Harris and Wilson (1978) Model Revisited: The Spatial Period-doubling Cascade in an Urban Retail Model

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Harris and Wilson (1978) Model Revisited: The Spatial Period-Doubling Cascade in an Urban Retail Model*

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Abstract

Harris and Wilson (1978)'s retail location model is one of the pioneering works in regional sciences. This model considers the combination of the "fast" and "slow" dynamics to describe spontaneous spatial pattern formation processes in the economic landscape. Although the model was proposed some time ago, its comparative static (bifurcation) properties have not yet been sufficiently explored. We employ a simple analytical approach developed by Akamatsu et al. (2012) to reveal previously unknown bifurcation properties of the model in a space with a large number of locations. It is analytically shown that the spatial structure's evolutionary path exhibits a remarkable property, namely a "spatial period-doubling cascade," which cannot be observed in the popular two-location setup. Furthermore, we discuss strong linkages between the model and "new economic geography" models in terms of their model structures and bifurcation properties. These results offer a new theoretical perspective for understanding agglomeration and spatial structure evolution.

Keywords: agglomeration, multiple agglomerations, stability, bifurcation, new economic geography model
JEL Classification: R12, R13, C62, F12, F15

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1 Introduction

Economic activities are highly localized in space. For any spatial scale, such as countries, regions, or cities, unequal spatial concentrations of population, firms, or shops can be observed. Inspired by such localizations, numerous scholars, including location theorists, geographers, economists, and physicists, have attempted to explain why and how these spatial structures emerge and evolve over time. Besides underlying spatial heterogeneities or locally embedded contexts (e.g., natural advantages such as rivers or harbors, institutional regulations, and other cultural contexts), one of the basic factors fostering the emergence of spatial inequality is the existence of various forms of agglomeration economies—spatial increasing returns to scale whose origins are usually explained by mutually reinforcing externalities (see Duranton and Puga, 2004, for a survey).

Over the past three decades, researchers have emphasized the importance of the interplay between such agglomeration economies and dynamic self-organization processes in shaping spatial structure. A pioneering model in the field of geography is that of Harris and Wilson (1978) (hereafter "the HW model"). Based on a static urban retail model of Huff (1963) and Lakshmanan and Hansen (1965), the paper formulated a simple, dynamic model of agglomeration with spatial returns to scale. Their innovation was that the spatial pattern of retailers dynamically evolved to reflect their profitability. As early as the 1970s, their research emphasized that these type of models inevitably encounter problems such as (i) multiple equilibria, (ii) path dependence, that is, strong dependence on the initial condition, and (iii) catastrophic phase transitions (bifurcations), all of which are popular ideas in regional sciences today. The most striking problem is the third one: gradual changes of structural parameters (e.g., transport cost) may destabilize previously stable spatial configurations, resulting in the *emergence* of other spatial structures—including lumpy, spatially unequal agglomerations.[1]

Numerous explorations of the model's properties have since been conducted by geographers (e.g., Clarke, 1981; Wilson, 1981; Rijk and Vorst, 1983a, 1983b, to note a few). There have been, however, sizeable obstacles hindering the detailed analysis of the model. The model's three combined characteristics already mentioned prevent us from analyzing the model's intrinsic bifurcation properties. There is, fundamentally, only one possible way to analytically study quantitative properties of equilibria in the model beyond the qualitative properties [e.g., the existence or (non-)uniqueness of equilibria]: the two-location setup.

For analytical tractability, there is a long tradition in regional sciences to elucidate the essential properties of dynamic spatial agglomeration models in a two-location setup. For the HW model, the existence and uniqueness of equilibrium points are rigorously addressed in a general case by Rijk and Vorst (1983a,b). The studies, however, depend on a two-location setup to draw further concrete implications from the model, such as the critical points at which catastrophic bifurcations occur. To the authors' knowledge, no sufficient analytical studies have addressed quantitative results under a multi-locational setting beyond the two-location setup. Other studies by Wilson, such as Wilson (1981), employ a graphical trick that focuses only on a single location at once to draw useful insights into the model's bifurcation

[1] The study by Papageorgiou and Smith (1983) is a pioneering work of such an approach in economics. The study demonstrated that the emergence of spatial agglomeration can be explained by *instability* of the uniform, flat-earth equilibrium.

mechanism. Such an analysis is, however, insufficient if we are interested in the spatial system's behavior as a whole.[2]

Although the two-location setup is an effective starting point, it has several limitations. First, these models are, by definition, incapable of describing or explaining rich varieties of *polycentric* spatial concentrations of economic activities observed in the real world (Anas et al., 1998); basically, these models can express only binary states: complete dispersion or agglomeration. Second, the extent to which implications of the two-location setting can be generalized to a multi-locational version of the model is unclear. For instance, indistinguishable models in a two-location world can exhibit significantly different bifurcation patterns in a multi-locational world.[3] The two-location setup is too degenerated in the spatial dimension and lacks sufficient resolution for the modeler to determine the difference. Because there are undoubtedly many locations in the real world, not just two, heavy reliance on the two-location setup requires resolution.[4]

This paper advances the discussion one step forward. We identify the intrinsic bifurcation properties of the HW model in a *multi-location setting beyond two*. To this end, we utilize a technique proposed by Akamatsu et al. (2012), which is tailored for the analytical treatment of general spatial agglomeration models.

The contribution of this paper is twofold. First, by allowing arbitrary numbers of zones, provided that the zones are located on a symmetric circumference, we analytically follow a spatial structure's complete evolutionary path through the process of changing a structural parameter. Specifically, starting from a uniform spatial distribution of retailers, we consider the process of *improvement in the global level of transportation technologies* that is captured by a transport cost parameter. We derive the closed-form and semi-closed-form formulae for the critical points of the transport cost parameter, i.e., points where catastrophic phase transitions occur; we also identify the characteristics of these bifurcations, that is, the emergent spatial configuration after each bifurcation. We demonstrate that the spatial structure's evolutionary path in the HW model exhibits a remarkable *recursive* property called the *spatial period-doubling cascade*; after each bifurcation, the number of locations with positive mass of retailers (market centers) is reduced by half, doubling the spacing between them.[5]

Second, we address a strong connection between the HW model and the new economic geography (NEG) literature, which originated from Krugman (1991)'s core–periphery (CP) model. We make a side-by-side comparison of the bifurcation properties of the HW model and Pflüger (2004)'s NEG model.[6] In terms of similarities, Akamatsu et al. (2012) demonstrated

[2]Geographers resorted to numerical approaches once the difficulties and limitations in analytical treatments of their models were realized. The next section reviews related studies.

[3]Recently, Akamatsu et al. (2015) showed that a fundamental difference exists in agglomeration patterns of Forslid and Ottaviano (2003)'s model and Helpman (1998)'s model in a multi-region economy with more than two locations. The former admits stable polycentric agglomeration patterns, unlike the latter. Helpman's model allows only unimodal, mono-centric spatial agglomeration pattern to emerge, if any. The result questions recent empirical studies that utilize a Helpman (1998)-type model to fit the polycentric patterns of the real world (e.g., Redding and Sturm, 2008; Allen and Arkolakis, 2014). In the two-location world, however, these models are indistinguishable.

[4]See Behrens and Thisse (2007) for a thorough discussion on the "dimensionality issue" and its empirical relevance.

[5]See Definition 1 and Figure 5 in 4.5.

[6]We choose Pflüger (2004) for the sake of analytical tractability. The bifurcation properties of Pflüger

that Pflüger's (Pf's) NEG model also exhibit the spatial period-doubling cascade in a multi-region economy; for dissimilarities, we show that, depending on parameter values, the HW model can exhibit an *imperfect* spatial period-doubling cascade, unlike Pf's model. These (dis-)similarities are explained in terms of the models' properties of agglomeration and dispersion forces. The key is how these forces depend on the underlying distance structure, which is reflected in each model's *net agglomeration forces*.

The remainder of this paper is organized as follows. Section 2 reviews the related literature. Section 3 introduces the HW model. In Section 4, employing Akamatsu et al. (2012)'s approach, we study the model's bifurcation properties in the course of decreasing transportation costs. Section 5 discusses the relationship between the HW and Pflüger (2004)'s model. Section 6 offers concluding remarks.

2 Related Literature

A recent review of the HW model can be found in the study by Wilson (2008), where a generalization of Harris and Wilson (1978)'s modeling strategy is termed the *Boltzmann–Lotka–Volterra* (BLV) method.[7] The BLV method is a synthesis of the *fast dynamic* ("Boltzmann" component) and *slow dynamic* ("Lotka–Volterra" component). The fast dynamic describes the short-run spatial interaction patterns (i.e., *flows* between nodes, such as the trip distribution patterns between the origin–destination pairs) and the short-run payoff landscape, whereas the slow dynamic describes the gradual evolution in the spatial distribution of mobile factors that govern the flow generation/attraction processes (i.e., *stocks* at nodes, such as population at origins and destinations[8]). The entropy-maximizing framework, which was introduced to regional sciences by Wilson (1967) and further developed in Wilson (1970a), is one of the most unified flow-based *static* spatial interaction modeling paradigms. Accounting for flow-dependent disequilibrium evolutions of stock values at the nodes, BLV formalism adds a *dynamic* aspect to these models. The HW model is a canonical example of the BLV method.

The BLV method is sufficiently general to include a large number of modeling techniques in regional sciences as subsets. A good example is Krugman (1991)'s CP model, which opened up a new branch of economics, namely NEG. NEG models differ from classical location theory models because they are able "to combine old ingredients in a new recipe" (Ottaviano and Thisse, 2005) that employs a full-fledged general equilibrium framework; NEG models succeed in bonding up firm-level increasing returns and transportation costs between regions and factor mobility into compact, simplified general equilibrium models. From the BLV perspective, models in the NEG literature are a subclass of the BLV method whose fast dynamics are based on conventional microeconomic modeling techniques. The NEG literature also emphasizes, similar to geography, self-organization and phase transitions

(2004)'s model are essentially similar to those of other prominent NEG models.

[7]For convenience, we review the BLV method in Appendix A.

[8]Studies by Wilson and other geographers may reflect the *synergetics* and related literatures in the 1970s and 1980s (Haken, 1973). Synergetics emphasize the combination of the fast and slow dynamic and changing structural parameters in the processes of spatio-temporal pattern formation, which strongly resembles the approaches employed by regional scientists, including Wilson. Haken discusses possible applications of his theory to the field of geography (Haken, 1985).

in spatial structures. [9] The dependence on the two-region setup is more prevalent in the NEG literature because of economists' desire for clear exposition. Krugman (1991) also relied on two-location models to reinforce his ideas. There have been few analytical studies under a multi-location setup.

Compared with NEG models, the BLV method allows a considerably wide class of short-run spatial interactions because it does not require general equilibrium condition. Hence, it is likely that, depending on the model's specifications, the BLV method can still provide wide varieties of numerous new insights into the nature of spatial structures' self-organizing processes—the method could identify "how the main forces acting at each spatial scale interact to generate the space-economy" (Thisse, 2010) beyond the scope of conventional modeling techniques, such as those employed in NEG models. The power of the method is, however, yet to be fully demonstrated due to insufficient understanding of BLV models' analytical properties.

To explore more general properties of BLV models (including the HW model) beyond the two-location setup, geographers have heavily relied on computer simulations. Clarke and Wilson (1983) and Clarke and Wilson (1985) report results from their extensive numerical studies on BLV models. Systematically changing model structural parameters, the authors ran numerous experiments to determine the type of spatial structures that eventually emerge, at what point phase transitions occur, or how initial conditions affect resulting spatial patterns (see Clarke et al., 1998; Wilson, 2010; Wilson and Dearden, 2011; Dearden and Wilson, 2015, for recent explorations). Because we can easily add any "realistic" condition to numerical simulations, they enjoy great generalities, including two-dimensional space or systems with multi-class mobile agents. An interesting finding from these studies is the self-organization of hexagonal spatial agglomeration patterns, which are quite similar to those proposed in the classical central place theory of Christaller (1933) and Lösch (1940). The great generality of computer simulations has, however, inevitably limited the clarity of these studies' implications. For example, Weidlich and Haag (1987), Weidlich and Munz (1990), Munz and Weidlich (1990) also showed the emergence of hexagonal agglomeration pattern numerically using a combination of the fast and slow dynamic. However, the model is so complicated that it is practically impossible to determine why and how such a result is obtained. In fact, there has been no accepted rigorous proof for the emergence of the Christaller–LÃűsch hexagon from the BLV models. Lacking a concrete understanding of the bifurcation mechanisms that govern models in hand may considerably limit the conclusions' effectiveness as well as their potential empirical applications.

Numerical and analytical approaches for the BLV models—including the HW model—to date seem to fall on the two extremes of the trade-off between generality and clarity: the former numerical approach assumes generality, whereas the latter assumes clarity. This study is an attempt to bridge this gap with regard to "clarity".

[9] Almost no cross-references have been made between BLV and NEG-type models. Considering the strong similarities between their methodologies, this fact is surprising and seems inappropriate. Our sub-aim is to add a cross-reference to acknowledge the contributions from both sides.

3 The Model

In this section, we construct the retailer model by Harris and Wilson (1978). Although we use a slightly different interpretation on the variables for consistency with the economic literature, the essential economic intuitions and mathematical properties are unchanged.

3.1 The Fast Dynamic: Spatial Interactions

Consider a city that is discretized into K zones and associated centroids. We denote the set of these discrete zones by $\mathcal{K} \equiv \{0, 1, ..., K-1\}$. Generalized transport costs between the centroids of zones are given exogenously by $\boldsymbol{T} \equiv [t_{ij} \mid i, j \in \mathcal{K}]$. Each zone contains a continuum of retailing firms, each operating a retailing shop. The number of retailers at zone i is denoted by $h_i \geq 0$. A fixed portion of consumers reside in each zone. Consumers are assumed to inelastically buy retail goods. The total per capita consumer demand for shopping activity is an exogenously given constant O_i in each zone $i \in \mathcal{K}$. In the HW model, we are interested in the *equilibrium*—a precise definition is given later in this section—spatial distribution $\boldsymbol{h}^*$ of the retailers.

We assume that in the short run, consumers' shopping behavior is captured by a set of origin-constrained gravity equations. We denote the spatial distribution of retailers by a K-dimensional vector $\boldsymbol{h} \equiv [..., h_i, ...]^\top \in \mathbb{R}_+^K$. In the short-time scale, $\boldsymbol{h}$ is assumed to be a *fixed constant*. The consumer demand $S_{ij}(\boldsymbol{h})$ from zone i to j, measured as a cash flow, is modeled as a set of origin-constrained gravity equations

$$S_{ij}(\boldsymbol{h}) = \frac{1}{\Delta_i(\boldsymbol{h})} h_j^\alpha \exp[-\beta t_{ij}] O_i \qquad \forall i, j \in \mathcal{K} \tag{1}$$

where $\Delta_i(\boldsymbol{h})$ is a normalizing function

$$\Delta_i(\boldsymbol{h}) \equiv \sum_{k \in \mathcal{K}} h_k^\alpha \exp[-\beta t_{ik}] \qquad \forall i \in \mathcal{K}$$

that ensures the conservation of demand from each zone (i.e., $\sum_{j \in \mathcal{K}} S_{ij} = O_i$ for all i). In $S_{ij}(\boldsymbol{h})$, the parameters $\alpha, \beta > 0$ are assumed to be exogenous. The term h_i^α is interpreted as the *attractiveness* of retailers in zone i where α determines the economy of scale. When $\alpha < 1$, it represents diminishing returns with respect to scale h_i; $\alpha = 1$ represents constant returns; and $\alpha > 1$ indicates increasing returns. As will be discussed, an interesting case arises when $\alpha > 1$. On the other hand, β dictates how fast demand decreases with travel cost t_{ij}. Therefore, $\exp[-\beta t_{ij}]$, as a whole, is interpreted as impedance of interactions from zone i to j.[10] Thus, β can be interpreted as a *global transportation cost parameter* that controls spatial interaction levels in the city. Later, we will assess the effect of gradually lowering β (i.e., improvements in the transportation technology). Note that in the HW model, the price of retail goods is absent.[11] This specification of the spatial interaction $\boldsymbol{S}(\boldsymbol{h})$ between zones is equivalent to an entropy maximization procedure; it corresponds to the "Boltzmann" component of the BLV method (see Wilson, 2008, and Appendix A).

