Minimal conditions for parametric continuity and stable policy in extreme settings

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Abstract

In civil conflicts, warring factions commit atrocities for arbitrarily small gains in territory. On the product of territory-atrocity pairs, such (revealed) preferences are lexicographic. In such settings, an outside policy maker faces a nonmetrizable parameter space. Taking preferences over actions given the parameter as primitive, we develop tools for evaluating and approximating policy. We identify simple and testable conditions for a utility representation and a pseudometric, both of which are continuous on a parameter space with minimal structure. We then bring our results to bear upon the Syrian conflict and propose a policy that is stable relative to a sufficiently moderate (ϵ-lexicographic) warring faction.

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1 Introduction

During the current civil war in Syria, participants have often revealed extreme preferences over their strategy space. Even the more moderate factions have committed atrocities in return for small observable gains. Lexicographic orderings seem to provide a reasonable summary of behaviour in such settings. They provide are a particularly simple way factions to be or at least appear decisive.

Now consider a policy maker, Angela, who is trying to bring about less extreme behaviour and, eventually, a peaceful settlement. As an outsider, Angela has only imprecise measurements of the facts on the ground and her ostensible policy instruments are blunt or coarse to say the least: for simplicity, air strikes or negotiations. Nonetheless, policy is required, so what should Angela do?

The model we develop in the present paper allows us to explore the potential for policies that are stable or robust relative to a topology that is derived from a moderate lexicographic ordering of Angela's parameter space. To those with more extreme orderings, her policy will appear unstable and unpredictable. This raises the cost of committing attrocities and encourages these factions to become, or at least appear, more moderate.

By moderate, we mean that atrocities are only committed for an $\epsilon > 0$ gain in territory; for a fixed gain in territory, no atrocity is committed. Yet even such orderings are nonmetrizable and thus beyond the scope of related models of parametric continuity [8, 14, 13, 2]. These models provide a way of representing preferences over actions that vary with the parameter with utility function that is continuous. Whilst this way of summarising preferences would provide a useful check for a policy maker such as Angela, the motivation for continuity is particularly strong when the optimal policy must be approximated as is likely to be the case in fog of war.

Technical contribution Our main theorem identifies minimal conditions for parametric continuity of a utility representation of preferences on a fixed and countable set of actions. In settings with imprecise information, this is essential since it allows Angela to implement a good approximation of the optimal policy: any closed set in the parameter space is the intersection of a countable collection of open sets. This countable intersection property characterises what are known as perfectly normal topological spaces. In our lead example we go somewhat further, for although parameter space in nonmetrizable, distance to the optimal policy is well-defined.

Our key requirement is that strict preference for one action over another is robust to perturbations in the parameter. We refer to this property as *pairwise stability* (of strict preference). This condition is simple, and since it applies to pairs, it is testable. Moreover, in our second theorem, we show that it yields a continuous value function and an upper hemicontinuous policy correspondence provided the set of actions is also discrete.

Existing models require that the parameter space is metrizable. As a result, they exclude lexicographic orderings of the parameter space. These models also require that the graph of the (weak) preference correspondence (from the parameter space into set of action pairs) is closed. We show that pairwise stability is equivalent to the latter given our restriction on the set of actions.

Aside from providing the ability to test or approximate optimal policies, perfectly normal parameter spaces provide an upper bound in the following sense: every other parameter space supports preferences that satisfy the axioms, but have no utility representation that is continuous in the parameter.

2 Parameters, pairwise stability and choice

Let A denote a nonempty set of *actions* and let Θ a nonempty set of *parameters*. Recall that Θ (and where necessary A) becomes a topological space provided it is endowed with a certain collection τ of subsets. τ is a topology if it is closed under finite intersection and arbitrary unions. $G \in \tau$ is then called an *open* set. As usual, explicit reference to τ is typically suppressed.

A basic topological requirement on Θ (or any other topological space) that we take for granted in the sequel is that each singleton set $\{\theta\}$ in Θ is the complement of an open set and therefore *closed*: the $\mathcal{T}_1$ separation axiom of topology. For the results in the present section, we will also require that Θ is Hausdorff: if $\theta \neq \theta'$, then there exist disjoint open neighbourhoods N and N' of θ and θ' respectively. (This ensures that limits are well-defined.)

2.1 Parameter dependent preferences

We take as primitive, statements of the form "at θ, action b is strictly preferred to action a". Such statements are summarised by a family of binary relations $\prec_\theta$ on A, one for each $\theta \in \Theta$. $\prec_\theta$ is a subset of $A^2 = A \times A$ and is referred to as *preferences at* θ or *given* θ. The collection $\{\prec_\Theta\} \overset{\text{def}}{=} \{\prec_\theta : \theta \in \Theta\}$ is the object we refer to as *preferences*.

2.2 Ordering given the parameter

For the case where Θ is a singleton and strict preference is primitive, the following *ordering* condition is standard and debated elsewhere. It is a necessary condition for the existence of a utility function at each θ.

Axiom $\mathcal{O}$. *Both* asymmetry *and* negative transitivity *at each* θ :

$\mathcal{O}_1$ *For every* $(a, b, \theta) \in A^2 \times \Theta$*, if* $a \prec_\theta b$*, then not* $b \prec_\theta a$*;*

$\mathcal{O}_2$ *For every* $(a, b, c, \theta) \in A^3 \times \Theta$*, if* $a \prec_\theta b$*, then* $a \prec_\theta c$ *or* $c \prec_\theta b$*.*

Results in the present section only appeal to $\mathcal{O}_1$. It is common to write $a \sim_\theta b$ whenever a and b are such that neither $a \prec_\theta b$ nor $b \prec_\theta a$. In the presence of $\mathcal{O}$, this notation is consistent with the fact that $\{\sim_\Theta\}$ is a collection of indifference or equivalence relations on A. By the same token, when $\mathcal{O}$ holds, the weak preference relation $\precsim_\theta$, which is the disjoint union of $\prec_\theta$ and $\sim_\theta$, is complete and transitive at each θ.[†]

2.3 Pairwise stability of strict preference

Strict preference is *pairwise stable at* θ provided that, for every $a, b \in A$ such that $a \prec_\theta b$, there exists an open neighbourhood N of θ in Θ such that $a \prec_\eta b$ for every η in N. Since the arbitrary union of open sets is open, the following axiom captures (global) *pairwise stability* of strict preference.

Axiom $\mathcal{PS}$. *For every* $a, b \in A$*, the set* $\{\theta : a \prec_\theta b\}$ *is open in* Θ*.*

[†]Recall that $\precsim_\theta$ is *complete* if, for all a, b and c in A, $a \precsim_\theta b$ or $b \precsim_\theta a$; $\precsim_\theta$ is *transitive* if $a \precsim_\theta b \precsim_\theta c$ implies $a \precsim_\theta c$.

This axiom also appears in Gilboa and Schmeidler [5, 3], where the objective is to obtain a representation that is linear in the parameter. Our goal is to explore the implications for parametric continuity of the utility representation and policy. We begin by showing that, regardless of the topology on A, $\mathcal{PS}$ implies that the (strict) preference correspondence is lower hemicontinuous (l.h.c.). Recall that this holds when $\{\theta : \prec_\theta \cap B \neq \varnothing\}$ is open for every open $B \subseteq A^2$.

Proposition 1. *If preferences satisfy $\mathcal{PS}$, then the correspondence $\theta \mapsto \prec_\theta$ is lower hemicontinuous. If A is discrete, then the converse is also true.*

Even in the presence of $\mathcal{O}$, $\mathcal{PS}$ does not imply that the weak preference correspondence $\theta \mapsto \precsim_\theta$ is upper hemicontinuous. The latter requires that $\{\theta : \precsim_\theta \cap F \neq \varnothing\}$ is closed for every closed $F \subseteq A^2$. Indeed, let F be any closed, infinite subset of A^2, then although $\mathcal{O}$ and $\mathcal{PS}$ together ensure the set $\{\theta : a \precsim_\theta b\}$ is closed, the union over the pairs $a \times b \in F$, need not be closed. For the case where A is discrete, we now show that the axioms imply that the graph $\{(\theta, a, b) : a \precsim_\theta b\}$ of weak preferences is closed in $\Theta \times A^2$.

Proposition 2. † *Let A be discrete, let Θ be Hausdorff, and let preferences satisfy $\mathcal{O}_1$. Then $\mathcal{PS}$ holds if and only if the graph of $\theta \mapsto \precsim_\theta$ is closed.*

For the case where A is discrete, proposition 2 highlights an important connection between our model and existing theoretical results where the assumption that weak preferences define a closed correspondence is standard.

Axiom $\mathcal{CG}$. *The set $\{(\theta, a, b) : a \precsim_\theta b\}$ is closed in $\Theta \times A^2$.*

† See page 29 for proof.

Experiments in the field and the laboratory frequently deal with discrete action sets. As such, proposition 2 provides a simple and intuitive way of testing the hypothesis that weak preferences define a closed correspondence.

2.4 Pairwise stability and discrete choice

When A is discrete, $\mathcal{PS}$ alone is enough to yield upper hemicontinuity of a choice correspondence $C : \Theta \to 2^A$ that selects the set of undominated actions that are feasible at each θ. As well as being a requirement for many fixed point theorems, upper hemicontinuity of choice allows us to derive a continuous value function in theorem 3.

Lemma 1. [†] *Let A be discrete and let Θ be Hausdorff. For any continuous and compact-valued feasibility correspondence $\Phi : \Theta \to 2^A$, if preferences satisfy $\mathcal{PS}$, then the policy $C(\theta) \stackrel{\text{def}}{=} \{a \in \Phi(\theta) : \text{there is no } b \in \Phi(\theta) \text{ with } a \prec_\theta b\}$ is compact-valued and u.h.c. on Θ.*

When A is discrete and Θ is connected, the requirement that the feasibility correspondence is continuous (both u.h.c. and l.h.c.) is severe, for it implies that Φ is constant.[‡] In Sah and Zhao [19], for discrete A and Θ equal to the unit interval I, envelope theorems are derived. Similarly, in Milgrom and Segal [17], $\Theta = I$ in the canonical case, though no structure on A is assumed. In both of these papers, the feasibility correspondence is absent.

