replaceable equipment:*

$$d\Pi^* = C^* \;\;\&\;\; dT^* = \frac{\sigma C^*}{\alpha'(T^*)},$$

*where:*

$$C^* = \frac{A(0)}{1 - e^{-\sigma T^*}} = A(0)(1 + e^{-\sigma T^*} + e^{-2\sigma T^*} + \cdots)$$

*2. For scrapping finitely replaceable equipment:*

$$d\Pi_n = C_n,\; C_0 = A(0) \;\;\&\;\; dT_n = \frac{\sigma C_{n-1}}{\alpha'(T_n)},\; dT_0 = 0,$$

*where:*

$$C_n = A(0) + e^{-\sigma T_n} C_{n-1} = A(0)[1 + e^{-\sigma T_n} + \cdots + e^{-\sigma(T_n + \cdots + T_1)}].$$

***Proof.***

*Concerning the profits it is a direct consequence of the envelope theorem applied to the functions*

$$\Pi(T) = A(T)/(1 - e^{-\sigma T}),\; \Pi_n(T) = A(T) + e^{-\sigma T}\Pi_{n-1}$$

*Concerning the durations we use in addition the defining relations:*

$$\alpha(T) = \sigma\Pi(T),\; \alpha(T) = \sigma\Pi_{n-1}.$$

*The profit reduction per period is more pronounced in the steady state policy. We note that by their definition we have:* $\alpha'(T^*) < 0$, $\alpha'(T_n) < 0$.

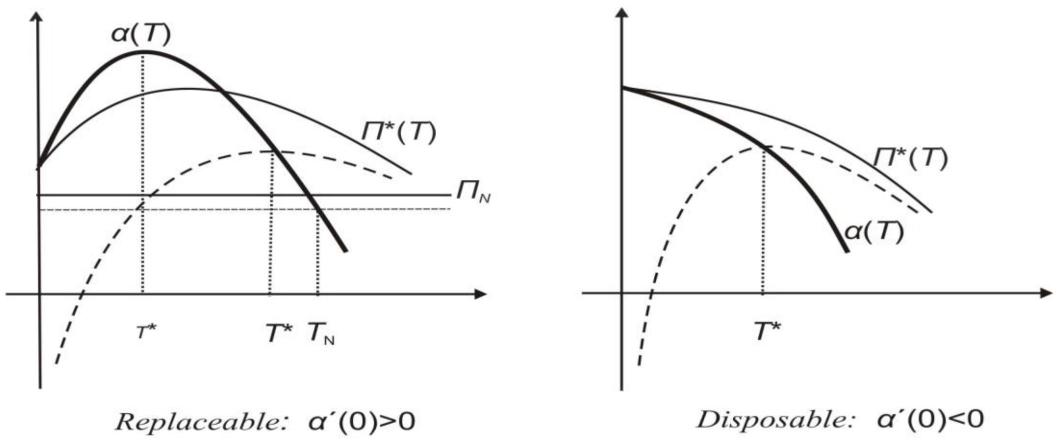


Figure 2

**3. An example**

In Bitros and Flytzanis (2007) we analyzed the properties of optimal utilization and maintenance policies in a one period model by using techniques of optimal control. Considering now the problem of multi-period replacements we will examine the simplest version of that model for which we can compute directly all the relevant quantities. The specifications are as follows:

$$A(T) = \max \int_0^T e^{-\sigma t} q(S) dt + e^{-\sigma T} S - P \quad \text{with} \quad \dot{S} = -wS,\ q = rS^{\varepsilon} \Rightarrow \dot{q} = \varepsilon q,\ P = S_0\ . \tag{11}$$

Thus the value $S$ of capital stock changes at the rate $w$, while the value $q$ of the services it provides changes at the rate $\varepsilon w$. We have no transactions cost. We compute the functions:

$$S = S_0 e^{-wT},\ q = q_0 e^{-\varepsilon wT},\ Q = \frac{q_0}{\varepsilon w + \sigma}[1 - e^{-(\varepsilon w + \sigma)T}],\ \text{where } q_0 = rS_0^{\varepsilon} \tag{12}$$

$$A(T) = \frac{q_0}{\varepsilon w + \sigma}[1 - e^{-(\varepsilon w + \sigma)T}] + S_0 e^{-(w+\sigma)T} - S_0 \Rightarrow A(\infty) = \frac{q_0}{\varepsilon w + \sigma} - S_0 \tag{13}$$

$$\alpha(T) = e^{\sigma T} A'(T) = q_0 e^{-\varepsilon wT} - (w + \sigma) S_0 e^{-wT} \quad \Rightarrow \alpha(0) = q_0 - (w + \sigma) S_0 \tag{14}$$

$$\alpha'(T) = -\varepsilon w q_0 e^{-\varepsilon wT} - w(w + \sigma) S_0 e^{-wT} \quad \Rightarrow \alpha'(0) = -\varepsilon w q_0 - w(w + \sigma) S_0 \tag{15}$$