[10]Note that a change of coordinate $\beta := \log \tau$ with $\tau > 1$ yields the iceberg transport technology usually assumed in NEG models because then we can write $\exp[-\beta t_{\text{ik}}] = \tau^{-t_{\text{ik}}}$.

[11]If we allow some reduced form, the price of retail goods (and land rent) can be easily added to the model.

3.2 Short-Run Retailer Profits

The following three subsections focus on the long-run equilibrium for $\boldsymbol{h}$. We define retailer profit and then specify the incentive landscape induced by the short-run spatial interaction. First, summing up the consumers' demands from all other zones, the total revenue $S_i(\boldsymbol{h})$ of all retailers locating in zone i is given by

$$S_i(\boldsymbol{h}) \equiv \sum_{j \in \mathcal{K}} S_{ji}(\boldsymbol{h}) = \sum_{j \in \mathcal{K}} \frac{1}{\Delta_j(\boldsymbol{h})} h_i^{\alpha} \exp[-\beta t_{ji}] O_j.$$

We assume that revenue is equally distributed for all firms in the zone. In other words, after choosing zone i for the shopping destination, visiting consumers select every shop (firm) in zone i with equal probability of $1/h_i$. Then, the revenue of a single firm in zone i is given by $S_i(\boldsymbol{h})/h_i$. We also assume that in each zone, a firm must only pay a fixed entry cost $\kappa_i > 0$. Then, the profit of a firm in zone i is[12]

$$\Pi_i(\boldsymbol{h}) = \frac{S_i(\boldsymbol{h})}{h_i} - \kappa_i = \sum_{j \in \mathcal{K}} \frac{1}{\Delta_j(\boldsymbol{h})} h_i^{\alpha - 1} \exp[-\beta t_{ji}] O_j - \kappa_i. \tag{2}$$

Using the above profit function Π_i, the "profit function" of firms in the original paper by Harris and Wilson is $\hat{\Pi}_i \equiv h_i \Pi_i = S_i - \kappa_i h_i$. This original profit function is considered to be the aggregated profit of zone i in our model. In effect, the original model assumes a single large firm that operates oligopolistically a large retailing shop in each zone. These large firms are assumed to change h_i, which is interpreted as the *capacity* of the shop zone i (e.g., floorspace of the shop), in response to the profit $\hat{\Pi}_i$.

It is convenient to define the *spatial discounting matrix* $\boldsymbol{D}$ for further analysis. The spatial discounting matrix is a K-by-K matrix, whose (i, j)-th element is defined by $d_{ij} \equiv \exp[-\beta t_{ij}]$. Using $\boldsymbol{D}$, we have a useful vector-form expression of the profit function

$$\boldsymbol{\Pi}(\boldsymbol{h}) = \boldsymbol{M}^{\top} \boldsymbol{O} - \boldsymbol{\kappa} \tag{3a}$$

$$\boldsymbol{M} \equiv \mathrm{diag}[\boldsymbol{\Delta}]^{-1} \boldsymbol{D} \, \mathrm{diag}[\boldsymbol{h}]^{\alpha - 1} \tag{3b}$$

$$\boldsymbol{\Delta} \equiv \boldsymbol{D} \, \mathrm{diag}[\boldsymbol{h}]^{\alpha} \mathbf{1} \tag{3c}$$

where $\boldsymbol{O} \equiv [..., O_i, ...]^{\top}$ and $\boldsymbol{\kappa} \equiv [..., \kappa_i, ...]^{\top}$; the vector $\mathbf{1}$ denotes a vector of appropriate dimension whose elements are all one. Throughout this paper, $\mathrm{diag}[\boldsymbol{a}]$ denotes a diagonal matrix whose diagonal entries are given by the vector $\boldsymbol{a}$ and off-diagonals are all zero.

The vector-form expression using $\boldsymbol{D}$ has some utility other than its simplistic appearance. By its definition, transport cost structure $\boldsymbol{T}$ and associated impedance structure $\{d_{ij}\}$ of the spatial interaction are completely encapsulated in $\boldsymbol{D}$. In other words, $\boldsymbol{D}$ contains all the relevant information of the underlying physical space. From this, the profit function's functional form, in relation to $\boldsymbol{D}$, reveals the way in which it depends on the physical space. In our payoff function, the first term arises from spatial interactions between zones (i.e., consumers' shopping behavior). The second term is a zone-specific term that does not depend on the distance structure because (as assumed) it is a per-zone fixed cost. The former depends

[12] A problem occurs, namely, division by zero, when $\alpha < 1$ and $h_i = 0$. However, throughout this paper, we focus on the case $\alpha > 1$, which does not cause any mathematical problem.

on $\boldsymbol{D}$, whereas the latter does not. Moreover, using the vector-form expression reveals *how* the former depends on $\boldsymbol{D}$ in a macroscopic manner compared with looking directly at the element-wise equation. In Section 5, we show the true value of this vector-form expression by comparing two distinct agglomeration models.

3.3 Firms' Entry–Exit Behavior and the Equilibrium Condition

The previous subsection defined the model's payoff structure. We now formulate the equilibrium condition. We assume that the retailing market contains infinitely many potential entrants seeking profit opportunities. We assume that if the retailers' profit is non-negative in some zones, new firms enter these zones. Therefore, in the long run, retailer profits are exhausted by retailing firms' entry–exit behavior. In effect, for a spatial distribution of firms $\boldsymbol{h}$ to be stationary, we require the following zero-profit condition:

$$\begin{cases} \Pi_i(\boldsymbol{h}) = 0 & \text{if } h_i > 0 \\ \Pi_i(\boldsymbol{h}) \leq 0 & \text{if } h_i = 0 \end{cases} \qquad \forall i \in \mathcal{K} \tag{4}$$

We call a spatial distribution of firms $\boldsymbol{h}$ that satisfies the above condition an *equilibrium*.

The HW model is an open-city model (Fujita, 1989); the total number of retailers at an equilibrium is thus determined from the equilibrium condition itself. Note that the equilibrium condition is equivalently expressed in the following complementarity condition:

$$h_i \Pi_i(\boldsymbol{h}) = 0,\ h_i \geq 0,\ \Pi_i(\boldsymbol{h}) \leq 0 \qquad \forall i \in \mathcal{K} \tag{5}$$

Adding up, we have the following equation that holds at any equilibrium:

$$\sum_{i \in \mathcal{K}} h_i \Pi_i(\boldsymbol{h}) = \sum_{i \in \mathcal{K}} S_i(\boldsymbol{h}) - \sum_{i \in \mathcal{K}} \kappa_i h_i = 0$$

The first term in the middle equation is the total demand from all consumers in the city. It thus reduces to the following relation:

$$\sum_{i \in \mathcal{K}} O_i - \sum_{i \in \mathcal{K}} \kappa_i h_i = 0 \tag{6}$$

This conservation equation, which constrains the total number of firms at any equilibrium, is equivalent to the "balancing condition" of Harris and Wilson (1978).

3.4 The Slow Dynamic: Adjustment and the Stability of Equilibria

Because of the returns to scale modeled in the spatial interaction function (consumers' demand), the HW model admits multiple equilibria in a wide range of parameter values, particularly when $\alpha > 1$. Therefore, we must select the set of reasonable—under some criteria—equilibria from among them. This paper focuses on the set of *stable* equilibria under an adjustment dynamic of $\boldsymbol{h}$. The dynamic introduced here corresponds to the slow dynamic or the "Lotka–Volterra" component of the BLV method.

The adjustment dynamic $\boldsymbol{F}$ is the slow dynamic that determines the long-run spatial distribution of firms; so far, we have not discussed *how* the equilibrium condition (4) is achieved. In this paper, consistent with Harris and Wilson (1978), we assume that the spatial pattern $\boldsymbol{h}$ gradually evolves in proportion to both profit $\boldsymbol{\Pi}(\boldsymbol{h})$ and the state $\boldsymbol{h}$ itself. We assume that the time evolution of $\boldsymbol{h}$ is governed by the following dynamic:

$$\dot{\boldsymbol{h}} = \boldsymbol{F}(\boldsymbol{h}) \equiv \mathrm{diag}[\boldsymbol{h}] \cdot \boldsymbol{\Pi}(\boldsymbol{h}) = [S_i(\boldsymbol{h}) - \kappa_i h_i].$$

Note that the dynamic is consistent with the aforementioned entry–exit behavior of firms; therefore, the set of stationary points for the dynamic coincides with the set of equilibrium points: every stationary point satisfies the equilibrium condition (4) and vice versa.

To define the stability of a given equilibrium, we employ stability under small perturbations (i.e., local stability). We first define the adjustment dynamic $\boldsymbol{F}$ for the state variable $\boldsymbol{h}$, which includes all equilibrium points in the set of its stationary point. We restrict our attention to the neighborhood of an equilibrium $\boldsymbol{h}^*$. The stability of $\boldsymbol{h}^*$ is then defined in the sense of linear asymptotic stability under $\boldsymbol{F}$. The theory of dynamical systems posits that a stationary point $\boldsymbol{h}^*$ of $\boldsymbol{F}$ is linearly asymptotically stable if all the eigenvalues of the Jacobian matrix at the point, $\nabla \boldsymbol{F}(\boldsymbol{h}^*) \equiv [\partial F_i(\boldsymbol{h}^*)/\partial h_j]$, have negative real parts; $\boldsymbol{h}^*$ is linearly unstable if at least one of the eigenvalues has a positive real part. These facts facilitate the investigation of a given equilibrium's stability by analyzing the eigenvalues of the Jacobian matrix $\nabla \boldsymbol{F}(\boldsymbol{h}^*)$.

4 Lowering Transport Costs and the Evolution of Spatial Structure

4.1 Changing Structural Parameters and Bifurcations

In the previous section, we formulated the HW model; in addition, the definition of its equilibria and their stability have been introduced. *The effect of changes in exogenous structural parameters,* such as α, β, $\boldsymbol{O}$, $\boldsymbol{\kappa}$, and $\boldsymbol{T}$. This is because a change in a parameter, say, α or β, may lead to catastrophic phase transitions.

Such comparative static (bifurcation) analyses have been conducted by Harris and Wilson (1978) or Clarke (1981), and it is suggested that catastrophic phase transitions will occur in spatial patterns at some critical parameter values. Their analyses are, however, not fully systematic ones that employ graphical tricks focusing on a single zone at once. Although one can draw some qualitative conclusions from such analyses, they do not provide concrete insights into the systemic behavior of multiple zones together. Clarke and Wilson (1985), and more recently Dearden and Wilson (2015), report the result of extensive numerical assessments on the effect of changing α and β. They revealed the properties of equilibrium spatial distributions at different (α, β) pairs. Such an approach provides qualitative insights but not clear-cut conclusions.

This section, through an explicit stability analysis, unveils the previously unknown (at least in an mathematical sense) bifurcation properties of the HW model *in line with a decreasing transportation cost parameter* β.[13] As discussed in the introduction, we employ a method proposed by Akamatsu et al. (2012). The key components of the method are (i) the spatial discounting matrix $\boldsymbol{D}$, (ii) the racetrack zone system, and (iii) discrete Fourier transformation

[13]Other possible parameter paths are discussed in Appendix B.

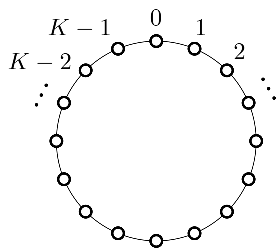


Figure 1: The racetrack economy ($K = 16$). Small white circles represent possible entry locations (i.e., zones) for a retailing firm. Lines connect neighboring zones.

(DFT). Combining these three components enables us to analytically derive eigenpairs for the Jacobian matrix $\nabla \boldsymbol{F}$ of the adjustment dynamics at the equilibrium points of interest. Hence, we can explicitly study the (in-)stability of equilibria and bifurcations.

4.2 Racetrack Zone System and the Flat-earth Equilibrium

Following the classical tradition of geography, we consider a homogeneous space over which all the underlying parameters are uniform: $O_i = O$, $\kappa_i = \kappa$. Let us further assume that all zones are equivalent, and no zone enjoys better access to consumers. Specifically, for the underlying physical space, we assume the *racetrack zone system*, or *racetrack economy*, in which all K zones are placed equidistantly on a circumference (see Figure 1). For instance, in a line segment, zones near the boundaries have fewer opportunities to access consumers compared with the central portion. In a racetrack economy, however, every zone has the same level of accessibility to other zones. This assumption may also be interpreted as an approximation of an infinite line.

In the racetrack zone system, the travel cost t_{ij} between zones i and j is defined as the shortest path length on the circumference, i.e.,

$$t_{ij} = (2\pi/K) \cdot m(i,j)$$
$$m(i,j) \equiv \min .\{|i-j|, K-|i-j|\}.$$

Under this setting, the (i,j)th entry of spatial discounting matrix $\boldsymbol{D}$, d_{ij}, is given by

$$d_{ij} = \exp[-\beta t_{ij}] = r^{m(i,j)},$$

where r denotes the *spatial discounting factor* that captures accessibility between two consecutive zones on the circumference:

$$r \equiv \exp[-\beta(2\pi/K)].$$

By definition, r is a strictly decreasing function of β. Corresponding to $\beta \in (0,\infty)$, the feasible range of r is $(0,1)$: $\beta = 0 \Leftrightarrow r = 1$ and $\beta \to \infty \Leftrightarrow r \to 0$. Because our focus is decreasing β, it corresponds to increasing r. For convenience, we use r, not β, in the remainder of the paper. We interpret $r \in (0,1)$ as a *global freeness parameter of spatial interaction between zones*.

In the racetrack zone system, the spatial discounting matrix $\boldsymbol{D}$ has a special structure called *circulant* property. A *circulant matrix* $\boldsymbol{A}$ is a square matrix in which each row is the previous row cycled forward one step; the entries in each row are a cyclic permutation of the entries in the first row $\boldsymbol{a}_0$. For example, when $K = 4$, the spatial discounting matrix $\boldsymbol{D}$ is

$$\boldsymbol{D} = \begin{bmatrix} 1 & r & r^2 & r \\ r & 1 & r & r^2 \\ r^2 & r & 1 & r \\ r & r^2 & r & 1 \end{bmatrix}.$$

Clearly, the kth row ($k = 1, 2, 3$) is obtained by rotating the first row $\boldsymbol{d}_0 = [1, r, r^2, r]$ to the right k times. The fact that $\boldsymbol{D}$ is a circulant parametrized by a single global transport cost parameter r plays a key role in the analysis of this paper; as shown below, it simplifies the stability analysis of equilibria.

We assume that the spatial distribution of retailers is initially uniform; this symmetry means that the uniform distribution of retailers is always an equilibrium. We call this spatial structure the "flat-earth equilibrium" and denote it by $\bar{\boldsymbol{h}} = [h, h, ..., h]^\top$. From the conservation condition, we have $h = O/\kappa$.

We assume that $r \approx 0$ in the first place (i.e., transport costs are quite large). Starting from $\bar{\boldsymbol{h}}$ and small r, we follow the evolution of spatial structure in line with increasing accessibility r. In other words, we start from a "corner-shop" spatial economy (Wilson and Oulton, 1983) wherein consumers travel short distances and all shopping demands are met by local retail shops in each zone. The gradual increase in r enables consumers to travel longer distances, changing the balance of the agglomeration and dispersion forces.