It turns out that the same space that we use for modeling our main application will also accommodate a continuous and variable feasibility cor-

[†] See page 30 for proof.

[‡] Recall that if Θ is connected, the only subsets that are both open and closed are $\varnothing$ and Θ itself; thus $\{\theta : \Phi(\theta) \subseteq B\}$ is either empty or equal to Θ for every $B \in 2^A$.

respondence. It is the following minimal extension of the unit interval.

2.5 The split interval

The canonical example of the parameter space we adopt is the *split interval*.

Definition. *Consider the usual unit interval I. Split each $r \in I$ into a pair of elements $r \times 0$ and $r \times 1$. The resulting set $\mathbb{I} \overset{\text{def}}{=} I \times \{0, 1\}$ is an ordered product with the standard lexicographic ordering: $\theta <^{\text{lex}} \eta$ in $\mathbb{I}$ if and only if $\theta_1 < \eta_1$ or $[\theta_1 = \eta_1$ and $\theta_2 = 0 < 1 = \eta_2]$.*

As with the unit interval, the order topology generated by $<^{\text{lex}}$ on the split interval consists of unions over the basic open order intervals $(\zeta, \eta) \overset{\text{def}}{=} \left\{\theta : \zeta <^{\text{lex}} \theta <^{\text{lex}} \eta\right\}$. For arbitrary $r < s$ in I, fig. 1 depicts the "clopen" interval $[r \times 1, s \times 0]$ in $\mathbb{I}$. These intervals allow for a feasibility correspondence that is both continuous and variable.

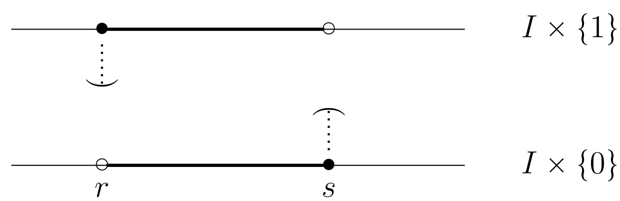

Figure 1: The closed interval $[r \times 1, s \times 0]$ is equal to the open interval $(r \times 0, s \times 1)$ because both $(r \times 0, r \times 1)$ and $(s \times 0, s \times 1)$ are empty intervals.

The split interval is a salient example of a compact, Hausdorff space that is perfectly normal, but nonmetrizable [10]. Recall that Θ is metrizable if there exists a metric $d : \Theta \times \Theta \to \mathbb{R}_+$ such that every open set $G \subseteq \Theta$ is a union of open balls $\{\theta : d(\theta, \eta) < \epsilon\}$ such that $\eta \in \Theta$ and $\epsilon > 0$.

Definition. *A* perfectly normal *space is both* perfect *and* normal. *A space is* perfect *if every closed subset is the intersection of a countable collection of open sets. A space is* normal *if, for every F and F' closed and disjoint, there exist disjoint open neighbourhoods N and N' of F and F' respectively.*

Note that metrizable spaces are perfectly normal. For when F and F' are closed and disjoint, there exists $\epsilon > 0$ with $d(\theta, \theta') \geqslant \epsilon$ for every $\theta \in F$ and $\theta' \in F'$. Moreover, for any closed set F, let $\{G_n : n \in \mathbb{N}\}$ be the collection of $1/n$ neighbourhoods of F and $F = \bigcap_1^\infty G_n$. These concepts are key to the results that now follow.

3 Parametric continuity of utility

In the present section, we first explore the possibility of representing preferences that satisfy $\mathcal{PS}$ with utility function that varies continuously with the parameter. We show that such representations are only guaranteed if the parameter space is perfectly normal. We then highlight the limitations of perfectly normal spaces and strengthen $\mathcal{PS}$ with a view to representing preferences when the parameter space is normal. This latter step substantially reduces the gap between assumptions of the previous section and those required for a utility representation. Recall that in the previous section, Θ was a Hausdorff space. Normal spaces clearly extend the Hausdorff assumption from singletons to arbitrary closed sets.

Throughout this section, although no external topology on A is required, our proof makes fundamental use of the assumption that A countable. The results are therefore purely concerned with continuity of utility in the parameter. This absence of structure on A is in the spirit of epistemic game theory [**Rubinstein˙Modeling˙bounded˙rationality**, 1, 3], where players choose pure strategies but have uncertainty about opponents' choices. This absence of structure also provides a partial axiomatic foundation for the modern approach to envelope theorems of Milgrom and Segal [17], a point that we discuss further in subsection 3.3.

3.1 Perfectly normal parameter spaces

Throughout this paper, a *utility function* $U(\cdot,\theta) : A \to \mathbb{R}$ for $\prec_\theta$ (or a *representation* of $\prec_\theta$) has the following property: for every $a, b \in A$, $a \prec_\theta b$ if and only if $U(a,\theta) < U(b,\theta)$. Formally, U is a *parametrically continuous (utility) representation* if it satisfies conditions 1 and 2 of the next theorem.

Theorem 1. *Let A be countable and let Θ be perfectly normal. $\mathcal{O}$ and $\mathcal{PS}$ hold for $\{\prec_\Theta\}$ if and only if there exists a function $U : A \times \Theta \to \mathbb{R}$ such that*

1. *for every $\theta \in \Theta$, $U(\cdot,\theta)$ is a utility function for $\prec_\theta$;*

2. *for every $a \in A$, $U(a,\cdot)$ is continuous on Θ.*

Proof of theorem 1. We proceed by induction on A. For the initial case (step 1), we take $|A| = 2$. This argument is useful for the discussion that follows, and present it next. Step 2 (the inductive step) appeals to Michael's selection theorem and appears on page 31 in the appendix.

Step 1. Let $A = \{a, b\}$. By $\mathcal{O}_1$ and $\mathcal{PS}$, $F = \{\theta : a \sim_\theta b\}$ is closed in Θ. Since Θ is perfect, there exists $\{G_n : n \in \mathbb{N}\}$ of open sets satisfying $\bigcap_1^\infty G_n = F$. For each n, note that F and $\Theta - G_n$ are disjoint and the latter is also closed. Since Θ is normal, the Urysohn lemma guarantees the existence of a continuous, real-valued function on Θ such that $f_n(\theta) = 0$ on F, $f_n(\theta) = 1$ on $\Theta - G_n$, and $0 \leqslant f_n(\theta) \leqslant 1$ otherwise. Note that these latter inequalities are not strict in general.

Let $f = \sum_1^\infty 2^{-n} f_n$ and note that $f : \Theta \to I$ is the continuous and uniform limit of the partial sums $\sum_1^m 2^{-n} f_n$. Moreover, since every $\theta \in \Theta - F$ belongs to some $\Theta - G_n$, $f(\theta) = 0$ if and only if $a \sim_\theta b$. Let $U(a, \cdot)$ be the zero function on Θ. We obtain a utility function for each $\prec_\theta$ by taking

$$U(b, \theta) \stackrel{\text{def}}{=} \begin{cases} f(\theta) & \text{if } a \prec_\theta b, \\ -f(\theta) & \text{otherwise.} \end{cases}$$

For continuity of $U(b, \cdot)$ note that: by $\mathcal{PS}$, for every θ such that $b \prec_\theta a$, there is an open neighbourhood N_θ such that $b \prec_\eta a$ for every $\eta \in N_\theta$; moreover $U(b, \cdot) = -f(\cdot)$ on N_θ; and finally, f is continuous on N_θ.

This completes step 1, the proof of theorem 1 continues on page 31. □

Step 1 in the proof of theorem 1 yields the "only if" part of the following well-established, alternative route to perfect normality. (For the converse, note that if $f : \Theta \to \mathbb{R}$ is a continuous function, then $G_n = \{\theta : |f(\theta)| < 1/n\}$ is an open neighbourhood of $F = f^{-1}(0)$ for each $n \in \mathbb{N}$ and $\bigcap_1^\infty G_n = F$.)

Proposition 3. *Θ is perfectly normal if and only if every closed subset F of Θ is a* zero set*. That is, for some continuous $f : \Theta \to \mathbb{R}$, $f^{-1}(0) = F$.*

Proposition 3 tells us that if a space is not perfectly normal, then it contains a closed subset that is not a zero set. This leads to the following justification of our claim that perfect normality is a minimal requirement for parametric continuity without further axioms on preferences.

Proposition 4. † *If* Θ *is not perfectly normal, then there are preferences that have no parametrically continuous representation and satisfy both* $\mathcal{O}$ *and* $\mathcal{PS}$*.*

An example of a compact, Hausdorff (and hence normal) parameter space that is not perfectly normal is the lexicographically ordered unit square $\mathbb{S}$. The order topology on $\mathbb{S}$ is generated by the same lexicographic ordering that characterises the split interval $\mathbb{I}$. In fact, $\mathbb{I}$ is an example of a closed subset of $\mathbb{S}$ that is not a zero set.‡ So if preferences are such that $\{\theta \in \mathbb{S} : a \sim_\theta b\} = \mathbb{I}$, then there is no function that satisfies conditions 1 and 2 of theorem 1.

In fact the same argument shows that, if the discrete dimension of the split interval contains three or more elements, we obtain a parameter space which fails to be perfect. That is, even the lexicographically ordered space $I \times \{0, 1, 2\}$ fails to be perfect. This surprising fact shows how restrictive even perfectly normal parameter spaces can be.

A second drawback of perfectly normal spaces for economic applications is that their product need not be perfectly normal. Indeed, the product $\mathbb{I}^2$ of the split interval with itself contains a copy of the Sorgenfrey plane (defined in the proof of proposition 5) which fails to be normal. If a space contains a subset that is not perfectly normal, then the space itself fails to be perfectly normal. In fact, with the usual product topology, even the compact set $\{0, 1\}^{\mathbb{R}}$

†See page 34 for proof.