We note that the equipment is profitable if:

$$\alpha(0) > 0 \Rightarrow q_0 > (w + \sigma) S_0 \tag{16}$$

In general $w$ and $r$ could be of either sign. We will examine only the usual case where the equipment downgrades: $w > 0$, and then for profitability we must also have $r > 0$. This excludes equipment of type *A*. The equipment will be of:

$$\text{Type } B \text{ if } \alpha'(0) > 0 : \varepsilon q_0 < (w+\sigma)S_0 \tag{17}$$

$$\text{Type } C \text{ if } \alpha'(0) < 0 : \varepsilon q_0 > (w+\sigma)S_0 \ . \tag{18}$$

Also, the equipment will be scrappable if the following equation has a positive solution $T$, which then will be the scrapping duration:

$$\alpha(T) = q_0 e^{-\varepsilon w T} - (w+\sigma)S_0 e^{-wT} = 0 \Rightarrow T_0 = \frac{1}{(\varepsilon-1)w}\ln\frac{q_0}{(w+\sigma)S_0} \ . \tag{19}$$

We distinguish the following cases:

$\varepsilon \geq 1$*: The value of the services deteriorates at least as fast as the value of the capital stock.* The equipment is of type *C*, steady state disposable and scrapping infinitely replaceable. It is non-scrappable if $\varepsilon = 1$, scrappable if $\varepsilon > 1$.

$\varepsilon < 1$ *: The value of capital stock deteriorates faster than the value of the services.* This is the usual case and we examine it in more detail. The equipment will be:

Profitable of type *C* if:

$$\{\alpha(0) > 0, \alpha'(0) < 0\} \Rightarrow \{q_0 > (w+\sigma)S_0, \ \ \varepsilon q_0 > (w+\sigma)S_0\}$$

Profitable of type *B* if :

$$\{\alpha(0) > 0, \alpha'(0) > 0\} \Rightarrow \{q_0 > (w+\sigma)S_0, \ \ \varepsilon q_0 < (w+\sigma)S_0\} \ .$$

Expressing the conditions in terms of the discount rates, we find two critical values:

$$\alpha(0) = 0 \Rightarrow \sigma_0 = \frac{q_0 - wS_0}{S_0} \ \ \& \ \ \alpha'(0) = 0 \Rightarrow \sigma_\infty = \frac{\varepsilon q_0 - wS_0}{S_0} \ , \tag{20}$$

where $\sigma_0 > \sigma_\infty$, with the following properties:

$\sigma > \sigma_0$: The equipment is non-profitable

$\sigma < \sigma_\infty$: The equipment is profitable of type *C*, steady state disposable and scrapping infinitely replaceable. Also it is non-scrappable.

$\sigma_0 > \sigma > \sigma_\infty$: The equipment is profitable of type *B*, with:

$$\alpha'(T) = 0 \Rightarrow \tau^* = \frac{1}{(\varepsilon-1)w}\ln\frac{\varepsilon q_0}{(w+\sigma)S_0} \ \ \& \ \ \frac{\alpha(0)}{\sigma} = \frac{q_0 - (w+\sigma)S_0}{\sigma} \tag{21}$$

Hence the equipment is steady state strongly replaceable and scrapping finitely $N$ – replaceable, with

$$\tau^* = \frac{1}{(\varepsilon-1)w}\ln\frac{\varepsilon q_0}{(w+\sigma)S_0} \le T^* < T_N$$
$$\frac{\alpha(0)}{\sigma} = \frac{q_0-(w+\sigma)S_0}{\sigma} < \{\Pi^*, \Pi_N\} < \frac{\alpha^*}{\sigma} \tag{22}$$

Also it is non-scrappable, with:

$$T_0 = \infty,\ \Pi_0 = A(\infty) = \frac{q_0}{\varepsilon w+\sigma} - S_0\,. \tag{23}$$

Proceeding, we can determine further critical values for the discount rate, between the above two, with increasing replaceability. In particular, the condition:

$$\frac{\alpha(0)}{\sigma} = \Pi_0 \Rightarrow \sigma_1 = \varepsilon\frac{q_0 - wS_0}{S_0}\,, \tag{24}$$

distinguishes the following sub-cases:

$\sigma_0 > \sigma > \sigma_1$: The equipment is non-replaceable and hence durable.