4.3 Emergence of Agglomeration: Destabilization of the Flat-Earth Equilibrium

We first investigate the stability of the flat-earth equilibrium $\bar{\boldsymbol{h}}$. To analytically examine the equilibrium's stability, we must derive the eigenvalues $\boldsymbol{g}$ of the Jacobian matrix for the adjustment dynamic under the configuration. The Jacobian matrix $\nabla \boldsymbol{F}(\bar{\boldsymbol{h}})$ is obtained as

$$\nabla \boldsymbol{F}(\bar{\boldsymbol{h}}) = h \nabla \boldsymbol{\Pi}(\bar{\boldsymbol{h}}) = (\alpha\kappa) \left(-\bar{\boldsymbol{D}}^2 + \hat{\alpha} \boldsymbol{I} \right) \tag{7}$$

where $\nabla \boldsymbol{\Pi}$ denotes the Jacobian matrix of profit, $\boldsymbol{I}$ denotes the identity matrix, $\hat{\alpha} \equiv 1 - \alpha^{-1}$, and $\bar{\boldsymbol{D}} \equiv \boldsymbol{D}/d$ with $d \equiv \boldsymbol{d}_0 \cdot \boldsymbol{1}$ denotes the row-normalized spatial discounting matrix [for the computation of $\nabla \boldsymbol{F}(\bar{\boldsymbol{h}})$, see Appendix C.1]. Because $\boldsymbol{I}$ and $\bar{\boldsymbol{D}}$ are both circulant matrices, $\nabla \boldsymbol{F}(\bar{\boldsymbol{h}})$ is also circulant.

Although the above formula is simple, we cannot obtain the eigenvalues and associated eigenvectors of $\nabla \boldsymbol{F}(\bar{\boldsymbol{h}})$ analytically under general, asymmetric distance structures between zones. In such a setting, stability analysis at any equilibrium point is purely a numerical task and does not provide any clear insight into the model's essential bifurcation properties. This is one of the most significant reasons why many previous analytical studies in geography and NEG have focused on spatially degenerated two-location models with great symmetry.

In the racetrack zone system, however, we can analytically obtain eigenpairs of $\nabla \boldsymbol{F}(\bar{\boldsymbol{h}})$ because of the circulant property. The eigenvalues and eigenvectors of a K-dimensional

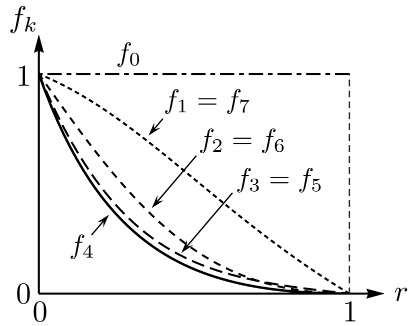


Figure 2: The eigenvalues $\boldsymbol{f}$ of $\bar{\boldsymbol{D}}$ for $K = 8$ as functions of r.

circulant matrix are obtained analytically using the DFT matrix of the same dimension. Specifically, we can easily show the following relation (see Appendix C.1):

$$\mathrm{diag}[\boldsymbol{g}] = (\alpha\kappa)\left(-\,\mathrm{diag}[\boldsymbol{f}]^2 + \hat{\alpha}\,\mathrm{diag}[\mathbf{1}]\right) \tag{8}$$

where $\boldsymbol{g}$ and $\boldsymbol{f}$ denote the eigenvalues of $\nabla \boldsymbol{F}(\bar{\boldsymbol{h}})$ and $\overline{\boldsymbol{D}}$, respectively. Comparing equations with (7), we notice that the functional relation between vectors $\boldsymbol{g}$, $\boldsymbol{f}$, and $\mathbf{1}$ are exactly the same as that between $\nabla \boldsymbol{F}$, $\overline{\boldsymbol{D}}$, and $\boldsymbol{I}$.

To obtain $\boldsymbol{g}$, we need to know the eigenvalues $\boldsymbol{f}$ of $\overline{\boldsymbol{D}}$ in a racetrack system. We have the following characterization of $\boldsymbol{f}$, which is illustrated in Figure 2 for the case $K = 8$.

Lemma 1 (Akamatsu et al. (2012), Lemma 4.2)**.** All eigenvalues $\boldsymbol{f} \equiv [f_0, f_1, ..., f_{K-1}]$ of $\overline{\boldsymbol{D}}$ are obtained analytically by the DFT of the first row $\boldsymbol{d}_0$; the kth eigenvector associated with f_k is the kth column vector $\boldsymbol{z}_k$ of the DFT matrix. Furthermore, the following hold true:

(a) First, $f_0 = 1$. For the others, each f_k is a monotonically decreasing function of spatial discounting factor r and takes a value of $(0, 1)$.

(b) Assume that K is even. Then, the minimal eigenvalue $\min_k\{f_k\}$ is always f_M ($M \equiv K/2$). The minimal eigenvalue f_M and the associated eigenvector is

$$f_M = \begin{cases} C(r) & \text{if } K = 2 \\ \{C(r)\}^2 & \text{if } K \geq 4 \end{cases} \tag{9}$$

$$\boldsymbol{z}_M \equiv [(-1)^j] = [1, -1, \ldots]^\top. \tag{10}$$

where $C(r) \equiv (1 - r)/(1 + r)$.

Summarizing the above discussion, we have the following lemma concerning the eigenvalues $\boldsymbol{g}$ of the Jacobian matrix of the dynamic at flat-earth equilibrium $\bar{\boldsymbol{h}}$:

Lemma 2. Let $\boldsymbol{g}$ be the eigenvalues of the Jacobian matrix of the adjustment dynamic, $\nabla \boldsymbol{F}(\bar{\boldsymbol{h}})$, at the flat-earth equilibrium $\bar{\boldsymbol{h}}$ in the economy. Then, $\boldsymbol{g}$ is

$$g_k = (\alpha\kappa)\cdot G(f_k) \tag{11a}$$

$$G(f) = -f^2 + \hat{\alpha} \tag{11b}$$

The kth eigenvector associated with g_k is the kth column vector $\boldsymbol{z}_k$ of the DFT matrix.

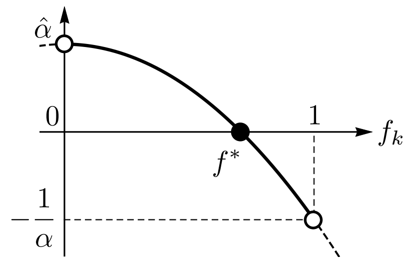


Figure 3: The net agglomeration force $G(f_k)$ of the HW model ($\alpha > 1$).

The eigenvalues $\boldsymbol{g}$ of the Jacobian matrix have many economic meanings; they can be interpreted as the *net agglomeration force* at work in the HW model. Note that the sign of g_k solely depends on the sign of $G(f_k)$. The function $G(f)$ captures the agglomeration and dispersion forces of the HW model. The first term of $G(f)$ is always negative; it is a dispersion force arising from competition between retailers at different locations, which is modeled in the gravity equations of consumer demand. The second term, $\hat{\alpha}$, captures the scale effect of α. It is positive if $\alpha > 1$, zero if $\alpha = 1$, and negative if $\alpha < 1$. When $\alpha > 1$, the second term is interpreted as an agglomeration force. Thus, $G(f)$ represents net agglomeration force in the sense that it is the pure agglomeration force (second term) minus the pure dispersion force (first term).

We see that if $\alpha \leq 1$, the function $G(f)$ always takes a negative value for all $f \in (0, 1)$; this reflects the fact that when $\alpha \leq 1$ no agglomeration forces exist in the model. The flat-earth equilibrium $\bar{\boldsymbol{h}}$ is always asymptotically stable at any r if $\alpha \leq 1$. Because we are interested in the process of symmetry breaking and self-organization, the remainder of the paper focuses on the case $\alpha > 1$ (Figure 3).

In the remainder, we assume that K is even and investigate the bifurcation from $\bar{\boldsymbol{h}}$ using Lemma 1 and Lemma 2. When transport costs are high, the value of f_k is near 1 (Figure 2). We can easily verify from Figure 3 that g_k is negative for all k; the flat-earth equilibrium is therefore stable. Since f_k is decreasing in r and $G(f)$ is decreasing in f, it follows that g_k is increasing in r. As long as $\alpha > 1$, the destabilization of $\bar{\boldsymbol{h}}$ will result. The first eigenvector that becomes positive is, of course, the maximal eigenvalue among $\boldsymbol{g}$. The eigenvalue is always g_M; this follows from the fact that f_M is always the minimal among $\boldsymbol{f}$ (Figure 2 and Figure 3). Thus, when r increases, the bifurcation from the flat-earth equilibrium occurs at $r^{(0)}$ that satisfies $G(f_M(r^{(0)})) = 0$. This bifurcation moves the spatial distribution of retailers in the direction of $\boldsymbol{z}_M$ in (10).

From the above discussion, we obtain a complete characterization of the first bifurcation from the flat-earth equilibrium as follows:

Proposition 1 (The first bifurcation)**.** Assume that $\alpha > 1$ and K is even.[14] We start from a high level of transportation costs ($r \approx 0$) in which the flat-earth equilibrium $\bar{\boldsymbol{h}}$ is stable.

[14]This assumption is a canonical one to utilize Lemma 1 (b). When we assume that for an odd K, the first bifurcation's properties will be changed; it is of course not so simple as Proposition 1. However, the qualitative properties of the emergent spatial pattern after the bifurcation will be similar: the average distance between market centers approximately doubles after the bifurcation. Showing this is a numerical task that is beyond the focus of this paper.

Further, we consider a steady decrease in transportation costs, i.e., a steady increase in r. Then the following hold true:

(a) $\bar{\boldsymbol{h}}$ becomes unstable at $r^{(0)} \equiv f_M^{-1}(\sqrt{\hat{\alpha}})$.[15]

(b) The bifurcation at $r^{(0)}$ leads to a spatial structure

$$\boldsymbol{h} = \bar{\boldsymbol{h}} + \delta \boldsymbol{z}_M = [h+\delta, h-\delta, h+\delta, h-\delta, \dots]^\top$$

with $\delta \in (0, h)$, in which the number of retailers in alternate regions increase.

(c) $r^{(0)}$ decreases as α increases.

Proof. For a complete proof including the exact formula of $r^{(0)}$, see Appendix C.2. □

Figure 4 illustrates the flat-earth equilibrium and resulting pattern after the first bifurcation. Intuitively, the increase in α (i.e., stronger agglomeration force) leads to faster agglomeration in line with increasing r.

The maximality of g_M has an intuitive interpretation. The following relation holds true by definition:

$$g_k \cdot \boldsymbol{z}_k = \nabla \boldsymbol{F}(\bar{\boldsymbol{h}}) \cdot \boldsymbol{z}_k = h \nabla \boldsymbol{\Pi}(\bar{\boldsymbol{h}}) \cdot \boldsymbol{z}_k.$$

Each element of $\nabla \boldsymbol{\Pi}(\bar{\boldsymbol{h}}) \cdot \boldsymbol{z}_k$ represents increased profit for a retailer when the retailers relocate in the direction of $\boldsymbol{z}_k$; thus, each g_k is a marginal increase in total profit in each zone. If g_k is positive, when the state slightly changes in the direction of $\boldsymbol{z}_k$, zones with increased numbers of retailers gain more profit and zones with decreased numbers of retailers lose. This encourages further deviation in the direction of $\boldsymbol{z}_k$. The opposite holds true if g_k is negative; when $g_k < 0$, a deviation in the direction $\boldsymbol{z}_k$ reduces profit at *every* location, thereby producing inertia in retail firms' location choice. When r increases, the eigenvalue g_M is the first to vanish; its maximality means that the inertia in this direction is weakest.

A remaining question is *why* inertia in the $\boldsymbol{z}_M$ direction is the weakest. Given that the agglomeration force is constant and does not depend on the transport cost parameter, the bifurcation, or endogenous agglomeration, is induced by some decline in the dispersion force. In the HW model, the dispersion force arises from the competition over fixed consumer demand. Lower transport costs enable consumers to travel longer distances, thus reducing spatial isolation, which is reflected by the decrease in $\boldsymbol{f}$. Consequently, the *latent* market territory for which each market center can serve consumer demands gradually expand as r increases, strengthening competition between market centers. Possible entrance locations are given by the discrete, equidistantly placed zones; actual expansion of the market territory occurs when this latent market territory exactly doubles, leading the alternating market centers to disappear. This transition is expressed by the direction $\boldsymbol{z}_M$. This result is reminiscent of the arguments of classical central place theory in which market centers are equidistantly spaced at an equilibrium.

[15] The critical value $r^{(0)}$ is analytically obtained. Yet, the exact expression of f_M depending upon K (see Lemma 1), we omit the exact formula to avoid unnecessarily complicating the presentation.

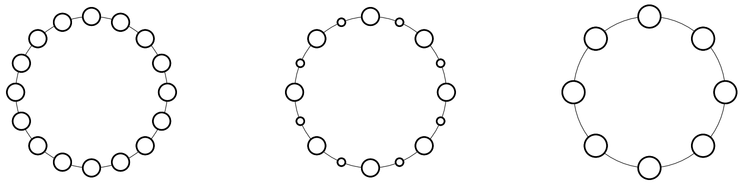


Figure 4: The spatial patterns relevant to the first bifurcation ($K = 16$).

4.4 The Stability of $K/2$-Centric Pattern and the Second Bifurcation

After the first bifurcation from the flat-earth equilibrium, δ increases rapidly in accordance with the increase in r. We expect that the spatial configuration will converge to another symmetric equilibrium pattern in which only alternate zones in $\mathcal{K}$ have retailers. We denote the $K/2$-centric spatial configuration by $\boldsymbol{h}^{(1)}$ Figure 4 (c). The bracketed superscript indicates the extent of the symmetry breaking inducted by increasing r. We define $\boldsymbol{h}^{(1)}$ by

$$\boldsymbol{h}^{(1)} \equiv \left[2h, 0, 2h, 0, \ldots, 2h, 0\right]^\top$$

in which the even-numbered zones have twice as many firms as the flat-earth equilibrium while the other zones have none because of the preceding bifurcation. It is immediate that the spacial configuration satisfies the equilibrium condition (4) and the conservation equation (6). Compared with the "corner-shop" interpretation of $\bar{\boldsymbol{h}}$, the spatial pattern $\boldsymbol{h}^{(1)}$ may be interpreted as a city with slight heterogeneity because some zones lack retailers.

This outcome has an apparent symmetry and resemblance to the flat-earth equilibrium; ignoring zones without retailers yields a "flat-earth" equilibrium with the number of zones reduced to $K/2$, doubling the distance ("spatial period") between them. The similarity encourages us to hypothesize that the bifurcation from $\boldsymbol{h}^{(1)}$, analogous to the bifurcation from $\bar{\boldsymbol{h}}$, further halves the number of market centers and leads to a $K/4$-centric pattern:

$$\boldsymbol{h}^{(2)} \equiv [4h, 0, 0, 0, 4h, 0, 0, 0, \ldots, 4h, 0, 0, 0]^\top.$$

of course provided that K is a multiple of 4. This intuition can be proven by analytically deriving the eigenvalues of $\nabla \boldsymbol{F}(\boldsymbol{h}^{(1)})$ and the associated eigenvectors; there is again a single function $G^{(1)}(r)$ that represents the net agglomeration force at this configuration. We have the following proposition:

Proposition 2 (The second bifurcation)**.** Assume that K is a multiple of 4. Let $\boldsymbol{g}$ be the eigenvalues of $\nabla \boldsymbol{F}(\boldsymbol{h}^{(1)})$. Then,

(a) There is a function $G^{(1)}(r)$ of r that satisfies $\operatorname{sgn}(\max_k g_k) = \operatorname{sgn}(G^{(1)}(r))$.

(b) If $\alpha < 2$, then $\boldsymbol{h}^{(1)}$ is stable at sufficiently small r; if $\alpha > 1$, then $\boldsymbol{h}^{(1)}$ is unstable at sufficiently large r.

(c) Consider the steady increase in r. If $1 < \alpha < 2$, then $\boldsymbol{h}^{(1)}$ becomes unstable at the critical value $r^{(1)}$ that is given as the unique analytical solution for the equation $G^{(n)}(r) = 0$.