‡If G is an open subset of $\mathbb{S}$ that contains $\mathbb{I}$, then G is the union of basic open order intervals the form $(\theta_1 \times \theta_2, \theta_1' \times \theta_2')$ such that $\theta_1 < \eta < \theta_1'$ and $\eta \in \mathbb{I}$. Since every such basic open interval contains uncountably many points η such that $\eta_2 \notin \{0, 1\}$, any countable intersection of such G still contains (uncountably many) elements of $\mathbb{S} - \mathbb{I}$.

of functions $g : \mathbb{R} \to \{0,1\}$ fails to be perfectly normal: each singleton set in $\{0,1\}^{\mathbb{R}}$ is not a zero set.†

In contrast, the product of an arbitrary number of compact Hausdorff spaces is compact Hausdorff. Moreover, since every compact Hausdorff space is normal, there is a rich supply of normal parameter spaces that are useful for economic applications that are neither metrizable nor perfectly normal.

3.2 Extension to normal parameter spaces

When Θ is not perfect, preferences may still be "perfectly pairwise stable".

Axiom $\mathcal{PS}^*$. *For every $a, b \in A$, $\{\theta : a \prec_\theta b\}$ is the open union of an increasing and countable collection of sets that are closed in Θ.*

When preferences satisfy $\mathcal{PS}^*$, there exists a countable collection $\{F_n\}$ of closed subsets of $\{\theta : a \prec_\theta b\}$, such that, for each $\eta \in \{\theta : a \prec_\theta b\}$, there exists $m \in \mathbb{N}$ such that $\eta \in F_m$. $\mathcal{PS}^*$ clearly helps to make identification of preferences tractable. In a consumer choice setting, a sequence of questions of the form "Do you strictly prefer b to a at every wealth level in the closed interval $[\theta_1, \theta_1']$?" allows us to approximate preferences to an arbitrary degree. Moreover, $\mathcal{PS}^*$ extends our model to normal parameter spaces.

Theorem 2. *Let A be countable and let Θ be normal. Preferences have a parametrically continuous utility representation if and only if $\mathcal{O}$ and $\mathcal{PS}^*$ hold.*

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†In the product topology, all but finitely many factors of the product are the whole of $\{0,1\}$. Thus, any intersection of a countable number of open sets, has all but countably many factors equal to the whole of $\{0,1\}$.

Proof of theorem 2 (from page 13). In the next subsection, we show that $\mathcal{PS}^*$ generates a pseudometric on Θ. Like a metric, the pseudometric we consider $p : \Theta^2 \to \mathbb{R}$ satisfies positivity, symmetry and the triangle inequality. But in contrast, it allows for $p(\zeta, \eta) = 0$ when $\zeta \neq \eta$. Such pairs are simply incomparable under p. Denote this incomparability relation on Θ by $\bowtie$.

The pseudometric we consider is generated by preferences, so that $\zeta \bowtie \eta$ implies [for every $a, b \in A$, $a \prec_\zeta b$ if and only if $a \prec_\eta b$]. This allows us to pass to the quotient space Θ_p that identifies points that are incomparable under p. (By the next subsection, it will be clear that $\bowtie$ is an equivalence relation.) By construction, Θ_p is metrizable and thus perfectly normal. Thus, by theorem 1, there exists a parametrically continuous utility representation $U : A \times \Theta_p \to \mathbb{R}$. Finally, the extension from Θ_p to Θ is simple: take U to be constant on each equivalence class of $\bowtie$. □

3.3 Pseudometrics for nonmetrizable spaces

Whilst this subsection is essential to the proof of theorem 2, it is also motivated by the search for envelope theorems that are so useful in applications see Milgrom and Segal [17] and Sah and Zhao [19]. In fact the pseudometric p that we construct next (and appeal to in the proof of theorem 2) allows us to define standard derivatives on the quotient space generated by the incomparability classes of p.

Lemma 2. [†] *Let A be countable and let Θ be normal. If preferences satisfy $\mathcal{O}$ and $\mathcal{PS}^*$, then there exists a continuous pseudometric $p : \Theta^2 \to \mathbb{R}_+$ such that $p(\zeta, \eta) = 0$ implies [for every $a, b \in A$, $a \prec_\zeta b$ if and only if $a \prec_\eta b$].*

[†] See page 34 for proof.

In the special case where Θ is the split interval, p of lemma 2 is a minor modification of the standard metric on I. In general however, since neither the pseudometric we construct, nor the utility representations of theorems 1 and 2 are unique upto a positive affine transformation, it makes little sense to speak of sufficient conditions for Lipschitz or absolute continuity that is so vital to the results of Milgrom and Segal [17]. To derive such a rich model from preferences, further conditions on preferences or on A are required.

At this point, it is worth noting that we view the present model as providing a starting point for utility representation and (pseudo)metrizing the parameter space. Our main goal is put utility and choice on a similar footing vis-á-vis preferences. In this view a more refined and numerically meaningful utility and pseudometric would be calibrated at a later stage in the modelling process, or revealed through experience.

In connection with this, one route to a pseudometric that is unique upto a single positive affine transformation is provided by the model of Gilboa and Schmeidler [5, 3, 4, 6]. However, the price of this uniqueness is that the representation is necessarily linear in the parameter. Moreover, the diversity axiom (for every $(a, b, c, d) \in A^4$ consisting of distinct actions, there exists θ such that $a \prec_\theta b \prec_\theta c \prec_\theta d$) is often too strong, a point that we discuss further in O'Callaghan [18]. Indeed, one of our original motivations was to weaken the axioms of Gilboa and Schmeidler and accommodate the nonlinearity in the parameter that we allow for here.

4 Joint continuity of utility

For the case where A is discrete, the next result shows that the same conditions (on preferences and the parameter space) that we assumed in the last section suffice for a utility representation that is jointly continuous. Recall that $U : A \times \Theta \to \mathbb{R}$ is jointly continuous if it is continuous on $A \times \Theta$.

Theorem 3. † *Let A be discrete and let Θ be normal. $\mathcal{O}$ and $\mathcal{PS}^*$ hold and A is countable if and only if preferences have a jointly continuous representation $U : A \times \Theta \to \mathbb{R}$. Moreover, if $\Phi : \Theta \to 2^A - \varnothing$ is a continuous, compact-valued correspondence, then the following value function is continuous on Θ*

$$V(\cdot) \stackrel{\text{def}}{=} \max \{U(a, \cdot) : a \in \Phi(\cdot)\} .$$

Note that when A is discrete, the requirement that it is also countable is necessary for a real-valued representation. Theorem 3 highlights a natural source of applications for the results of the preceding section. Building on the discussion of subsection 3.3, it also provides a partial axiomatic foundation for the model of Sah and Zhao [19]. Sah and Zhao identify conditions for envelope theorems to hold when A is discrete and each $U(a, \cdot)$ is continuous and concave on Θ [19, p.628].

4.1 Extension to uncountable A

When A is uncountable, $\mathcal{CG}$ of subsection 2.3 appears to be indispensible. It is possible that, analogous to $\mathcal{PS}^*$, a strengthening of $\mathcal{CG}$ to "preferences

†See page 35 for proof.

have a perfectly closed graph", might allow for parameter spaces that are normal. However, such a condition is still further removed from behaviour and would take us beyond the scope of the present paper.

In the present subsection, we confirm that Levin [13, Theorem 1] has an immediate extension to the case where Θ is perfectly normal. We also show that whenever $A \times \Theta$ is not perfectly normal, there exist preferences that satisfy $\mathcal{O}$, $\mathcal{CG}$ and have no jointly continuous utility representation.

Levin [13, Theorem 1] yields a jointly continuous representation for A second countable and locally compact (each point in A has a compact neighbourhood) and, of course, assumes $\mathcal{CG}$. The fact that Levin assumes that Θ is metrizable turns out to be unnecessary. The key to this extension is that the product of a second countable space and a perfectly normal space is perfectly normal [20, p.249]. The proof then follows directly from Levin [13].

Theorem 4. *Let A be second countable and locally compact and let Θ be perfectly normal. $\mathcal{O}$ and $\mathcal{CG}$ hold if and only if preferences have a jointly continuous utility representation.*

Without the requirement that $A \times \Theta$ is perfectly normal, we have the following result which is analogous to proposition 4.

Proposition 5. [†] *If $A \times \Theta$ is not perfectly normal, then there are preferences that have no jointly continuous representation and satisfy both $\mathcal{O}$ and $\mathcal{CG}$.*

We close with the observation that when $A \times \Theta$ is not perfectly normal, the proof of Levin may still hold, the only difference is that the representation is only guaranteed to be *separately continuous*: $U(a, \cdot)$ is continuous for each

[†] See page 36 for proof.

a and $U(\cdot, \theta)$ is continuous for each θ. In view of proposition 5, this comment applies to every space that meets the criteria of Caterino, Ceppitelli, and Maccarino [2], but fails to be perfectly normal. A full investigation of separate continuity would also take us beyond the scope of this paper.

5 Application to policy in Syria

Let $t \in I$ denote the proportion of Syrian territory controlled by a given warring faction. The discrete binary variable d describes whether or not a given faction has commited an atrocity. If $d = 0$, then the faction has commited an atrocity. Each faction lexicographically orders the resulting product $I \times \{0, 1\}$. As we now show, some lexicographic orderings are far less extreme than others. In all cases, factions are decisive (indifference sets are singletons) and are guided by simple, unambiguous rules.

Let $<^{0\times 1}$ coincide with the ordering that generates the split interval of subsection 2.5. That is, $t \times d <^{0\times 1} t' \times d'$ if and only if $t < t'$ or $[t = t'$ and $d < d']$. In other words, a faction with ordering $<^{0\times 1}$ is willing to commit an atrocity for any gain in territory; moreover, holding territory fixed, this faction prefers not to commit an atrocity. More moderate, yet equally decisive is the ϵ-lexicographic ordering $<^{\epsilon\times 1}$ and corresponding space $\mathbb{I}_{\epsilon\times 1}$ for some $\epsilon > 0$. This faction is only willing to commit an atrocity for sufficiently large gain in territory. Formally, $t \times d <^{\epsilon\times 1} t' \times d'$ if and only if one of the following mutually exclusive conditions hold:

1. $t < t'$ and $d = d'$ (more territory is better for fixed atrocity value);

2. $t + \epsilon < t'$ and $d' < d$ (willing to commit an atrocity for ϵ gain);

3. $|t - t'| \leqslant \epsilon$ and $d < d'$ (no atrocity is better for similar territory).