$\sigma_1 > \sigma > \sigma_\infty$: The equipment is replaceable.

For an example, we consider equipment with the following technical characteristics:

$$\{S_0 = 1,\ r = 0.2,\ w = 0.1,\ \varepsilon = 0.4\} \Rightarrow \{\sigma_0 = 0.1,\ \sigma_1 = 0.04,\ \sigma_\infty = -0.02\}$$

Since $\sigma_\infty < 0$, the equipment is of type *B***,** if profitable. In particular, considering the transitory policy, the equipment is:

Profitable if $\sigma < 0.1$, non-replaceable if $\sigma \ge 0.04$, replaceable for smaller $\sigma$.

We compute the relevant quantities for two values of $\sigma$.

**1.** $\sigma = 0.06 > \sigma_1 \Rightarrow \alpha(0)/\sigma = 0.67,\quad \alpha^*/\sigma = 1.25,\quad \tau^* = 8$

| $T^*$, $\Pi^*$ | $T_0^*$, $\Pi_0^*$ | $T_1$, $\Pi_1$ | $T_2$, $\Pi_2$ | $T_3$, $\Pi_3$ |
|---|---|---|---|---|
| 21 1.1 | ∞ 1 | 25 1.1 | 22 1.1 | 21 1.1 |

$\Rightarrow N = 0,\ \ \Pi_0 = 1$

**2.** $\sigma = 0.03 < \sigma_1 \Rightarrow \alpha(0)/\sigma = 2.33,\ \ \alpha^*/\sigma = 2.89,\ \ \tau^* = 11.5$

| $T^*$, $\Pi^*$ | $T_0^*$, $\Pi_0^*$ | $T_1^*$, $\Pi_1^*$ | $T_2^*$, $\Pi_2^*$ | $T_3$, $\Pi_3$ |
|---|---|---|---|---|
| 14 2.8 | ∞ 1.8 | 29 2.3 | 22 2.5 | 19 2.6 |

$\Rightarrow N = 2,\ \ \Pi_2 = 2.5$

We indicate with asterisks the profitable replacements in the sense of initially replaceable as used in this work. The last profitable replacement is given by the first time the profit crosses the level $\alpha(0)/\sigma$. Thus in the first example the steady state policy realizes total profit $\Pi^* = 1.1$ with replacement duration $T^* = 21$, while the transitory policy total profit $\Pi_0^* = 1$ is realized without any replacements, because the equipment is scrapping durable. The second replacement, without being profitable by our definition, is eventually profitable recovering the remaining part of the profit: $1.1 - 1 = 0.1$, with replacement duration $T_1 = 25$. In the second calculation, we half the discount rate and the equipment becomes $2$ – replaceable. Now the steady state policy total profit $\Pi^* = 2.8$ is realized with replacement duration $T^* = 14$, while the transitory replacement policy total profit $\Pi_2 = 2.5$ is realized with two replacements of duration $T_2 = 22$ and $T_1 = 29$, and then $T_0 = \infty$. The 3rd replacement is not profitable by our definition.

**4. Some implications**

We summarize the basic differences between the two approaches to replacement, i.e. the steady state replacement policy and the transitory replacement policy.

1. Concerning the profit horizon, in the steady state case it is taken invariably as infinite, while in the transitory case it is determined by the parameters of the equipment and of the market. This in itself is an important finding because by adopting, for example, in capital budgeting the transitory approach to replacement we can allow endogenously for the influence of profit horizon on the selection of projects.
2. As indicated in the example, the steady state policy binds the operator to undertaking an infinite number of replacements, often with small profit in each consecutive replacement period. This implies in turn that re-investment opportunities last forever. So the steady state policy is overly restrictive because it precludes abandonment or scrapping at any time. On the contrary, the transitory replacement policy may allow the owner to realize almost the same total profit as under the steady state policy, but with few or even without any replacements, depending on the parameters: $\{r, w, \varepsilon, \sigma\}$.
3. The steady state policy predicts consistently shorter replacement durations than does the transitory replacement policy, with sometimes very small profits per replacement period. In some cases this difference may be extreme, as some steady state replaceable equipment may be classified as durable non-replaceable in the context of transitory replacement policy.