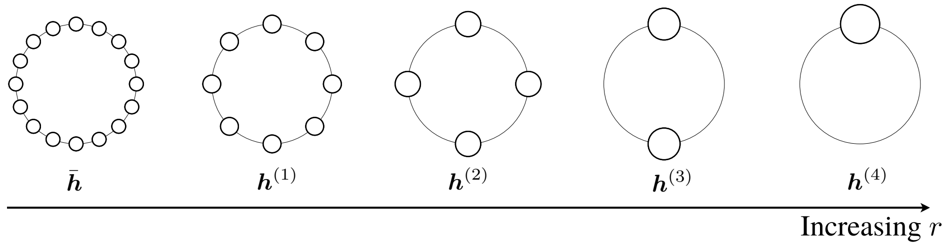


Figure 5: The spatial period-doubling cascade ($K = 16$).

(d) The bifurcation leads to a spatial pattern

$$\boldsymbol{h} = \boldsymbol{h}^{(1)} + \delta\{\boldsymbol{h}^{(2)} - \boldsymbol{h}^{(1)}\}$$

with $\delta \in (0, 1)$, in which retailers in alternate market centers increase in number.

(e) $r^{(1)}$ decreases as the increasing return parameter α increases.

Proof. See Appendix C.3 for the proof and the exact formulae of $G^{(1)}(r)$ and $r^{(1)}$. □

The condition $\alpha < 2$ requires that the increasing return effect should not be too strong so that $\boldsymbol{h}^{(1)}$ can be a stable equilibrium for some r; otherwise, $\boldsymbol{h}^{(1)}$ cannot be stable at any transport cost level r. As discussed for the first bifurcation, the condition $\alpha > 1$ requires that the increasing return effect should *exist* so that $\boldsymbol{h}^{(1)}$ becomes unstable according to an increase in r.

4.5 Recursive Emergence of Spatial Structure: The Spatial-Period Doubling Cascade

Our analysis so far has shown that under suitable values of α, the first and second bifurcation lead to a spatially alternate pattern, each time reducing the number of the market centers (zones with retailers) by half: $K \to K/2 \to K/4$. This property implies that this recursive bifurcation process will continue. For clarity in presentation, we assume in the following that $K = 2^J$ for some positive integer J.[16] We define the *spatial period-doubling cascade* as follows:

Definition 1 (The spatial period-doubling cascade)**.** We consider a steady decrease of transport costs in a racetrack zone system with $K = 2^J$ for some positive integer J. A series of bifurcations from the flat-earth equilibrium is called the *spatial period-doubling cascade* when (a) every bifurcation exactly halves the number of market centers, doubling the spacing between neighboring ones, and (b) the recursive process continues until a mono-centric pattern is attained.

[16]If we assume that K is not a power of 2 (e.g., $K = 12$), we may observe some stable equilibria without rotational symmetries (but with some reflectional symmetry). In such a case, however, bifurcation behavior will be much complicated and a numerical bifurcation analysis should be conducted, as Ikeda et al. (2012a) did for Krugman (1991)'s model with $K = 6$. See also Footnote 14 for the effect of assuming an odd K.

Figure 5 depicts this bifurcation for $K = 2^4 = 16$. The potential existence of the spatial period-doubling cascade path is proved for general spatial agglomeration models using group-theoretic bifurcation theory (Ikeda et al., 2012a, Proposition 5). Its actual existence depends on individual models and parameter values. Akamatsu et al. (2012) has shown that a multi-regional extension of the Pflüger (2004)'s NEG model actually exhibits the spatial period-doubling cascade for a wide range of parameters. Below, we prove that the HW model also satisfies this property with suitable values of α. In Section 5, we will discuss how this similarity stems from the properties of the net agglomeration forces operating in these models.

To verify the occurrence of the spatial period-doubling cascade, we first derive the eigenvalues $\boldsymbol{g}$ of $\nabla \boldsymbol{F}$ at the $K/2^n$-centric equilibrium $\boldsymbol{h}^{(n)}$ $(n = 1, 2, ..., J-1)$ and examine the resulting bifurcation. The equilibrium pattern $\boldsymbol{h}^{(n)}$ is[17]

$$\boldsymbol{h}^{(n)} \equiv \Big[\underbrace{2^n h, 0, 0, \ldots, 0}_{2^n}, \underbrace{2^n h, 0, 0, \ldots, 0}_{2^n}, \ldots, \underbrace{2^n h, 0, 0, \ldots, 0}_{2^n}\Big]^\top.$$

The following proposition characterizes the bifurcation from the $K/2^n$-centric equilibrium $\boldsymbol{h}^{(n)}$:

Proposition 3 (The bifurcation from $\boldsymbol{h}^{(n)}$)**.** Assume that $K = 2^J$ with some positive integer J. Let $\boldsymbol{g}$ be the eigenvalues of $\nabla \boldsymbol{F}(\boldsymbol{h}^{(n)})$. Then,

(a) There is a function $G^{(n)}(r)$ of r that satisfies $\operatorname{sgn}(\max_k g_k) = \operatorname{sgn}(G^{(n)}(r))$.

(b) If $\alpha < 2^n$, then $\boldsymbol{h}^{(n)}$ is stable at sufficiently small r; if $\alpha > 1$, then $\boldsymbol{h}^{(n)}$ is unstable at sufficiently large r.

(c) Consider the steady increase in r. If $1 < \alpha < 2^n$, then $\boldsymbol{h}^{(n)}$ becomes unstable at the critical value $r^{(n)}$ that is given as the unique solution for the equation

$$G^{(n)}(r) = 0 \tag{12}$$

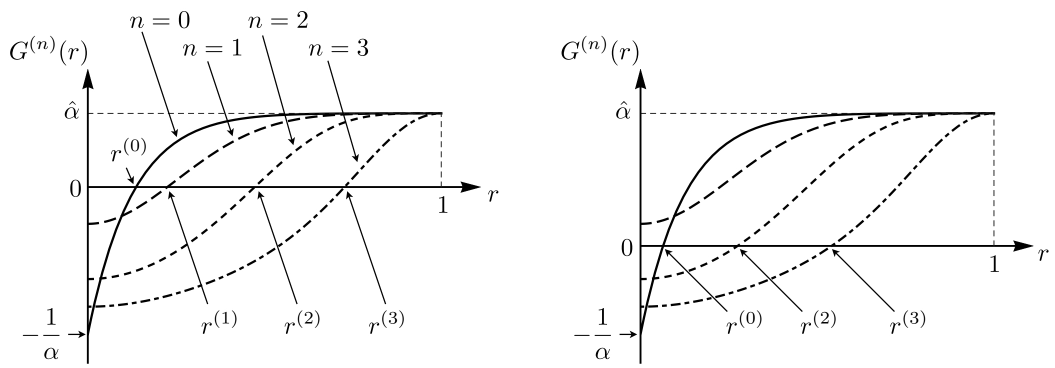
(d) The bifurcation leads to a spatial pattern

$$\boldsymbol{h} = \boldsymbol{h}^{(n)} + \delta\{\boldsymbol{h}^{(n+1)} - \boldsymbol{h}^{(n)}\}$$

with $\delta \in (0, 1)$, in which the number of retailers in the alternate market centers increase.

Proof. See Appendix C.5. □

In Proposition 3, the function $G^{(n)}(r)$ is interpreted as representing the largest net agglomeration force at $\boldsymbol{h}^{(n)}$. We see that Proposition 3 is a generalization of Proposition 2, although in this case, the equation is not analytically solvable for $n \geq 2$. The condition $1 < \alpha < 2^n$ again ensures that the bifurcation from $\boldsymbol{h}^{(n)}$ occurs. Proposition 3 indicates that any bifurcation from a symmetric equilibrium $\boldsymbol{h}^{(n)}$ is of a period-doubling nature, leading to another symmetric equilibrium $\boldsymbol{h}^{(n)} + 1$. Basic economic intuitions are similar to those of the



(a) $1 < \alpha < \bar{\alpha}$

(b) $2 < \alpha < 2^2$

Figure 6: All $G^{(n)}(r)$ $(n = 0, 1, 2, 3)$ and the critical points $r^{(n)}$ for $K = 16$.

first bifurcation; it reflects gradual expansion of market territory and increasing competition between market centers.

We now have the (semi-)closed-form formulae of the critical points $\{r^{(n)}\}$. We have yet to show, however, under what condition $\boldsymbol{h}^{(n)} + 1$ actually emerges after the bifurcation from $\boldsymbol{h}^{(n)}$. Specifically, $\boldsymbol{h}^{(n)} + 1$ can be already unstable before $r^{(n)}$ is attained. If the critical values $\{r^{(n)}\}$ are increasing in n, each bifurcation reduces the number of market centers (zones with positive numbers of retailing firms) by half—the successive emergence of $\boldsymbol{h}^{(n)}$ is ensured. The required condition for the spatial period-doubling cascade is

$$r^{(n)} < r^{(n+1)} \qquad \forall n \leq J - 2 \tag{13}$$

To verify the property, we should at least require $r^{(0)} < r^{(1)}$, provided that these bifurcations occur (i.e., $1 < \alpha < 2$). The inequality is analytically solvable; we can derive the threshold value $\overline{\alpha} \in (1, 2)$ such that if $\alpha < \overline{\alpha}$, then $r^{(0)} < r^{(1)}$ holds true. Given $\overline{\alpha}$, we have the following proposition:

Proposition 4 (The occurrence of the spatial period-doubling cascade)**.** Let $\overline{\alpha}$ be the solution for $r^{(0)}(\alpha) = r^{(1)}(\alpha)$. Assume that $1 < \alpha < \overline{\alpha}$ and $K = 2^J$ with some integer $J \geq 2$.[18] Then, the HW model exhibits the spatial period-doubling cascade.

Proof. See Appendix C.8 for the proof and analytical formula for $\bar{\alpha}$. □

The proof of Proposition 4 in Appendix C.8 is seemingly tedious, but the intuition is straightforward. When $1 < \alpha < \overline{\alpha}$ holds true, $1 < \alpha < 2^n$ holds for all $n = 1, 2, ..., J-1$; thus, all the relevant bifurcations will occur. The remaining task is to show that the relations hold true. Because $r^{(n)}$ are the solutions for $G^{(n)}(r) = 0$, we compare $G^{(n)}(r)$ for different n, as Figure 6 illustrates. In Figure 6a, we can graphically see that $r^{(n)} < r^{(n)} + 1$ for all n if $r^{(0)} < r^{(1)}$.

[17]We easily verify that these spatial configurations actually satisfy the equilibrium condition.

[18]Note that we exclude $K = 2$ because then the spatial period-doubling cascade trivially occurs.

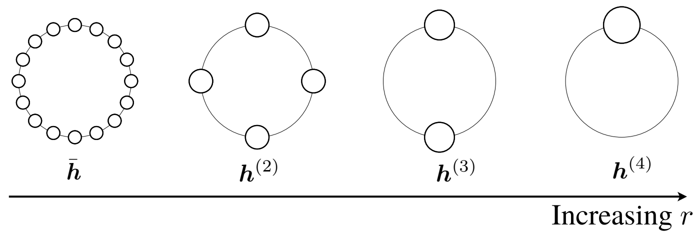


Figure 7: The spatial period-doubling cascade with "skipping" ($K = 16$).

4.6 The Spatial Period-Doubling Cascade with "Skipping"

In the previous subsection, we proved that when $1 < \alpha < \overline{\alpha}$, the HW model exhibits the spatial period-doubling cascade, which was previously addressed in Pf's NEG model. When $\alpha > \overline{\alpha}$, the HW model indicates somewhat different bifurcation properties compared with the simple spatial period-doubling cascade. We have the following proposition:

Proposition 5 (The spatial period-doubling cascade with skipping)**.** Assume that $\alpha \geq \bar{\alpha}$ and $K = 2^J$ with some positive integer $J \geq 2$.[19] Then, the first bifurcation from the flat-earth equilibrium results in the emergence of $\boldsymbol{h}^{(\hat{n})}$ with $\hat{n} \geq 2$. A further decrease in transport cost triggers a series of spatial period-doubling bifurcations originating from $\boldsymbol{h}^{(\hat{n})}$.

Proof. See Appendix C.9. For the intuition, Figure 6b would suffice. □

Figure 6b illustrates the patterns of $G^{(n)}(r)$ in such a case. Figure 7 illustrates the evolution of spatial structure under the same setting as Figure 6b. In this figure, $\boldsymbol{h}^{(1)}$ is already unstable at $r^{(0)}$; destabilization of $\bar{\boldsymbol{h}}$ results in emergence of a deformed spatial configuration $\boldsymbol{h}^{\times(1)} \equiv \boldsymbol{h}^{(1)} + \delta\{\boldsymbol{h}^{(2)} - \boldsymbol{h}^{(1)}\}$ with $\delta \in (0, 1)$. As r increases, $\boldsymbol{h}^{\times(1)}$ converges to $\boldsymbol{h}^{(2)}$, which becomes the origin and a spacial period-doubling cascade starts from $\boldsymbol{h}^{(2)}$. Therefore, a symmetric $K/2$-centric pattern $\boldsymbol{h}^{(1)}$ is "skipped" in this evolutionary path.

Under the two-zone setup (when $K = 2$), "skipping" of a possible equilibrium pattern in the line of decreasing transport costs does not occur because we have only two alternative equilibrium patterns: the flat-earth equilibrium $\bar{\boldsymbol{h}} \equiv [h, h]^\top$ and the so-called core–periphery equilibrium $\boldsymbol{h}^{(1)} \equiv [2h, 0]^\top$. Recalling the "spatial resolution" issue of two-location models discussed in the introduction, the "skip" in the HW model should be a concrete example that indicates the insufficiency of analytical approaches resorting to the two-location setup. Although qualitative bifurcation properties of models would be, perhaps, similar in the multi-location world, hidden rich implications that cannot be delineated in the two-location world probably exist.

4.7 Sustainability of Agglomeration in the HW model

We have so far focused on the bifurcation properties in the direction of monotonically increasing r (decreasing transport cost). We are also interested in the effect of *increasing* transportation costs after an agglomeration is established—or *sustainability* of a given equilibrium. Typical results from the NEG literature reveal that when transport costs gradually

[19]We require $J \geq 2$ because there should at least one equilibrium pattern that will be "skipped".

increase again, a critical point, often termed a *sustain point*, exists for these costs at which once-established agglomerations are no longer stable. This property is illustrated by a tomahawk diagram in Krugman (1991) where a hysteresis is present.

In the HW model, however, such sustain points are absent. More precisely, an established equilibrium, for example, $\boldsymbol{h}^{(n)}$, is always asymptotically stable in the whole range $r \in (0, r^{(n)})$. Therefore, when starting from a value of r where $\boldsymbol{h}^{(n)}$ is stable with decreasing r (increasing transport costs), no destabilization of the equilibrium occurs—$\boldsymbol{h}^{(n)}$ is fully sustained. Once an agglomeration is formed, no path leads to escape from the agglomeration. This extreme hysteresis property reflects the characteristics of the retailing activity's consumer demand function. Consumer demand for a firm completely vanishes if the number of firms in the location equals zero. Potential entrants are thus prevented from locating in the zone because profit there is negative due to the fixed entry cost κ. Therefore, once a zone is abandoned by retailers, it will never obtain a new retailer regardless of the extent of transport costs, provided that it is finite.[20]

4.8 Numerical Examples

Some numerical examples of spatial structural evolution in the HW model can illustrate the theoretical results. Figure 8 depicts typical examples of bifurcation diagrams for different values of α. The number of zones $K = 16$. Values of α are selected to cover all essential cases that emerge: (1) relatively small α where the spatial period-doubling cascade occurs, (2) relatively large α where the spatial period-doubling cascade with skipping occurs, and (3) an intermediate case.

In each sub-figure in Figure 8, black thick lines represent the number of firms located at the largest market center at stable equilibria for different r. Placed above are the spatial agglomeration patterns that will emerge when r gradually increases from $r = 0$ to $r = 1$. Each critical value of $r^{(n)}$ for the bifurcation from $\boldsymbol{h}^{(n)}$, if any, is obtained by solving (12). As discussed, the HW model exhibits an extreme hysteresis property: every symmetric agglomeration pattern $\boldsymbol{h}^{(n)}$ is asymptotically stable—or "sustained"—in the whole range $(0, r^{(n)})$.