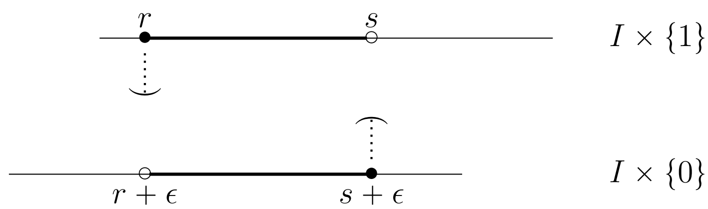

Figure 2: The moderate ϵ-lexicographic ordering finds the closed order interval $[r \times 1, (s+\epsilon) \times 0]$ equal to the open order interval $((r+\epsilon) \times 0, s \times 1)$.

It is straightforward to check that the indifference sets of $<^{\epsilon \times 1}$ are singletons. Morever, as ϵ tends to 0, this moderate ϵ-lexicographic ranking tends to $<^{0 \times 1}$. Figure 2 shows that $\mathbb{I}_{\epsilon \times 1}$ contains a copy of the split interval and is therefore nonmetrizable for every $\epsilon < 1$.

As ϵ tends to 1, $<^{\epsilon \times 1}$ tends to the lexicographic ordering that has dominant second dimension: every $t' \times d'$ with $d' = 1$ dominates every $t \times d$ with $d = 0$. The ordering $<^{1 \times 1}$ is that of a rare pacificist who would not commit an atrocity for any proportion of Syrian territory. Of all the above orderings on Θ, it is only the latter that generates a metrizable topology. Indeed, $\mathbb{I}_{1 \times 1}$ is homeomorphic to the union of two disjoint, compact intervals in $\mathbb{R}$.

At the other extreme are the orderings $<^{\epsilon \times 0}$, that would give up ϵ territory in order to commit an atrocity. For $\epsilon > 0$, such orderings would appear unsustainable, for they would run out of territory. Perhaps nomadic terrorists or suicide-bombers are a good match for such preferences. On the other hand, a faction that reveals $<^{0 \times 0}$ might be hard to distinguish from one with the ordering $<^{0 \times 1}$ that generates the split interval. Both these orderings commit atrocities for any gain in territory. The difference is that, holding territory fixed, $<^{0\times 0}$ would rather commit an atrocity.

Remark 1. *Whilst we only consider hypothetical orderings, the ordering* $<^{0\times 0}$ *seems to be a reasonable model of the preferences revealed by ISIS (Islamic State of Iraq and Syria). In May 2015, The Guardian reported that ISIS controls* 50% *of Syrian land following its capture of Palmyra.*[†] *ISIS then proceeded to commit numerous atrocities in Palmyra. These facts suggest that, at that time, ISIS was located at* $1/2 \times 0$ *in* $I \times \{0, 1\}$.

5.1 The parameter space

Note that, for any $0 \leqslant \epsilon, \epsilon' < 1$, the cartesian product $\mathbb{I}_\epsilon \times \mathbb{I}_{\epsilon'}$ is compact Hausdorff (and hence normal) but not perfectly so. This follows from the fact that $\mathbb{I}_\epsilon \times \mathbb{I}_{\epsilon'}$ contains a copy of the Sorgenfrey plane as defined in the proof of proposition 5. As consequence, whenever Angela's policy varies across factions, the only results that apply are lemma 1 and theorems 2 and 3. That is only results that allow for normal parameter spaces.

We now show that when Θ is the finite product of $\{\mathbb{I}_{\epsilon_n} : n = 1, \ldots, m\}$, ordered, a and preferences are *convex in the parameter* if $\{\theta : a \prec_\theta b\}$ is an order interval for every $a, b \in A$. Convexity in the parameter is often natural when dimensions in the parameter space index public orderings of the action or commodity space. Salient examples are prices, wealth or prestige [**RR˙Back˙to˙fundamentals**].

Proposition 6. [‡] *Let* Θ *be ordered and first countable (so that every singleton*

[†] www.theguardian.com/world/2015/may/21/isis-palmyra-syria-islamic-state
[‡] See page 37 for proof.

is a zero set). If preferences satisfy $\mathcal{PS}$ and are convex in the parameter, then $\mathcal{PS}^*$ holds.

5.2 Policy, utility and value

Consider the collection $\mathbb{P} = \{<^{\epsilon \times d} : \epsilon \times d \in I \times \{0,1\}\}$ of preferences described in the preceding subsection. It seems reasonable to suppose that (ceteris parabus) Angela's ranking of $\mathbb{P}$ would be such that $<^{\epsilon \times 0}$ lies below $<^{\epsilon' \times 1}$ for every $\epsilon, \epsilon' \in I$.

5.3 Extremism begets extremism

Suppose there are m factions, each controlling $t_n < t_{n+1}$ territory. To begin with, we suppose that the policy maker is

Although we do not consider such spaces in this paper, similar ϵ-lexicographic orderings are clearly possible for generalisations of the split interval that have more than two elements in discrete dimension. In particular, let $E \subseteq I$ denote the collection of factions. That is each $\epsilon \in E$, corresponds to an ordering $<^{\epsilon}$ and a space $\mathbb{I}_{\epsilon}$. Since each $\mathbb{I}_{\epsilon}$ is compact Hausdorff, so is the product over $\{\mathbb{I}_{\epsilon} : \epsilon \in E\}$. Thus, Θ is normal, but certainly not perfectly so. (Even the product of two copies of the split interval is not perfectly normal for it contains a subspace that is homeomorphic to the Sorgenfrey plane that is described in the proof of proposition 5.)

We suppose that Angela wishes to implement a policy that results in the product over $\{\mathbb{I}_{\epsilon} : \epsilon \geqslant \epsilon^*$ for every $\epsilon \in E\}$. The hope is that the threshold ϵ^* is high enough to encourage no faction to seek to increase its territory (except through peaceful means).

5.4 Evaluating policy

Let $\Theta \stackrel{\text{def}}{=} \mathbb{I}_\epsilon$ and let $\Phi : \Theta \to 2^A - \varnothing$ be Angela's feasibility correspondence. In line with the view that When , theorem 1 ensures that whenever Angela's preferences satisfy $\mathcal{O}$ and $\mathcal{PS}$ and A is countable, there exists a utility representation that represents her preferences and is continuous on Θ.

5.5 A simple three stage game

Suppose that there are m factions each of which controls territory $t_n < t_{n+1}$ for $n < m$ such that $\sum_1^m t_n = 1$. To simplify, suppose that there exists $0 < \epsilon' < 1$, such that no faction hopes to gain more than ϵ' territory from all the other factions put together. That is, each feasibility correspondence $\Phi_n : \prod_1^m \mathbb{I}_n \to \mathbb{I}_n$ is constrainted to lie within distance ϵ' of t_n in the territory dimension. In the atrocity dimension, we suppose that whenever n chooses to increase its territory, it faces a 50% chance of committing an atrocity.

In view of this, Angela wishes to implement a policy that ensures the revealed ordering of every faction is at least as moderate as the $<^{\epsilon'}$ ordering. She does so by announcing ϵ' at time $t = 0$. The announcement informs each faction n that Angela's preferences over $A = \{a, b\}^m$ (where a stands for airstrikes and b stands for negotiations) satisfy the following conditions:

1. they are pairwise stable on $\Theta = (\mathbb{I}_{\epsilon'})^m$;

2. for every $t_n \times 1 < t' < (t_n + \epsilon') \times 0$

Given lemma 1, this is equivalent to her policy being an upper hemicontinuous correspondence $C : \Theta \to 2^A$.

At $t = 1$, j faces the situation on the ground. We suppose that whenever j chooses to increase its territory, it faces a 50% chance of committing an atrocity. This has the effect that, if j chooses $r' > r$ such that $r' - r \leqslant \epsilon'$, he faces airstrikes with probability $\frac{1}{2}$.

We suppose that together, the situation on the ground, the tremble probability and the information contained in Angela's announcement mean that j faces a continuous and compact valued feasibility constraint $\Phi_j : \mathbb{I}_0 \to 2^{\mathbb{I}_{\epsilon'}} - \varnothing$ from which he chooses the θ' that maximises $<^j$. Finally, at $t = 2$, Angela implements $C(\theta')$.

Angela wishes to specify her policy for the faction j, where j controls $0 < r < 1$ of Syrian territory, has already committed an atrocity and has the ordering $<^j$ described above. At time $t = 0$, Angela makes an announcement $\epsilon' > 0$.

whether or not fight owill make their decision as to whether they will take the opportunities to take more land or not.

Angela seeks to implement a policy such that only the ϵ^* faction's Suppose that Angela knows that if every faction were to $\epsilon \in I$ such that Let $A_1 = I$ index the collection of possible ϵ-lexicographic orderings on Θ. We suppose that Angela will choose an optimal ϵ (and hence an optimal topology on Θ). She does so knowing the response that the warring factions will

and let $A_2 = \{a, b\}$, where a denotes "airstrikes" and b denotes "negotiations".

We suppose that Angela will

Angela's value function is separately continuous.

If Angela satisfies conditions $\mathcal{O}$ and $\mathcal{PS}$, then for each $a, b \in A$, $F \stackrel{\text{def}}{=} \{\theta :$

$a \sim_\theta b\}$ is a closed subset of Θ.

Case 1. Suppose that $F = I' \times \{0,1\}$ for some closed interval $I' \subseteq I$. In this case, Angela is indecisive for every $\theta \in F$ and strictly prefers to order airstrikes for every θ such that $\sup I' < \theta_1$. For any $\theta_1 < \inf I'$, she strictly prefers to negotiate a truce. In this case, Angela appears to reveal that she is only concerned with the first dimension of the parameter space.