4. As the parameters change, in the case of steady state replacements the economy adjusts gradually, e.g. by changing the replacement period. In the case of transitory replacement policy, except for this smooth change, we have also sudden changes when the parameters cross certain critical values where an additional replacement policy becomes profitable or ceases to be so. Thus at some parameter values, e.g. the interest rate, we would observe a burst or a slump in the demand for replacement investment much like the "spikes" discovered in recent years by researchers studying investment at the plant level
5. How the technical characteristics of the equipment combine with the structure of the market to determine the value of parameter $\varepsilon$ is of critical importance for differentiating between the two types of replacement policies. In particular, in the simple case $\varepsilon = 1$, the equipment is disposable and the two replacement policies become indiscernible, whereas in the usual case where $\varepsilon < 1$, the value of productive services deteriorates slower than the value of equipment and the two policies may lead to substantially different economic implications at both the firm and the economy levels.[5]

## 5. Conclusions

Past studies of the optimal lifetime of assets have assumed that the duration of the re-investment process is determined by the owner of the asset on the basis of his perception on how long the investment opportunity remains profitable. As a result, whereas researchers in the field of capital budgeting have emphasized the so-called "abandonment" model, in which the owner's profit horizon is limited to a single investment cycle, economic theorists have adopted the so-called "steady state" replacement model in which reinvestments take place indefinitely at equal time intervals. Our analysis showed that these two polar classes of models constitute sub-cases of a more general model in which the owner's profit horizon is chosen jointly along with the other decision variables. Moreover, comparisons between our model with that of steady state replacement revealed that there are considerable differences. In particular, we found that: i) the two models lead to different estimates concerning the profit horizon, the duration of replacements, the timing of abandonment or scrapping, and the impact of productive capacity and market structure on service lives, as these are determined by various parameters, ii) even though the steady state replacement policy may result in higher total profit, it does so at great expense in flexibility, because the replacements are built into the model from the beginning, and iii) the transitory replacement policy seems more realistic in that the replacements are undertaken only if forced on the decision maker by decreasing profits.

Clearly, it is important to investigate which of the two models is closer to actual practice, since as indicated above they lead to very different predictions concerning the dependence of re-investment policies on the market parameters.

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## Endnotes

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[1] Users of equipment may be either owners or renters. The difference between the two is that when the owner decides to stop operations he scraps his equipment, whereas the renter is obliged to replace it. This leads to two distinct sets of replacement policies, one with terminal scrapping and the other with terminal replacement. We examine only the problem with terminal scrapping faced by owners.

[2] Note the difference between the terms "*Profitable*" and "*Profit making*". Profitable is a piece of equipment, which yields a positive sum of operating and scrapping revenue flows, whereas a piece of equipment is profit making if it yields only a positive operating revenue flow. Expressed in a different way, profitability does not necessarily imply operating profits. It may have operating losses balanced by capital gains, e.g. as happens with antiques", or more generally when we have upgrading.

[3] We assume that replacement is equivalent to scrapping the old equipment and buying new, i.e. we have ignored incentives in the form of discounts for replacements, except maybe for a fixed discount independent of the state of the equipment.

[4] Under the previous policy of *transitory replacements with terminal scrapping*, the investor is presumed to choose the number of replacements as well as their durations. In doing so he decides about the profit horizon of the investment process on the basis of the parameters of the various functions involved, including those that are determined by current and future market conditions. Unlike this policy, researchers in economics, finance and other related areas are content to adopt the *steady state policy* by assumng that the profit horizon is infinite and that replacements take place at uniform time intervals. Even though the reasons for these assumptions are not spelled out, the rationale may be traced back to Blackwell's theorem, a suitable form of which in the infinite horizon problem yields an optimal replacement policy with equal lifetimes. But following the arguments by Lehrer and Shmaya (2006) the latter may not be a better approach than the one we have proposed, since once the process of infinite equidistant replacements has started, it does not afford the investor a costless option to stop re-investing. For, if for some reason the investor has to stop re-investing, he may have to absorb such a high cost in terms of foregone profits that the steady state may turn out to be inferior to the transitory one. Of course, in any case, if initial expectations about market conditions prove erroneous at some future date, the investor can always revise his perpetual equidistant replacement policy or even decide to quit altogether because it is not an indissoluble contract.

[5] For an account of the range and the significance of the implications that arise in this regard, see Bitros (2007, 2008) and Bitros and Flytzanis (2007).