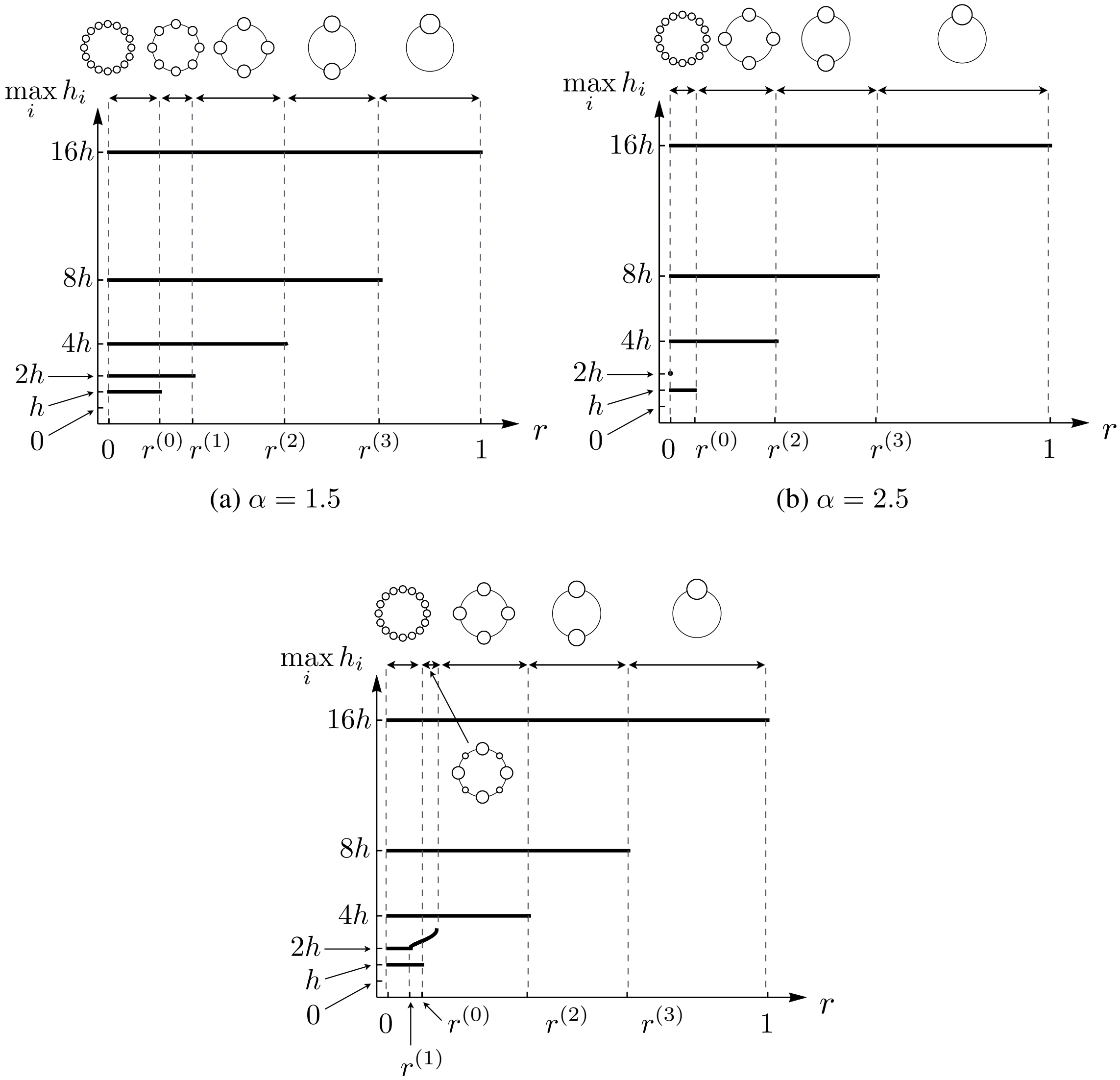
Figure 8a depicts the case $\alpha = 1.5$. The value satisfies the condition $1 < \alpha < \bar{\alpha}$. Therefore, from Proposition 4, we expect that spatial period-doubling cascade occurs. The spatial structure's evolutionary path follows the pattern

$$\bar{\boldsymbol{h}} \to \boldsymbol{h}^{(1)} \to \boldsymbol{h}^{(2)} \to \boldsymbol{h}^{(3)} \to \boldsymbol{h}^{(4)}$$

and each bifurcation at the critical values $r^{(n)}$ $(n = 0, 1, 2, 3)$ doubles the number of firms in each market center while halving the number of market centers.

Figure 8b shows the case $\alpha = 2.5$. This is an example of the spatial period-doubling cascade *with skipping* because $\alpha > \bar{\alpha}$. Moreover, $\alpha > 2$ holds true. Accordingly, the eight-centric pattern $\boldsymbol{h}^{(1)}$ cannot be a stable equilibrium for essentially all values of r, except for the extreme case $r = 0$. The first n that satisfies $r^{(0)} < r^{(n)}$ is $n = 2$. The emergent spatial

[20]Some readers may think that this extreme path-dependence is an implausible or even unnatural property for an agglomeration model. If undesirable, an extra spatial interaction term can be added such that there is always positive consumer demand for every location, as in many NEG models.



(a) $\alpha = 1.5$

(b) $\alpha = 2.5$

(c) $\alpha = 1.95$

Figure 8: Typical examples of the bifurcation diagram for different valuesof α ($K = 16$).

structure after the first bifurcation is, hence, the four-centric pattern $\boldsymbol{h}^{(2)}$. In the course of increasing r, the path is

$$\bar{\boldsymbol{h}} \to \boldsymbol{h}^{(2)} \to \boldsymbol{h}^{(3)} \to \boldsymbol{h}^{(4)}$$

in which $\boldsymbol{h}^{(1)}$ is completely skipped compared with the first case.

Figure 8c presents the case $\alpha = 1.95$. This case is an intermediate case between (a) and (c). In this case, $\alpha > \bar{\alpha}$ holds, but α is not large enough (i.e., $\alpha < 2$) to destabilize $\boldsymbol{h}^{(1)}$ for all r. After the first bifurcation, a deformed eight-centric pattern emerges. Thus the evolutionary behavior is quite similar to the spatial period-doubling cascade, except that $\boldsymbol{h}^{(1)}$ itself is skipped in the line of increasing r; after the first bifurcation, a deformed eight-centric pattern emerges.

5 Connection to the New Economic Geography Models

Krugman (1991)'s CP model proposed one of the most successful micro-founded modeling paradigms of economic agglomeration. The "Dixit–Stiglitz, icebergs, evolution, and the computer" modeling technique (Fujita et al., 1999) introduced by the paper have opened up a fruitful field called NEG today. So far this paper has demonstrated that the HW model exhibits the spatial period-doubling cascade. As we summarize 5.1 below, Akamatsu et al. (2012) showed that Pflüger (2004)'s NEG model also exhibit this quality. We discuss the source of this similarity in 5.2.

5.1 Pflüger's NEG Model and Its Bifurcation Properties

This subsection briefly introduces a NEG model by Pflüger (2004) (the Pf model) for use in later comparisons. The Pf model enjoys analytical tractability compared with Krugman's original CP model yet preserves basic properties such as the *forward linkage* (price index effect) and *backward linkage* (demand effect).

Let $\boldsymbol{h}$ be the spatial distribution of mobile consumers, whose ith element represents the number of mobile consumers at location i. The total number of mobile consumers is an exogenously given constant H (i.e., closed economy). In the *short-run*, $\boldsymbol{h}$ is fixed as mobile consumers cannot change their location. Given $\boldsymbol{h}$, the general equilibrium condition determines the indirect utility level v_i of consumers at each location i. If fortunate, we have a closed-form expression of the indirect utility $\boldsymbol{v}(\boldsymbol{h})$ as a function of $\boldsymbol{h}$:

$$\boldsymbol{v}(\boldsymbol{h}) = \frac{\mu}{\sigma - 1} \log[\boldsymbol{D}^\top \boldsymbol{h}] + \frac{\mu}{\sigma} \boldsymbol{M}^\top (\boldsymbol{h} + l\boldsymbol{1}) \tag{14a}$$

$$\boldsymbol{M} \equiv \{\text{diag}[\boldsymbol{D}^\top \boldsymbol{h}]\}^{-1} \boldsymbol{D} \tag{14b}$$

where $\mu > 0$, $\sigma > 1$, $l > 0$ are constants interpreted as consumers' expenditure ratio on manufacturing goods, elasticity of substitution of manufacturing goods, and number of immobile consumer at each location, respectively.[21] The second term in (14a) is similar to the first term of the profit function (3a) of the HW model because both reflect total inflow at each location that is induced by gravity equations of trade flows.

In the *long-run*, the mobile consumers are free to choose a residing location. The long-run equilibrium condition is the standard one

$$\begin{cases} V = v_i(\boldsymbol{h}) & \text{if } h_i > 0 \\ V \geq v_i(\boldsymbol{h}) & \text{if } h_i = 0 \end{cases} \qquad \forall i \in \mathcal{K}$$

where V an endogenously determined equilibrium utility level. For the long-run adjustment dynamic, NEG models often assume the so-called *replicator dynamic*:

$$\dot{h}_i = h_i \cdot \{v_i(\boldsymbol{h}) - \bar{v}(\boldsymbol{h})\}$$

[21]For the details and derivation of the vector-matrix representation of the multi-regional Pf model, consult Akamatsu et al. (2012). Note that $\log[\boldsymbol{a}]$ for a vector $\boldsymbol{a}$ with strictly positive elements is defined as a component-wise $\log[\cdot]$ operation: $\log[\boldsymbol{a}] \equiv [\log[a_i]]$.

where $\bar{v}(\boldsymbol{h}) \equiv H^{-1}\sum_i h_i v_i(\boldsymbol{h})$ denotes the average indirect utility level.

The above construction indicates that the model belongs to the BLV formalism category. The short-run equilibrium condition in the Pf model corresponds to the "Boltzmann" component, whereas the long-run evolutionary dynamic corresponds to the "Lotka–Volterra" component. The most characteristic modeling philosophy of NEG models in general, as emphasized by many scholars including Krugman (e.g., Krugman, 2011), is that these models' "Boltzmann" component is equipped with a full-fledged general equilibrium model according to a combination of the Dixit–Stiglitz model and the iceberg transportation cost. As a reduced-form model, however, NEG models comfortably fall into the general framework of the BLV method.

Assume again that the underlying physical space is a circumference with K discrete locations; further assume that K is even. The bifurcation properties of the Pf model on a circumference has been extensively studied by Akamatsu et al. (2012). As previously, we start from high transport costs (i.e., $r \approx 0$) at which the flat-earth equilibrium is stable and gradually increases r. Then, the destabilization of the flat-earth equilibrium $\bar{\boldsymbol{h}} \equiv [h, h, ..., h]^\top$ (for the Pf model, $h = H/K$), if any, is period doubling and results in a $K/2$-centric pattern as in the HW model; mobile consumers in alternate locations disappear, and remaining locations double in the number of inhabitants. Akamatsu et al. (2012) further proves that the spatial period-doubling bifurcation occurs in the Pf model if K is a power of two. The bifurcation properties of the Pf model are thus similar to those of the HW model.

5.2 Harris and Wilson's Model v.s. Pflüger's Model

The source of similarity between the Pf and HW models will now be discussed. For simplicity, we focus only on the first bifurcation. Again, K is assumed to be even.

Although apparent differences exist in the equilibrium conditions and dynamics, it is sufficient to compare the Jacobian matrices of the payoff functions at $\bar{\boldsymbol{h}}$ (i.e., $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$ and $\nabla \boldsymbol{\Pi}(\bar{\boldsymbol{h}})$) and their eigenvalues to compare the character of the first bifurcation; at the flat-earth equilibrium, the Jacobian matrix of the dynamic and that of the payoff function coincide up to a constant multiple in both models.

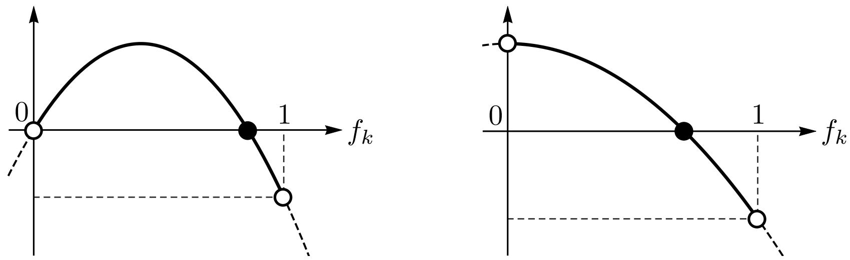
For the Pf model, the Jacobian matrix of the indirect utility function $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$ is

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = \frac{1}{h}\left(a\bar{\boldsymbol{D}} - b\bar{\boldsymbol{D}}^2\right),$$

where $a \equiv \mu(\sigma-1)^{-1} + \sigma^{-1}$ and $b \equiv \mu\sigma^{-1}(1 + lh^{-1})$. For the HW model, the Jacobian matrix $\nabla \boldsymbol{\Pi}(\bar{\boldsymbol{h}})$ of the profit function at $\bar{\boldsymbol{h}}$ is given as follows:

$$\nabla \boldsymbol{\Pi}(\bar{\boldsymbol{h}}) = \frac{O\alpha}{h^2}\left(\hat{\alpha}\boldsymbol{I} - \bar{\boldsymbol{D}}^2\right).$$

The Jacobian matrices take relatively simple form with respect to $\bar{\boldsymbol{D}}$; they are just quadratics. The functional form of the Jacobian matrix reflects each model's interplay of agglomeration and dispersion forces. Yet, as discussed in 4.3, asymmetry in the underlying distance structure prevents analytically obtaining the eigenvalues for these matrices. Imposing the racetrack topology, the true utility of Akamatsu et al. (2012)'s method is that we can extract intrinsic properties of the model without unnecessary numerical complications.



(a) $G^{\mathrm{Pf}}(f_k)$ (b) $G^{\mathrm{HW}}(f_k)$

Figure 9: The net agglomeration forces $G^{\mathrm{HW}}(f_k)$ and $G^{\mathrm{Pf}}(f_k)$ as a function of f_k.

On the racetrack, it is immediate that for $\nabla \boldsymbol{v}\left(\bar{\boldsymbol{h}}\right)$ its eigenvalues $\boldsymbol{e}^{\mathrm{Pf}}$ are

$$e_k^{\mathrm{Pf}} = \frac{1}{h} \cdot G^{\mathrm{Pf}}(f_k) \tag{15a}$$
$$G^{\mathrm{Pf}}(x) \equiv ax - bx^2 \tag{15b}$$

where f_k denotes the kth eigenvector of $\bar{\boldsymbol{D}}$. Similarly, the eigenvalues $\boldsymbol{e}^{\mathrm{HW}}$ of $\nabla \boldsymbol{\Pi}(\bar{\boldsymbol{h}})$ are

$$e_k^{\mathrm{HW}} = \frac{O\alpha}{h^2} \cdot G^{\mathrm{HW}}(f_k) \tag{16a}$$
$$G^{\mathrm{HW}}(x) \equiv \hat{\alpha} - x^2 \tag{16b}$$

In both models, the associated eigenvector for the kth eigenvalue e_k is the kth column vector of the K-dimensional DFT matrix.