Case 2. Suppose Angela is always decisive. That is, F is empty and there is a unique cutoff $k \in I$ such that $a \prec_\theta b$ if and only if $\theta \leqslant^{\text{lex}} k \times 0$. In this case, airstrikes are chosen whenever θ satisfies $k \times 0 <^{\text{lex}} \theta$. Note that the reason such preferences satisfy $\mathcal{PS}$ is that $\{\theta : \theta \leqslant^{\text{lex}} k \times 0\}$ is both open and closed. It is closed by definition of the order topology and open because it is equal to $\{\theta : x <^{\text{lex}} k \times 1\}$. (Since $\{0,1\}$ is discrete, there is no element of Θ that lies between $k \times 0$ and $k \times 1$.) In this case, Angela reveals that she cares about the second dimension: at proportion k she strictly prefers to negotiate only if the number of ISIS members is low.

Utility representation The lexicographically ordered set Θ is a well known example of a perfectly normal topological space that is not metrizable. Theorem 1 ensures that there is a continuous function that characterises Angela's preferences. The fact that Θ is not metrizable means that other, similar models such as Mas-Colell [14], Levin [13], and Caterino, Ceppitelli, and Maccarino [2] do not apply.[†] It is straightforward to generalise this example to include mixed strategies, for some countable set of mixtures of a and b.

[†]In fact, Θ is not submetrizable either and this excludes the model of Caterino, Ceppitelli, and Maccarino [2]. Submetrizability requires the existence of a function that is a continuous bijection to a metrizable space. Though this bijection need not be a homeomorphism.

Indeed, this example can easily be merged into the preceding abstract consumer choice problem. It is not hard to see the value of continuity in view of this. For instance, a comparative statics analysis where Angela's preferences vary with ϵ might be fruitful in more general settings with uncertainty.

Generalising the example The next proposition confirms the somewhat surprising fact that, without further assumptions regarding Angela's preferences, the present example cannot be extended to the case where second dimension contains any other elements $0 < \theta_2 < 1$. The issue is that any such parameter space fails to be perfectly normal. In this case, corollary **??** shows that preferences satisfying $\mathcal{O}$ and $\mathcal{PS}$ may fail to yield a continuous representation. For the purposes of the following propososition, let $F = I' \times \{0, 1\}$ for some nondegenerate, closed interval $I' \subset I$ just we assumed in case 1 of the present example.

Proposition 7. *For* $\Theta = I \times^{\text{lex}} \left\{0, \frac{1}{2}, 1\right\}$ *and* $\{\theta : a \sim_\theta b\} = F$. *Even when preferences satisfy* $\mathcal{O}$ *and* $\mathcal{PS}$*, there is no utility representation that is continuous in the parameter.*[†]

As we show next, when Θ is of the form of proposition 7, one way to guarantee that preferences have a representation is to consider preferences that are "convex in the parameter".

Convexity in the parameter The following, final axiom of the paper allows us to go beyond perfectly normal spaces in the case that Θ is an ordered topological space.

[†]See page 37 for proof.

Axiom $\mathcal{V}$. *Let Θ be an ordered set, then $\{\prec_\theta\}_{\theta\in\Theta}$ is convex in the parameter whenever, for each $a, b \in A$, the set $\{\theta : a \sim_\theta b\}$ is an order interval in Θ.*

The preferences in proposition 7 are not convex because for every $\theta, y \in \{\theta : a \sim_\theta b\} = I \times \{0, 1\}$ such that $\theta <^{\text{lex}} \eta$, there exists, $\zeta \in I \times \{1/2\}$ such that $\theta <^{\text{lex}} \zeta <^{\text{lex}} \eta$. Indeed, as the proof of proposition 7 shows, it is precisely the absence of the elements of $I \times \{1/2\}$ that prevent $I \times \{0, 1\}$ from being a zero set of Y.

Proposition 8. *If $A = \{a, b\}$, Θ is ordered and first countable and preferences satisfy $\mathcal{O}$, $\mathcal{PS}$ and $\mathcal{V}$, then preferences have a continuous representation.*

Proposition 8 provides a limited way of going beyond the confines of perfect normal parameter spaces. An example of an ordered space that is not first countable is the following. Consider the lexicographically ordered set $\{0, \ldots, \omega_1\} \times^{\text{lex}} [0, 1)$ where again the first dimension is dominant, but this time it is discrete, well-ordered and ω_1 is the smallest uncountable ordinal number. In this case, there is no continuous function that is zero precisely at $\omega_1 \times 0$.

6 Application to consumer choice

Let $A \subset \mathbb{R}_+^{n-1}$ be a discrete and countable set of commodities. This might, for instance, be the set $\mathbb{Z}_+^{n-1}$ of vectors with nonnegative integer-valued entries, or with rational entries that have decimal expansions restricted to (at most) 10 decimal places. Whilst this assumption is atypical, it has received relatively recent attention in [19, 17]. It has also been motivated in the game theoretic setting by [3, 1] and is standard in empirical settings where discrete choice econometric models (see [15]) are often used.

Let Θ be the Cartesian product of the set price-wealth vectors $\mathbb{R}^n_{++}$ with some other set of parameters Θ. The latter may include other socio-economic data in econometric settings or characterise frames in experiments.

Lemma 3. *If Θ is perfectly normal, then so is $\mathbb{R}^n_{++} \times \Theta$.*[†]

Each element of Θ is denoted by $\theta = (p, w, \theta)$, where p is the vector of prices $(p_1, \ldots, p_{n-1})$ and w denotes wealth. Angela's ability to choose elements of A is constrained by her budget. This is a standard, u.h.c. correspondence $\mathcal{B} : \Theta \to \mathbb{A}$, $\theta \mapsto \{a \in A : p \cdot a \leqslant w\}$.

With a view to ensuring the existence of a maximal element, we assume $\mathcal{B}(\theta)$ is compact for each θ. Since A is discrete, this holds if and only if $\mathcal{B}(\theta)$ is finite for each $\theta \in \Theta$.

Lemma 4. *$\mathcal{B}$ is u.h.c.*[‡]

We augment this to obtain a stable feasibility constraint $\Phi : \Theta \to \mathbb{A} \cup \varnothing_A$

$$\Phi(\theta) = \begin{cases} \varnothing_A & \text{if } p \cdot a = w \text{ for some } a \in A, \\ \mathcal{B}(\theta) & \text{otherwise.} \end{cases}$$

Preferences For each $\theta \in \Theta$, we assume Angela is able to rank the elements of A according to $\prec_\theta$ with a view to identifying the best element(s) in $\mathcal{B}(\theta)$. Thus $\{\prec_\theta\}_{\theta \in \Theta}$ satisfies $\mathcal{O}$. This yields a representation $U : A \times \Theta \to \mathbb{R}$

[†] See page 39 for proof.
[‡] See page 39 for proof.

satisfying condition 1 of theorem 1. If Angela is indifferent between two or more best elements, all such elements belong to her demand correspondence at θ. The latter is a map $\mathcal{D} : \Theta \to 2^A - \varnothing$ such that $\mathcal{D}(\theta) \subseteq \mathcal{B}(\theta)$ for all $\theta \in \Theta$. The standard model assumes that Θ is a singleton and that, for all $\theta, y \in \Theta$, $\prec_\theta = \prec_\eta$. The most natural generalisation allows preferences to vary across Θ. This possibility is considered in the example of the next subsection. For the present purposes, the only additional assumption we require is that $\mathcal{PS}$ holds.

Parametric continuity For the purposes of conducting a comparative statics analysis a minimal requirement is that there exists a continuous indirect utility function $V : \Theta \to \mathbb{R}$, $\theta \mapsto \max\{U(a, \theta) : a \in \mathcal{B}(\theta)\}$, and that $\mathcal{D}$ is u.h.c. The latter ensures that the demand correspondence is continuous whenever it is a function. Since we have assumed that A is countable and discrete, Θ is perfectly normal, $\mathcal{B}$ is u.h.c. and compact-valued and preferences satisfy $\mathcal{O}$ and $\mathcal{PS}$, theorem **??** yields the desired properties for V and $\mathcal{D}$.

7 Summary

We have given conditions on preferences and the parameter space for a general model of parametric continuity of preference. The main theorem shows that preferences satisfying the axioms can be represented by a function that is a utility given the parameter and is continuous on the parameter space.

Whilst the main drawback of the present model is that the set of actions must be countable, this assumption has allowed us to obtain the minimal conditions for parametric continuity. Firstly, the axioms on preferences are necessary and sufficient for parametric continuity of the representation. Secondly, if the parameter space is not perfectly normal, then there exist preferences that vary continuously with the parameter, but have representation that is continuous in the parameter.

When the set of actions is discrete, $\mathcal{PS}$, the axiom that captures pairwise stability of strict preference, is both necessary and sufficient for joint continuity of the representation on the product of actions and parameters. This yields a generalisation of existing results from the literature on jointly continuous representations to the case where the parameter space is perfectly normal. Via a simple extension of Berge's theorem of the maximum, this joint continuity allowed us to derive (i) a value function that is continuous and (ii) a choice correspondence that is upper hemicontinuous.

The applications demonstrated that there are novel settings to which the present results appear uniquely suited.

A Proofs

Proof. The map $\theta \mapsto \prec_\theta$ defines a correspondence on Θ with values in $2^{A\times A}$. This is l.h.c. provided that, for every open $G \subseteq A \times A$, the set $\{\theta : \prec_\theta \cap G \neq \varnothing\}$ is open. This latter set is just the union of $\{\theta : a \prec_\theta b\}$ such that $a \times b \in G$. Thus, $\mathcal{PS}$ implies l.h.c. of $\theta \mapsto \prec_\theta$.

When A is discrete, every $B \subseteq A \times A$ is both open and closed. If $\theta \mapsto \prec_\theta$ is l.h.c., take $B = \{a \times b\}$. Then $\{\theta : \prec_\theta \cap B \neq \varnothing\}$ is open and equal to $\{\theta : a \prec_\theta b\}$. Thus, when A is discrete the converse holds, as required. $\square$

Proof of proposition 2 (from page 6). Suppose $\mathcal{PS}$ holds. Let D be a directed set and let $\eta_\nu \times a_\nu \times b_\nu \to \eta \times a \times b$ be a net such that $a_\nu \precsim_{\eta_\nu} b$ for each $\nu \in D$. Then there exists $\mu \in D$ such that for every $\nu \geqslant \mu$, $\eta_\nu \times a_\nu \times b_\nu = \eta_\nu \times a \times b$. Now suppose that $b \prec_\eta a$, so that the graph of $\theta \mapsto \precsim_\theta$ is not closed. Then by $\mathcal{PS}$, there exists a neighbourhood N of θ such that $a \prec_\theta b$ for every $\theta \in N$. But since the net converges to $\eta \times a \times b$, there exists m' such that $\eta_\nu \in N$ for every $\nu \geqslant \mu'$. But then for every $\nu \geqslant \max\{\mu, \mu'\}$, we have both $a \precsim_{\eta_\nu} b$ (because $\eta_\nu \times a \times b$ belongs to the graph of $\precsim_\Theta$) and $b \prec_{\eta_\nu} a$ (because $\eta_\nu \in N$): a contradiction of $\mathcal{O}_1$.

Now suppose that the graph of $\theta \mapsto \precsim_\theta$ is closed. Then for fixed $a, b \in A$, the set $\{(\theta, a, b) : a \precsim_\theta b\}$ is closed. By $\mathcal{O}_1$, this is equivalent to $\mathcal{PS}$. □

Proof of lemma 1 (from page 7). u.h.c. of C follows from **AB˙Hitchhikers˙guide**: the intersection of a closed correspondence and a compact-valued u.h.c. correspondence is u.h.c. By assumption, Φ is u.h.c. and compact-valued, and since it is a feasibility constraint, at each θ, $C(\theta)$ is indeed equal to the intersection $C(\theta) \cap \Phi(\theta)$. Therefore, C is u.h.c. provided it is a closed correspondence: that is, provided the graph $\operatorname{gr} C \stackrel{\text{def}}{=} \{(\theta, a) : a \in C(\theta)\}$ is closed.

First note that Φ is itself a closed correspondence by **AB˙Hitchhikers˙guide** which requires A Hausdorff, Φ u.h.c. and compact-valued. .Let $(\theta_n, a_n)_{n \in D}$ be a net with values in $\operatorname{gr} C$ and limit equal to (η, a). Since $C(\theta) \subseteq \Phi(\theta)$ for every $\theta \in \Theta$, $a \in \Phi(\theta_n)$ for every n. Since Φ is a closed correspondence, a is feasible at θ. Since A is discrete, the singleton set $\{a\}$ is the smallest open neighbourhood of any $a \in A$. Since $(\theta_n, a_n) \to (\theta, a)$, there exists $m \in D$ such that $(\theta_n, a_n) = (\theta_n, a)$ for every $n \geqslant m$.

By way of contradiction, suppose $a \prec_\eta b$ for some $b \in \Phi(\eta)$. (So that $a \notin$

$C(\eta)$ and $\operatorname{gr} C$ is not closed.) Since Φ is l.h.c., $\Phi^{-}(b) \stackrel{\text{def}}{=} \{\theta : \Phi(\theta) \cap \{b\} \neq \varnothing\}$ is open. Since $\eta \in \Phi^{-}(b)$, there is a neighbourhood N of η such that b is feasible on N. By $\mathcal{PS}$, there exists a neighbourhood N' of η such that $a \prec_{\theta} b$ for every $\theta \in N'$. Let $N'' = N \cap N'$. Since $\theta_n \to \eta$, there exists m' such that $\theta_n \in N''$ for every $n \geqslant m'$. Let $m'' = \max\{m, m'\}$. Then for every $n \geqslant m''$ both $a, b \in \Phi(\theta_n)$ and $a \prec_{\theta_n} b$. But then we arrive at a contradiction, for every $n \geqslant m''$, $a_n = a$ is suboptimal. That is, contrary to our assumption, we have shown that $(\theta_n, a_n) \notin \operatorname{gr} C$ for every $n \geqslant m''$.

Finally, the fact that C is compact-valued follows because $C(\theta)$ is a closed subset of the compact set $\Phi(\theta)$ for each θ. □

REMAINING STEPS IN THE PROOF OF THEOREM 1 (FROM PAGE 10).

STEP 1. See page 10 for the initial step in the induction on A.

STEP 2 (INDUCTIVE STEP). Let $\{1, 2, 3 \ldots\}$ be an arbitrary enumeration of A, and let $[j] \subseteq A$ denote the first j elements of the enumeration. Fix $j \in A$. The induction hypothesis ensures the existence of a function $U^{j-1} : [j-1] \times \Theta \to [-1, 1]$ that satisfies (1) and (2) of theorem 1. For each $a \in [j-1]$ take $U^{j}(a, \cdot) \stackrel{\text{def}}{=} U^{j-1}(a, \cdot)$. It remains to show that we can find an extension of U^{j} to $[j]$ that satisfies (1) and (2) of theorem 1.

The required function $U^{j}(j, \cdot)$ will coincide with f in the following version of Michael's selection theorem [16, Theorem 3.1"'].

THEOREM (Good and Stares [7]). *Θ is perfectly normal if and only if, whenever $g, h : \Theta \to \mathbb{R}$ are respectively upper and lower semi-continuous functions and $g \leqslant h$, there is a continuous $f : \Theta \to \mathbb{R}$ such that $g \leqslant f \leqslant h$ and $g(\theta) < f(\theta) < h(\theta)$ whenever $g(\theta) < h(\theta)$.*

In our setting, g and h will be envelope functions. To ensure they are well-defined, we introduce two fictional actions $\underline{a}$ and $\overline{a}$. These satisfy the property: $\underline{a} \precsim_\theta k \precsim_\theta \overline{a}$ for all $(k, \theta) \in [j] \times \Theta$. Accordingly, we define $[j-1]' = [j-1] \cup \{\underline{a}, \overline{a}\}$, and let $U^j(\underline{a}, \cdot) \equiv -1$ and $U^j(\overline{a}, \cdot) \equiv +1$. Both are clearly continuous functions on Θ. Moreover, for all $\theta \in \Theta$, the following functions are well-defined.

$$g(\theta) \stackrel{\text{def}}{=} \max \left\{ U^j(k, \theta) : k \precsim_\theta j \text{ and } k \in [j-1]' \right\},$$
$$h(\theta) \stackrel{\text{def}}{=} \min \left\{ U^j(k, \theta) : j \precsim_\theta k \text{ and } k \in [j-1]' \right\}.$$

In the three claims that follow, we prove that g and h satisfy the conditions for Michael's selection theorem. In particular $g \leqslant h$; $g(\theta) < h(\theta)$ whenever $j \not\sim_\theta k$ for every $k \in [j-1]'$; g is upper semicontinuous and h is lower semicontinuous. The inductive step is then completed by letting $U^j(j, \cdot) = f$, where f satisfies the conditions of Michael's selection theorem. Clearly, U^j satisfies 1 and 2 of theorem 1. Moreover, U^j takes values in $[-1, 1]$.

Claim 1. *For all* $\theta \in \Theta$, $g(\theta) \leqslant h(\theta)$.

Proof of claim 1. Fix θ. By construction, there exist $k, l \in [j-1]'$ satisfying $g(\theta) = U^j(k, \theta)$ and $h(\theta) = U^j(l, \theta)$. By definition, $k \precsim_\theta j$ and $j \precsim_\theta l$. By $\mathcal{O}_2$, $k \precsim_\theta l$ and the inductive hypothesis then ensures that $g(\theta) \leqslant h(\theta)$. □

Claim 2. *For all* $\theta \in \Theta : g(\theta) = h(\theta)$ *iff* $k \sim_\theta j$ *for some* $k \in [j-1]$.

Proof of claim 2. If $g(\theta) = h(\theta)$, then, by construction, there is some $k \in [j-1]' \cap \{l : l \precsim_\theta j\} \cap \{l : j \precsim_\theta l\}$. By $\mathcal{O}_1$, for every such k, $k \sim_\theta j$. Conversely, if $k \sim_\theta j$, then both $k \precsim_\theta j$ and $j \precsim_\theta k$. □

Claim 3. *$g : \Theta \to \mathbb{R}$ is upper semicontinuous.*

A symmetric argument to the one that follows, but with inequalities and direction of weak preference reversed, shows that h is lower semicontinuous.

Proof of claim 3. Recall (or see [11, p.101]) that g is upper semicontinuous provided the set $\{\theta : r \leqslant g(\theta)\}$ is closed for each $r \in \mathbb{R}$. Note that by the construction of g,

$$\{\theta : r \leqslant g(\theta)\} = \bigcup_{k \in [j-1]'} \left(\{\theta : r \leqslant U^j(k,\theta)\} \cap \{\theta : k \precsim_\theta j\}\right).$$

Recall that the finite union of closed sets is closed. Moreover, since $U^j(k,\cdot)$ is continuous, $\{\theta : r \leqslant U^j(k,\theta)\}$ is closed (preimage of a closed set is closed); and $\{\theta : k \precsim_\theta j\}$ is closed by $\mathcal{O}_1$ and $\mathcal{PS}$. □

Step 3 (The countably infinite case). The above argument holds for each j in $\mathbb{N}$.† For countably infinite A, we choose $U : A \times \Theta \to \mathbb{R}$ such that its graph satisfies $\operatorname{gr} U = \bigcup_{j \in \mathbb{N}} \operatorname{gr} U^j$. Since Michael's selection theorem is used at each j, for this step we appeal to the axiom of dependent choice. Alternatively, following [12, p.23], let $U(j,\cdot) = U^j(j,\cdot)$ for each $j \in \mathbb{N}$, and again appeal to the axiom of (dependent) choice.

Step 4 (Necessity of the axioms). The necessity of $\mathcal{O}_1$ and $\mathcal{O}_2$ is well-known and the following argument confirms that $\mathcal{PS}$ is necessary.