As Section 4 discussed, the sign of the eigenvalues solely depends on the sign of $G(f)$. Figure 9 illustrates $G^{\mathrm{HW}}(f_k)$ and $G^{\mathrm{Pf}}(f_k)$. In either of (15) and (16), the positive term (the first term) represents the agglomeration force of the model while the negative term (the second) represents the dispersion force. Each of (15) and (16) can be interpreted as a *net agglomeration force* as a whole, by which we mean (agglomeration force) $-$ (dispersion force). Each e_k can thus be seen as the net strength of the agglomeration force in the deviation direction of $\boldsymbol{z}_k$. If $e_k < 0$ for all k, there is no incentive exists for the mobile agents to disturb the flat-earth equilibrium $\bar{\boldsymbol{h}}$; if some e_k is positive for some direction, the agglomeration force in this direction overcomes the dispersion force and $\bar{\boldsymbol{h}}$ becomes unstable.[22]

We first discuss the similarities and differences of the net agglomeration forces G^{HW} and G^{Pf}. First, in both models, $G(f)$ is a concave quadratic of f_k because of the dispersion force (second term). The similarity stems from the fact that both models have a spatial competition effect (or, equivalently, a market crowding effect) captured by gravity equations of consumer demand. This works as a dispersion force and appears as the negative second order term of $\bar{\boldsymbol{D}}$ in the Jacobian matrix of the payoff function [$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$ or $\nabla \boldsymbol{\Pi}(\bar{\boldsymbol{h}})$]. We see that the dispersion force (i.e., $-bf_k^2$ or $-f_k^2$) weakens in line with increasing r, because each f_k is decreasing in r.

[22] For economic interpretations of the eigenvalues and the eigenvectors, see also 4.3.

In contrast, the other term differs between models. The first term represents the agglomeration forces in each model. In the Pf model, the first term is linear in f_k; in the HW model, it is a constant with respect to f_k. This difference reflects the properties of the agglomeration forces. In the Pf model, the agglomeration force is a space-dependent one—price index effect and demand effects, which is strongly affected by the spatial distribution of mobile consumers, hence the distance between locations. On the other hand, a space-independent agglomeration force is at work in the HW model: the relative attractiveness of a zone i is determined exclusively by $(h_i)^\alpha$, which does not require any spatial dimension but only the number of firms in the zone. Reflecting this, the agglomeration force in the Pf model (i.e., af_k) gradually decreases in line with increasing r while that of the HW model (i.e., $\hat{\alpha}$) remains constant.

The models' bifurcation properties reflect the above (dis-)similarities in net agglomeration forces. Both models exhibit spatial period-doubling behavior induced by the maximality of e_M ($M = K/2$), whose associated eigenvector $\boldsymbol{z}_M$ is the period-doubling direction. The maximality of e_M in line with increasing r is the consequence of the *shape of* $G(f)$: in both models, $G(f)$ is decreasing for large f (i.e., small r). The increase in r results in decreases in all f_k from 1 to 0, where the minimal one is f_M. Thus, e_M is the maximal eigenvalue. Figure 9 provides some graphical intuitions. This common property of $G^{\text{HW}}(f)$ and $G^{\text{Pf}}(f)$ arises from the spatial competition effect modeled in the gravity equation. We may thus hypothesize that the existence of spatial competition is a necessary condition for the emergence of the spatial period-doubling cascade in line with decreasing transportation costs.

On the other hand, only the HW model exhibits spatial period skipping. In the Pf model, if the parameters are such that the destabilization of the flat-earth equilibrium occurs, then the model *always* exhibits a complete spatial period-doubling cascade. In contrast to the HW model, there is no possibility of skipping because the HW model's agglomeration force is not space dependent. Even when transport costs are quite high ($r \approx 0$), increasing α directly destabilizes $\boldsymbol{h}^{(n)}$ with $n \geq 1$ (Figure 6b) because of the strong agglomeration economy; the flat-earth equilibrium is stable by chance, and when $\alpha \to \infty$, the flat-earth equilibrium becomes unstable for $r \in (0, 1)$. In addition, in the two-location setting, we do not have a means to qualitatively distinguish between the HW and Pf models when we gradually increase r. Both models exhibit a transition from the flat-earth equilibrium $\bar{\boldsymbol{h}} \equiv [h, h]^\top$ to the CP equilibrium $\boldsymbol{h}^{(1)} \equiv [2h, 0]^\top$. Again, this demonstrates the limitation of the two-location assumption.

6 Concluding Remarks

This paper has analytically unveiled previously unknown bifurcation properties of Harris and Wilson (1978)'s dynamic retail location model under a multi-locational (more than two) setting. We demonstrated that Akamatsu et al. (2012)'s approach can be used to reveal the model's agglomeration/dispersion forces and analytically trace the recursive emergence of spatial structures. Specifically, we analytically derived the critical points of a structural parameter, namely, the transport cost parameter r, at which the first and second bifurcation occur. Moreover, we demonstrated that the spatial structure's evolutionary path exhibits a striking property called the *spatial period-doubling cascade*, previously found only in

Pflüger (2004)'s NEG model. We clarified the similarities and differences between Harris and Wilson (1978)'s model and Pf's NEG model: both models exhibit the spatial period-doubling cascade, but only the HW model exhibits extreme sustainability of equilibria and a skip in equilibrium patterns. In many spatial agglomeration models, the Jacobian matrix of the dynamic or payoff function is related to the spatial discounting matrix in a relatively simple way. However, it is merely a negative quadratic for HW and Pf models. In such a situation, by imposing the racetrack topology as a testbed, the method developed by Akamatsu et al. (2012) allows to analytically reveal the proposed model's intrinsic bifurcation properties.

Clarke and Wilson (1985) revealed that in a two-dimensional plain, the HW model exhibits remarkable spatial patterns that resemble the hexagonal agglomeration patterns proposed in classical central place theory of Christaller (1933) and Lösch (1940). The underlying mechanism for this striking result, however, had not been fully explained due to lack of analytical equipment. Recent studies by Ikeda et al. (2012b), Ikeda et al. (2014a), Ikeda et al. (2014b, 2016) reveal that two-dimensional NEG models admit hexagonal spatial configurations as stable equilibria. Judging from the strong linkage between NEG models and the HW model, it is likely that we can rigorously explain Clarke and Wilson (1985)'s numerical results for the HW model; such an explanation will be a materialization of Harris (1985)'s idea, where he hints the applicability of the HW model, which is an intra-urban model, to an interregional context so that we can explain central place systems.

Another interesting research direction is analyzing models with multiple types of agents. Although the present framework assumes multilocations, it does not enable different sized regions to emerge. To endogenously produce various sized agglomerations, we should extend our framework to include multiple types of agents to produce an *endogenous hierarchy of locations* as in the study by Tabuchi and Thisse (2011) which employ a NEG model. This is, however, beyond the scope of this paper.

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A Introduction to the Boltzmann–Lotka–Volterra Method

For self-containedness, we outline the BLV method by Wilson (2008) and discuss its economic interpretations in this appendix.

Consider a space equipped with K discrete locations. We assume an index set of locations $\mathcal{K} \equiv \{0, 1, \dots, K-1\}$. To introduce a spatial dimension on $\mathcal{K}$, the transport cost patterns between the locations are exogenously given by $\boldsymbol{T} = [t_{ij} \mid i, j \in \mathcal{K}]$. Each element of $\boldsymbol{T}$ is the so-called generalized transport cost: it is assumed that $\boldsymbol{T}$ includes any cost associated with the travel between locations. We want to model spatial structures, possibly spatial agglomeration patterns, at *equilibria* (in some sense) on this space. Let $\boldsymbol{h} \equiv [\dots, h_i, \dots]^\top \in \mathbb{R}_+^K$ be a non-negative K-dimensional real that represents the spatial structure. $h_i \geq 0$ may be interpreted as the number of residents, shops, or firms at location i depending on the modeler's interest. Of significance here is that $\boldsymbol{h}$ is endogenously determined by the model.

The "Boltzmann" component, or the *fast dynamic*, of the BLV method models the *short-run* equilibrium. In the short-run, $\boldsymbol{h}$ is assumed to be fixed. Modeled here is the *spatial interaction pattern* $\boldsymbol{S} \equiv [S_{ij} \mid i, j \in \mathcal{K}] \in \mathbb{R}_+^{K \times K}$ between locations. $\boldsymbol{S}$ is a matrix whose elements can be interpreted as, for example, the monetary flows or the trip patterns from origins i to destinations j. $\boldsymbol{S}$ should depend on $\boldsymbol{h}$ to obtain non-trivial results. As the analysis of the HW model will show, $\boldsymbol{S}(\boldsymbol{h})$ may be subject to some constraints (e.g., the conservation of trip demands at the origins, the total transport cost spent by the spatial interaction). The BLV method assumes that $\boldsymbol{S}(\boldsymbol{h})$ arises as a result of an entropy-maximizing problem with such constraints. Let the set of all feasible spatial interaction patterns at $\boldsymbol{h}$ be $\mathcal{S}(\boldsymbol{h}) \subseteq \mathbb{R}_+^{K \times K}$. Then, the "most probable" spatial interaction pattern is obtained by solving the following problem (Wilson, 1970a, 1970b):

$$\max_{\boldsymbol{S} \in \mathcal{S}(\boldsymbol{h})} . \ \mathcal{H}(\boldsymbol{S}) \equiv -\sum_{i \in \mathcal{K}} \sum_{j \in \mathcal{K}} S_{ij} \log[S_{ij}]$$

where $\mathcal{H}(\cdot)$ is the Boltzmann–Shannon entropy.

The "Lotka–Volterra" component of the BLV method, or the *slow dynamic*, on the other hand, models the long-run equilibration of $\boldsymbol{h}$. Given the spatial distribution $\boldsymbol{h}$ and the associated spatial interaction pattern $\boldsymbol{S}(\boldsymbol{h})$ in the short-run, the modeler specifies the *payoff function* $\boldsymbol{v}(\boldsymbol{h}, \boldsymbol{S}(\boldsymbol{h})) = [\dots, v_i(\boldsymbol{h}, \boldsymbol{S}(\boldsymbol{h})), \dots]^\top$ that essentially maps the state $\boldsymbol{h}$ to the incentive landscape on $\mathcal{K}$. In the long-run, $\boldsymbol{h}$ is allowed to dynamically evolve depending on the payoff $\boldsymbol{v}(\boldsymbol{h}, \boldsymbol{S})$ and the state $\boldsymbol{h}$ itself. The dynamic change of $\boldsymbol{h}$ is assumed to be governed by some evolutionary dynamic $\boldsymbol{F}$ that takes $\boldsymbol{h}$ and $\boldsymbol{v}(\boldsymbol{h}, \boldsymbol{S})$ as the inputs:

$$\dot{\boldsymbol{h}} = \boldsymbol{F}(\boldsymbol{h}, \boldsymbol{v}(\boldsymbol{h}, \boldsymbol{S}))$$

Typically, the locations with relatively larger values of $v_i(\boldsymbol{h})$ are assumed to glow faster, and vice versa, under the dynamic $\boldsymbol{F}$. Such dynamics are very similar to—or perhaps, to some extent, inspired by—population dynamics model in mathematical and theoretical biology, hence the name "Lotka–Volterra." The *equilibrium* for a BLV model is defined as a stationary point of the above long-run dynamic: a point $\boldsymbol{h}^*$ that satisfies $\boldsymbol{F}(\boldsymbol{h}^*) = \boldsymbol{0}$. This combination of the fast and slow dynamic is the basic theoretical framework of the BLV method. Although we used the terminologies "location," "flows," or "space" to fix the ideas consistent with the focus of this paper (the retailer model), the applicability of the BLV method is not limited to spatial problems.[23]

Implicit in the BLV method is utility/profit-maximizing economic agents who select their own strategies and the standard spatial equilibrium condition based on no arbitrage. As it is noted by Wilson (2008), there is a close linkage between such formalism and the BLV method. Actually, it is just a matter of perception, or the preference of the modeler. From an economics perspective, $\boldsymbol{h}$ can be interpreted as an aggregated variable determined by the infinitesimally small *agents' choices* (e.g., the choices by consumers or firms): h_i is the number of agents selecting location i. The payoff function $v_i(\boldsymbol{h})$ is the indirect utility (for consumers) or profit (for firms) enjoyed by those who are choosing location $i \in \mathcal{K}$ at the given spatial distribution, or state of the system, $\boldsymbol{h}$. In formal economic models such as those of NEG, the spatial interaction pattern $\boldsymbol{S}(\boldsymbol{h})$ and payoff vector $\boldsymbol{v}(\boldsymbol{h}, \boldsymbol{S})$ are obtained from the result of market interaction between utility/profit-maximizing agents. The "Boltzmann" component of the BLV method models this mechanism in a reduced-form way that is characterized by the entropy-maximization problem.[24]

In the BLV method, equilibria are defined as the stationary points of the slow dynamic. The standard spatial equilibrium condition in economics is different. It is formulated as the following no arbitrage condition

$$\begin{cases} V = v_i(\boldsymbol{h}) & \text{if } h_i > 0 \\ V \geq v_i(\boldsymbol{h}) & \text{if } h_i = 0 \end{cases} \qquad \forall i \in \mathcal{K}$$

where V is an equilibrium payoff. When the total number of agents is an exogenous constant H, that is, when the following conservation equation

$$\sum_{i \in \mathcal{K}} h_i = H,$$

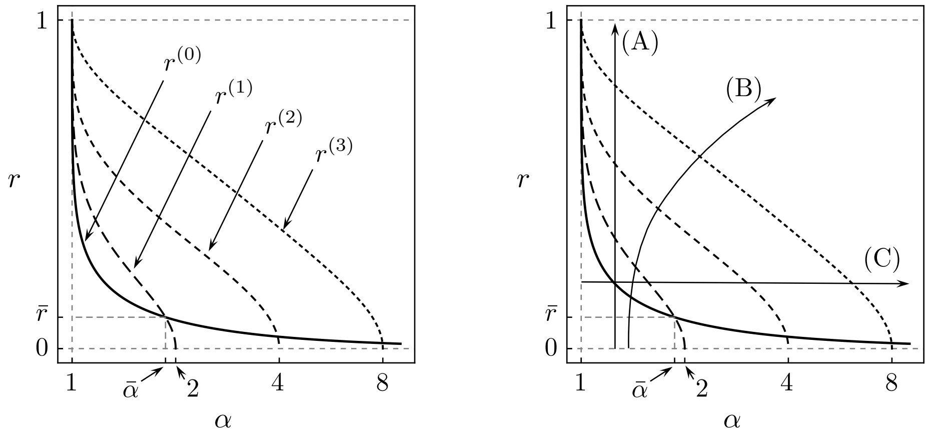
holds, V is endogenously determined to satisfy the equilibrium condition. In the literature of Urban Economics (Fujita, 1989), such formulation is called the *closed-city* model reflecting no change in the number of agents in the system. A slightly different formulation is also possible. When V is exogenously given (e.g., when V is assumed to be zero), whereas the total number of agents H is endogenously determined. In contrast to the closed-city model, this type of formulation is called the *open-city* model because the total number of agents may vary from one equilibrium to another from the migration of agents from/to the outside of the system. For either type of model, we define the equilibrium condition and the dynamic such that the two conditions are mutually consistent.

B Other Parameter Paths

The bifurcation analysis in the main text exclusively focused on a specific class of parameter paths: *increasing* r (or, equivalently, decreasing β) *with a predetermined* α. We aim to make

[23] For instance, Wilson (2008) discusses an application of the method to the analysis of scale-free networks.

[24] Note that there is concrete correspondence between the entropy-maximization problem and utility-maximization in the representative consumer model with the constant elasticity of substitution (CES) preference (Anderson et al., 1992). The former is, therefore, actually a reduced-form model of the latter. It is also equivalent to a random utility model (namely logit model).



(a) The critical values $r^{(n)}(\alpha)$.

(b) Examples of parameter path.

Figure 10: Critical values and examples of parameter path of structural evolution ($K = 16$).

a clear comparison with NEG models. There are other types of parameter paths, discussed briefly in this appendix. We allow the two main parameters of the HW model, namely r (β) and α, to change (these are the relevant parameters for the bifurcation properties of the HW model, since O and κ change only the scale of $\boldsymbol{h}$). We demonstrate that, even in this case, the spatial period-doubling cascade occurs along typical parameter paths; this demonstrates the robustness of the conclusions in the main text.