Take any $U : A \times \Theta \to \mathbb{R}$ satisfying (1) and (2) of theorem 1. Fix $a, b \in A$. Let $G = \{\theta : U(a,\theta) - U(b,\theta) < 0\}$. Since the difference of two continuous functions is continuous, G is open. Moreover, $G = \{\theta : a \prec_\theta b\}$.

†I thank Atsushi Kajii for bringing this subtle issue to my attention.

This completes the proof of theorem 1. $\square$

Proof of proposition 4 (from page 12). Let $A = \{a, b\}$ and suppose that $F = \{\theta : a \sim_\theta b\}$ for some closed set F that is not a zero set. Such an F exists whenever Θ fails to be perfectly normal. (See **??** for an explicit example of such a set.) Since preferences satisfy $\mathcal{O}_1$ and there are only two actions, there exists a representation of preferences. Take $U : A \times \Theta \to \mathbb{R}$ to be any such representation and define $f : \Theta \to \mathbb{R}$ to be the map $\theta \mapsto U(a, \theta) - U(b, \theta)$. Since U is a representation, $f(\theta) = 0$ if and only if $\theta \in F$. Thus $f^{-1}(0) = F$ and, since F is not a zero set, $f = U(a, \cdot) - U(b, \cdot)$ is discontinuous. By the algebra of continuous functions, at least one of $U(a, \cdot)$ and $U(b, \cdot)$ is discontinuous. $\square$

Proof of lemma 2 (from page 14). For any given $a, b \in A$, the set $F_{ab} = \{\theta : a \sim_\theta b\}$ is closed by $\mathcal{PS}^*$. Moreover, $\mathcal{O}_1$ and $\mathcal{PS}^*$ ensure the existence of a countable and decreasing sequence of open sets with intersection equal to F_{ab}. Since Θ is normal, the argument of step 1 of theorem 1 ensures the existence of a continuous function $f_{ab} : \Theta \to [-1, 1]$ such that $f_{ab}^{-1}(0) = F_{ab}$ and $0 < f_{ab}(\theta)$ if and only if $a \prec_\theta b$. Let $p_{ab}(\zeta, \eta) \stackrel{\text{def}}{=} |f_{ab}(\zeta) - f_{ab}(\eta)|$ for each $\zeta, \eta \in \Theta$.

Clearly $p_{ab} : \Theta^2 \to \mathbb{R}$ inherits positivity, symmetry and the triangle inequality from $|\cdot|$ on $\mathbb{R}$. Moreover, $p_{ab}(\zeta, \eta) = 0$ implies [$a \prec_\zeta b$ if and only if $a \prec_\eta b$]. (The latter holds because whenever $b \precsim_\zeta a$ and $a \prec_\eta b$, we have $f_{ab}(\zeta) \leqslant 0 < f_{ab}(\eta)$, so that $p_{ab}(\zeta, \eta) \neq 0$.)

The above argument generates a collection of continuous pseudometrics $\Pi \stackrel{\text{def}}{=} \{p_{ab} : a, b \in A\}$ on Θ. Crucially for the next step, A is countable: the collection of pseudometrics is then countable; only countable intersections of perfect sets are countable. For an arbitrary enumeration $\{p_1, p_2, \dots\}$ of Π, take $p \overset{\text{def}}{=} \sum_1^\infty 2^{-n} p_n$. Clearly, if $p(\zeta, \eta) = 0$, then $p_{ab}(\zeta, \eta) = 0$ for every $a, b \in A$. By the preceding paragraph therefore, it only remains to check that p is indeed a continuous pseudometric. Since each p_n is nonnegative and symmetric with values in $[0, 2]$, so is p. Moreover, for each m, the partial sum $\sum_1^m 2^{-n} p_n(\theta, \eta)$ satisfies the triangle inequality by induction: the sum of two pseudometrics preserves this inequality. The sandwich or squeeze lemma for sequences then ensures $p(\theta, \eta)$ also satisfies the triangle inequality. Continuity follows by uniform convergence of the continuous partial sums to p. This completes the proof of the lemma. □

Proof of theorem 3 (from page 16). U is jointly continuous by the following argument. Fix $(a, \theta) \in A \times \Theta$ and consider, for some directed set D, a net $E = ((a_\nu, \theta_\nu))_{\nu \in D}$ in $A \times \Theta$ with limit (a, θ). We show that $U(a_\nu, \theta_\nu) \to U(a, \theta)$. Recall that (a, θ) is the limit of E if and only if, for every neighborhood N of (a, θ), there exists $\mu \in D$ such that for every $\nu \geqslant \mu$, $(a_\nu, \theta_\nu) \in N$. Since A is discrete, $\{a\}$ is open and for some N_θ open in Θ, the set $\{a\} \times N_\theta$ is an (open) neighborhood of (a, θ) in the product topology on $A \times \Theta$. Thus, there exists μ such that for every $\nu \geqslant \mu$, $U(a_\nu, \theta_\nu) = U(a, \theta_\nu)$. Finally, part 2 of theorem 1 ensures that $U(a, \theta_\nu) \to U(a, \theta)$.

For continuity of V, let $U^* : \Theta \times A \to \mathbb{R}$ satisfy $U^*(\theta, a) \overset{\text{def}}{=} U(a, \theta)$ for every $(\theta, a) \in \Theta \times A$. By the preceding paragraph, U^* is continuous on $\Theta \times A$. In lemma 1, we derived a u.h.c. choice correspondence C that coincides with $\operatorname{argmax}\{U(a, \cdot) : a \in \Phi(\cdot)\}$. Finally, note that $V = U^* \circ \operatorname{gr} C$. V is then u.h.c. as the continuous composition of u.h.c. correspondences [**AB˙Hitchhikers˙guide**], and since it is single-valued, it in fact continuous.

□

Proof. [PROOF OF PROPOSITION 5 (FROM PAGE 17)] By assumption, there exists a closed, nonzero subset F of $A \times \Theta$. Let $\{(a, \theta) : a \sim_\theta b\} = F$ and let preferences satisfy $\mathcal{O}$ and $\mathcal{CG}$ on $A - \{b\}$. Then every representation has $U(a, \theta) - U(b, \theta) = 0$ for every $(a, \theta) \in F$. Let U' be the following transformation of U. For every $a \in A$, $U'(a, \cdot) = U(a, \cdot) - U(b, \cdot)$. Then $U' : A \times \Theta \to \mathbb{R}$ satisfies $U'(F) = 0$. That is, $(a, \theta) \in F$ implies $U'(a, \theta) = 0$. Let $b \prec_\theta a$ for every (a, θ) in the open set $(A \times \Theta) - F$. Since F is closed and $b \precsim_\theta a$ for every $(a, \theta) \in A \times \Theta$, preferences satisfy $\mathcal{O}$ and $\mathcal{CG}$ on all of A. Since $U'(b, \cdot)$ is identically equal to zero, $0 < U'(a, \theta)$ for every $(a, \theta) \notin F$. Since $F = (U')^{-1}(0)$ is not a zero set, U' is discontinuous on $A \times \Theta$.

For an explicit example consider the Sorgenfrey line $\mathbb{L}$. This is the unit interval I where the basic open sets are half-open intervals $[r, s)$ such that $r < s$ in I. $\mathbb{L}$ is a well-known example of a perfectly normal, separable space that is not second countable and such that the Sorgenfrey plane $\mathbb{L}^2$ is not normal. Take A to be the discrete union of $\mathbb{L}$ and $\{b\}$ for some $b \notin \mathbb{L}$ and take $\Theta = \mathbb{L}$. Finally, take F to be the anti-diagonal of $\mathbb{L}^2$ and let $\{\prec_\Theta\}$ be such that for each $-r \in \Theta$, r is the worst element in $A - \{b\}$; $\prec_{-r}$ assigns higher order to elements that are further from r according to the standard metric on $\mathbb{R}$; and, moreover, for each feasible $\epsilon > 0$, $-\epsilon + r \sim_{-r} \epsilon + r$. Finally, for every rational number $q \in \mathbb{L}$, let $b \sim_{-q} q$; and for every irrational number $s \in \mathbb{L}$, suppose that $b \prec_{-s} s$.

Clearly $\{\prec_\Theta\}$ satisfies $\mathcal{O}$. To check $\mathcal{CG}$, suppose otherwise that $a_\nu \sim_{\theta_\nu} b$ for every ν and $(a_\nu, \theta_\nu) \to (a, \eta)$ such that $b \prec_\eta a$. Then by construction, each θ_ν is a rational number and $a_\nu = -\theta_\nu$. Moreover, since $a_\nu \to a$ and $\theta_\nu \to \eta$, we have $a = -\eta$. Since the anti-diagonal of $\mathbb{L}^2$ is a discrete, there exists a finite number μ such that $(a_\nu, \theta_\nu) = (a, \eta)$ for every $\nu \geqslant \mu$, a contradiction of the assumptions regarding the sequence. $\square$

Proof of proposition 6 (from page 20). Let θ' be an element of the closure of $G = \{\theta : a \prec_\theta b\}$. Since G is convex in the order, we may, w.l.o.g., suppose that θ' is a least upper bound for G. Since Θ is first countable ([**SS˙Counterexample**]) there exists a countable collection $\{N_n\}$ of open neighbourhoods of θ' such that, for every neighbourhood N of θ', $N_n \subseteq N$ for some n. Since singleton sets are closed, it is not hard to show that $\{\theta'\} = \bigcap_1^\infty N_n$.

Let $F_n = G \cap (\Theta - N_n)$ for each n. Then each F_n is a closed subset of G and the union of these sets is precisely G. Finally, if it is not the case that $F_n \subseteq F_{n+1}$ for each n, let F'_n be a new sequence such that $F'_1 = F_1$ and $F'_m = \bigcup_1^m F_n$ for each $m \geqslant 2$. Since $\{F'_n\}$ is an increasing and countable collection of sets that are closed in Θ, $\mathcal{PS}^*$ holds. $\square$

Proof of proposition 7 (from page 25). We first show that $F := I \times \{0, 1\}$ is closed in Θ, so that $\mathcal{PS}$ holds. It suffices to show that F contains each of its limit points. Let θ be a limit point of F.