Figure 10 (a) plots the bifurcation points $\{r^{(n)}\}$ as the functions of α with $K = 16$, where we take a logarithmic scale for the α axis. Every $r^{(n)}$ is a monotonically decreasing function of α, as discussed in Proposition 1 and Proposition 2 for $n = 0, 1$. Below each curve $r^{(n)}(\alpha)$, the $K/2^n$-centric pattern $\boldsymbol{h}^{(n)}$ is stable (i.e., $\boldsymbol{h}^{(0)} \equiv \bar{\boldsymbol{h}}$). We define $\overline{r} \equiv r^{(0)}(\overline{\alpha}) = r^{(1)}(\overline{\alpha})$. If we fix r instead of α, then these curves can be considered to be the critical values of α.

We again focus on the symmetry-breaking process, i.e., the structural evolution from the flat-earth equilibrium $\bar{\boldsymbol{h}}$, which is stable when either α or r is very small. We thus consider a steady increase in α and/or r in this appendix. Note that the maximal eigenvalue remains the same even when we allow both structural parameters to change. Every bifurcation, which is induced by crossing threshold lines in Figure 10, is thus period doubling.

Figure 10 (b) illustrates typical instances of possible parameter paths. Path (A) is an instance such as discussed in the main text: the value of α is fixed and we consider a steady increase in r. The path sequentially crosses all $r^{(n)}$ because $1 < \alpha < \overline{\alpha}$ (Proposition 4). Observe that if $\overline{\alpha} < \alpha < 4$, then $\boldsymbol{h}^{(1)}$ is already unstable when $\bar{\boldsymbol{h}}$ becomes stable; $\boldsymbol{h}^{(1)}$ is skipped (Proposition 5).

Parameter paths like (B) and (C) were not discussed in the main text. Path (C) demonstrates a steady increase in α with a fixed value of r. Whenever $r > \overline{r}$ holds, the path sequentially crosses all $r^{(n)}$; the spatial period-doubling cascade occurs when α steadily increases. On the other hand, when the fixed value of r is very small ($r < \overline{r}$), then the dispersion force is very strong. In such a case, when α becomes large enough to destabilize $\bar{\boldsymbol{h}}$, some skip may occur. This result is analogous to the intuition of Proposition 5, while in the present case, the dispersion force, rather than the agglomeration force, is too strong.

On path (B), both α and r change. (B) is also a spatial period-doubling cascade path. There are many possible paths like (B) and we cannot cover them all. But Figure 10 shows that in many cases, the HW model exhibits the spatial period-doubling cascade.

C Technical Appendix

C.1 Jacobian Matrices and Their Eigenpairs at the Flat-earth Equilibrium

At any configuration $\boldsymbol{h}$, the Jacobian matrix for the adjustment dynamic is given as follows:

$$\nabla \boldsymbol{F}(\boldsymbol{h}) = \operatorname{diag}[\boldsymbol{\Pi}(\boldsymbol{h})] + \operatorname{diag}[\boldsymbol{h}] \nabla \boldsymbol{\Pi}(\boldsymbol{h})$$

where $\nabla \boldsymbol{\Pi}(\boldsymbol{h})$ is the Jacobian matrix of firms' profit at $\boldsymbol{h}$:

$$\begin{aligned}
&\nabla \boldsymbol{\Pi}(\boldsymbol{h}) = \operatorname{diag}[\tilde{\boldsymbol{M}}^\top \boldsymbol{O}] - \alpha \boldsymbol{M}^\top \operatorname{diag}[\boldsymbol{O}] \boldsymbol{M} \\
&\tilde{\boldsymbol{M}} \equiv (\alpha - 1) \operatorname{diag}[\boldsymbol{\Delta}]^{-1} \boldsymbol{D} \operatorname{diag}[\boldsymbol{h}]^{\alpha - 2}
\end{aligned}$$

At the flat-earth equilibrium, we obtain the following expression

$$\begin{aligned}
&\nabla \boldsymbol{F}(\bar{\boldsymbol{h}}) = h \nabla \boldsymbol{\Pi}(\bar{\boldsymbol{h}}) \\
&\nabla \boldsymbol{\Pi}(\bar{\boldsymbol{h}}) = \frac{\alpha O}{h^2} \left(-\bar{\boldsymbol{D}}^2 + \hat{\alpha} \boldsymbol{I} \right).
\end{aligned}$$

Note that $\boldsymbol{\Pi}(\bar{\boldsymbol{h}}) = \boldsymbol{0}$. Plugging $h = O/\kappa$, we obtain (7).

Let $\boldsymbol{Z} = [z_{jk}]$ be the K-dimensional DFT matrix. Its typical element z_{jk} is given by $z_{jk} = \omega^{jk}$, where $\omega \equiv \exp[\mathrm{i}(2\pi/K)]$ where i being the imaginary unit. For the eigenvalues and eigenvectors of a K-dimensional circulant matrix, we have the following fact:

Lemma C.1.1 (e.g., Horn and Johnson (2012))**.** Let $\boldsymbol{A}$ be a K-dimensional circulant matrix with its first row vector being $\boldsymbol{a}_0 = [a_{0,i}]$. Then, the matrix $\boldsymbol{A}$ is of full rank and diagonalizable by similarity transformation by the K-dimensional discrete Fourier transformation matrix $\boldsymbol{Z}$: let $\boldsymbol{\lambda} \equiv [\lambda_0, \lambda_1, \ldots, \lambda_{K-1}]^\top$ be the eigenvalues of $\boldsymbol{A}$, then $\operatorname{diag}[\boldsymbol{\lambda}] = \boldsymbol{Z}\boldsymbol{A}\boldsymbol{Z}^{-1}$ holds. Moreover, the eigenvalues $\boldsymbol{\lambda}$ are given by the discrete Fourier transformation of $\boldsymbol{a}_0$. That is, $\boldsymbol{\lambda}$ satisfies

$$\boldsymbol{\lambda} = \boldsymbol{Z} \boldsymbol{a}_0^\top.$$

The eigenvectors of $\boldsymbol{A}$ are given by column vectors of $\boldsymbol{Z}$, i.e., the eigenvector associated with λ_k is

$$\boldsymbol{z}_k = [1, \omega^k, \omega^{2k}, \ldots, \omega^{k(K-1)}]^\top \qquad k = 0, 1, \ldots, K-1$$

The Jacobian matrix at the flat-earth equilibrium is diagonalizable by similarity transformation using $\boldsymbol{Z}$, and the eigenvalues of $\nabla \boldsymbol{F}(\bar{\boldsymbol{h}})$ are obtained by the discrete Fourier transformation of its first row; we can diagonalize $\nabla \boldsymbol{F}(\bar{\boldsymbol{h}})$ by multiplying the K-dimensional DFT matrix $\boldsymbol{Z}$ from the left and $\boldsymbol{Z}^{-1}$ from the right side of (7) to obtain (8) and thus Lemma 2.

C.2 Proof for Proposition 1

If for some k we have $g_k > 0$ for all r and k, the flat-earth equilibrium $\bar{\boldsymbol{h}}$ cannot be stable at any level of transport cost r. On the other hand, $g_k < 0$ for all r and k implies that $\bar{\boldsymbol{h}}$ is locally asymptotically stable for all r. Thus, we require that α lies in some specific range to ensure $G(f) = 0$ has at least one solution in $f \in (0, 1)$, so that some $\boldsymbol{g}$ changes its sign at some $f \in (0, 1)$. Elementary algebra shows the following lemma:

Lemma C.2.1. The parameter α should satisfy

$$\alpha > 1$$

to ensure that (a) the flat-earth equilibrium $\bar{\boldsymbol{h}}$ is stable for some small r (large f_k) and (b) a destabilization of $\bar{\boldsymbol{h}}$ occurs according to the increase of r (decrease of f_k).

Proof. First, $G(f)$ is strictly decreasing for $f \in [0, 1]$ and $f_k(r)$ is strictly decreasing for $r \in (0, 1)$; thus $G(f_k(r))$ is strictly increasing for $r \in (0, 1)$. Therefore, each $G(f_k(r)) = 0$ has a unique solution if and only if $G(0) = \hat{\alpha} = 1 - \alpha^{-1} > 0$ and $G(1) = -\alpha^{-1} < 0$. □

We thus require $\alpha > 1$. We also assume that K is an even to apply Lemma 1 (b). Then, the maximal eigenvalue and its associated eigenvector are

$$g_M(r) = (\alpha\kappa) \cdot G(f_M(r))$$
$$\boldsymbol{z}_M = [(-1)^j]$$

Solving $G(f_M(r)) = 0$ yields $r^{(0)} = f_M^{-1}(\sqrt{\hat{\alpha}})$. The exact formula is

$$r^{(0)} = \begin{cases} C(\sqrt{\hat{\alpha}}) & \text{if } K = 2 \\ C(\sqrt[4]{\hat{\alpha}}) & \text{if } K \geq 4 \end{cases}.$$

where $C(x) \equiv (1 - x)/(1 + x)$. It is immediate that $\mathrm{d}r^{(0)}/\mathrm{d}\alpha < 0$.

C.3 Proof for Proposition 2

First, we have the following lemma regarding the eigenvalues at $\boldsymbol{h}^{(1)}$:

Lemma C.3.1. Let $\boldsymbol{g}^{(1)}$ be the eigenvalues of the Jacobian of the adjustment dynamic at $K/2$-centric equilibrium $\boldsymbol{h}^{(1)}$. The maximal eigenvalue $g^{(1)}_{\text{max}} \equiv \max_k g^{(1)}_k$ is given by:

$$g^{(1)}_{\text{max}} = (\alpha\kappa) \cdot G^{(1)}(r)$$
$$G^{(1)}(r) \equiv -\frac{1}{2} \{f(r)\}^2 + \hat{\alpha}$$
$$f(r) \equiv \begin{cases} C(r^2) & \text{if } K = 4 \\ \{C(r^2)\}^2 & \text{if } K \geq 8 \end{cases}$$

and the associated eigenvector is

$$\boldsymbol{z} \equiv [\underbrace{1, 0, -1, 0}, \underbrace{1, 0, -1, 0}, \ldots, \underbrace{1, 0, -1, 0}]^\top.$$

Proof. Appendix C.4 □

As in $\bar{\boldsymbol{h}}$, depending on the value of α, there is a possibility that destabilization of $\boldsymbol{h}^{(1)}$ does not occur. Following the same line of logic as the uniform equilibrium $\bar{\boldsymbol{h}}$, we conclude the following:

Lemma C.3.2. For the $K/2$-centric equilibrium $\boldsymbol{h}^{(1)}$ to be stable at some r and bifurcation to occur according to the increase of r, the parameter α should satisfy

$$1 < \alpha < 2$$

Proof. $G^{(1)}(r)$ is a monotonically increasing function of r. Therefore, destabilization of $\boldsymbol{h}^{(1)}$ occurs if and only if $G^{(1)}(0) < 0$ and $G^{(1)}(1) = -1/2 + \hat{\alpha} > 0$. □

Solving $G^{(1)}(r) = 0$, we obtain $r^{(1)}$ as follows:

$$r^{(1)} = \begin{cases} \sqrt{C(\sqrt{2\hat{\alpha}})} & \text{if } K = 4 \\ \sqrt{C(\sqrt[4]{2\hat{\alpha}})} & \text{if } K \geq 8 \end{cases}$$

It is immediate to verify $\mathrm{d}r^{(1)}/\mathrm{d}\alpha < 0$.

C.4 Proof for Lemma C.3.1

The discussion here is quite similar to that of Akamatsu et al. (2012). To analytically derive the eigenvalues of the Jacobian matrix $\nabla \boldsymbol{F}(\boldsymbol{h}^{(1)})$, we must change the coordinate system because $\nabla \boldsymbol{F}(\boldsymbol{h}^{(1)})$ itself is not a circulant matrix. Specifically, we must consider a permutation of row/column indices. We define permutation matrix $\boldsymbol{P}$ by

$$\boldsymbol{P} \equiv \left[\begin{array}{c} \boldsymbol{P}_{\text{even}} \\ \hdashline \boldsymbol{P}_{\text{odd}} \end{array}\right] \tag{17}$$

in which $\boldsymbol{P}_{\text{even}}$ and $\boldsymbol{P}_{\text{even}}$ are both a $K/2$-by-K matrix defined by $\boldsymbol{P}_{\text{even}} = [\delta_{i,j/2}]$ and $\boldsymbol{P}_{\text{odd}} = [\delta_{i,(j-1)/2}]$, where $\delta_{\cdot,\cdot}$ is Kronecker's delta. $\boldsymbol{P}_{\text{even}}$ extracts even-numbered zones; $\boldsymbol{P}_{\text{odd}}$ extracts odd-numbered zones.

Define $\boldsymbol{D}^{\times}$ by $\boldsymbol{D}^{\times} \equiv \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{\top}$. We have

$$\boldsymbol{D}^{\times} = \left[\begin{array}{c:c} \boldsymbol{D}^{(0)} & \boldsymbol{D}^{(1)} \\ \hdashline \boldsymbol{D}^{(1)\top} & \boldsymbol{D}^{(0)} \end{array}\right]$$

where $\boldsymbol{D}^{(0)}$ and $\boldsymbol{D}^{(1)}$ are both $K/2$-by-$K/2$ circulant with the first row given by

$$\boldsymbol{d}_0^{(0)} = [1, r^2, r^4, \ldots, r^M, \ldots, r^2],$$
$$\boldsymbol{d}_0^{(1)} = [r, r^3, \ldots, r^{M-1}, r^{M-1}, \ldots, r].$$

with $M = K/2$. The matrix $\boldsymbol{D}^{(0)}$ captures the distance structure between market centers (or *core zones*) while $\boldsymbol{D}^{(1)}$ captures that of core zones and *periphery zones* without retailers.

Because both $\boldsymbol{D}^{(i)}$ are circulant matrices, from Lemma C.1.1 we can diagonalize $\boldsymbol{D}^{\times}$ by a block diagonal matrix $\boldsymbol{Z}^{\times} \equiv \mathrm{diag}[\boldsymbol{Z}_{[K/2]}, \boldsymbol{Z}_{[K/2]}]$ with $\boldsymbol{Z}_{[K/2]}$ being a $K/2$-dimensional DFT matrix.

In a similar spirit, permutation by $\boldsymbol{P}$ enables us to block diagonalize $\nabla \boldsymbol{F}(\boldsymbol{h}^{(1)})$. Specifically, some algebra show that if we define $\nabla \boldsymbol{F}^{\times}(\boldsymbol{h}^{(1)}) \equiv \boldsymbol{P} \nabla \boldsymbol{F}(\boldsymbol{h}^{(1)}) \boldsymbol{P}^{\top}$, it is

$$
\begin{aligned}
\nabla \boldsymbol{F}^{\times}(\boldsymbol{h}^{(1)}) &= \left[\begin{array}{c:c} \boldsymbol{V} & \boldsymbol{0} \\ \hdashline \boldsymbol{0} & -\kappa \boldsymbol{I} \end{array}\right] \\
\boldsymbol{V} &= (\alpha\kappa) \left\{ -\frac{1}{2} \left(\{\bar{\boldsymbol{D}}^{(0)}\}^2 + \bar{\boldsymbol{D}}^{(1)} \bar{\boldsymbol{D}}^{(1)\top} \right) + \hat{\alpha} \boldsymbol{I} \right\}
\end{aligned}
$$

Again we can diagonalize $\nabla \boldsymbol{F}^{\times}(\boldsymbol{h}^{(1)})$ by $\boldsymbol{Z}^{\times}$, because every sub-matrix of $\nabla \boldsymbol{F}^{\times}(\boldsymbol{h}^{(1)})$ is circulant. Let $\boldsymbol{e}$ be the eigenvalues of $\boldsymbol{V}$. A straightforward computation yields

$$
\begin{aligned}
e_k &= (\alpha\kappa) \cdot \hat{G}(f_k^{(0)}, f_k^{(1)} f_k^{(1\top)}) \qquad\qquad k = 0, 1, \ldots, K/2 - 1 \\
\hat{G}(x, y) &\equiv -\frac{1}{2}(x^2 + y) + \hat{\alpha}
\end{aligned}
$$

where $\boldsymbol{f}^{(0)}, \boldsymbol{f}^{(1)}, \boldsymbol{f}^{(1\top)}$ are the eigenvalues of $\bar{\boldsymbol{D}}^{(0)}, \bar{\boldsymbol{D}}^{(1)}, \bar{\boldsymbol{D}}^{(1)\top}$, respectively. Placing this back into the original coordinating system, the eigenvalues $\boldsymbol{g}^{(1)}$ of $\nabla \boldsymbol{F}(\boldsymbol{h}^{(1)})$ are obtained as

$$
g_k = \begin{cases} e_{k/2} & \text{if } k \text{ is an even} \\ -\kappa & \text{if } k \text{ is an odd} \end{cases}. \tag{18}
$$

Moreover, we show that

$$
\arg\max_k .\, e_k = \arg\min_k .\, \{f_k^{(0)}\}^2 + f_k^{(1)} f_k^{(1\top)} = M/2
$$

and that

$$
f_{M/2}^{(0)} = \begin{cases} C(r^2) & \text{if } K = 4 \\ \{C(r^2)\}^2 & \text{if } K \geq 8 \end{cases}, \quad f_{M/2}^{(1)} f_{M/2}^{(1\top)} = 0
$$

We thus obtain $g_{\max}^{(1)}$ in Lemma C.3.1. The corresponding eigenvector is M-th column vector of $\boldsymbol{Z}^{(1)} \equiv \boldsymbol{P}^{\top} \boldsymbol{Z}^{\times} \boldsymbol{P}$.

C.5 Proof for Proposition 3

We have the following lemmas:

Lemma C.5.1. Let $\boldsymbol{g}^{(n)}$ be the eigenvalues of the Jacobian matrix of the adjustment dynamic at $K/2^n$-centric equilibrium $\boldsymbol{h}^{(n)}$ $(n = 1, 2, 3, \ldots, J-1)$. The maximal eigenvalue $g_{\max}^{(n)} \equiv \max_k g_k^{(n)}$ is given by

$$
g_{\max}^{(n)} = (\alpha\kappa) \cdot G^{(n)}(r) \tag{19a}
$$

$$
G^{(n)}(r) \equiv -\tilde{G}^{(n)}(r) + \hat{\alpha} \tag{19b}
$$

where $\tilde{G}^{(n)}$ is a monotonically decreasing function of r, and its associated eigenvector is

$$
\boldsymbol{z}_M^{(n)} \equiv \Big[\underbrace{1, 0, 0, \ldots, 0}_{2^n}, \underbrace{-1, 0, \ldots, 0}_{2^n}, \ldots, \underbrace{-1, 0, \ldots, 0}_{2^n}\Big]^{\top}
$$

Proof. See Appendix C.6 for the proof including the exact definition of $\tilde{G}^{(n)}(r)$. □

Lemma C.5.2. For the $K/2^n$-centric equilibrium $\boldsymbol{h}^{(n)}$ $(n = 1, 2, \dots, J-1)$ to be stable at some r, and bifurcation to occur according to the decrease in r, the parameter α should satisfy

$$1 < \alpha < 2^n$$

Proof. See Appendix C.7. □

Combining and summarizing the above two lemmas, the proposition follows.

C.6 Proof for Lemma C.5.1

In this appendix, we derive the analytical expression of the eigenvalues $\boldsymbol{g}^{(n)}$ of the Jacobian matrix of the dynamic at K_n-centric equilibrium. Here $K_n \equiv K/2^n$ is the number of market centers (the number of zones with $h_i > 0$). As in Appendix C.3, we should first permute in advance $\nabla \boldsymbol{F}(\boldsymbol{h}^{(n)})$ to ensure block circulant property and, thus, the eigenvalues are obtained by DFT.

We define the permutation matrix $\boldsymbol{P}^{(n)}$ $(n = 1, 2, \dots, J-1)$ by

$$\boldsymbol{P}^{(n)} \equiv \mathrm{diag}\left[\underbrace{\hat{\boldsymbol{P}}^{(n)}, \hat{\boldsymbol{P}}^{(n)}, \dots, \hat{\boldsymbol{P}}^{(n)}}_{2^n}\right], \; \hat{\boldsymbol{P}}^{(n)} \equiv \left[\begin{array}{c} \boldsymbol{P}^{(n)}_{\text{even}} \\ \hdashline \boldsymbol{P}^{(n)}_{\text{odd}} \end{array}\right]$$

For an example, see Appendix C.3, which is the case $n = 1$. The matrices, $\boldsymbol{P}^{(n)}_{\text{even}}$ and $\boldsymbol{P}^{(n)}_{\text{odd}}$ are a K_n-by-K_{n-1} matrix that extracts even-numbered zones and odd-numbered zones in the *core zones*, respectively. Hence, $\boldsymbol{P}^{(n)}$ is a $2^n/2$-by-$2^n/2$ block diagonal matrix whose diagonal entries are a K_{n-1}-by-K_{n-1} matrix.

Using $\boldsymbol{P}^{(n)}$, define permutation matrix $\boldsymbol{P}$ by

$$\boldsymbol{P} \equiv \boldsymbol{P}^{(n)}\boldsymbol{P}^{(n-1)} \cdots \boldsymbol{P}^{(1)}.$$

The recursive definition of the permutation matrix $\boldsymbol{P}$ reflects the recursive spatial period-doubling bifurcations. Applying a similarity transformation using $\boldsymbol{P}$ to $\nabla \boldsymbol{F}(\boldsymbol{h}^{(n)})$, we obtain the 2^n-by-2^n block diagonal matrix whose blocks are all the K_n-by-K_n matrix:

$$\nabla \boldsymbol{F}^{\times}(\boldsymbol{h}^{(n)}) \equiv \boldsymbol{P} \nabla \boldsymbol{F}(\boldsymbol{h}^{(n)}) \boldsymbol{P}^{\top} = \left[\begin{array}{c:c:c:c} \boldsymbol{V} & & & \\ \hdashline & -\kappa \boldsymbol{I} & & \\ \hdashline & & \ddots & \\ \hdashline & & & -\kappa \boldsymbol{I} \end{array}\right]$$

$$\boldsymbol{V} \equiv (\alpha\kappa)\left(-\frac{1}{2^n}\sum_{i=0}^{2^n-1} \bar{\boldsymbol{D}}^{(i)}\bar{\boldsymbol{D}}^{(i)\top} + \left(1 - \frac{1}{\alpha}\right)\boldsymbol{I}\right)$$

We thus see $K - 2^n$ out of K eigenvalues are all $-\kappa$. Hence, $\boldsymbol{V}$ is the only matrix that affects the stability. Let the eigenvalues of $\boldsymbol{V}$ be $\boldsymbol{e}$; only the maximal eigenvalue in $\boldsymbol{e}$ should be required to assess the stability of $\boldsymbol{h}^{(n)}$.

In $\boldsymbol{V}$, the matrices $\bar{\boldsymbol{D}}^{(i)}$ are K_n-by-K_n matrices that is obtained by row-wise normalization of $\boldsymbol{D}^{(i)}$, where $\boldsymbol{D}^{(i)}$ are first block-row matrices of $\boldsymbol{D}^{\times} \equiv \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{\top}$. Each $\bar{\boldsymbol{D}}^{(i)}$ is circulant with a first row vector $\boldsymbol{d}_0^{(i)}$ explicitly given by

$$d_{0,k}^{(i)} = \begin{cases} r^{i+k2^n} & \text{if } 0 \leq k \leq K_n/2 \\ r^{(K_n-k)2^n-i} & \text{if } K_n/2 \leq k \leq K_n - 1 \end{cases} \qquad i = 0, 1, \ldots, 2^n$$

All the matrices in the right hand side being circulant matrix, $\boldsymbol{V}$ is also a circulant matrix. Thus, its eigenvalues are obtained by applying DFT by $\boldsymbol{Z}_{[K_n]}$. After some tedious computation using explicit formula for $\boldsymbol{d}_0^{(i)}$, we obtain the analytical expression of the eigenvalues $\boldsymbol{e}$ of $\boldsymbol{V}$. Moreover, we prove that the maximal eigenvalue among $\boldsymbol{e}$ is $e_{K_n/2}$. The associated eigenvector for $e_{K_n/2}$ is the M-th column of the matrix $\boldsymbol{P}^{\top}\boldsymbol{Z}^{\times}\boldsymbol{P}$ where $\boldsymbol{Z}^{\times} \equiv \boldsymbol{I} \otimes \boldsymbol{Z}_{[K_n]}$, where $\otimes$ denotes the Kronecker product.

Explicitly writing down $e_{K_n/2}$, we have the following result, proving Lemma C.5.1.

$$\max_k g_k^{(n)} = e_{K_n/2} = G^{(n)}(r) = (\alpha\kappa) \cdot \hat{G}^{(n)}(r)$$

where

$$\hat{G}^{(n)}(r) \equiv -\tilde{G}^{(n)}(r) + \hat{\alpha}$$

$$\tilde{G}^{(n)}(r) \equiv \begin{cases} \dfrac{1}{2^n}\{\psi_0^{(n)}(r)\}^2 + \dfrac{1}{2^n}\psi_0^{(n)}(r)\displaystyle\sum_{j=1}^{2^n-1}\psi_j^{(n)}(r) & \text{if } n \leq J-2 \\ \dfrac{1}{2^n}\{\psi_0^{(n)}(r)\}^2 + \dfrac{1}{2^n}\displaystyle\sum_{j=1}^{2^n-1}\psi_j^{(n)}(r) & \text{if } n = J-1 \end{cases}$$

$$\psi_j^{(n)}(r) \equiv \left(\frac{1-r^{2^n-2j}}{1+r^{2^n-2j}}\right)^2 \qquad \forall j \neq 0$$

$$\psi_0^{(n)}(r) \equiv \left(\frac{1-r^{2^n}}{1+r^{2^n}}\right)^{p(n)}, \qquad p(n) = \begin{cases} 1 & \text{if } n = J-1 \\ 2 & \text{otherwise} \end{cases}$$

For a detailed discussion and proof for maximality of $e_{K_n/2}$, consult Appendix G and J of Akamatsu et al. (2012). Note that there is a typo in Akamatsu et al. (2012)'s Lemma 6.5 for the case $n = J - 1$. Combining (J.6) and (J.7) in Appendix J of the paper, we obtain an equation similar to the above.

C.7 Proof for Lemma C.5.2

Note that $G^{(n)}(r)$ $(n = 1, 2, \ldots, J-2)$ are equivalently written as

$$G^{(n)}(r) = -\frac{1}{2^n}\psi_0^{(n)}(r)\left(\psi_0^{(n)}(r) + \epsilon_n(r)\right) + \hat{\alpha} \tag{20a}$$

$$\epsilon_n(r) \equiv 2\sum_{k=1}^{(2^n/2)-1} C_k(r), \quad C_k(r) \equiv \left(\frac{1-r^{2k}}{1+r^{2k}}\right)^2 \tag{20b}$$

Because it is evident that $\mathrm{d}C_k/\mathrm{d}r \leq 0, \mathrm{d}\psi_0^{(n)}(r)/\mathrm{d}r \leq 0$ with equality holding iff $r = 1$, $G^{(n)}(r)$ as a whole is strictly increasing for $r \in (0, 1)$. Therefore, to prove that the equation $G^{(n)}(r) = 0$ has a unique solution in $r \in (0, 1)$, it is sufficient to show that $G^{(n)}(0) < 0$ and $G^{(n)}(1) > 0$. We see $G^{(n)}(0) = \hat{\alpha} + 2^{-n} - 1 = -\alpha^{-1} + 2^{-n}$ and $G^{(n)}(1) = \hat{\alpha}$. The case $n = J - 1$ is also immediate. Combining these relations, we have Lemma C.5.2.

C.8 Proof for Proposition 4

Assume that $K = 2^J$ with some integer $J \geq 2$. The critical value $\bar{\alpha}$, which is the solution to the equation $r^{(0)}(\alpha) = r^{(1)}(\alpha)$ is given by

$$\bar{\alpha} = \begin{cases} \rho^{-6}(1+\rho)(1+\rho^2) \approx 1.87 & \text{if } K = 4 \\ 6^{-1}(4 + \rho_+ + \rho_-) \approx 1.93 & \text{if } K \geq 8 \end{cases} \tag{21}$$

where $\rho \equiv \sqrt[4]{2},\ \rho_\pm \equiv (73 \pm 6\sqrt{87})^{1/3}$. If $\alpha < \bar{\alpha}$ holds, then $r^{(0)} < r^{(1)}$ holds.

The remaining task is showing $r^{(n)} < r^{(n+1)}$ for $n \geq 1$. As in Lemma C.5.2, every $\boldsymbol{h}^{(n)}$ destabilizes if $1 < \alpha < \bar{\alpha}$ since $\bar{\alpha} < 2$. Observe that critical points for these bifurcations $\{r^{(n)}\}$ are solutions to $G^{(n)}(r) = 0$. Because $G^{(n)}(r)$ are strictly increasing in $r \in (0, 1)$, if we can show

$$G^{(n)}(r) > G^{(n+1)}(r)$$

for all r, this implies $r^{(n)} < r^{(n+1)}$. Using the same notation as Appendix C.7 and using $\psi_0^{(n+1)} \geq \psi_0^{(n)}$ (with equality iff $r = 1$), we conclude for $r \in (0, 1)$

$$\begin{aligned} &G^{(n)}(r) - G^{(n+1)}(r) > \frac{1}{2^{n+1}} \psi_0^{(n)}(r)\Big(\Delta\psi_0(r) + \Delta\epsilon(r)\Big) \\ &\Delta\psi_0(r) \equiv \psi_0^{(n+1)}(r) - 2\psi_0^{(n)}(r), \\ &\Delta\epsilon(r) \equiv \epsilon_{n+1}(r) - 2\epsilon_n(r) \end{aligned}$$

We see

$$\begin{aligned} \Delta\epsilon(r) &= (\epsilon_{n+1} - \epsilon_n) - \epsilon_n \\ &= 2 \sum_{k=2^n/2}^{(2^{n+1}/2)-1} C_k + 2 \sum_{k=1}^{(2^n/2)-1} C_k \\ &= 2C_{2^n/2} + 2 \sum_{k=1}^{(2^n/2)-1} \Big(C_{k+(2^n/2)} - C_k\Big) \end{aligned}$$

and because $C_k \geq C_l$ for $k > l$ (with equality iff $r = 1$), we see that $\Delta\epsilon(r)$ is non-negative. Moreover, because $\psi_0^{(n)} = C_{2^n/2}$ for $n \neq J - 1$, we have for such n

$$\Delta\psi_0(r) + \Delta\epsilon(r) = \epsilon_{n+1}(r) + 2 \sum_{k=1}^{(2^n/2)-1} \Big(C_{k+(2^n/2)} - C_k\Big) \geq 0$$

with equality, again, iff $r = 1$. This inequality and non-negativity of $\Delta\epsilon(r)$ imply that $G^{(n)} - G^{(n+1)} > 0$ holds for all $r \in (0, 1)$, thereby leading to Proposition 4.

C.9 Proof for Proposition 5

First assume that α satisfies $2^{\bar{n}-1} \leq \alpha < 2^{\bar{n}}$ for some $\bar{n} \geq 2$. Then, $\boldsymbol{h}^{(1)}, \ldots, \boldsymbol{h}^{(\bar{n}-1)}$ are unstable for all r. Hence, possible emergent patterns after the first bifurcation are $\boldsymbol{h}^{(\bar{n})}, \boldsymbol{h}^{(\bar{n}+1)}, \ldots$. The emergent pattern is $\boldsymbol{h}^{(\hat{n})}$, where $\hat{n}$ is the first $n \geq \bar{n}$ that satisfies $r^{(\hat{n})} > r^{(0)}$. On the other hand, we see from Appendix C.8 that for all n that is stable for some r, the bifurcation points are increasing in n (i.e., $r^{(n)} < r^{(n+1)}$). Thus, after the emergence of $\boldsymbol{h}^{(\hat{n})}$, a spatial period-doubling cascade starts.