We first show that although $w = \inf^{\text{lex}} F$ belongs to F, it is not a limit point. We do so by identifying a neighbourhood N_w containing no element of F except w. W.l.o.g. let N_w be the order interval (η, ζ) for some $\eta <^{\text{lex}} w <^{\text{lex}} w_1 \times 1/2$, where w_1 is the first element of $w = w_1 \times 0$. Clearly, N_θ is open and contains no element of F other than θ. A similar argument shows that the supremum m of F is not a limit point either.

Next take any θ satisfying $w <^{\text{lex}} x <^{\text{lex}} m$ and $\theta_2 = 0$ and let $N_\theta = (\eta, \zeta)$ for some $\eta <^{\text{lex}} x <^{\text{lex}} \zeta$. We show that there exists $\theta' \in F$ such that $\eta <^{\text{lex}} x' <^{\text{lex}} \theta$. Since $\eta <^{\text{lex}} \theta$, the definition of the lexicographic order ensures that $\eta_1 < \theta_1$. Let $\theta_2 = 0$ and take $\theta'_1 = w_1$ if $\eta <^{\text{lex}} w$ and $\theta'_1 = {}^1\!/\!_2(\eta_1 + \theta_1)$ otherwise. Then θ' belongs to $F \cap N_\theta$. A similar argument holds for θ such that $\theta_2 = 1$.

We now show that there is no continuous representation of preferences. Let $U : A \times \Theta \to \mathbb{R}$ be a representation of $\{\prec_\theta\}_{\theta \in \Theta}$. (Such a representation exists since $A = \{a, b\}$ and $\mathcal{O}_1$ holds.) W.l.o.g., let $U(a, \cdot) \equiv 0$ and let $f := U(b, \cdot)$. Then $f^{-1}(0) = F$. We suppose that f is continuous and derive a contradiction. Since f is continuous, it has the property that $G_n = \left\{\theta : |f(\theta)| < \frac{1}{n}\right\}$ is open in Y for each $n \in \mathbb{Z}_{++}$. Moreover, $F \subset G_n$ for each n. The contradiction we seek is $\bigcap_1^\infty G_n \neq F$.

We claim that each G_n contains all but finitely many elements of $I \times \{\frac{1}{2}\}$. This will suffice for our purpose because the union of countably many finite sets is countable, so that the intersection $\bigcap_1^\infty G_n$ contains (uncountably many) elements of $I \times \{\frac{1}{2}\}$.

Recall that, since I is compact in $\mathbb{R}$, each of its open covers has a finite subcover. Every open set G such that $F \subset G$ is the union of some collection of basic intervals $\{(\eta^k, \zeta^k)\}$ in Θ. Consider the collection of projected intervals $\{(\eta_1^k, \zeta_1^k)\}$ in $\mathbb{R}$. This collection covers I, for otherwise, there exists $\theta \in I$ such that $\theta_1 \notin F$. By the compactness of I, we can find a finite subcover $\{(\eta^k, \zeta^k)\}_{k=1}^n$ of F. Now, for every $r \in \mathbb{R}$ satisfying $\eta_1^k <^{\text{lex}} r <^{\text{lex}} \zeta_1^k$, the point $\theta = r \times {}^1\!/\!_2$ belongs to $(\eta^k, \zeta^k) \subset G$. □

Proof of proposition 8 (from page 26). By the axioms, let $F = \{\theta :$

$a \sim_\theta b\}$ be a closed, convex subset of Θ and suppose that $a \prec_\eta b$ if and only if $\eta < \theta$ for every $\theta \in F$. Let the set $\{\eta : a \prec_\eta b\}$ be nonempty. Let θ denote the least element of F. θ is well-defined by virtue of the fact that F is closed and convex. Since Θ is first countable, there exists a countable sequence of open sets $G_n = \{\zeta : \eta^n < \zeta\}$ such that $\bigcap_1^\infty G_n = \{\zeta : \theta \leqslant \zeta\}$. Note that $\Theta - G_n$ is a closed subset of $\{\theta : a \prec_\theta b\}$. Since every ordered space is normal, the argument of step 1 of theorem 1 shows that we may find a continuous and nonnegative function f such that $f(\theta) > 0$ if and only if $a \prec_\theta b$. In the same way, the greatest element in F yields a nonnegative function g such that $g(\theta) > 0$ if and only if $b \prec_\theta a$. The desired representation $U : A \times \Theta \to \mathbb{R}$ follows directly from theorem 1. All remaining cases are either trivial or similar. □

Proof of lemma 3 (from page 27). By [20, p.249], the cartesian product of a second countable space with a perfectly normal space is perfectly normal. $\mathbb{R}$ is second countable as it has a countable basis: the open intervals with rational endpoints. □

Proof of lemma 4 (from page 27). Since $\mathcal{B}$ is independent of $\theta \in \Theta$, it suffices to consider sequences in $\mathbb{R}^n_{++}$. We prove that $\mathcal{B}$ satisfies the following definition for upper hemicontinuity: for any sequence (p^k, w^k) in $\mathbb{R}^n_{++}$ with limit (p, w) and open $G \subseteq A$ such that $\mathcal{B}(p, w) \subseteq G$, there exists $l \in \mathbb{N}$ such that for all $k \geqslant l$, $\mathcal{B}(p^k, w^k) \subseteq G$.†

Let $B(p, w) \subseteq G$ for some arbitrary nonempty subset of A. Seeking a contradiction, we suppose there exists a sequence $(p^k, w^k) \to (p, w)$ with the

†This definition is equivalent to the earlier definition by [9, Lemma 2.1.1].

following property: for every $l \in \mathbb{N}$, there exists $k \geqslant l$ such that $a^k \in \mathcal{B}(p^k, w^k)$ such that $a^k \notin G$. Since $B(p, w) \subseteq G$ this is equivalent to $r_k = p^k \cdot a^k - w^k \leqslant 0$ and $s_k = p \cdot a^k - w > 0$ respectively.

We pass to the sequence of such k. Note that since $(p^k, w^k) \to (p, w)$, $r_k - s_k \to 0$. Then $r_k \leqslant 0 < s_k$ for all k implies $s_k \to 0$. Fix $\epsilon > 0$, then there are infinitely many k such that $0 < s_k < \epsilon$. Thus, the set $\mathcal{B}(p, w + \epsilon)$ contains every element in $\mathcal{B}(p, w)$ and, for k sufficiently large, every a^k that defines s_k. Moreover, $\mathcal{B}(p, w + \epsilon)$ is finite, so that the sequence (s_k) is finite-valued. But since $s_k \to 0$, there exists $l \in \mathbb{N}$ such that for every $k \geqslant l$, $s_k = 0$. This contradiction completes the proof. □

References

[1] Robert Aumann and Adam Brandenburger. "Epistemic Conditions for Nash Equilibrium". English. In: *Econometrica* 63.5 (1995), ISSN: 00129682.

[2] Alessandro Caterino, Rita Ceppitelli, and Francesca Maccarino. "Continuous utility functions on submetrizable hemicompact k-spaces". In: *Applied General Topology* 10 (2 2009), pp. 187–195.

[3] Itzhak Gilboa and David Schmeidler. "A derivation of expected utility maximization in the context of a game". In: *Games and Economic Behavior* 44.1 (2003), pp. 172–182. ISSN: 0899-8256. DOI: 10.1016/S0899-8256(03)00015-0.

[4] Itzhak Gilboa and David Schmeidler. *A theory of case-based decisions.* Cambridge University Press, 2001. ISBN: 9780521003117.

[5] Itzhak Gilboa and David Schmeidler. "Act similarity in case-based decision theory". In: *Economic Theory* 9.1 (1997), pp. 47–61.

[6] Itzhak Gilboa and David Schmeidler. "Inductive Inference: An Axiomatic Approach". In: *Econometrica* 71.1 (2003), pp. 1–26.

[7] Chris Good and Ian Stares. "New proofs of classical insertion theorems". In: *Comment. Math. Univ. Carolinae* 41.1 (2000), pp. 139–142.

[8] Werner J. Hildenbrand. "On Economies with Many Agents". In: *Journal of Economic Theory* 2 (1970), pp. 161–188.

[9] Tatsuro Ichiishi. *Game theory for economic analysis*. Academic Press, Inc., 1983. ISBN: 0-12-370180-5.

[10] Roy A. Johnson. "A Compact Non-Metrizable Space Such That Every Closed Subset is a G-Delta". In: *The American Mathematical Monthly* 77.2 (1970), pp. 172–176. ISSN: 00029890, 19300972.

[11] John L. Kelley. *General topology*. Springer Science & Business Media, 1975.

[12] David M. Kreps. *Notes on the theory of choice*. Underground classics in economics. Westview Press, 1988. ISBN: 9780813375533.

[13] Vladimir L. Levin. "A Continuous Utility Theorem for Closed Preorders on a σ-Compact Metrizable Space". In: *Soviet Math. Doklady* 28 (1983), pp. 715–718.

[14] Andreu Mas-Colell. "On the Continuous Representation of Preorders". In: *International Economic Review* 18 (1977), pp. 509–513.

[15] Daniel L. McFadden. "Econometric analysis of qualitative response models. Handbook of Econometrics". In: vol. II. Elsevier Science Publishers BV, 1984. Chap. 24.

[16] Ernest Michael. "Continuous Selections I". In: *The Annals of Mathematics* 63(2) (1956), pp. 361–382.

[17] Paul Milgrom and Ilya Segal. "Envelope theorems for arbitrary choice sets". In: *Econometrica* 70.2 (2002), pp. 583–601.

[18] Patrick H. D. O'Callaghan. "Context and Decision: three approaches to representing preferences". In: *University of Warwick PhD Thesis* (2012).

[19] Raaj Kumar Sah and Jingang Zhao. "Some envelope theorems for integer and discrete choice variables". In: *International Economic Review* 39.3 (1998).

[20] Vladimir V. Tkachuk. *A Cp-Theory Problem Book*. Springer, 2011.